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Classification of three-state Hamiltonians solvable by Coordinate Bethe Ansatz

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Abstract

We classify 'all' Hamiltonians with rank 1 symmetry and nearest neighbour interactions, acting on a periodic three-state spin chain, and solvable through (generalization of) the coordinate Bethe ansatz (CBA). We obtain in this way four multi-parametric extensions of the known 19-vertex Hamiltonians (such as Zamolodchikov-Fateev, Izergin-Korepin, Bariev Hamiltonians). Apart from the 19-vertex Hamiltonians, there exists 17-vertex and 14-vertex Hamiltonians that cannot be viewed as subcases of the 19-vertex ones. In the case of 17-vertex Hamiltonian, we get a generalization of the genus 5 special branch found by Martins, plus three new ones. We get also two 14-vertex Hamiltonians.

We solve all these Hamiltonians using CBA, and provide their spectrum, eigenfunctions and Bethe equations. A special attention is made to provide the specifications of our multi-parametric Hamiltonians that give back known Hamiltonians.

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1 Introduction

In his seminal paper [1], Bethe provided a method, called now coordinate Bethe ansatz (CBA), to compute the eigenvalues and the eigenfunctions for the Heisenberg (or XXX 1/2-spin) models [2]. The same idea has been also used intensively in the context of the Bose gas with δ-interaction when the particles have no spin and with periodic boundary condition [3]. When they carry a spin [4, 5], the famous Yang-Baxter equation shows up through the same techniques, and actually it appeared for the first time in this context. When open boundaries are imposed, the procedure needs to be modified but still applies [6,7], even when the boundaries are not diagonal [8]. The Hubbard model [9] is another example where the CBA has been used successfully [10]. Remark also that, depending on the context, different generalizations of CBA have been considered, see e.g. [11–14].

However this method has been considered as deprecated in the eighties in favor of the quantum inverse scattering method (QISM) [15, 16]. This latter method is more algebraic and provides a full set of commuting Hamiltonians. It lies on the explicit numerical solution of the Yang-Baxter equation and on representations of the underlying algebra. Nevertheless, to the best of our knowledge, one does not know, in general, if the set of models solved by QISM and the ones solved by CBA are equivalent.

In this paper, we classify the most general Hamiltonians with nearest neighbourg interaction and acting on a three-state spin chain with rank 1 symmetry (i.e. the nineteen-vertex models) that can be solved by the CBA. We also compare our results with the classification of the solutions of the Yang-Baxter equation for the nineteen vertex model [17–19]. To solve this type of models by CBA, the historical method must be generalized following the lines of [11,20] where particular 19-vertex models[1] (the Izergin-Korepin and the Zamolodchikov-Fateev models) or of [13] where higher spin chain have been solved. This type of computation has been initiated in [21] but the huge algebraic equations the authors got did not allow them to provide a classification. Here, with the use of formal mathematical software, we succeed in obtaining a complete classification. We recover as subcases all the models discovered by solving the Yang-Baxter equation and, as an important by-product, we get the eigenvalues and eigenfunctions, which were not known previously, for the models obtained in [19] (see section 4.1). We also obtain four 17-vertex models, one of them being a generalization of the special branch found in [19], and two new 14-vertex models. These 17-vertex and 14-vertex Hamiltonians, are *not* subcases of 19-vertex Hamiltonians.

The paper is organized as follows. In section 2, we present the general Hamiltonian we want to solve and the symmetries one may consider. We give the outlines of the coordinate Bethe ansatz in section 3: we derive the Bethe ansatz equations (BAE) and determine sets of constraint equations to be satisfied by the parameters entering the Hamiltonian. Our results are collected in the proposition 3.1. In section 4, we provide the complete classification of the Hamiltonians we can solve by CBA and give in each case the eigenvalues and the eigenfunctions. Finally, in section 5, we present simplified versions of the Hamiltonians, including explicit 9×9 matricial expressions with physically relevant parameters, and, when possible, connect them with known models.

Associated to this classification, we constructed an interactive web page [22] that can test any 19-vertex Hamiltonian to determine if it is solvable by the coordinate Bethe ansatz. If so,

[1] See remark 4.2 for the name "19-vertex".

it provides also the connection with the models we present in this paper, as well as the physical data of the model.

2 General settings

2.1 Hamiltonian

We consider a $U(1)$-invariant Hamiltonian H acting on a spin chain of length L, where each site carries a $\mathbb{C}^3$ vector space (i.e. we deal with three-state models). We assume nearest neighbor interactions, that is

$$H = \sum_{j=1}^{L} H_{j,j+1}, \tag{2.1}$$

and periodic conditions, i.e. the site $L+1$ is identified with the site 1. As usual in such presentations the indices $(j, j+1)$ indicate where $H_{j,j+1}$ acts non trivially. The $U(1)$ generator corresponds to the S^z component of the total spin[2], $S^z = \sum_{j=1}^{L} s_j^z$. On each site, we choose as basis vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{with} \quad s^z|j\rangle = j\,|j\rangle. \tag{2.2}$$

Under these requirements, the most general two-site Hamiltonian takes the form

$$\begin{aligned} H_{12} &= \sum_{i_1,i_2,j_1,j_2\in\{0,1,2\}} h_{i_1\,i_2}^{j_1\,j_2}\, E_{i_1,j_1}\otimes E_{i_2,j_2} \\ &= pE_{01}\otimes E_{10} + qE_{10}\otimes E_{01} + t_1E_{21}\otimes E_{01} + s_1E_{12}\otimes E_{10} + t_2E_{01}\otimes E_{21} + s_2E_{10}\otimes E_{12} \\ &\quad + t_3E_{12}\otimes E_{21} + s_3E_{21}\otimes E_{12} + t_pE_{02}\otimes E_{20} + s_pE_{20}\otimes E_{02} + \sum_{i,j} v_{ij}E_{ii}\otimes E_{jj}, \end{aligned} \tag{2.3}$$

where E_{ij} denote the elementary 3×3 matrices with entry 1 in position (i,j) and zero elsewhere. In matricial form, it reads

$$H_{12} = \begin{pmatrix} v_{00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_{01} & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_{02} & 0 & t_2 & 0 & t_p & 0 & 0 \\ 0 & q & 0 & v_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_2 & 0 & v_{11} & 0 & s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & v_{12} & 0 & t_3 & 0 \\ 0 & 0 & s_p & 0 & t_1 & 0 & v_{20} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_3 & 0 & v_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_{22} \end{pmatrix}. \tag{2.4}$$

We aim at finding the most general Hamiltonian of the form (2.4) that is solvable by generalized coordinate Bethe ansatz (CBA). This will lead to an exhaustive classification of the possible sets of constraints on the parameters entering H, see section 4.

[2]Strictly speaking, the spin is $L - S^z$.

Before performing this calculation, we use the symmetries of the problem to keep only physically relevant parameters.

2.2 Symmetries and transformations

The Hamiltonian (2.4) exhibits some symmetries that allow us to simplify it.

- **Telescopic terms.** For any matrix A, let us consider the following transformation of the local Hamiltonian:
$$H'_{j,j+1} = H_{j,j+1} + A_j - A_{j+1}. \tag{2.5}$$
Then the periodicity condition implies that
$$H' = \sum_{j=1}^{L} H'_{j,j+1} = \sum_{j=1}^{L} H_{j,j+1} = H. \tag{2.6}$$
Demanding the $U(1)$ invariance to be preserved forces the matrix A to be diagonal: $A = \mathrm{diag}(a_1, a_2, a_3)$.
The transformation (2.5) for diagonal matrix, which involves only two independent parameters, say $a_1 - a_2$ and $a_1 - a_3$, leads to the following invariant combinations of the parameters:
$$\begin{aligned} &V = v_{01} + v_{10} - 2v_{00}\,, \quad X_{11} = v_{11} - v_{00} - V\,, \quad Y = v_{02} + v_{20} - 2v_{00} - 2V\,, \\ &X_{12} = v_{12} + v_{20} - v_{10} - v_{00} - 2V\,, \quad X_{21} = v_{21} + v_{02} - v_{01} - v_{00} - 2V\,, \\ &X_{22} = v_{22} - v_{00} - 2V. \end{aligned} \tag{2.7}$$
Note that this choice is not unique: in fact, the combinations above appear naturally when dealing with the coordinate Bethe ansatz, see section 3.

- As already mentioned, we are considering **U(1)-invariant Hamiltonians.** This implies in particular that the entries of H_{12} satisfy $h_{i_1\,i_2}^{j_1\,j_2} = 0$ if $i_1 + i_2 \neq j_1 + j_2$, as it can be checked from eq. (2.4). In order to get a symmetry of rank one only, one has to impose $(t_1, t_2, s_1, s_2) \neq (0, 0, 0, 0)$, condition that we assume to be satisfied in the whole paper. Indeed, the rank of the symmetry algebra determines the form of the CBA one should use. Hence, it is necessary to fix this rank. The only diagonal generators that commute with the Hamiltonian are then the identity matrix $\mathbb{I}$ and the S^z component of the total spin given in section 2.1. This property can be used to set the zero of the energy for example. A particular interesting choice is to consider $H_{12} - \frac{1}{2}V(s_1^z + s_2^z)$.
Of course, a further diagonal element can be removed from the Hamiltonian using the identity. In the following we choose $v_{00} = 0$.

One can then consider the following transformations:

- **Parity transformation (P):** $h_{i_1\,i_2}^{j_1\,j_2} \to h_{i_2\,i_1}^{j_2\,j_1}$ (that is $H_{12} \to H_{21}$), which corresponds to the following correspondence between the parameters (X_{11}, Y, X_{22} are invariant):
$$p \leftrightarrow q, \quad t_1 \leftrightarrow t_2, \quad s_1 \leftrightarrow s_2, \quad t_3 \leftrightarrow s_3, \quad t_p \leftrightarrow s_p, \quad X_{12} \leftrightarrow X_{21} \tag{2.8}$$

The Hamiltonians H_{12} and H_{21} lead to systems where the chain is oriented from right to left instead of left to right. Therefore, the set of solutions that lead to solvable Hamiltonian has therefore to be invariant under the correspondence (2.8).

- **Time reversal (T):** $h_{i_1\,i_2}^{j_1\,j_2} \to h_{j_1\,j_2}^{i_1\,i_2}$ (that is $H_{12} \to H_{12}^t$), which corresponds to the following correspondence between the parameters (all diagonal terms are invariant):

$$p \leftrightarrow q, \quad t_1 \leftrightarrow s_1, \quad t_2 \leftrightarrow s_2, \quad t_3 \leftrightarrow s_3, \quad t_p \leftrightarrow s_p \tag{2.9}$$

- **Charge conjugation (C):** $h_{i_1\,i_2}^{j_1\,j_2} \to h_{2-i_1\,2-i_2}^{2-j_1\,2-j_2}$ (i.e. indices 0 and 2 are exchanged and index 1 is invariant), which corresponds to the following correspondence between the parameters:

$$\begin{aligned}
&p \leftrightarrow s_3, \quad q \leftrightarrow t_3, \quad t_1 \leftrightarrow t_2, \quad s_1 \leftrightarrow s_2, \quad t_p \leftrightarrow s_p,\\
&V \leftrightarrow -V - Y - 2X_{22} + X_{12} + X_{21}, \quad X_{11} \leftrightarrow X_{11} + Y + X_{22} - X_{12} - X_{21},\\
&Y + X_{22} \leftrightarrow 5(Y + X_{22}) - 4(X_{12} + X_{21}), \quad Y - X_{22} \leftrightarrow Y - X_{22},\\
&X_{12} + X_{21} \leftrightarrow 6(Y + X_{22}) - 5(X_{12} + X_{21}), \quad X_{12} - X_{21} \leftrightarrow X_{21} - X_{12}
\end{aligned} \tag{2.10}$$

The action of the charge conjugation is equivalent to choose as pseudo-vacuum $|\widetilde{\Omega}\rangle = \bigotimes_{i=1}^{L} |2\rangle$ instead of $|\Omega\rangle = \bigotimes_{i=1}^{L} |0\rangle$, exchanging the roles of the vectors $|0\rangle$ and $|2\rangle$. The solution to the problem obtained thanks to the coordinate Bethe ansatz can be reproduced *mutatis mutandis*, but taking into account the correspondence (2.10). We will use this property in section 4.3.

Action of these three transformations on solvable Hamiltonians is displayed in table 1, see appendix A. In the following we will work modulo these transformations.

3 Coordinate Bethe ansatz

We construct Hamiltonian eigenvectors using a generalization of the original coordinate Bethe ansatz, following the techniques developed in [11, 13].

3.1 Results

Since the S^z component of the total spin commutes with the Hamiltonian, one can decompose the space of states $\mathcal{H}$ into subspaces with fixed S^z-eigenvalue

$$\mathcal{H} = \bigoplus_{M=0}^{2L} \mathcal{V}_M\,, \qquad S^z\,\varphi_M = M\,\varphi_M\,, \quad \forall \varphi_M \in \mathcal{V}_M,$$

and look for eigenvectors of H in a given subspace $\mathcal{V}_M$.

For $M = 0$, we have a one-dimensional subspace corresponding to a particular eigenvector of the Hamiltonian, called the pseudo-vacuum, defined here as $|\Omega\rangle = \bigotimes_{i=1}^{L} |0\rangle$. It is easy to see that, since we made the choice $v_{00} = 0$, $|\Omega\rangle$ is an eigenvector of H with eigenvalue zero.

Then, in $\mathcal{V}_M$, one considers states with M pseudo-excitations obtained by acting with the raising operator on the pseudo-vacuum. More precisely, an elementary state with M pseudo-excitations is given by

$$|x_1,\dots,x_M\rangle = \underbrace{|0\rangle\otimes\cdots\otimes|0\rangle}_{x_1-1}\otimes|m_1\rangle\otimes\underbrace{|0\rangle\otimes\cdots\otimes|0\rangle}_{x_{m_1+1}-x_{m_1}-1}\otimes|m_2\rangle\otimes\underbrace{|0\rangle\otimes\cdots\otimes|0\rangle}_{x_{m_1+m_2+1}-x_{m_1+m_2}-1}\otimes|m_3\rangle\otimes\dots \tag{3.1}$$

where $1\le x_1\le x_2\le ...\le x_M\le L$.

The x_j's are the locations of the pseudo-excitations along the chain, and $m_k\in\{1,2\}$ such that $\sum m_k = M$. For $j = 1+m_1+\cdots+m_{k-1}$, one has $m_k = 2$ if $x_{j+1} = x_j$ and $m_k = 1$ otherwise.

These states form a basis of the subspace $\mathcal{V}_M$ of states with a given number M of pseudo-excitations.

An eigenstate Ψ_M for the Hamiltonian in a given sector with M pseudo-excitations is obtained as a linear combination of the elementary states (3.1) with coefficients $a(x_1,\dots,x_M)$, which are complex-valued functions to be determined:

$$\Psi_M = \sum_{1\le x_1\le\cdots\le x_M\le L} a(x_1,\dots,x_M)|x_1,\dots,x_M\rangle. \tag{3.2}$$

We assume a plane wave decomposition for the functions $a(x_1,\dots,x_M)$:

$$a(x_1,\dots,x_M) = \sum_{\sigma\in\mathfrak{S}_M} A_\sigma^{(j_1,\dots,j_P)}(k_1,\dots,k_M)\exp\left(\sum_{n=1}^M ik_{\sigma(n)}x_n\right) = \sum_{\sigma\in\mathfrak{S}_M} A_\sigma^{(j_1,\dots,j_P)}(\vec{k})e^{i\vec{k_\sigma}\cdot\vec{x}}. \tag{3.3}$$

Here $\mathfrak{S}_M$ is the permutation group of M elements and $A_\sigma^{(j_1,\dots,j_P)}(k_1,\dots,k_M)$ are functions on the symmetric group algebra depending on some parameters k to be determined later (these are solutions of the so-called Bethe ansatz equations). The indices $(j_1,\dots,j_P)$ correspond to double excitations, i.e. indices such that $x_{j_k+1} = x_{j_k}$ for $k = 1,\dots,P$. When there are no double excitations, the indices $(j_1,\dots,j_P)$ are of course omitted.

Proposition 3.1 *The Hamiltonian H given in* (2.4) *is solvable by CBA provided its parameters obey the constraints given in* (3.18), (3.21) *and* (3.24). *The complete set of solutions to these equations is given in section 4, modulo the symmetries mentioned in section 2.2 and displayed in table 1.*

For CBA-solvable Hamiltonians H, the state (3.2) *is an eigenvector of H, with energy*

$$E_M = MV + \sum_{n=1}^M (q\,e^{ik_n} + p\,e^{-ik_n}) \tag{3.4}$$

provided the coefficients A_σ *and* $A_\sigma^{(j_1,...,j_P)}$, $P = 1, ..., \left[\frac{M}{2}\right]$, *are related by*

$$\frac{A_{\sigma T_j}(\vec{k})}{A_\sigma(\vec{k})} \equiv S(k_{\sigma(j)}, k_{\sigma(j+1)}) = -\frac{\Lambda(k_{\sigma(j)}, k_{\sigma(j+1)})}{\Lambda(k_{\sigma(j+1)}, k_{\sigma(j)})}\,, \tag{3.5}$$

$$\frac{A_\sigma^{(j)}(\vec{k})}{A_\sigma(\vec{k})} \equiv N(k_{\sigma(j)}, k_{\sigma(j+1)}) = \frac{(e^{ik_{\sigma(j)}} - e^{ik_{\sigma(j+1)}})(p + qe^{ik_{\sigma(j)}+ik_{\sigma(j+1)}})(t_2 + t_1 e^{ik_{\sigma(j)}+ik_{\sigma(j+1)}})}{2\,\Lambda(k_{\sigma(j+1)}, k_{\sigma(j)})}, \tag{3.6}$$

$$\frac{A_\sigma^{(j_1,...,j_P)}(\vec{k})}{A_\sigma(\vec{k})} = \prod_{n=1}^{P} N(k_{\sigma(j_n)}, k_{\sigma(j_n+1)}), \tag{3.7}$$

where $T_j \in \mathfrak{S}_M$ *denotes the transposition* $(j, j+1)$ *and*

$$\Lambda(k_j, k_n) = e^{ik_n}(s_1 + s_2 e^{ik_j+ik_n})(t_2 + t_1 e^{ik_j+ik_n}) \tag{3.8}$$
$$- \Big(Y e^{ik_j+ik_n} - q e^{ik_j+ik_n}(e^{ik_j} + e^{ik_n}) - p(e^{ik_j} + e^{ik_n}) + s_p e^{2ik_j+2ik_n} + t_p\Big)\big(X_{11} e^{ik_n} - q e^{ik_j+ik_n} - p\big).$$

The momenta k_j *must also obey the Bethe equations*

$$e^{ik_j L} = \prod_{n\neq j} S(k_n, k_j)\,, \qquad j = 1, ..., M. \tag{3.9}$$

Remark that when $p = q = 0$, the energy depends only on the number of pseudo-excitations. In this case, one needs to consider another vacuum to get a complete spectrum, see section 4.3. When p and q are both non vanishing, the energy can be rewritten as

$$E_M = MV + \sqrt{pq}\sum_{n=1}^{M}(z_n\sqrt{\theta} + \frac{1}{z_n\sqrt{\theta}}) \quad \text{where} \quad \theta = q/p \quad \text{and} \quad z_n = e^{ik_n}. \tag{3.10}$$

In this case, after eliminating the constant term MV thanks to the S_z operator and rescaling of the Hamiltonian by $\sqrt{pq}$, the energy clearly depends only on the parameter θ (and those of the S-matrix through the Bethe equations).

3.2 Proofs

Since H is a sum of two-site operators $H_{j,j+1}$, one has to single out only the following configurations:

1. **Configurations leading to the determination of eigenvalues and eigenvectors:**
   - the x_j's are far from each other ("generic case"),
   - $x_{j+1} = x_j + 1$ for some j and the other x_n's are far from each other,
   - $x_{j+1} = x_j$ for some j and the other x_n's are far from each other,
   - $x_{j_k+1} = x_{j_k}$, $x_{j_k} + 1 < x_{j_k+2}$ for $k = 1, ..., P$, and the other x_n's are far from each other,
2. **Configurations leading to constraints on the parameters of the models:**
   - $x_{j+1} = x_j$ and $x_{j+2} = x_j + 1$ for a given j , the other x_n's are far from each other,
   - $x_{j+1} = x_j$ and $x_{j-1} = x_j - 1$ for a given j, the other x_n's are far from each other,
   - $x_{j-1} = x_j$ and $x_{j+1} = x_{j+2} = x_j + 1$, the other x_n's are far from each other.

3. **Configurations leading to Bethe equations and/or already known eqs:**
   - $x_1 = 1$ and the other x_n's are far from each other,
   - $x_M = L$ and the other x_n's are far from each other,
   - $x_1 = 1$, $x_M = L$ and the other x_n's are far from each other,
   - $x_1 = x_2 = 1$ (or equivalently $x_{M-1} = x_M = L$) and the other x_n's are far from each other,
   - $x_1 = x_2 = 1$ and $x_M = L$ (or equivalently $x_1 = 1$ and $x_{M-1} = x_M = L$), and the other x_n's are far from each other,
   - $x_1 = x_2 = 1$ and $x_{M-1} = x_M = L$, and the other x_n's are far from each other.

3.2.1 Configurations leading to energy and eigenstates

• **Configuration where the x_j's are generic**, i.e. are far from each other and from the edges: $1 < x_1 \ll ... \ll x_n \ll x_{n+1} \ll ... \ll x_M < L$. Projecting the Schrödinger equation on it, we get

$$\sum_{\sigma\in\mathfrak{S}_M} A_\sigma(\vec{k}) e^{i\vec{k_\sigma}\cdot\vec{x}} \left(MV + \sum_{n=1}^{M} (qe^{ik_{\sigma(n)}} + pe^{-ik_{\sigma(n)}}) \right) = E_M \sum_{\sigma\in\mathfrak{S}_M} A_\sigma(\vec{k}) e^{i\vec{k_\sigma}\cdot\vec{x}} \tag{3.11}$$

which leads to the value (3.4) of the energy of the state Ψ_M.

• **Configuration where $x_{j+1} = x_j + 1$ for a given j (not on the edges)**, the other x_n's being far from each other and from the edges. Then one gets

$$\sum_{\sigma\in\mathfrak{S}_M} e^{i\vec{k_\sigma}\cdot\vec{x}} \left(A_\sigma(\vec{k}) \left(X_{11} - qe^{ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} \right) + A^{(j)}_\sigma(\vec{k}) \left(s_2 e^{ik_{\sigma(j)}} + s_1 e^{-ik_{\sigma(j+1)}} \right) \right) = 0 \tag{3.12}$$

Note that, since $x_{j+1} = x_j + 1$, one has here

$$e^{i\vec{k_\sigma}\cdot\vec{x}} = e^{ik_{\sigma(j+1)}} \exp\left(i(k_{\sigma(j)} + k_{\sigma(j+1)})x_j + \sum_{n\neq j,j+1} ik_{\sigma(n)}x_n \right),$$

which implies a symmetrization in the exchange $j \leftrightarrow j+1$ before projecting onto the independent states (3.1). Hence one gets, where $T_j \in \mathfrak{S}_M$ denotes the transposition $(j, j+1)$,

$$\begin{aligned} &A_\sigma(\vec{k}) e^{ik_{\sigma(j+1)}} \left(X_{11} - qe^{ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} \right) + A_{\sigma T_j}(\vec{k}) e^{ik_{\sigma(j)}} \left(X_{11} - qe^{ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j)}} \right) + \\ &\left(A^{(j)}_{\sigma T_j}(\vec{k}) + A^{(j)}_\sigma(\vec{k}) \right) \left(s_2 e^{ik_{\sigma(j+1)} + ik_{\sigma(j)}} + s_1 \right) = 0. \end{aligned} \tag{3.13}$$

• **Configuration where $x_{j+1} = x_j$ for a given j (not on the edges)**, the other x_n's being far from each other and from the edges. Then one has

$$\begin{aligned} \sum_{\sigma\in\mathfrak{S}_M} e^{i\vec{k_\sigma}\cdot\vec{x}} \Big(&A^{(j)}_\sigma(\vec{k}) \big(Y - qe^{ik_{\sigma(j)}} - qe^{ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} \\ &+ s_p e^{ik_{\sigma(j)} + ik_{\sigma(j+1)}} + t_p e^{-ik_{\sigma(j)} - ik_{\sigma(j+1)}} \big) + A_\sigma(\vec{k}) \big(t_1 e^{ik_{\sigma(j+1)}} + t_2 e^{-ik_{\sigma(j)}} \big) \Big) = 0. \end{aligned} \tag{3.14}$$

After symmetrization in $(j, j+1)$ as above, one obtains

$$\big(A^{(j)}_\sigma(\vec{k}) + A^{(j)}_{\sigma T_j}(\vec{k})\big)\big(Y - qe^{ik_{\sigma(j)}} - qe^{ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} + s_p e^{ik_{\sigma(j)}+ik_{\sigma(j+1)}} + t_p e^{-ik_{\sigma(j)}-ik_{\sigma(j+1)}}\big) + A_\sigma(\vec{k})\big(t_1 e^{ik_{\sigma(j+1)}} + t_2 e^{-ik_{\sigma(j)}}\big) + A_{\sigma T_j}(\vec{k})\big(t_1 e^{ik_{\sigma(j)}} + t_2 e^{-ik_{\sigma(j+1)}}\big) = 0. \tag{3.15}$$

Without any loss of generality, we choose[3] to impose $A^{(j)}_{\sigma T_j}(\vec{k}) = A^{(j)}_\sigma(\vec{k})$. Then, using eqs. (3.13) and (3.15), one gets relations (3.5) and (3.6).

• **Configuration where $x_{j_k+1} = x_{j_k}$ for $k = 1, ..., P$** and the other x_n's being far from each other and from the edges. One gets

$$\sum_{\sigma\in\mathfrak{S}_M} e^{i\vec{k_\sigma}\cdot\vec{x}}\left\{A^{(j_1,...,j_P)}_\sigma(\vec{k})\Big(PY + \sum_{k=1}^{P} t_p e^{-ik_{\sigma(j_k)}-ik_{\sigma(j_k+1)}} - pe^{-ik_{\sigma(j_k)}} - pe^{-ik_{\sigma(j_k+1)}} + s_p e^{ik_{\sigma(j_k)}+ik_{\sigma(j_k+1)}} - qe^{ik_{\sigma(j_k)}} - qe^{ik_{\sigma(j_k+1)}}\Big) + \sum_{k=1}^{P} A^{(j_1\dots\check{j}_k\dots j_P)}_\sigma(\vec{k})(t_2 e^{-ik_{\sigma(j_k)}} + t_1 e^{ik_{\sigma(j_k+1)}})\right\} = 0 \tag{3.16}$$

where $A^{(j_1\dots\check{j}_k\dots j_P)}_\sigma(\vec{k})$ means that $x_{j_n+1} = x_{j_n}$ for $n = 1, ..., P$ and $n \neq k$.
Morevover, after projection onto the states (3.1), one needs to symmetrize (independently) on each pair $(j_n, j_n + 1)$. One is led to a recursion relation linking $A^{(j_1,...,j_P)}_\sigma(\vec{k})$ and $A^{(j_1,...,j_{P-1})}_\sigma(\vec{k})$ that can be solved, and one gets (3.7).

3.2.2 Configurations leading to constraints on parameters

• **Configuration where $x_{j+1} = x_j$ and $x_{j+2} = x_j + 1$ for a given j (not on the edges)**, the other x_n's being far from each other and from the edges. One obtains

$$\sum_{\sigma\in\mathfrak{S}_M} e^{i\vec{k_\sigma}\cdot\vec{x}}\Big(A^{(j)}_\sigma(\vec{k})\big(X_{21} - qe^{ik_{\sigma(j)}} - qe^{ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j+2)}} + t_p e^{-ik_{\sigma(j)}-ik_{\sigma(j+1)}}\big) + A^{(j+1)}_\sigma(\vec{k}) s_3 e^{ik_{\sigma(j+1)}} + A_\sigma(\vec{k}) t_2 e^{-ik_{\sigma(j)}}\Big) = 0. \tag{3.17}$$

Here one has $e^{i\vec{k_\sigma}\cdot\vec{x}} = e^{ik_{\sigma(j+2)}} \exp\left(\sum_{n\neq j,j+1,j+2} ik_{\sigma(n)}x_n + \sum_{n=j,j+1,j+2} ik_{\sigma(n)}x_j\right)$ given the configuration. Therefore, projecting onto the states (3.1), it is now necessary to symmetrize on $(j, j+1, j+2)$. Taking into account the relations (3.5) and (3.6) that allow one to express all A functions in terms of $A_\sigma(\vec{k})$ only, one gets now

$$\sum_{\sigma\in\mathfrak{S}_3} \mathcal{E}_{21}(k_{\sigma(j)}, k_{\sigma(j+1)}, k_{\sigma(j+2)}) = 0, \tag{3.18}$$

[3]Another possible choice [21] is to impose $A^{(j)}_{\sigma T_j}(\vec{k}) = S(k_{\sigma(j)}, k_{\sigma(j+1)})\, A^{(j)}_\sigma(\vec{k})$. One goes from one choice to another through the renormalisation $A^{(j)}_\sigma(\vec{k}) \;\to\; (k_{\sigma(j)} - k_{\sigma(j+1)})\Lambda(k_{\sigma(j)}, k_{\sigma(j+1)})\, A^{(j)}_\sigma(\vec{k})$.

where

$$\mathcal{E}_{21}(k_{\sigma(j)},k_{\sigma(j+1)},k_{\sigma(j+2)}) = A_\sigma(\vec{k})\, e^{ik_{\sigma(j+2)}}\Big(N(k_{\sigma(j)},k_{\sigma(j+1)})\big(X_{21} - qe^{ik_{\sigma(j)}} - qe^{ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j+2)}} + t_p e^{-ik_{\sigma(j)}-ik_{\sigma(j+1)}}\big) + N(k_{\sigma(j+1)},k_{\sigma(j+2)})\, s_3 e^{ik_{\sigma(j+1)}} + t_2 e^{-ik_{\sigma(j)}}\Big). \tag{3.19}$$

Then projecting the above constraint onto the monomials in the variables $e^{ik_{\sigma(\ell)}}$, $\ell = j,\, j+1,\, j+2$, one gets a first set of 32 constraint equations. For sake of simplicity, we avoid writing these equations here.

• **Configuration where $x_{j+1} = x_j$ and $x_{j-1} = x_j - 1$ for a given j (not on the edges)**, the other x_n's being far from each other and from the edges. In the same way, when one considers such a configuration, one obtains

$$\sum_{\sigma\in\mathfrak{S}_M} e^{i\vec{k_\sigma}\cdot\vec{x}}\Big(A^{(j)}_\sigma(\vec{k})\big(X_{12} - qe^{ik_{\sigma(j-1)}} - qe^{ik_{\sigma(j)}} - qe^{ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} + s_p e^{ik_{\sigma(j)}+ik_{\sigma(j+1)}}\big) + A^{(j-1)}_\sigma(\vec{k})\, t_3 e^{-ik_{\sigma(j)}} + A_\sigma(\vec{k})\, t_1 e^{ik_{\sigma(j+1)}}\Big) = 0. \tag{3.20}$$

Again, after projection onto the states (3.1), one is left to symmetrize on $(j-1, j, j+1)$,

$$\sum_{\sigma\in\mathfrak{S}_3} \mathcal{E}_{12}(k_{\sigma(j-1)},k_{\sigma(j)},k_{\sigma(j+1)}) = 0 \tag{3.21}$$

where the expression $\mathcal{E}_{12}$ is given by

$$\mathcal{E}_{12}(k_{\sigma(j-1)},k_{\sigma(j)},k_{\sigma(j+1)}) = e^{-ik_{\sigma(j)}}\Big(N(k_{\sigma(j)},k_{\sigma(j+1)})\big(X_{12} - qe^{ik_{\sigma(j-1)}} - qe^{ik_{\sigma(j)}} - qe^{ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} + s_p e^{ik_{\sigma(j)}+ik_{\sigma(j+1)}}\big) + N(k_{\sigma(j-1)},k_{\sigma(j)})t_3 e^{-ik_{\sigma(j)}} + t_1 e^{ik_{\sigma(j+1)}}\Big). \tag{3.22}$$

The projection of the constraint equation onto the monomials in the variables $e^{ik_{\sigma(\ell)}}$, $\ell = j-1,\, j,\, j+1$ leads to a second set of 32 constraint equations.

• **Configuration where $x_{j-1} = x_j$ and $x_{j+1} = x_{j+2} = x_j + 1$**, the other x_n's being far from each other and from the edges. One gets

$$\sum_{\sigma\in\mathfrak{S}_M} e^{i\vec{k_\sigma}\cdot\vec{x}}\Big(A^{(j-1,j+1)}_\sigma(\vec{k})\big(X_{22} + t_p e^{-ik_{\sigma(j-1)}-ik_{\sigma(j)}} - qe^{ik_{\sigma(j-1)}} - qe^{ik_{\sigma(j)}} - qe^{ik_{\sigma(j+1)}} - qe^{ik_{\sigma(j+2)}} + s_p e^{ik_{\sigma(j+1)}+ik_{\sigma(j+2)}} - pe^{-ik_{\sigma(j-1)}} - pe^{-ik_{\sigma(j)}} - pe^{-ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j+2)}}\big) + A^{(j+1)}_\sigma(\vec{k})\, t_2 e^{-ik_{\sigma(j-1)}} + A^{(j-1)}_\sigma(\vec{k})\, t_1 e^{ik_{\sigma(j+2)}}\Big) = 0. \tag{3.23}$$

Since now $e^{i\vec{k_\sigma}\cdot\vec{x}} = e^{ik_{\sigma(j+1)}+ik_{\sigma(j+2)}} \exp\Big(\sum_{n\neq j-1,j,j+1,j+2} ik_{\sigma(n)}x_n + \sum_{n=j-1,j,j+1,j+2} ik_{\sigma(n)}x_j\Big)$, one symmetrizes on $(j-1, j, j+1, j+2)$, and gets

$$\sum_{\sigma\in\mathfrak{S}_4} \mathcal{E}_{22}(k_{\sigma(j-1)},k_{\sigma(j)},k_{\sigma(j+1)},k_{\sigma(j+2)}) = 0, \tag{3.24}$$

where

$$\mathcal{E}_{22} = A_\sigma(\vec{k})\, e^{ik_{\sigma(j+1)}+ik_{\sigma(j+2)}}\Big(N(k_{\sigma(j-1)},k_{\sigma(j)})N(k_{\sigma(j+1)},k_{\sigma(j+2)})\big(X_{22} + t_p e^{-ik_{\sigma(j-1)}-ik_{\sigma(j)}} \\ - qe^{ik_{\sigma(j-1)}} - qe^{ik_{\sigma(j)}} - qe^{ik_{\sigma(j+1)}} - qe^{ik_{\sigma(j+2)}} + s_p e^{ik_{\sigma(j+1)}+ik_{\sigma(j+2)}} - pe^{-ik_{\sigma(j-1)}} - pe^{-ik_{\sigma(j)}} \\ - pe^{-ik_{\sigma(j+1)}} - pe^{-ik_{\sigma(j+2)}}\big) + N(k_{\sigma(j+1)},k_{\sigma(j+2)})\, t_2 e^{-ik_{\sigma(j-1)}} + N(k_{\sigma(j-1)},k_{\sigma(j)})\, t_1 e^{ik_{\sigma(j+2)}}\Big). \tag{3.25}$$

The projection of the constraint equation onto the monomials in the variables $e^{ik_{\sigma(\ell)}}$, $\ell = j-1, j, j+1, j+2$ finally leads to a third set of constraint equations.

The solutions to the sets of equations (3.18), (3.21) and (3.24) give necessary conditions to be satisfied among the parameters defining the two-site Hamiltonian (2.4) to ensure the solvability of the chain. This leads to a classification of three-state integrable models as shown in the next section.

3.2.3 Configurations leading to the Bethe equations

We now concentrate on configurations with at least one excitation lying on the chain edges 1 and/or L. Using the periodicity condition of the chain, this will allow us to derive the equations that determine the admissible values of the parameters k entering into the definition of the plane wave (3.3), i.e. the Bethe ansatz equations.

• **Configuration where** $x_1 = 1$ and the other x_n's are far from each other and from the edges: $1 = x_1 \ll ... \ll x_n \ll x_{n+1} \ll ... \ll x_M < L$. Then one gets provided that $p \neq 0$,

$$\sum_{\sigma\in\mathfrak{S}_M} A_\sigma(\vec{k}) \left(\exp\Big(\sum_{n=2}^{M} ik_{\sigma(n)}x_n\Big) - \exp\Big(ik_{\sigma(M)}L + \sum_{n=2}^{M} ik_{\sigma(n-1)}x_n\Big)\right) = 0. \tag{3.26}$$

Performing the transformation $\sigma \to \sigma T_1 \dots T_{M-1}$ in the second term, one gets $A_{\sigma T_1 \dots T_{M-1}}(\vec{k}) = e^{-ik_{\sigma(1)}L} A_\sigma(\vec{k})$. Taking into account (3.5), we finally obtain the Bethe ansatz equations (3.9).

In the same way, one can consider a configuration where $x_M = L$ and the other x_n's are far from each other and from the edges: $1 < x_1 \ll ... \ll x_n \ll x_{n+1} \ll ... \ll x_M = L$. Then one gets provided that $q \neq 0$,

$$\sum_{\sigma\in\mathfrak{S}_M} A_\sigma(\vec{k}) \left(\exp\Big(ik_{\sigma(M)}(L+1) + \sum_{n=1}^{M-1} ik_{\sigma(n)}x_n\Big) - \exp\Big(ik_{\sigma(1)} + \sum_{n=1}^{M-1} ik_{\sigma(n+1)}x_n\Big)\right) = 0. \tag{3.27}$$

Now, performing the transformation $\sigma \to \sigma T_{M-1} \dots T_1$ in the second term, one gets

$$A_{\sigma T_{M-1}\dots T_1}(\vec{k}) = e^{ik_{\sigma(M)}L} A_\sigma(\vec{k}),$$

which leads also to equation (3.9).

Note that since we excluded the values $p = q = 0$, the BAE (3.9) holds in any case.

• **Configuration where $x_1 = 1$, $x_M = L$ and the other x_n's are far from each other:** $1 = x_1 \ll ... \ll x_n \ll x_{n+1} \ll ... \ll x_M = L$. One obtains

$$\sum_{\sigma\in\mathfrak{S}_M} A_\sigma(\vec{k}) \exp\Big(ik_{\sigma(1)} + ik_{\sigma(M)}L + \sum_{n=2}^{M-1} ik_{\sigma(n)}x_n\Big)\big(X_{11} - qe^{ik_{\sigma(M)}} - pe^{-ik_{\sigma(1)}}\big)$$
$$+ A_\sigma^{(M-1)}(\vec{k}) \exp\Big(ik_{\sigma(M-1)}L + ik_{\sigma(M)}L + \sum_{n=2}^{M-1} ik_{\sigma(n-1)}x_n\Big)$$
$$+ A_\sigma^{(1)}(\vec{k}) \exp\Big(ik_{\sigma(1)} + ik_{\sigma(2)} + \sum_{n=2}^{M-1} ik_{\sigma(n+1)}x_n\Big) = 0. \tag{3.28}$$

One then performs the transformations $\sigma \to \sigma T_1 \dots T_{M-1}$ (second term) and $\sigma \to \sigma T_{M-1} \dots T_1$ (third term) and uses the relations (3.5) and (3.6). After the necessary symmetrization on the pair $(1, M)$ and projection onto the states (3.1), one is left with an equation expressed in terms of $A_\sigma(\vec{k})$ only, the S-matrix and the decay coefficient N. Plugging the BAE (3.9) into the obtained equation, it appears that no further condition is required.

• **Other "edge configurations".** They correspond to the following cases:

(i) $x_1 = x_2 = 1$ (or equivalently $x_{M-1} = x_M = L$),

(ii) $x_1 = x_2 = 1$ and $x_M = L$ (or equivalently $x_1 = 1$ and $x_{M-1} = x_M = L$),

(iii) $x_1 = x_2 = 1$ and $x_{M-1} = x_M = L$, and the other x_n's are far from each other.

The approach is similar to the previous case. After obtaining the Schrödinger equation for the particular configuration under consideration using the periodicity conditions, one performs the suitable transformations on the permutations and write all functions $A(\vec{k})$ in terms of the running $A_\sigma(\vec{k})$ only, products of S-matrices and decay coefficients N. If necessary, symmetrize on the indices which are left after the projection on the elementary states (3.1). In each case, plugging the BAE (3.9) into the equation that is finally obtained leads to some constraint equation belonging to one of the sets (3.18), (3.21) or (3.24). No further condition is eventually needed.

4 Solutions of the constraint equations

In this section, we present all the non trivial solutions to equations (3.18), (3.21) and (3.24), described in section 3. It provides a classification of three-state models solvable by CBA. We used a formal calculation software to solve completely these equations, and found 4 nineteen-vertex, 4 seventeen-vertex and 2 fourteen-vertex models, up to the transformations under parity, time reversal and charge conjugation, see section 2.2. If one includes the images of the irreducible solutions under these transformations, one gets 22 solutions, see table 1.

Remark 4.1 Of course, when directly solving the equations (3.18), (3.21) and (3.24), one finds many more solutions, but most of them are subcases of these 10 "irreducible" solutions. We developed a sofware that, starting from any given Halmiltonian of the form (2.4), can analyze

whether the Hamiltonian is solvable by CBA, and, if so, to which one of the 10 irreducible solutions it corresponds. This program is freely accessible on our web page [22]. Let us stress that the correspondence may be sensitive to the choice of free parameters that is used. This is taken into account by the software. However, in this article we made (arbitrarily) one specific choice. The other ones are found through the image under parity, charge conjugation and/or time reversal transformations of the choice we present here. We illustrate this in a particular case, see section 4.1.4.

In the following, we classify the models that have $(t_1, t_2, s_1, s_2) \neq (0, 0, 0, 0)$ *and* $(p, q, t_3, s_3) \neq (0, 0, 0, 0)$*. Because we work modulo P, C, T transformation, it is enough to present the solutions with* $(p, q) \neq (0, 0)$ *and* $(t_1, t_2) \neq (0, 0)$*:*
(i) since we are considering $U(1)$ invariant models, to get a symmetry of rank 1 only (not rank 2), we are led to $(t_1, t_2, s_1, s_2) \neq (0, 0, 0, 0)$. Now suppose that we get a solution with $(t_1, t_2) = (0, 0)$. Then, this solution has $(s_1, s_2) \neq (0, 0)$. But the image of this solution under time reversal is also a solution and has $(t_1, t_2) \neq (0, 0)$ and $(s_1, s_2) = (0, 0)$: since we are working modulo this transformation, we can choose to present solutions with $(t_1, t_2) \neq (0, 0)$;
(ii) to be able to construct the CBA on the vacuum $|\Omega\rangle$ or $|\tilde{\Omega}\rangle$, one needs to have $(p, q, t_3, s_3) \neq (0, 0, 0, 0)$. Now, since charge conjugation (2.10) exchanges (p, q) and (t_3, s_3), we can suppose $(p, q) \neq (0, 0)$.

These requirements exclude all the cases obtained in [17], but models 7 and 10: the remaining ones are models with rank 2 symmetry, or diagonal Hamiltonians, or not solvable through CBA. They exclude also the model based on Temperley-Lieb algebra [23], for which another type of CBA is needed [12].

We introduce the following reduced parameters:

$$\tau_p = t_p/p\,, \quad \tau_2 = t_2/p\,, \quad \tau_3 = t_3/p\,, \quad \theta = q/p\,, \quad \Upsilon = Y/p\,, \quad \sigma = s_1 t_2/p^2\,, \quad \mu = t_1/t_2. \tag{4.1}$$

These reduced parameters are the only ones that are part of the physical data of the models: scattering matrix S, decay coefficient N, energy E and BAEs. Hence the other ones can be eliminated from the model through gauge transformations and/or telescoping terms, as it is done in section 5. We chose to present here our "raw" Hamiltonians to be easily compared with any given Hamiltonian.

These "raw" Hamiltonians are defined whatever the values of the free parameters, provided they lead to well-defined expressions for the other parameters. The reduced parameters are valid for generic values of the free parameters and can be ill-defined for some specific values, see remark 5.1 in section 5.

We define J as one solution of the equation $J^2 + J + 1 = 0$.

4.1 Nineteen vertices

In this subsection, we focus on the solutions for which all off-diagonal parameters entering in the Hamiltonian are non zero.

Remark 4.2 We will call the corresponding Hamiltonian a "19-vertex" one, since we get 19 non-vanishing entries for H_{12} when adding the 9 diagonal parameters to the 10 off-diagonal ones.

Note however that one can always cancel some of the diagonal entries using the symmetries as discussed in section 2.2.

We would like to stress also that this name 19-vertex is not related to the terminology used for R-matrix formalism.

4.1.1 Generalized Zamolodchikov-Fateev model (gZF)

The parameters which are left free are p, t_p, t_2, s_1. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$q = s_3 = \frac{p^3}{t_p^2}\,, \quad t_1 = \frac{p^2 t_2}{t_p^2}\,, \quad t_3 = p\,, \quad s_2 = \frac{p^2 s_1}{t_p^2}\,, \quad s_p = \frac{p^4}{t_p^3}, \tag{4.2}$$

while on the diagonal we get:

$$X_{11} = 0\,, \quad Y = \frac{2p^2}{t_p}\,, \quad X_{12} = X_{21} = \frac{3p^2 - s_1 t_2}{t_p}\,, \quad X_{22} = \frac{4p^2 - 2s_1 t_2}{t_p}. \tag{4.3}$$

The S-matrix depends only on the reduced parameters τ_p and σ:

$$S(z_1, z_2) = -\frac{z_1 z_2 - \tau_p(z_1 + z_2 - \sigma z_2) + \tau_p^2}{z_1 z_2 - \tau_p(z_1 + z_2 - \sigma z_1) + \tau_p^2} \tag{4.4}$$

and the decay coefficient N reads

$$N(z_1, z_2) = \frac{\tau_2 \tau_p (z_1 - z_2)}{2(z_1 z_2 - \tau_p(z_1 + z_2 - \sigma z_1) + \tau_p^2)}. \tag{4.5}$$

Remark 4.3 The PT-invariant models of ref. [18], branch 1A, are obtained as particular cases of this one. More precisely, setting

$$p = \frac{2k^2}{k^4 - 1}\,, \quad t_p = \frac{-2\epsilon_1 k^2}{k^4 - 1}\,, \quad t_2 = s_1 = \pm e^{-\frac{i\pi}{4}(1-\epsilon_1)} \frac{2k}{k^2 - 1}\,, \tag{4.6}$$

one recovers the branch 1A Hamiltonians $H_{1A}^{\pm}(k, \epsilon_1)$ of ref. [18] which is associated to the Zamolodchikov-Fateev model [24].

4.1.2 Generalized Izergin-Korepin model (gIK)

The parameters which are left free are p, t_p, t_2. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$s_p = v^4 \frac{p^4}{t_p^3}\,, \quad s_3 = q = v^2 \frac{p^3}{t_p^2}\,, \quad t_3 = p\,, \quad t_1 = u_{\pm}^{-1} \frac{p^2 t_2}{t_p^2} \tag{4.7}$$

$$s_1 = v(v-1)\frac{p^2}{t_2}\,, \quad s_2 = u_{\mp}^{-1} v(v-1) \frac{p^4}{t_2 t_p^2}, \tag{4.8}$$

while on the diagonal we get:

$$X_{11} = v(v+1)\,\frac{p^2}{t_p}\,, \quad Y = (v^2+1)\,\frac{p^2}{t_p}\,, \quad X_{22} = 2(v+1)\,\frac{p^2}{t_p}, \tag{4.9}$$

$$X_{12} = (v^2+1-u_{\mp}^{-1})\,\frac{p^2}{t_p}\,, \quad X_{21} = (v^2+1-u_{\pm}^{-1})\,\frac{p^2}{t_p}\,, \tag{4.10}$$

where v is a free parameter and $u_\pm$ are the two solutions of the equation

$$v^4\,Z^2 + (1+2v-v^2)Z + 1 = 0. \tag{4.11}$$

The S-matrix depends only on the reduced parameter τ_p and v:

$$S(z_1,z_2) = -\frac{(v^2 z_1 z_2 - \tau_p(z_1+vz_2)+\tau_p^2)(v^2 z_1 z_2 - \tau_p(1+v)z_2+\tau_p^2)}{(v^2 z_1 z_2 - \tau_p(z_2+vz_1)+\tau_p^2)(v^2 z_1 z_2 - \tau_p(1+v)z_1+\tau_p^2)} \tag{4.12}$$

and the decay coefficient N reads

$$N(z_1,z_2) = \frac{\tau_2\tau_p(z_1-z_2)(u_\pm^{-1}z_1z_2+\tau_p^2)}{2(v^2 z_1 z_2 - \tau_p(z_2+vz_1)+\tau_p^2)(v^2 z_1 z_2 - \tau_p(1+v)z_1+\tau_p^2)}. \tag{4.13}$$

Remark 4.4 The PT-invariant models of ref. [18], branch 2A, which is also linked to the Izergin-Korepin model [25], are obtained as particular cases of this one. More precisely, setting

$$p = \frac{2k^2}{k^4-1}\,, \quad t_p = \frac{2k^4}{(k^6+\epsilon_1)(k^2-\epsilon_1)}\,, \quad t_2 = \mp e^{-\frac{i\pi}{4}(1-\epsilon_1)}\frac{2k}{k^6+\epsilon_1}\,, \tag{4.14}$$

one recovers the branch 2A Hamiltonians $H_{2A}^{\pm}(k,\epsilon_1,\epsilon_2)$ of ref. [18]. Note that each branch 2A Hamiltonian is related to the two models corresponding to the choices $u_\pm$ with $d = \epsilon_1 k^{-2\epsilon_2}$ or $d = \epsilon_1 k^{2\epsilon_2}$ through the following parametrization

$$v = \frac{d}{d^2-d+1}\,, \quad u_- = -(d^2-d+1)^2\,, \quad u_+ = -\frac{(d^2-d+1)^2}{d^4}\,. \tag{4.15}$$

4.1.3 Generalization of the Bariev model (gB)

This models appears to be a multi-parametric interpolation of three known models, one of them being the Bariev model, see remarks 4.5 and 4.6.

The parameters which are left free are p, q, t_1, t_2, t_p. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$s_1 = J\,\frac{Jt_1^2t_p^2 - pqt_2^2}{t_1t_2^2}\,, \quad s_2 = J^2\,\frac{Jt_1^2t_p^2 - pqt_2^2}{t_2^3}\,, \quad s_3 = -J^2\,\frac{pt_1}{t_2}\,, \quad t_3 = -J\,\frac{qt_2}{t_1}\,, \quad s_p = J\,\frac{t_1^2t_p}{t_2^2}, \tag{4.16}$$

while on the diagonal we get:

$$Y = \frac{p^2t_1^2t_2 + Jpqt_1t_2^2 + J^2q^2t_2^3 - J^2t_1^3t_p^2}{t_1^2t_2t_p}\,, \quad X_{22} = \frac{p^2t_1^2t_2 + Jpqt_1t_2^2 + J^2q^2t_2^3}{t_1^2t_2t_p}\,, \quad X_{11} = J^2\,\frac{t_1t_p}{t_2}, \tag{4.17}$$

$$X_{12} = \frac{p^2t_1^2t_2 + Jpqt_1t_2^2 + J^2q^2t_2^3 + t_1^3t_p^2}{t_1^2t_2t_p}\,, \quad X_{21} = \frac{p^2t_1^2t_2 + Jpqt_1t_2^2 + J^2q^2t_2^3 + Jt_1^3t_p^2}{t_1^2t_2t_p}. \tag{4.18}$$

The S-matrix depends only on the reduced parameters τ_p, θ and μ:

$$S(z_1, z_2) = -\frac{\Lambda(z_1, z_2)}{\Lambda(z_2, z_1)}$$

where

$$\begin{aligned}\Lambda(z_1, z_2) = J\mu^4\tau_p^2 z_1^2 z_2^2 - \mu^2\tau_p\theta z_1 z_2(z_1 + z_2) - J^2\mu^3\tau_p z_1 z_2^2 + (\mu - \theta)(\mu - J^2\theta) z_1 z_2 \\ + J^2\mu^3\tau_p^2 z_2^2 - \mu^2\tau_p(z_1 + z_2) - J\mu\tau_p\theta z_2 + \mu^2\tau_p^2\end{aligned} \tag{4.19}$$

and the decay coefficient N reads

$$N(z_1, z_2) = \frac{\tau_2\tau_p\mu^2(z_1 - z_2)(1 + \mu z_1 z_2)}{2\Lambda(z_2, z_1)}. \tag{4.20}$$

When $q = J\,\dfrac{t_1^2 t_p^2}{p t_2^2}$ (i.e. $\theta = J\mu^2\tau_p^2$), the S-matrix and the decay coefficient N simplify and one gets

$$S(z_1, z_2) = -\frac{J\mu^2\tau_p^2 z_1 z_2 - J^2\mu\tau_p z_2 + 1}{J\mu^2\tau_p^2 z_1 z_2 - J^2\mu\tau_p z_1 + 1} \tag{4.21}$$

and

$$N(z_1, z_2) = \frac{\tau_2\tau_p(z_1 - z_2)(1 + \mu z_1 z_2)}{2(z_1 - \tau_p)(z_2 - \tau_p)(J\mu^2\tau_p^2 z_1 z_2 - J^2\mu\tau_p z_1 + 1)}\,. \tag{4.22}$$

Remark 4.5 The Bariev model [21, 26] is obtained as a particular case of this one. More precisely, setting $p = q = 1$, $J = \jmath$, $t_1 = -\jmath^2\sqrt{t_p^2 - 1}$, $t_2 = \jmath\sqrt{t_p^2 - 1}$, where $\jmath = \exp(\frac{2i\pi}{3})$, one gets for the other parameters $t_3 = s_3 = 1$, $s_p = t_p$, $X_{11} = -t_p$, $Y = t_p + \dfrac{1}{t_p}$, $X_{12} = -\jmath t_p + \dfrac{1}{t_p}$, $X_{21} = -\jmath^2 t_p + \dfrac{1}{t_p}$, $X_{22} = \dfrac{1}{t_p}$ which are the values of ref. [26]. In that case, the S-matrix reads

$$S(z_1, z_2) = -\frac{\tau_p^2 z_1^2 z_2^2 - \tau_p z_1^2 z_2 + z_1 z_2 - \tau_p^2 z_2^2 - \tau_p z_1 + \tau_p^2}{\tau_p^2 z_1^2 z_2^2 - \tau_p z_1 z_2^2 + z_1 z_2 - \tau_p^2 z_1^2 - \tau_p z_2 + \tau_p^2} \tag{4.23}$$

and the decay coefficient N takes the form

$$N(z_1, z_2) = \frac{\tau_2\tau_p(z_1 - z_2)(1 - \jmath z_1 z_2)}{2(\tau_p^2 z_1^2 z_2^2 - \tau_p z_1 z_2^2 + z_1 z_2 - \tau_p^2 z_1^2 - \tau_p z_2 + \tau_p^2)}. \tag{4.24}$$

Remark 4.6 The PT-invariant models of ref. [18], branch 2B, are obtained as particular cases of this one. More precisely, setting

$$p = q = -2i\,, \quad t_1 = \pm\epsilon_1\epsilon_2 e^{-2i\epsilon_2\pi/3}\,, \quad t_2 = \pm\epsilon_1\epsilon_2 e^{2i\epsilon_2\pi/3}\,, \quad t_p = -i\epsilon_1\sqrt{3} \tag{4.25}$$

one recovers the branch 2B Hamiltonians $H_{2B}^{\pm}(\epsilon_1, \epsilon_2)$ of ref. [18] (with $\epsilon_1, \epsilon_2 \in \{-1, +1\}$).

The main branch genus 5 model of ref. [19] is also a particular case of this one. More precisely, setting

$$p = -\epsilon_2\,, \quad q = -\epsilon_1\,, \quad t_1 = \frac{\sqrt{3} \mp i}{2}\sqrt{\epsilon_1\epsilon_2 - 1}\,, \quad t_2 = \frac{\sqrt{3} \pm i}{2}\sqrt{\epsilon_1\epsilon_2 - 1} \tag{4.26}$$

one recovers the main branch genus 5 Hamiltonians $H_{MB5}^{\pm}(\epsilon_1, \epsilon_2)$ of ref. [19] (we remind that here ϵ_1 and ϵ_2 are free parameters).

4.1.4 Generalization of the Hamiltonian built on $\mathcal{U}_q(sl(2))$ special representation at roots of unity (SpR)

The parameters which are left free are p, q, t_p, t_2, t_3. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$t_1 = \frac{qt_2}{p}\,, \quad s_1 = \frac{pt_3}{t_2}\,, \quad s_2 = \frac{qt_3}{t_2}\,, \quad s_3 = \frac{qt_3}{p}\,, \quad s_p = \frac{q(t_3^2 - t_3 p + p^2)}{pt_p}, \tag{4.27}$$

while on the diagonal we get:

$$X_{11} = 0\,, \quad Y = X_{12} = X_{21} = X_{22} = \frac{t_3^2 - t_3 p + p^2}{t_p} + \frac{qt_p}{p}. \tag{4.28}$$

Remark 4.7 Note that eqs (4.27) and (4.28) look as if some of the free parameters, say t_p, cannot be set to zero. This is due to the choice of parametrization we made. However, one can choose alternative presentations. For instance, we could use as free parameters p, q, s_p, t_1, s_3. In that case, the remaining parameters take the form

$$t_2 = \frac{pt_1}{q}\,, \quad s_1 = \frac{ps_3}{t_1}\,, \quad s_2 = \frac{qs_3}{t_1}\,, \quad t_3 = \frac{ps_3}{q}\,, \quad t_p = \frac{p(s_3^2 - s_3 q + q^2)}{qs_p}, \tag{4.29}$$

and

$$X_{11} = 0\,, \quad Y = X_{12} = X_{21} = X_{22} = \frac{s_3^2 - s_3 q + q^2}{s_p} + \frac{ps_p}{q}. \tag{4.30}$$

Clearly, (4.29) shows that we can now set $t_p = 0$ without any problem. This new choice of parametrization is in fact the image under parity of the previous choice.

The S-matrix depends only on the reduced parameters τ_p and τ_3:

$$S(z_1, z_2) = -\frac{(\tau_3^2 - \tau_3 + 1)z_1 z_2 - \tau_p(z_1 + z_2 - \tau_3 z_2) + \tau_p^2}{(\tau_3^2 - \tau_3 + 1)z_1 z_2 - \tau_p(z_1 + z_2 - \tau_3 z_1) + \tau_p^2} \tag{4.31}$$

and the decay coefficient N reads

$$N(z_1, z_2) = \frac{\tau_2 \tau_p (z_1 - z_2)}{2((\tau_3^2 - \tau_3 + 1)z_1 z_2 - \tau_p(z_1 + z_2 - \tau_3 z_1) + \tau_p^2)}. \tag{4.32}$$

Remark 4.8 The PT-invariant models of ref. [18], branch 1B, which are linked to the models associated to special representation of $\mathcal{U}_q(sl(2))$ at roots of unity [27–29], are obtained as particular cases of this one. More precisely, setting

$$p = q = -2i\,, \quad t_2 = \pm 2\,, \quad t_3 = 2i\,, \quad t_p = -2i\epsilon_1\sqrt{3} \tag{4.33}$$

one recovers the branch 1B Hamiltonians $H_{1B}^{\pm}(\epsilon_1)$ of ref. [18].

4.2 Generalization of the special branch genus 5 model (SB_5)

The parameters which are left free are p, q, t_2, Y. The remaining parameters entering the off-diagonal part of the Hamiltonian are given by

$$t_p = s_p = 0\,, \quad t_1 = \frac{qt_2}{p}\,, \quad s_1 = -J^2\,\frac{p^2}{t_2}\,, \quad s_2 = -J\,\frac{pq}{t_2}\,, \quad t_3 = -J^2 p\,, \quad s_3 = -Jq \tag{4.34}$$

while on the diagonal we get:

$$X_{11} = 0\,, \quad X_{12} = X_{21} = X_{22} = Y. \tag{4.35}$$

The S-matrix depends only on the reduced parameters θ and Υ:

$$S(z_1, z_2) = -\frac{\theta z_1 z_2(z_1 - J^2 z_2) - \Upsilon z_1 z_2 + z_1 - J z_2}{\theta z_1 z_2(z_2 - J^2 z_1) - \Upsilon z_1 z_2 + z_2 - J z_1} \tag{4.36}$$

and the decay coefficient N reads

$$N(z_1, z_2) = -\frac{\tau_2(z_1 - z_2)(\theta z_1 z_2 + 1)}{2(\theta z_1 z_2(z_2 - J^2 z_1) - \Upsilon z_1 z_2 + z_2 - J z_1)}\,. \tag{4.37}$$

Remark 4.9 The special branch genus 5 model of ref. [19] is also a particular case of this one. More precisely, setting

$$p = t_2 = e^{\mp 2i\pi/3}\,, \quad q = -1\,, \quad Y = 4\Lambda \tag{4.38}$$

one recovers the special branch genus 5 Hamiltonians $H^{\pm}_{SB5}(\Lambda)$ of ref. [19].

4.3 Other models (17- and 14- vertex models)

The terminology "17-vertex" and "14-vertex" follows the one used for 19-vertex, see explanation in remark 4.2.

As explained above, eqs. (3.18), (3.21) and (3.24) provide a set of constraints (denoted $\mathcal{C}_{|\Omega\rangle}$ hereafter) on the parameters entering the Hamiltonian. Solving these equations, we get a set of solutions, each of them determining an Hamiltonian solvable through CBA. Then, the BAE (3.9) allows us to compute the eigenvalues (3.4) and eigenvectors (3.2) of the model using the S matrix and the decay coefficient N. The construction is based on the choice of a particular eigenvector of H: the pseudo-vacuum.

Since we perform a classification, one shall get the same set of solutions whatever the choice of the pseudo-vacuum. In the case of the three-state Hamiltonian we are studying, there are two pseudo-vacua $|\Omega\rangle = \bigotimes_{i=1}^{L} |0\rangle$ and $|\widetilde{\Omega}\rangle = \bigotimes_{i=1}^{L} |2\rangle$. Deploying the CBA machinery for each pseudo-vacuum leads to two distinct sets of constraint equations, $\mathcal{C}_{|\Omega\rangle}$ and $\mathcal{C}_{|\widetilde{\Omega}\rangle}$, the latter being obtained by applying the charge conjugation[4] (2.10) to the former. It follows in light of the foregoing that each solution of $\mathcal{C}_{|\Omega\rangle}$ should satisfy the equations coming from $\mathcal{C}_{|\widetilde{\Omega}\rangle}$. As it can be checked in Table A, it is indeed the case for the previous models.

However, there are cases where a solution to $\mathcal{C}_{|\Omega\rangle}$ does not solve identically $\mathcal{C}_{|\widetilde{\Omega}\rangle}$, but rather leads to additional constraints on the parameters. At this stage, the additional constraints could

[4] In fact, (2.10) is just the gauge transformation relating $|\Omega\rangle$ to $|\widetilde{\Omega}\rangle$.

be interpreted as a failure in the CBA method: eigenvectors built on $\mathcal{C}_{|\widetilde{\Omega}\rangle}$ are not eigenvectors of the Hamiltonian based on a solution of $\mathcal{C}_{|\Omega\rangle}$. In fact, it just indicates that the eigenvectors obtained by CBA on $\mathcal{C}_{|\Omega\rangle}$ do not provide a complete basis of eigenvectors. One needs to consider a second pseudo-vacuum $|\widetilde{\Omega}\rangle$ to get a (tentatively) complete basis. Thus, it is the full sets of constraints $\mathcal{C}_{|\Omega\rangle}$ and $\mathcal{C}_{|\widetilde{\Omega}\rangle}$ that need to be considered. That is what we did for the class of models presented in this section. In practice, it is simpler, but equivalent, to apply the transformation (2.10) to a given solution to the initial constraints $\mathcal{C}_{|\Omega\rangle}$, to impose the transformed solution to be also a solution of $\mathcal{C}_{|\Omega\rangle}$ (hence leading to more constraints on the parameters) and then to pull back the charge conjugation transformation (2.10) on the result to get the final answer.

4.3.1 Model $17V_1$

Solving the constraints $\mathcal{C}_{|\Omega\rangle}$, the parameters which are left free are $p, q, t_p, t_2, t_3, s_3, X_{22}$ and the constraints on the parameters are given by

$$s_1 = s_2 = 0\,, \quad s_p = \frac{pq}{t_p}\,, \quad t_1 = \frac{qt_2}{p} \tag{4.39}$$

and

$$X_{11} = 0\,, \quad Y = \frac{p^2}{t_p} + \frac{qt_p}{p}\,, \quad X_{12} = \frac{p^2}{t_p} + \frac{qt_p}{p} + \frac{pt_3}{t_p}\,, \quad X_{21} = \frac{p^2}{t_p} + \frac{qt_p}{p} + \frac{t_p s_3}{p}\,. \tag{4.40}$$

Solving now the constraints $\mathcal{C}_{|\widetilde{\Omega}\rangle}$, one gets two inequivalent possibilities:

- **Model** $17V_{1a}$**:** The additional conditions are

$$s_3 = \epsilon q\,, \quad t_3 = \epsilon p\,, \quad X_{22} = (1+\epsilon)Y, \quad \text{with} \quad \epsilon = \pm 1, \tag{4.41}$$

  the diagonal terms reading now

$$X_{12} = Y + \epsilon\,\frac{p^2}{t_p}\,, \quad X_{21} = Y + \epsilon\,\frac{qt_p}{p}. \tag{4.42}$$

- **Model** $17V_{1b}$**:** The additional conditions are

$$q = \frac{Ip^3}{t_p^2}\,, \quad t_3 = Ip\,, \quad s_3 = \frac{p^3}{t_p^2}\,, \quad X_{22} = (1+I)\,\frac{p^2}{t_p}\,, \tag{4.43}$$

  where I is one solution of $I^2 + 1 = 0$. It leads to a redefinition of the parameters:

$$s_p = \frac{Ip^4}{t_p^3}\,, \quad t_1 = \frac{Ip^2 t_2}{t_p^2}\,, \quad Y = (1+I)\,\frac{p^2}{t_p}\,, \quad X_{12} = (2I+1)\,\frac{p^2}{t_p}\,, \quad X_{21} = (I+2)\,\frac{p^2}{t_p}. \tag{4.44}$$

For both models $17V_{1a}$ and $17V_{1b}$ the S-matrix is trivial, $S(z_1, z_2) = -1$, and the decay coefficient N reads

$$N(z_1, z_2) = \frac{\tau_2 \tau_p (z_1 - z_2)}{2(z_1 - \tau_p)(z_2 - \tau_p)}. \tag{4.45}$$

4.3.2 Model $17V_2$

Solving the constraints $\mathcal{C}_{|\Omega\rangle}$, the parameters which are left free are p, q, t_p, t_2, t_3, s_3 and the constraints on the parameters are given by

$$s_1 = s_2 = 0\,, \quad s_p = \frac{pq}{t_p}\,, \quad t_1 = -\frac{p^2 t_2}{t_p^2} \tag{4.46}$$

and

$$X_{11} = Y = \frac{p^2}{t_p} + \frac{qt_p}{p}\,, \quad X_{12} = 2Y - \frac{qt_p t_3}{p^2}\,, \quad X_{21} = 2Y - \frac{p^2 s_3}{qt_p}\,, \quad X_{22} = 2Y. \tag{4.47}$$

Solving now the constraints $\mathcal{C}_{|\widetilde{\Omega}\rangle}$, the additional conditions are

$$s_3 = q\,, \quad t_3 = p\,, \tag{4.48}$$

the diagonal terms reading now

$$X_{12} = \frac{2p^2}{t_p} + \frac{qt_p}{p}\,, \quad X_{21} = \frac{p^2}{t_p} + \frac{2qt_p}{p}\,. \tag{4.49}$$

The S-matrix depends only on the reduced parameters τ_p and θ:

$$S(z_1, z_2) = -\frac{\theta\tau_p z_1 z_2 - (\theta\tau_p^2 + 1)z_2 + \tau_p}{\theta\tau_p z_1 z_2 - (\theta\tau_p^2 + 1)z_1 + \tau_p} \tag{4.50}$$

and the decay coefficient N reads

$$N(z_1, z_2) = \frac{-\tau_2(z_1 - z_2)(z_1 z_2 - \tau_p^2)}{2(\theta\tau_p z_1 z_2 - (\theta\tau_p^2 + 1)z_1 + \tau_p)(z_1 - \tau_p)(z_2 - \tau_p)}. \tag{4.51}$$

4.3.3 Model $14V_1$

Solving the constraints $\mathcal{C}_{|\Omega\rangle}$, the parameters which are left free are $p, t_p, t_2, t_3, X_{21}, X_{22}$ and the constraints on the parameters are given by

$$q = s_1 = s_2 = s_3 = s_p = 0\,, \quad t_1 = -\frac{p^2 t_2}{t_p^2}\,, \quad X_{11} = Y = \frac{p^2}{t_p}\,, \quad X_{12} = \frac{2p^2}{t_p}. \tag{4.52}$$

Solving now the constraints $\mathcal{C}_{|\widetilde{\Omega}\rangle}$, the additional conditions are

$$t_3 = \epsilon p\,, \quad X_{21} = X_{22} - \frac{p^2}{t_p} \quad \text{with} \quad \epsilon = \pm 1\,. \tag{4.53}$$

The S-matrix and the decay coefficient N read:

$$S(z_1, z_2) = -\frac{z_2 - \tau_p}{z_1 - \tau_p}\,, \quad N(z_1, z_2) = \frac{\tau_2(z_1 - z_2)(z_1 z_2 - \tau_p^2)}{2(z_1 - \tau_p)^2(z_2 - \tau_p)}. \tag{4.54}$$

4.3.4 Model $14V_2$

Solving the constraints $\mathcal{C}_{|\Omega\rangle}$, the parameters which are left free are p, t_p, t_1, t_2 and the constraints on the parameters are given by

$$q = s_1 = s_2 = s_3 = s_p = 0\,, \quad t_3 = -\frac{t_p^2 t_1}{p t_2} \tag{4.55}$$

and

$$X_{11} = 0\,, \quad X_{12} = Y = \frac{p^2}{t_p}\,, \quad X_{21} = X_{22} = \frac{p^2 t_2 - t_p^2 t_1}{t_p t_2}. \tag{4.56}$$

Solving now the constraints $\mathcal{C}_{|\widetilde{\Omega}\rangle}$, leads to one additional condition

$$t_1 = \frac{p^2 t_2}{t_p^2} \tag{4.57}$$

which gives $t_3 = -p$ and $X_{21} = X_{22} = 0$.

The S-matrix is trivial, $S(z_1, z_2) = -1$, and the decay coefficient N reads:

$$N(z_1, z_2) = \frac{\tau_2 (z_1 - z_2)(z_1 z_2 + \tau_p^2)}{2\tau_p (z_1 - \tau_p)(z_2 - \tau_p)}. \tag{4.58}$$

5 Reduced Hamiltonians

In this section, we use telescoping terms and gauge transformations (see section 2.2) to get a simple expression $\widetilde{H}$ for the Hamiltonians described in section 4. For all Hamiltonians, in a first step, we change the normalization and perform a gauge transformation:

$$H_{red} = \mathcal{N}_0\; G \otimes G \left(H - \frac{V}{2} \left(s_1^z + s_2^z \right) \right) G^{-1} \otimes G^{-1}\,, \tag{5.1}$$

where $\mathcal{N}_0$ is a constant, and G a 3×3 diagonal matrix. Their exact form depends on the model we consider. We get in this way Hamiltonians that depend only on the physical parameters. The transformation is valid only for generic values of the free parameters, see remark 5.1 below.

5.1 Nineteen vertices

5.1.1 Generalized Zamolodchikov-Fateev model

From the Hamiltonian H given in section 4.1.1, we perform the transformation (5.1) with

$$\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \mathrm{diag}\left(1, \left(\frac{t_2\, t_p^2}{s_1\, p^2}\right)^{1/4}, 1\right). \tag{5.2}$$

It leads to an Hamiltonian H_{red} depending on τ_p and σ only.

Remark 5.1 Note that this transformation is not valid for $s_1 = 0$, although the "raw" Hamiltonian is well-defined in this case, see section 4.1.1. This is due to the fact that the physical parameter σ vanishes for this particular value of s_1. One can however work on the raw Hamiltonian to get then a reduced Hamiltonian containing only the physical parameter τ_p.

To compare with existing models, we furthermore modify it to

$$\widetilde{H} = \frac{-2k^2}{k^4-1}\, H_{red} - \frac{k^4+1}{k^4-1}\,(s_1^z + s_2^z) \quad \text{with} \quad \sigma = \left(\frac{k^2+1}{k}\right)^2 \tag{5.3}$$

we get

$$\widetilde{H} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{k^4+1}{1-k^4} & 0 & \frac{2\tau_p\, k^2}{1-k^4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2\,k^4+2\,k^2+2}{1-k^4} & 0 & \frac{2k}{1-k^2} & 0 & \frac{2{\tau_p}^2 k^2}{1-k^4} & 0 & 0 \\
0 & \frac{2k^2}{(1-k^4)\tau_p} & 0 & \frac{k^4+1}{1-k^4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{2k}{1-k^2} & 0 & \frac{2\,k^4+2}{1-k^4} & 0 & \frac{2k{\tau_p}^2}{1-k^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{k^4+1}{1-k^4} & 0 & \frac{2\tau_p\, k^2}{1-k^4} & 0 \\
0 & 0 & \frac{2k^2}{(1-k^4){\tau_p}^2} & 0 & \frac{2k}{{\tau_p}^2(1-k^2)} & 0 & \frac{2\,k^4+2\,k^2+2}{1-k^4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2k^2}{(1-k^4)\tau_p} & 0 & \frac{k^4+1}{1-k^4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{5.4}$$

When $\tau_p = -1$, we recover the Zamolodchikov-Fateev model [24] (or spin-1 XXZ spin chain). When $\tau_p = -\epsilon_1$, $k = \exp(\frac{\gamma}{2} + \frac{i\pi}{4}(1-\epsilon_1))$ and $\epsilon_1 = \pm 1$ is left free, we get the models "branch 1A" described in [18]. The models 7 and 10 of [17] are also obtained in the same way.
To be complete, let us add that for $\tau_p = -1$, the Hamiltonian (5.4) is related to the one based on $\mathcal{U}_q(B_1^{(1)})$ given in [30] by $H^{\mathcal{U}_q(B_1^{(1)})}(1/k^2) = \widetilde{H}(k) + (\mathbb{I}\otimes e_{22} - e_{22}\otimes\mathbb{I}) + 2(\mathbb{I}\otimes e_{33} - e_{33}\otimes\mathbb{I})$ (the R-matrix of $\mathcal{U}_q(B_1^{(1)})$ we consider is normalized such that $R_{11}^{11} = 1$).

5.1.2 Generalized Izergin-Korepin model

From the Hamiltonian H given in section 4.1.2, the transformation (5.1) with

$$\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \text{diag}\left(1, \sqrt{\frac{t_2}{p}}\left(\frac{vu_-}{v-1}\right)^{1/4}, 1\right) \tag{5.5}$$

leads to an Hamiltonian H_{red} depending on τ_p and v only. To compare with existing model, we first make a change of variable $v = \dfrac{k}{k^2-k+1}$, $u_- = -(k^2-k+1)^2$ and define

$$\begin{aligned}\widetilde{H} = \frac{1}{(k^2-1)(k^2-k+1)}\Big(&-(k^2-k+1)^2 H_{red} + \tfrac{1}{2}(k^2+1)(k^2-k+1)\,(s_1^z+s_2^z) \\ &+ \tfrac{1}{2}(k^2-1)(k^2-k+1)\,(\mathbb{I}\otimes e_{22} - e_{22}\otimes\mathbb{I}) + \tfrac{1}{2}(k-1)^3(k+1)\,(\mathbb{I}\otimes e_{33} - e_{33}\otimes\mathbb{I})\Big)\end{aligned} \tag{5.6}$$

that is, with $\tau'_p = \tau_p/v$,

$$\widetilde{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{k^2}{k^2-1} & 0 & \frac{-k\tau'_p}{k^2-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(k^3-k^2+1)k}{(k^3+1)(k-1)} & 0 & \frac{-\tau'_p\sqrt{k}}{k^3+1} & 0 & \frac{-k^2{\tau'_p}^2}{(k^3+1)(k-1)} & 0 & 0 \\ 0 & \frac{-k}{(k^2-1)\tau'_p} & 0 & \frac{1}{k^2-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{k}}{\tau'_p\,(k^3+1)} & 0 & \frac{k^3-k^2+k-1}{k^3+1} & 0 & \frac{k^{5/2}\tau'_p}{k^3+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{k^2}{k^2-1} & 0 & \frac{-k\tau'_p}{k^2-1} & 0 \\ 0 & 0 & \frac{-k^2}{{\tau'_p}^2(k^3+1)(k-1)} & 0 & \frac{k^{5/2}}{\tau'_p\,(k^3+1)} & 0 & \frac{k^3-k+1}{(k^3+1)(k-1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-k}{(k^2-1)\tau'_p} & 0 & \frac{1}{k^2-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (5.7)$$

For generic values of k and τ_p, the Hamiltonian $\widetilde{H}$ is conjugated to the one based on $\mathcal{U}_q(A_2^{(2)})$, and given in [30] (the R-matrix of $\mathcal{U}_q(A_2^{(2)})$ we consider is normalized such that $R_{11}^{11} = 1$):

$$\widetilde{H}(k) = \widetilde{F}\, H^{A_2^{(2)}}(k)\, \widetilde{F}^{-1} \quad \text{with} \quad \widetilde{F} = \mathrm{diag}\left(u_1, \sqrt{u_1u_3}, u_3\right) \otimes \mathrm{diag}\left(u_1\frac{v}{\tau_p}, \sqrt{u_1u_3}, u_3\frac{\tau_p}{v}\right). \quad (5.8)$$

Note that the Hamiltonian $H^{A_2^{(2)}}(k)$ is related to the Izergin-Korepin model [25] through a constant gauge transformation and constant telescopic terms.

To be complete, let us add that the Hamiltonian $H^{A_2^{(2)}}$ is related to the Branch 2A of [18] through the following transformation[5]:

$$-2H^{A_2^{(2)}}(\epsilon_1 k^2) + (\mathbb{I}\otimes e_{22} - e_{22}\otimes\mathbb{I}) + 2\,(\mathbb{I}\otimes e_{33} - e_{33}\otimes\mathbb{I}) = F\,H^{2A}(k)\,F^{-1} \quad (5.9)$$

with

$$F = \mathrm{diag}\left(1, \frac{1}{\sqrt{\epsilon_1}}, \sqrt{-\epsilon_2\epsilon_3}, \sqrt{\epsilon_1}, \frac{1}{\sqrt{-\epsilon_2\epsilon_3}}, \sqrt{\epsilon_1}, \sqrt{-\epsilon_2\epsilon_3}, \frac{1}{\sqrt{\epsilon_1}}, 1\right). \quad (5.10)$$

Note that in the correspondence (5.9), H^{2A} corresponds to the branch 2A Hamiltonian H_{12} of ref. [18] for $\epsilon_2 = 1$, while it corresponds to H_{21} when $\epsilon_2 = -1$.

5.1.3 Generalized Bariev model

From the Hamiltonian H given in section 4.1.3, we perform the transformation (5.1) with

$$\mathcal{N}_0 = \frac{1}{p}\sqrt{\frac{t_2}{t_1}} \quad \text{and} \quad G = \mathrm{diag}\left(1, \sqrt{\frac{t_2}{p}}, 1\right). \quad (5.11)$$

We get an Hamiltonian H_{red} depending on τ_p, θ and μ only.

[5] We remind the correspondence with the notation of [18]: $u = exp(\lambda)$, $k = exp(\gamma/2 + \frac{i\pi}{4}(1-\epsilon_1))$.

This Hamiltonian can be related to the one of the Main Branch of ref. [19], H^{MB_5}. One defines, with $\delta = \dfrac{\mu^2 + J\theta\mu + J^2\theta^2}{4J^2\tau_p^2\mu^3}$,

$$\widetilde{H} = -\frac{J}{\tau_p\sqrt{\mu}}\, H_{red} + \delta\,(s_1^z + s_2^z - \mathbb{I}) + \tfrac{1}{2}\,(J - J^{-1})\,(\mathbb{I}\otimes e_{33} - e_{33}\otimes\mathbb{I}) \tag{5.12}$$

that is,

$$\widetilde{H} = \begin{bmatrix}
-\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{J}{\tau_p\mu} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\delta - J^2 & 0 & -\frac{J}{\tau_p\mu} & 0 & -J\mu^{-1} & 0 & 0 \\
0 & -\frac{J\theta}{\tau_p\mu} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\theta - J\tau_p^2\mu^2}{\tau_p\mu} & 0 & \delta - 1 & 0 & \frac{\theta - J\tau_p^2\mu^2}{J\tau_p\mu^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{J^2\theta}{\tau_p\mu^2} & 0 \\
0 & 0 & -J^2\mu & 0 & -\frac{J}{\tau_p} & 0 & -\delta - J & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \tau_p^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta
\end{bmatrix}. \tag{5.13}$$

Then one gets

$$\widetilde{H} = F\, H^{MB_5}\, F^{-1} \tag{5.14}$$

with $F = U \otimes U'$ where U and U' are expressed in terms of the free parameters u_1, u_3:

$$U = \mathrm{diag}\left(u_1, Z\left(\frac{\theta - J\tau_p^2\mu^2}{\mu}\right)^{1/4}\sqrt{u_1u_3}, u_3\right) \tag{5.15}$$

and

$$U' = \mathrm{diag}\left(-iJ\sqrt{\mu}\,u_1, \frac{1}{Z}\left(\frac{\theta - J\tau_p^2\mu^2}{\mu}\right)^{1/4}\sqrt{u_1u_3}, -\frac{u_3}{iJ\sqrt{\mu}}\right). \tag{5.16}$$

The parameters (ϵ_1, ϵ_2) entering into the definition of H^{MB_5} are given by one of these relations:

$$\begin{aligned}
&\epsilon_1 = \frac{i\theta}{\tau_p\mu^{3/2}}\,, \quad \epsilon_2 = -\frac{iJ^2}{\tau_p\mu^{1/2}} \quad \text{with the choice} \quad Z = 1 \\
&\epsilon_1 = -\frac{iJ^2}{\tau_p\mu^{1/2}}\,, \quad \epsilon_2 = \frac{i\theta}{\tau_p\mu^{3/2}} \quad \text{with the choice} \quad Z = \left(-\frac{J\theta}{\mu}\right)^{1/2}
\end{aligned} \tag{5.17}$$

and J is related to the parameter $\gamma_0 = \epsilon\frac{i\pi}{6}$ $(\epsilon = \pm 1)$ of H^{MB_5} by $J = -e^{-2\gamma_0}$.

5.1.4 Generalization of the Hamiltonian associated to special representation of $\mathcal{U}_q(sl(2))$ at roots of unity

From the Hamiltonian H given in section 4.1.4, the transformation (5.1) with

$$\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \mathrm{diag}\left(1, \sqrt{\frac{t_2}{p}}, 1\right) \tag{5.18}$$

produces an Hamiltonian H_{red} depending on τ_p, θ and τ_3 only, that is, with $\delta = \tau_3^2 - \tau_3 + 1 + \tau_p^2\theta$,

$$H_{red} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\delta & 0 & \tau_p & 0 & \tau_p^2 & 0 & 0 \\ 0 & \tau_p\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_p\tau_3\theta & 0 & 0 & 0 & \tau_p\tau_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\delta & 0 & \tau_p\tau_3 & 0 \\ 0 & 0 & \theta(\tau_3^2 - \tau_3 + 1) & 0 & \tau_p\theta & 0 & \frac{1}{2}\delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_p\tau_3\theta & 0 & \frac{1}{2}\delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta \end{bmatrix}. \tag{5.19}$$

The Branch 1B Hamiltonians of ref. [18] can be related to the Hamiltonian (5.18) through the following transformation:

$$\frac{2(k^2 - e^{2\gamma_0})}{(k^2-1)(1-e^{2\gamma_0})}\, H_{red} - \frac{k^2+1}{k^2-1}\,(s_1^z + s_2^z) = F\, H^{1B}(k, \epsilon_1, \epsilon_2)\, F^{-1} \tag{5.20}$$

with $\gamma_0 = \epsilon_1 \frac{i\pi}{3}$, $(\epsilon_1, \epsilon_2) \in \{-1, +1\}$, and $F = U \otimes U'$ where U and U' are expressed in terms of the free parameters u_1, u_3:

$$U = \mathrm{diag}\left(u_1, \sqrt{\frac{u_1 u_3}{\Delta}}, u_3\right) \quad \text{and} \quad U' = \mathrm{diag}\left(u_1\sqrt{\theta}, \sqrt{\frac{u_1 u_3}{\Delta}}, \frac{u_3}{\sqrt{\theta}}\right), \tag{5.21}$$

$\Delta = \dfrac{\epsilon_2}{e^{\gamma_0/2}\sqrt{\tau_3}}$, and the parameters θ, τ_p and τ_3 are linked to k by the relations

$$\tau_3 = \frac{e^{\gamma_0}(k^2-1)}{k^2 - e^{2\gamma_0}} \quad \text{and} \quad \tau_p\sqrt{\theta} = \frac{k(1-e^{2\gamma_0})}{k^2 - e^{2\gamma_0}}. \tag{5.22}$$

Note that the transformation (5.20) only holds when the parameters θ, τ_p and τ_3 are related to k by (5.22). Hence this model appears as a generalization of the Branch 1B models of [18].

5.2 Generalization of the special branch genus 5 model

From the Hamiltonian H given in section 4.2, we perform the transformation (5.1) with

$$\mathcal{N}_0 = \frac{1}{Y} \quad \text{and} \quad G = \mathrm{diag}\left(1, \sqrt{\frac{t_2}{p}}, 1\right) \tag{5.23}$$

that leads to an Hamiltonian H_{red} depending on θ and Υ only.

The genus 5 Special Branch Hamiltonian of ref. [19] can be related to the Hamiltonian (5.23). Indeed, one defines

$$\widetilde{H} = \frac{\Upsilon}{4J\sqrt{-\theta}}\left(4\, H_{red} - (s_1^z + s_2^z) + \mathbb{I}\right) \tag{5.24}$$

with $\gamma_0 = \pm\frac{i\pi}{3}$, $J = e^{-2\gamma_0}$, that is

$$\widetilde{H} = \begin{bmatrix} \frac{\Upsilon}{4J\sqrt{-\theta}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{J\sqrt{-\theta}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Upsilon}{4J\sqrt{-\theta}} & 0 & \frac{1}{J\sqrt{-\theta}} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{-\theta}}{J} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-\theta} & 0 & -\frac{\Upsilon}{4J\sqrt{-\theta}} & 0 & \frac{-J}{\sqrt{-\theta}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-J}{\sqrt{-\theta}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{-\theta}}{J} & 0 & \frac{\Upsilon}{4J\sqrt{-\theta}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{-\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\Upsilon}{4J\sqrt{-\theta}} \end{bmatrix}. \tag{5.25}$$

Then one gets

$$\epsilon\widetilde{H} = F\, H^{SB5}(\Lambda, e^{\gamma_0})\, F^{-1} \tag{5.26}$$

with $F = U \otimes U'$ where U and U' are expressed in terms of the free parameters u_1, u_3:

$$U = \mathrm{diag}\left(u_1, \sqrt{u_1 u_3}, u_3\right) \quad \text{and} \quad U' = \mathrm{diag}\left(\epsilon J^2\sqrt{-\theta}\, u_1, \sqrt{u_1 u_3}, \frac{\epsilon u_3}{J^2\sqrt{-\theta}}\right), \tag{5.27}$$

the parameter Λ being linked to Υ and θ by $\Lambda = \dfrac{\epsilon\Upsilon}{4J\sqrt{-\theta}}$ (with $\epsilon = \pm 1$).

5.3 Other models

5.3.1 Model $17V_{1a}$

From the Hamiltonian H given in section 4.3.1 (case $1a$), we perform the transformation (5.1)

$$\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \mathrm{diag}\left(1, \sqrt{\frac{t_2}{p}}, 1\right) \tag{5.28}$$

that leads to an Hamiltonian H_{red} depending on θ and τ_p only, that is,

$$H_{red} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1+\theta\tau_p^2}{2} & 0 & \tau_p & 0 & \tau_p^2 & 0 & 0 \\ 0 & \theta\tau_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1+\theta\tau_p^2}{2}+\epsilon & 0 & \epsilon\tau_p & 0 \\ 0 & 0 & \theta & 0 & \theta\tau_p & 0 & \frac{1+\theta\tau_p^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon\theta\tau_p & 0 & \frac{1+\theta\tau_p^2}{2}+\epsilon\theta\tau_p^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1+\epsilon)(1+\theta\tau_p^2) \end{bmatrix}. \quad (5.29)$$

5.3.2 Model $17V_{1b}$

From the Hamiltonian H given in section 4.3.1 (case 1b), we perform the transformation (5.1) with

$$\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \text{diag}\left(1, \sqrt{\frac{t_2}{p}}, 1\right) \quad (5.30)$$

that leads to an Hamiltonian H_{red} depending on τ_p only, that is

$$H_{red} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1+I) & 0 & \tau_p & 0 & \tau_p^2 & 0 & 0 \\ 0 & I\tau_p^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1+3I) & 0 & I\tau_p & 0 \\ 0 & 0 & I\tau_p^{-2} & 0 & I\tau_p^{-1} & 0 & \frac{1}{2}(1+I) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_p^{-1} & 0 & \frac{1}{2}(3+I) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1+I \end{bmatrix}. \quad (5.31)$$

5.3.3 Model $17V_2$

From the Hamiltonian H given in section 4.3.2, we perform the transformation (5.1) with

$$\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \text{diag}\left(1, \sqrt{\frac{t_2}{p}}, 1\right) \quad (5.32)$$

that leads to an Hamiltonian H_{red} depending on θ and τ_p only, that is

$$H_{red} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1+\theta\tau_p^2}{2} & 0 & \tau_p & 0 & \tau_p^2 & 0 & 0 \\ 0 & \tau_p\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\theta\tau_p^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3+\theta\tau_p^2}{2} & 0 & \tau_p & 0 \\ 0 & 0 & \theta & 0 & -\tau_p^{-1} & 0 & \frac{1+\theta\tau_p^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_p\theta & 0 & \frac{1+3\theta\tau_p^2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2(1+\theta\tau_p^2) \end{bmatrix}. \tag{5.33}$$

5.3.4 Model $14V_1$

From the Hamiltonian H given in section 4.3.3, we perform the transformation (5.1) with

$$\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \text{diag}\left(1, \sqrt{\frac{t_2}{p}}, 1\right) \tag{5.34}$$

that leads to an Hamiltonian H_{red} depending on τ_p, $\xi = X_{22}/p$ and $\epsilon = \pm 1$ only, that is

$$H_{red} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \tau_p & 0 & \tau_p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & \epsilon\tau_p & 0 \\ 0 & 0 & 0 & 0 & -\tau_p^{-1} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_p\,\xi - \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_p\,\xi \end{bmatrix}. \tag{5.35}$$

5.3.5 Model $14V_2$

From the Hamiltonian H given in section 4.3.4, we perform the transformation (5.1) with

$$\mathcal{N}_0 = \frac{t_p}{p^2} \quad \text{and} \quad G = \text{diag}\left(1, \sqrt{\frac{t_2}{p}}, 1\right) \tag{5.36}$$

that leads to an Hamiltonian H_{red} depending on τ_p only, that is

$$H_{red} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \tau_p & 0 & \tau_p^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & -\tau_p & 0 \\ 0 & 0 & 0 & 0 & \tau_p^{-1} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{5.37}$$

6 Conclusion

In this paper we have provided a classification of 'all' the Hamiltonians with rank 1 symmetry and nearest neighbour interactions, acting on a periodic three-state spin chain, and solvable through (a generalization of) the coordinate Bethe ansatz (CBA).

Of course, the search for an R-matrix formulation of the new models presented here should be done, but many directions of generalizations can also be planed. First of all, the case with rank 2 symmetry algebra can also be easily done using the same method. Next, the integrable Hamiltonians that are not solvable through CBA, such as the ones obtained from Temperley-Lieb algebras, should be classified too. Finally, a similar classification for models solvable through algebraic Bethe ansatz would help to give a better understanding of the connection between these two approaches.

There is also a natural question that arises from this classification: what possible extensions of this work can be envisioned for n-state Hamiltonians? A priori, the method becomes rather intricate when increasing the number of states on each site, so that there is few hope that this can be done in the same way. However, increasing the rank of the symmetry algebra at the same time could provide some simplification. This question is of relevance to recent developments in ultracold gases in optical lattices, such as the achievement of cooling down to quantum degeneracy five Ytterbium isotopes [31] which exhibit an enlarged $SU(6)$ symmetry.

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A P, C, T transformations

Model	# vertices	P action	C action	T action	Invariances
gZF	19	gZF	gZF	gZF	P, C, T
gIK	19	$\mathrm{gIK}\big\vert_{u_\pm \to u_\mp}$	$\mathrm{gIK}\big\vert_{u_\pm \to u_\mp}$	gIK	PC, T
gB	19	$\mathrm{gB}\big\vert_{J \to J^2}$	$\mathrm{gB}\big\vert_{J \to J^2}$	gB	PC, T
SpR	19	SpR	SpR	SpR	P, C, T
SB_5	17	$\mathrm{SB}_5\big\vert_{J \to J^2}$	$C(SB_5) = T(SB_5)\big\vert_{J \to J^2}$	$T(SB_5)$	PCT
$17V_{1a}$	17	$17V_{1a}$	$17V_{1a}$	$T(17V_{1a})$	P, C
$17V_{1b}$	17	$17V_{1b}\big\vert_{I \to -I}$	$C(17V_{1b})$	$T(17V_{1b})$	–
$17V_2$	17	$17V_2$	$17V_2$	$T(17V_2)$	P, C
$14V_1$	14	$P(14V_1)$	$C(14V_1) = P(14V_1)$	$T(14V_1)$	PC
$14V_2$	14	$P(14V_2)$	$C(14V_2) = P(14V_2)$	$T(14V_2)$	PC

Table 1: Actions of P, C, T on the Hamiltonians

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