Some properties of the lambda-mu-and-or-calculus

Karim Nour, Khelifa Saber

Some properties of the $\lambda\mu^{\wedge\vee}$-calculus

Karim NOUR & **Khelifa SABER**
LAMA - Equipe LIMD
Université de Savoie
Campus scientifique
73376 Le Bourget du Lac
Karim.Nour@univ-savoie.fr khelifa.saber@gmail.com

Abstract

In this paper, we present the $\lambda\mu^{\wedge\vee}$-calculus which at the typed level corresponds to the full classical propositional natural deduction system. Church-Rosser property of this system is proved using the standardization and the finiteness developments theorem. We define also the leftmost reduction and prove that it is a winning strategy.

1 Introduction

The $\lambda\mu^{\wedge\vee}$-calculus is an extension of the $\lambda\mu$-calculus associated by the Curry-Howard correspondence to the full classical natural deduction system, it was introduced by P. De Groote [7]. In the $\lambda\mu^{\wedge\vee}$-calculus as in any other abstract reduction system, termination, confluence and standardization appear among the principal properties. The question of termination, of course for the typed $\lambda\mu^{\wedge\vee}$-calculus, was studied by many authors [4], [7], [12] and [13], however the proof of strong normalization in [7] based on the CPS-transaltion did not work, correction for this proof was given in [9].

Confluence is a very important property, it guarantees the uniqueness of the normal form (if it exists) independently of the strategy of reduction, i.e, if we are allowed to write terms which necessarily do not finish under reduction, one expects at least that the possible result is independent of the strategy of reduction. There are different methods to prove the confluence property: parallel reduction, complete development, finiteness developments and standardization... For a strongly normalizable term rewriting system, one needs only to check the local confluence which suffices when combinated with Newman lemma.

Standardization is classical and a very convenient tool, the issue is the order in which the reduction steps are performed. In a standard reduction, this is done from *left* to *right*. According to the standardization theorem, any sequence of reductions can be transformed into a standard one. We find in current literature various equivalent definitions of this notion. In our work, we adopt the one given by R. David and W. Py [5] and [16], it is very convenient and has the advantage that we do not need to formalize the notion of residus "descendant" of a redex.

The presence of the permutative reduction of the form "$((t\ [x.u, y.v])\ \varepsilon)$ reduces to $(t\ [x.(u\ \varepsilon), y.(v\ \varepsilon)])$" has certain consequences not only on the termination of the system, but also on the standardization and the confluence, since the resulting rewriting system is not orthogonal. Therefore the treatment of these two notions is not trivial at all. Intuitively standard reduction contract redexes from *external* to *internal* and from *left* to *right*. However there is more freedom in the presence of permutative reductions, a permutative redex of the form $((\mu a.t\ [x.u, y.v])\ [r.p, s.q])\ \varepsilon$ may permute to $(\mu a.t\ [x.(u\ [r.p, s.q]), y.(v\ [r.p, s.q])])\ \varepsilon$ and to $(\mu a.t\ [x.u, y.v])\ [r.(p\ \varepsilon), s.(q\ \varepsilon)]$ and both possiblities as well as the embedded μ-redex should be treated equivalently. We can not favour one over the other and consider for example, the classical one as *external* or *the leftmost* one. That would be also the same thing for the two permutatives redexes.

In this work we use a definition "à la David" which captures this intuitive notion of standardization, when restricted to the λ-calculus (resp the $\lambda\mu$-calculus) corresponds exactly to the one given in [5] (resp [16]). As an application of this definition, we prove that leftmost reduction in a sense which we determinate later is a gaining strategy.

The finiteness developments theorem says that: if we mark a set R of redex occurences in a given term t and reduce only the marked redex occurences and redex occurences which descend from marked redex occurences, the reduction process always terminates. If we reduce every marked redex occurence, then the order in which such reductions are performed does not matter, R uniquely determines a term u to which t is reduced under any complete reduction of marked redex occurences. In addition, if we mark another set R' of redex occurences in t and follow this set through a complete R-reduction, the redex occurences from R' may be transformed by substitution or copied. However, it does not matter in what way we perform a complete R-reduction, the set of redex occurences in u which descend from R' is again uniquely determined. This theorem has important consequences like the confluence property. The proof for this theorem is difficult and required a standardization theorem, this is what we establish in the major part of this work.

This paper is presented as follows. Section 2 is an introduction to the typed $\lambda\mu^{\wedge\vee}$-calculus. Section 3, contains some useful technical results, in order to well define, head and leftmost reduction. In Section 4, we define the standard reductions and prove the standardization theorem. In Section 5, we introduce a marked version of $\lambda\mu^{\wedge\vee}$-calculus, to keep traces of the residu of redexes and prove the finiteness developments theorem. We close this section by the main theorem of this paper, i.e, the confluence property.

2 Notations and definitions

Definition 2.1 *We use notations inspired by the paper [2].*

1. *Types are formulas of propositional logic built from an infinite set of propositional variables and the constant $\perp$, using the connectors $\to$, $\wedge$ and $\vee$.*

2. *Let $\mathcal{X}$ and $\mathcal{A}$ be two disjoint infinite alphabets for distinguishing the λ-variables and μ-variables respectively. We code deductions by using a set of terms $\mathcal{T}$ which extends the λ-terms and is given by the following grammars:*

$$\mathcal{T} \ := \ \mathcal{X} \mid \lambda\mathcal{X}.\mathcal{T} \mid (\mathcal{T}\ \ \mathcal{E}) \mid \langle \mathcal{T}, \mathcal{T}\rangle \mid \omega_1\mathcal{T} \mid \omega_2\mathcal{T} \mid \mu\mathcal{A}.\mathcal{T} \mid (\mathcal{A}\ \ \mathcal{T})$$
$$\mathcal{E} \ := \ \mathcal{T} \mid \pi_1 \mid \pi_2 \mid [\mathcal{X}.\mathcal{T}, \mathcal{X}.\mathcal{T}]$$

An element of the set $\mathcal{E}$ is said to be an $\mathcal{E}$-term. An $\mathcal{E}$-term in the form $(\mathcal{T}\ \mathcal{E})$ or $(\mathcal{A}\ \mathcal{T})$ is called an application.

3. *The meaning of the new constructors is given by the typing rules below where Γ (resp Δ) is a context, i.e, a set of declarations of the form $x : A$ (resp $a : A$) where x is a λ-variable (resp a is a μ-variable) and A is a formula.*

$$\frac{}{\Gamma, x : A\ \vdash x : A\ ;\ \Delta}ax$$

$$\frac{\Gamma, x : A \vdash t : B; \Delta}{\Gamma \vdash \lambda x.t : A \to B; \Delta}\to_i \qquad \frac{\Gamma \vdash u : A \to B; \Delta \quad \Gamma \vdash v : A; \Delta}{\Gamma \vdash (u\ \ v) : B; \Delta}\to_e$$

$$\frac{\Gamma \vdash u : A; \Delta \quad \Gamma \vdash v : B; \Delta}{\Gamma \vdash \langle u, v \rangle : A \wedge B; \Delta} \wedge_i$$

$$\frac{\Gamma \vdash t : A \wedge B; \Delta}{\Gamma \vdash (t \ \ \pi_1) : A; \Delta} \wedge_e^1 \quad \frac{\Gamma \vdash t : A \wedge B; \Delta}{\Gamma \vdash (t \ \ \pi_2) : B; \Delta} \wedge_e^2$$

$$\frac{\Gamma \vdash t : A; \Delta}{\Gamma \vdash \omega_1 t : A \vee B; \Delta} \vee_i^1 \quad \frac{\Gamma \vdash t : B; \Delta}{\Gamma \vdash \omega_2 t : A \vee B; \Delta} \vee_i^2$$

$$\frac{\Gamma \vdash t : A \vee B; \Delta \quad \Gamma, x : A \vdash u : C; \Delta \quad \Gamma, y : B \vdash v : C; \Delta}{\Gamma \vdash (t \ \ [x.u, y.v]) : C; \Delta} \vee_e$$

$$\frac{\Gamma \vdash t : A; \Delta, a : A}{\Gamma \vdash (a \ \ t) : \bot; \Delta, a : A} \bot_i \quad \frac{\Gamma \vdash t : \bot; \Delta, a : A}{\Gamma \vdash \mu a.t : A; \Delta} \mu$$

4. *The cut-elimination procedure corresponds to the reduction rules given below. They are those we need to obtain the subformula property.*

   (a) *Logical reduction rules:*
   - $(\lambda x.u \ \ v) \triangleright_\beta u[x := v]$
   - $(\langle t_1, t_2 \rangle \ \ \pi_i) \triangleright_{\pi_i} t_i$
   - $(\omega_i t \ \ [x_1.u_1, x_2.u_2]) \triangleright_D u_i[x_i := t]$

   (b) *Permutative reduction rule:*
   - $((t \ \ [x_1.u_1, x_2.u_2]) \ \ \varepsilon) \triangleright_\delta (t \ \ [x_1.(u_1 \ \varepsilon), x_2.(u_2 \ \varepsilon)])$

   (c) *Classical reduction rule:*
   - $(\mu a.t \ \ \varepsilon) \triangleright_\mu \mu a.t[a :=^* \varepsilon]$
     where $t[a :=^* \varepsilon]$ *is obtained from* t *by replacing inductively each subterm in the form* $(a \ v)$ *by* $(a \ (v \ \varepsilon))$.

5. *Let* ε *and* ε' *be* $\mathcal{E}$*-terms. The notation* $\varepsilon \triangleright \varepsilon'$ *means that* ε *reduces to* ε' *by using one step of the reduction rules given above. Similarly,* $\varepsilon \triangleright^* \varepsilon'$ *means that* ε *reduces to* ε' *by using some steps of the reduction rules given above. The length of the sequence reductions* $\varepsilon \triangleright^* \varepsilon'$ *is the number of the one step* $\triangleright$ *reduction.*

6. *Let* ε *and* ε' *be* $\mathcal{E}$*-terms and* r *a redex of* ε*. The notation* $\varepsilon \triangleright^r \varepsilon'$ *means that* ε *reduces to* ε' *by reducing the redex* r.

7. *An* $\mathcal{E}$*-term* ε *is said to be normal iff it has no redexes.*

3 Characterization of the $\lambda\mu^{\wedge\vee}$-terms

In this section we develop some results already present in [4]. The presence of the critical pairs for the standardization, and the fact that a $\lambda\mu^{\wedge\vee}$-term can have more than one head, unlike the λ-calculus or the $\lambda\mu$-calculus, need such results before defining the notion of standardization, head or leftmost reductions.

Definition 3.1 1. *A term* t *is said to be simple if it is a variable or an application.*

2. *Let* $\mathcal{H} = \{*_1, ..., *_n, ...\}$ *be an infinite set of holes. The set of general contexts* $\mathcal{C}$ *is given by the following grammar:*

$$\mathcal{C} := \mathcal{H} \ \mid \lambda x.\mathcal{C} \ \mid \langle \mathcal{C}, \mathcal{C} \rangle \ \mid \omega_1 \mathcal{C} \ \mid \omega_2 \mathcal{C} \ \mid \mu a.\mathcal{C}$$

We consider only general contexts with different holes, i.e, if $*_{i_1}, ..., *_{i_n}$ *are the holes of a general context* $\mathbf{C}$*, then,* $*_{i_p} \neq *_{i_q}$ *for each* $p \neq q$.

3. *Let* $\mathbf{C}$ *be a general context with holes* $*_{i_1}, ..., *_{i_n}$ *and* f *a bijection from* $\mathbb{N}$ *to* $\mathbb{N}$*, then, the general context* $f(\mathbf{C})$ *is the context* $\mathbf{C}$ *with the holes* $*_{f(i_1)}, ..., *_{f(i_n)}$*, i.e,* $f(\mathbf{C})$ *is the general context* $\mathbf{C}$ *just with a different enumeration of its holes. Let* $\mathbf{C}, \mathbf{C}'$ *be two general contexts. We said that* $\mathbf{C}$ *is equivalent to* $\mathbf{C}'$ *and denote this by* $\mathbf{C} \simeq \mathbf{C}'$*, if there exists a bijection* f *from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that* $\mathbf{C}' = f(\mathbf{C})$*. Thus, if* $\mathbf{C}' \simeq \mathbf{C}$*, then* $\mathbf{C}$ *and* $\mathbf{C}'$ *have the same number of holes.*

4. *A context is an equivalent class for the previous equivalent relation. Then we can always suppose that the* n *holes of a context are* $*_1, ..., *_n$ *in this given order.*

5. *If* $\mathbf{C}$ *is a context with holes* $*_1, ..., *_n$ *and* $t_1, ..., t_n$ *are terms, then* $\mathbf{C}[t_1, ..., t_n]$ *is the term obtained by replacing each* $*_i$ *by* t_i*. The free variables of* t_i *can be captured in the term* $\mathbf{C}[t_1, ..., t_n]$.

Lemma 3.1 1. *Let* $t_1, ..., t_n, t'_1, ..., t'_m$ *be simple terms and* $\mathbf{C}, \mathbf{C}'$ *two contexts. If* $\mathbf{C}[t_1, ..., t_n] = \mathbf{C}'[t'_1, ..., t'_m]$*, then* $\mathbf{C} = \mathbf{C}'$.

2. *Each term* t *can be uniquely written as* $\mathbf{C}[t_1, ..., t_n]$*, where* $\mathbf{C}$ *is a context and* $t_1, ..., t_n$ *are simple terms.*

3. *Let* $t = \mathbf{C}[t_1, ..., t_i, ..., t_n]$ *be a term and* r *a redex of* t_i*. If* $t \triangleright^r t'$*, then* $t_i \triangleright^r \mathbf{C'_i}[t^i_1, ..., t^i_m]$ *and* $t' = \mathbf{C}'[t_1, ..., t_{i-1}, t^i_1, ..., t^i_m, t_{i+1}, ..., t_n]$ *where* $\mathbf{C}' = \mathbf{C}[*_i := \mathbf{C_i}]$.

Proof (1) By induction on $\mathbf{C}$. (2) By induction on t. (3) By induction on $\mathbf{C}$. □

Definition 3.2 *Let* $\bar{\varepsilon} = \varepsilon_1 ... \varepsilon_n$ *a sequence of* $\mathcal{E}$*-terms.*

1. *The length of* $\bar{\varepsilon}$ *is defined by* $lg(\bar{\varepsilon}) = n$.

2. *The sequence is said to be nice iff* ε_n *is the only* $\mathcal{E}$*-term which can be in the form* $[x.u, y.v]$.

3. *The sequence* $\bar{\varepsilon}$ *is said to be normal iff each* ε_i *is normal.*

4. *We write* $\bar{\varepsilon} \triangleright \bar{\varepsilon}'$ *iff* $\bar{\varepsilon} = \varepsilon_1 ... \varepsilon_i ... \varepsilon_n$*, and* $\bar{\varepsilon}' = \varepsilon_1 ... \varepsilon'_i ... \varepsilon_n$ *where* $\varepsilon_i \triangleright \varepsilon'_i$ *for* $1 \leq i \leq n$.

Lemma 3.2 (*and definition*) *Let* t *be a simple term.*

1. *The term* t *can be uniquely written as one of the figures below.*

2. *A head of* t *(denoted as* $hd(t)$*) is a set of subterms of* t*, it is defined by the figures below.*

3. *In the cases* (1), (2), (3), (4) *and* (5) *(resp.* (6)*) the sequence* $\bar{\varepsilon}$ *(resp.* $\varepsilon\bar{\varepsilon}$*) can be empty. In the cases* (5) *and* (6) *if* $\varepsilon\bar{\varepsilon}$ *is not nice, then* $\varepsilon\bar{\varepsilon} = \bar{r}\,[y.u, z.v]\varepsilon_i\bar{s}$.

	t	$hd\ (t)$
0	x	x
1	$(a\ u)\ \bar{\varepsilon}$	a
2	$((\lambda x.u\ \varepsilon)\ \bar{\varepsilon})$	$(\lambda x.u\ \varepsilon)$
3	$((\langle u_1, u_2\rangle\ \varepsilon)\ \bar{\varepsilon})$	$(\langle u_1, u_2\rangle\ \varepsilon)$
4	$((\omega_i u\ \varepsilon)\ \bar{\varepsilon})$	$(\omega_i u\ \varepsilon)$
5	$((\mu a.u\ \varepsilon)\ \bar{\varepsilon})$	$(\mu a.u\ \varepsilon)$ *or* *any permutative redex in the form* $((\mu a.u\ \bar{r}[y.v, z.w])\ \varepsilon_i)$
6	$((x\,\varepsilon)\ \bar{\varepsilon})$	x, *if the sequence* $\varepsilon\bar{\varepsilon}$ *is nice,* *else, any permutative redex in the form* $((x\ \bar{r}[y.u, z.v])\ \varepsilon_i)$

Proof By induction on the simple term t. We have either t is a variable, either $t = (a\,u)$ or $t = (u\ \varepsilon)$. Therefore we will examine the form of u.

- If u is not a simple term, then $u = \lambda x.v$, $u = \langle u_1, u_2\rangle$, $u = \omega_i v$ or $u = \mu a.v$. All these forms give us that t is respectively in the case (2), (3), (4) or (5).

- If u is a simple term, then induction hypothesis concludes.

The uniqueness is clear. □

Remark 3.1 *1. Observe that simple terms of the form* $((x\,\varepsilon)\,\bar{\varepsilon})$ *and* $((\mu a.u\,\varepsilon)\,\bar{\varepsilon})$ *can have more than one head.* $hd(t)$ *is a non-deterministic function to subterms of* t*.*

2. Terms of the form $(\lambda x.u\ \varepsilon)$, $(\langle u_1, u_2\rangle\ \varepsilon)$ *and* $(\omega_i v\ \varepsilon)$ *are not always redexes, for example* $(\lambda x.u\ \pi_i), (\omega_i v\ u)$, $(\langle u, v\rangle\,[x, p.y, q])$, *... and obviously they can not be typed. Therefore* $hd(t)$ *in the cases* (2), (3) *and* (4) *is not always a redex. This is why, in the rest our proofs such cases are not considered, not because these terms are not typed but because there is no redex to reduce (if there are, they are in the subterms, and inductions hypotheses allowed to conclude).*

The lemma 3.2 allows the $\lambda\mu^{\wedge\vee}$-term to be characterized in the following corollary, this characterization is useful for the standardization theorem section.

Corollary 3.1 *Any term can be written in the one of the following forms* $(a\,v)\,\bar{\varepsilon}$ *or* $(t\ \bar{\varepsilon})$*, where* $t = x$, $\lambda x.u$, $\langle u_1, u_2\rangle$, $\omega_i u$ *or* $\mu a.u$ *and* $\bar{\varepsilon}$ *is a finite sequence of* $\mathcal{E}$*-terms possibly empty.*

Proof Direct consequence of the lemma 3.2. □

4 Standardization theorem

The standardization theorem is a useful result stating that if $t \rhd^* t'$, then there is a sequence of reductions from t to t' "standard" in the sense that contractions are made from left to right, possibly with some jumps in between.

Intuitively the standard reduction contracts redexes from the external to the internal and from left to right. For λ-calculus, a standard reduction from the term $(\lambda x.u\,v)\varepsilon_1...\varepsilon_n$ either starts by reducing the head redex $(\lambda x.u\,v)$, else this one will never be converted after performing u, v or the arguments ε_i, i.e, from external to internal. An other characteristic, standard reduction from the term $((x\,u)\,v)$ has to reduce firstly in u and then in v, in this given order from left to right, this last notion is not captured by our definition given bellow since we consider that reduction in u and v are independent of each other.

One of the main consequences of standardization is that normal forms, if existing, can be reached by leftmost sequence reduction.This will be a simple consequence since standard reduction is defined such a way that leftmost reduction is just a particular standard one (definition of leftmost reduction is presented in next section).

The following definition of a standard sequence of reductions is somehow a generalization of those given by R. David and W. Py in [5] and [16], in the sense that if we restrict our definition to the λ or the $\lambda\mu$-calculus, it captures exactly their definitions .

Definition 4.1 *1. Let $\bar{w} = \bar{r}[x.p, y.q]\varepsilon\bar{s}$ where $\bar{r}$ and $\bar{s}$ are possibly empty. We define a new reduction relation $\succ$ by: $\bar{w} \succ \bar{r}[x.(p\,\varepsilon), y.(q\,\varepsilon)]\bar{s}$. As usual $\succ^*$ denotes the reflexive and transitive closure of $\succ$.*

2. Let ε, ε' be two $\mathcal{E}$-terms such that $\varepsilon \rhd \varepsilon_1 \rhd ... \rhd \varepsilon_n = \varepsilon'$. We denote $\kappa(\varepsilon \rhd^* \varepsilon') = (n, c)$ where c stands for the complexity of ε (n is the length of the reduction $\varepsilon \rhd^* \varepsilon'$). We say that this sequence reduction is standard and we write $\varepsilon \rhd^*_{st} \varepsilon'$ iff it is obtained as follows. This definition is given by induction on the ordered lexicographic pair (n, c). We use simultaneously the following abreviations:

$\bar{\varepsilon} = \varepsilon_1...\varepsilon_n \rhd^*_{st} \bar{\varepsilon}' = \varepsilon'_1...\varepsilon'_n$ means that $\varepsilon_i \rhd^*_{st} \varepsilon'_i$ for each $1 \leq i \leq n$.

(C_λ) *If $\varepsilon = (\lambda x.u\,\bar{\varepsilon})$, then $\varepsilon' = (\lambda x.u'\,\bar{\varepsilon}')$ with $u \rhd^*_{st} u'$ and $\bar{\varepsilon} \rhd^*_{st} \bar{\varepsilon}'$.*

(C_μ) *If $\varepsilon = (\mu a.u\,\bar{\varepsilon})$, then $\varepsilon' = (\mu a.u'\,\bar{\varepsilon}')$ with $u \rhd^*_{st} u'$ and $\bar{\varepsilon} \rhd^*_{st} \bar{\varepsilon}'$.*

(C_π) *If $\varepsilon = (\langle u_1, u_2\rangle\,\bar{\varepsilon})$, then ($\varepsilon' = \langle u'_1, u'_2\rangle\,\bar{\varepsilon}'$) with $u_i \rhd^*_{st} u'_i$ and $\bar{\varepsilon} \rhd^*_{st} \bar{\varepsilon}'$.*

(C_ω) *If $\varepsilon = (\omega_i u\,\bar{\varepsilon})$, then $\varepsilon' = (\omega_i\, u'\bar{\varepsilon}')$ with $u \rhd^*_{st} u'$ and $\bar{\varepsilon} \rhd^*_{st} \bar{\varepsilon}'$.*

(V_λ) *If $\varepsilon = (x\,\bar{\varepsilon})$, then $\varepsilon' = (x\,\bar{\varepsilon}')$ with $\bar{\varepsilon} \rhd^*_{st} \bar{\varepsilon}'$.*

(V_μ) *If $\varepsilon = (a\,u)$, then $\varepsilon' = (a\,u')$ with $u \rhd^*_{st} u'$.*

- *If $\varepsilon = (t\,\bar{\varepsilon})$, where t is a simple term in the form $(\lambda x.u\,v)$ (resp. $(\langle u_1, u_2\rangle\,\pi_i)$, $(\mu a.u\;\epsilon)$, $(\omega_i u\;[x_1.v_1, x_2.v_2])$, x, a), we denote by r its possible reductom $u[x := v]$ (resp. u_i, $\mu a.u[a :=^* \epsilon]$, $v_i[x_i := u]$).*

  ($\Rightarrow$) *Either $\varepsilon \rhd (r\,\bar{\varepsilon}) \rhd^*_{st} \varepsilon'$,*

  ($\succ_\delta$) *Either $\varepsilon \rhd (t\,\bar{\varepsilon}') \rhd^*_{st} \varepsilon'$ with $\bar{\varepsilon} \succ \bar{\varepsilon}'$.*

  ($\succ_\mu$) *Either $\varepsilon \rhd (\mu a.u\,\bar{\varepsilon}') \rhd^*_{st} \varepsilon'$ with $\epsilon\bar{\varepsilon} \succ \bar{\varepsilon}'$, and this only if $t = (\mu a.u\;\epsilon)$.*

  ($\succ_\omega$) *Or $\varepsilon \rhd (\omega_i u\,\bar{\varepsilon}') \rhd^*_{st} \varepsilon'$ with $[x_1.v_1, x_2.v_2]\bar{\varepsilon} \succ \bar{\varepsilon}'$, and this only if $t = (\omega_i u\;[x_1.v_1, x_2.v_2])$.*

- *If $\varepsilon = \pi_i$, then $\varepsilon' = \pi_i$.*
- *If $\varepsilon = [x_1.u_1, x_2.u_2]$, then $\varepsilon' = [x_1.u'_1, x_2.u'_2]$ with $u_i \rhd^*_{st} u'_i$.*

Remark 4.1 *1. In the rules $C_\lambda, ..., C_\omega$ the sequence $\bar{\varepsilon}$ is possibly empty and this corresponds to the cases where ε is not a simple term.*

2. Given a $\lambda\mu^{\wedge\vee}$-term of the form $(t\,\bar{\varepsilon})$ standard reduction strategies can start with various permutations using the rule $\succ_\delta$ ($\succ_\mu$, $\succ_\omega$ according to the form of t) until either we obtain a nice sequence and then reducing the possible head redex by the rule $\Rightarrow$ or start performing subterms of the arguments using the rules V_λ, V_μ or the rules $C_\lambda, ..., C_\omega$.

Lemma 4.1 *Assume that $\varepsilon \rhd^*_{st} \varepsilon'$.*

1. If $v \rhd^*_{st} v'$, then $\varepsilon[x := v] \rhd^*_{st} \varepsilon'[x := v']$.

2. If $\epsilon \rhd^*_{st} \epsilon'$, then $\varepsilon[a :=^* \epsilon] \rhd^*_{st} \varepsilon'[a :=^* \epsilon']$.

Proof Only the second assertion will be treated. By induction on $\kappa(\varepsilon \triangleright^*_{st} \varepsilon')$. We give just the case where the substitution intervenes. If $\varepsilon = (a\ u)$, then $\varepsilon' = (a\ u')$ where $u \triangleright^*_{st} u'$, thus, $\varepsilon[a :=^* \epsilon] = (a\ (u[a :=^* \epsilon]\ \epsilon))$ and $\varepsilon'[a :=^* \epsilon'] = (a\ (u'[a :=^* \epsilon']\ \epsilon'))$. By induction hypothesis, we have $u[a :=^* \epsilon] \triangleright^*_{st} u'[a :=^* \epsilon']$, then, by definition, $(a\ (u[a :=^* \epsilon]\ \epsilon)) \triangleright^*_{st} (a\ (u'[a :=^* \epsilon']\ \epsilon'))$. □

Theorem 4.1 (Standardization theorem) *If $\varepsilon \triangleright^* \varepsilon'$, then $\varepsilon \triangleright^*_{st} \varepsilon'$.*

Proof By induction on the length of the reduction $\varepsilon \triangleright^* \varepsilon'$ it suffices to prove the following lemma. □

Lemma 4.2 *If $\varepsilon \triangleright^*_{st} \varepsilon' \triangleright \varepsilon''$, then $\varepsilon \triangleright^*_{st} \varepsilon''$.*

Proof By induction on $\kappa(\varepsilon \triangleright^*_{st} \varepsilon') = (n, c)$. We examine how $\varepsilon \triangleright^*_{st} \varepsilon'$ following the different forms of ε.

- The cases where ε is not a simple term are direct consequences of the induction hypothesis (decreasing of c).
- If ε is a simple term, we will examine some cases, the others are similar or more simpler.

(C_λ) Let $\varepsilon = (\lambda x.u\ \bar{\varepsilon}) \triangleright^*_{st} (\lambda x.u'\ \bar{\varepsilon}') = \varepsilon' \triangleright \varepsilon''$, with $u \triangleright^*_{st} u'$ and $\bar{\varepsilon} \triangleright^*_{st} \bar{\varepsilon}'$ (here of course $\bar{\varepsilon}$ is not empty since ε is simple). We distinguish three cases:

- If $\varepsilon'' = (\lambda x.u''\ \bar{\varepsilon}')$ (resp. $(\lambda x.u'\ \bar{\varepsilon}'')$), where $u' \triangleright u''$ (resp. $\bar{\varepsilon}' \triangleright \bar{\varepsilon}''$), then the induction hypothesis concludes.
- If $\varepsilon'' = (u'[x := v']\ \varepsilon'_2 ... \varepsilon'_n)$, then $\bar{\varepsilon} = v\varepsilon_2...\varepsilon_n$ where $v \triangleright^*_{st} v'$, $\varepsilon_i \triangleright^*_{st} \varepsilon'_i$ and $\bar{\varepsilon}' = v'\varepsilon'_2...\varepsilon'_n$. Therefore, by the lemma 4.1, $u[x := v] \triangleright^*_{st} u'[x := v']$, thus $(u[x := v]\ \varepsilon_2...\varepsilon_n) \triangleright^*_{st} (u'[x := v']\ \varepsilon'_2...\varepsilon'_n)$. Finally, by the rule $(\Rightarrow)$, we have $\varepsilon \triangleright (u[x := v]\ \varepsilon_2...\varepsilon_n)$ $\triangleright^*_{st} (u'[x := v']\ \varepsilon'_2...\varepsilon'_n)$ is a standard sequence of reductions.
- The last case is $\varepsilon'' = (\lambda x.u'\ \bar{\varepsilon}'')$ where $\bar{\varepsilon}' \succ \bar{\varepsilon}''$. Therefore $\bar{\varepsilon} = \varepsilon_1...[y.p, z.q]\varepsilon_i...\varepsilon_n \triangleright^*_{st} \varepsilon'_1...[y.p', z.q']\varepsilon'_i...\varepsilon'_n$ and then $\bar{\varepsilon}'' = \varepsilon'_1...[y.(p'\ \varepsilon'_i), z.(q'\ \varepsilon'_i)]...\varepsilon'_n$. We have $[y.(p\,\varepsilon_i), z.(q\,\varepsilon_i)] \triangleright^*_{st} [y.(p'\ \varepsilon'_i), z.(q'\ \varepsilon'_i)]$, hence the rule $(\succ_\delta)$ allows to conclude: $\varepsilon \triangleright (\lambda x.u\,\varepsilon_1...[y.(p\,\varepsilon_i), z.(q\,\varepsilon_i)]...\varepsilon_n) \triangleright^*_{st}$ $(\lambda x.u'\,\varepsilon'_1...[y.(p'\ \varepsilon'_i), z.(q'\ \varepsilon'_i)]...\varepsilon'_n) = \varepsilon''$.

(C_μ) , (C_π) and (C_ω) are similar to the previous case.

(V_λ) Let $\varepsilon = (x\,\bar{\varepsilon})$, then $\varepsilon = (x\,\bar{\varepsilon}) \triangleright^*_{st} \varepsilon' = (x\,\bar{\varepsilon}') \triangleright (x\,\bar{\varepsilon}'') = \varepsilon''$, and

- Either $\bar{\varepsilon} = \varepsilon_1...\varepsilon_i...\varepsilon_n \triangleright^*_{st} \varepsilon'_1...\varepsilon'_i...\varepsilon'_n = \bar{\varepsilon}' \triangleright \varepsilon'_1...\varepsilon''_i...\varepsilon'_n = \bar{\varepsilon}''$, hence, by induction hypothesis, $\varepsilon_i \triangleright^*_{st} \varepsilon''_i$. Therefore, we have $\bar{\varepsilon} \triangleright^*_{st} \bar{\varepsilon}''$ and $\varepsilon = (x\,\bar{\varepsilon}) \triangleright^*_{st} (x\,\bar{\varepsilon}'') = \varepsilon''$.
- Or $\bar{\varepsilon} = \varepsilon_1...[y.p, z.q]\varepsilon_i...\varepsilon_n \triangleright^*_{st} \varepsilon'_1...[y.p', z.q']\varepsilon'_i...\varepsilon'_n = \bar{\varepsilon}' \succ$ $\varepsilon'_1...[y.(p'\ \varepsilon'_i), z.(q'\ \varepsilon'_i)]...\varepsilon'_n = \bar{\varepsilon}''$.
  We have $[y.(p\,\varepsilon_i), z.(q\,\varepsilon_i)] \triangleright^*_{st} [y.(p'\ \varepsilon'_i), z.(q'\ \varepsilon'_i)]$, therefore, by the rule $(\succ_\delta)$, $(x\,\bar{\varepsilon}) \triangleright (x\,\varepsilon_1...[y.(p\,\varepsilon_i), z.(q\,\varepsilon_i)]...\varepsilon_n) \triangleright^*_{st}$ $(x\,\varepsilon'_1...[y.(p'\ \varepsilon'_i), z.(q'\ \varepsilon'_i)]...\varepsilon'_n) = (x\,\bar{\varepsilon}'')$ is a standard sequence of reductions.

(V_μ) is a direct consequence of the induction hypothesis (decreasing of c).

$(\Rightarrow)$,$(\succ_\delta)$, $(\succ_\mu)$ and $(\succ_\omega)$ are direct consequences of the application of the induction hypothesis (decreasing of n).

□

Remark 4.2 *1. In [10], F. Joachimski and R. Matthes presented the ΛJ-calculus which is an extension of the λ-calculus by a generalized application that gives rise to permutative reductions, the resulting rewriting system is not orthogonal (this is also the case of the $\lambda\mu^{\wedge\vee}$-calculus). The definition of the standardization established here produced exactly the same treatment of the logical and the classical reductions ($\triangleright_\beta$, $\triangleright_{\pi_i}$, $\triangleright_D$ and $\triangleright_\mu$) like that of β-reduction in [10], this means that to avoid difficulties relative to μ-reduction, we need to treat it similarly as β-reduction. The ΛJ-calculus has served us as a model for studying our $\lambda\mu^{\wedge\vee}$-calculus. In fact, concerning the standardization, the restriction of the $\lambda\mu^{\wedge\vee}$-calculus to the $\lambda\mu$-one can serve as a model for the study of the rewriting system with permutative and structural reductions, as well as the ΛJ-calculus serves as a minimal model for the study of term rewriting systems with permutation.*

2. In the definition 4.1, we can be "more restrictive", by this we mean that, we have a strong standardization theorem when replacing the rules $\succ_\delta$ and $\succ_\omega$ by:

($\succ_\lambda$) $\varepsilon = ((\lambda x.u\, v)\,\bar{\varepsilon}) \triangleright ((\lambda x.u\, v)\,\bar{\varepsilon}') \triangleright^*_{st} ((\lambda x.u'\, v')\,\bar{\varepsilon}''') = \varepsilon'$ *with* $u \triangleright^*_{st} u'$, $v \triangleright^*_{st} v'$ *and* $\bar{\varepsilon} \succ \bar{\varepsilon}' \succ^* \bar{\varepsilon}'' \triangleright^*_{st} \bar{\varepsilon}'''$.

($\succ_\pi$) $\varepsilon = ((\langle u_1, u_2\rangle\, \pi_i)\,\bar{\varepsilon}) \triangleright ((\langle u_1, u_2\rangle\, \pi_i)\,\bar{\varepsilon}') \triangleright^*_{st} ((\langle u'_1, u'_2\rangle\, \pi_i)\,\bar{\varepsilon}''') = \varepsilon'$ *with* $u_i \triangleright^*_{st} u'_i$ *and* $\bar{\varepsilon} \succ \bar{\varepsilon}' \succ^* \bar{\varepsilon}'' \triangleright^*_{st} \bar{\varepsilon}'''$.

($\succ_D$) $\varepsilon = ((\omega_i u\,[x_1.v_1, x_2.v_2])\,\bar{\varepsilon}) \triangleright ((\omega_i u\,\bar{\varepsilon}') \triangleright^*_{st} (\omega_i u'\,\bar{\varepsilon}''') = \varepsilon'$ *with* $u \triangleright^*_{st} u'$ *and* $[x_1.v_1, x_2.v_2]\bar{\varepsilon} \succ \bar{\varepsilon}' \succ^* \bar{\varepsilon}'' \triangleright^*_{st} \bar{\varepsilon}'''$.

This can be explained by the fact that:

- *For the rules $\succ_\lambda$ and $\succ_\pi$, there are no interactions between v, π_i and $\bar{\varepsilon}$ via permutative reductions.*
- *For the rule $\succ_D$, even there are interactions between $[x_1.v_1, x_2.v_2]$ and $\bar{\varepsilon}$ via permutative reductions, after a $\triangleright_D$-reduction we will always get $(v_i\,\bar{\varepsilon})[x_i := u] = (v_i[x_i := u]\,\bar{\varepsilon})$ since x_i is not free in $\bar{\varepsilon}$.*

Contrary to the case $\varepsilon = ((\mu a.u\,[x.p, y.q])\,[r.k, s.l]\epsilon)$ in which there are more complications. Suppose that we give the priority to the classical redex, i.e, if it is not converted in the beginning then it will be never performed, thus the lemma 4.2 does not hold for this sequence of reductions:
$\varepsilon \triangleright ((\mu a.u\,[x.p, y.q])\,[r.(k\,\epsilon), s.(l\,\epsilon)])$
$\triangleright (\mu a.u\,[x.(p\,[r.(k\,\epsilon), s.(l\,\epsilon)]), y.(q\,[r.(k\,\epsilon), s.(l\,\epsilon)])])$
$\triangleright^*_{st} (\mu a.u\,[x.p', y.q']) = \varepsilon' \triangleright \mu a.u[a :=^* [x.p', y.q']]] = \varepsilon''$,
where $(p\,[r.(k\,\epsilon), s.(l\,\epsilon)]) \triangleright^*_{st} p'$ and $(q\,[r.(k\,\epsilon), s.(l\,\epsilon)]) \triangleright^*_{st} q'$, for the simple reason which is: we do not know how these two standard sequence reductions are made $(p\,[r.(k\,\epsilon), s.(l\,\epsilon)]) \triangleright^*_{st} p'$ and $(q\,[r.(k\,\epsilon), s.(l\,\epsilon)]) \triangleright^*_{st} q'$ (it depends on p and q). To resolve this problem we have to consider the rule $\succ_\mu$.

3. In the rest of this paper, we consider only typed terms, although the results that we prove here (normalisation of the leftmost reduction and the finiteness developments theorem) do not use neither the strong normalization theorem (hard to prove in this context) nor the confluence property (which is here a consequence the finiteness developements theorem). Untyped terms will not be considered for reasons of ease of reading. Note that techniques that we develop here allow to prove our results for all the calculus, but this requires modifications of some definitions (the head of a term, the head normal form, ...) and the treatment of more cases in proofs.

5 Head and leftmost reductions

In the λ-calculus, the leftmost reduction of a term consists in reducing its lefmost redex. Despite its laziness, this reduction has the advantage that is a winning strategy. By this, we means that: if a term t is normalizable, then leftmost reduction of t always reaches its normal form. In the $\lambda\mu^{\wedge\vee}$-calculus, a term can have more than one head, the thing which prevents us from defining the leftmost reduction in the sameway as in λ-calculus. Therefore, we start this section by introducing the notion of head reduction (which is also different from the one of the λ-calculus). This allows, first, to define the leftmost reduction as an iteration of head reduction. Then, to prove that a sequence of leftmost reductions is a standard one. Finally, to prove that the definition of the leftmost reduction which we give, provides a gaining strategy too.

Definition 5.1 *1. Let $\mathbf{C}[t_1,...,t_n]$ as in (2) of the lemma 3.1. A one step head reduction of a simple term t consists in reducing a head redex if any. We denote $t \triangleright_{hd} t'$ if t is reduced to t' by a head reduction. A one step head reduction of a term $\mathbf{C}[t_1,...,t_n]$ corresponds to a one step head reduction of one of the simple terms t_i $(1 \leq i \leq n)$. We denote by $\triangleright^*_{hd}$ the reflexive and transitive closure of $\triangleright_{hd}$.*

2. A simple head normal form is a simple term in the form $(x\ \bar{\varepsilon})$ where $\bar{\varepsilon}$ is a nice sequence, or in the form $(a\ u)$, the elements of the sequence $\bar{\varepsilon}$ (resp u) are called the arguments of the head variable x (resp a). The sequence $\bar{\varepsilon}$ is a nice one, and these are the only cases where we cannot reduce in the head, because there is no head since the arguments of the sequence cannot interacte between them via permutative reductions. A head normal form is a term in the form $\mathbf{C}[t_1,...,t_n]$ where all the t_i are simple head normal forms.

Remark 5.1 *Observe that there is no unicity of "the" head normal form.*
Take the simple term $t = ((x\,[y.u, z.v])\,[r.p, s.q]\varepsilon)$, then
$t \triangleright^*_{hd} (x\,[y.((u\,[r.p, s.q])\,\varepsilon), z.((v\,[r.p, s.q])\,\varepsilon)]) = t_1$,
$t \triangleright^*_{hd} (x\,[y.(u\,[r.(p\,\varepsilon), s.(q\,\varepsilon)]), z.(v\,[r.(p\,\varepsilon), s.(q\,\varepsilon)])]) = t_2$,
and both of t_1 and t_2 are head normal froms.

Definition 5.2 *A leftmost reduction of a term t consists to apply a head reduction on t until its head normal form $\mathbf{C}[t_1,...,t_n]$ (if it exists) and reiterate it on the arguments of t_i. We denote $t \triangleright^*_L t'$ if t is reduced to t' by a leftmost reduction. When an argument of a head variable is an $\mathcal{E}$-term in the form $[x.u, y.v]$, the reduction consists simply to reduce in u and v. If $t \triangleright^*_L t'$, then we denote $\kappa(t \triangleright^*_L t') = (n, c)$ where n stands for the length of the reduction $t \triangleright^*_L t'$ and c the complexity of t.*

The following theorem says that every sequence of leftmost reductions is a standard one.

Theorem 5.1 *If $t \triangleright^*_L t'$, then this sequence of reductions is a standard one.*

Proof By induction on $\kappa(t \triangleright^*_L t')$.

- The cases where t is not a simple term are direct consequences of the induction hypothesis. For example, if $t = \lambda x.u$, then $t' = \lambda x.u'$ where $u \triangleright^*_L u'$. Therefore, by induction hypothesis (decreasing of the complexity c), $u \triangleright^*_L u'$ is a standard sequence reductions, thus $\lambda x.u \triangleright^*_L \lambda x.u'$ is a standard one too.

- The cases where t is a simple term:

- If $t = (\mu a.u\,\varepsilon)\,\bar{\varepsilon}$, then either $t \rhd_{hd} (\mu a.u[a :=^* \varepsilon]\,\bar{\varepsilon}) \rhd_L^* t'$ or $t \rhd_{hd} ((\mu a.u\,\epsilon)\,\bar{r}) \rhd_L^* t'$ where $\varepsilon\bar{\varepsilon} \succ \epsilon\bar{r}$. Therefore, by induction hypothesis (decreasing of the length lg), $(\mu a.u[a :=^* \varepsilon]\,\bar{\varepsilon}) \rhd_L^* t'$ and $((\mu a.u\,\epsilon)\,\bar{r}) \rhd_L^* t'$ are two standard sequence reductions, thus, by definition, both of $t \rhd_{hd} (\mu a.u[a :=^* \varepsilon]\,\bar{\varepsilon}) \rhd_L^* t'$ and $t \rhd_{hd} ((\mu a.u\,\epsilon)\,\bar{r}) \rhd_L^* t'$ are standard sequence reductions too.
- The other cases are similar to the previous.

□

This theorem gives another way to define a leftmost reduction similarly to the definition of the standard reduction. The following definition (theorem) can be easily proven to be "equivalent" to the one given above.

Theorem 5.2 (*definition*) *Let ε, ε' be two $\mathcal{E}$-terms (resp. $\bar{\varepsilon}$ and $\bar{\varepsilon}'$ two finite sequences of $\mathcal{E}$-terms) such that $\varepsilon \rhd^* \varepsilon'$ (resp. $\bar{\varepsilon} \rhd^* \bar{\varepsilon}'$). We say that $\varepsilon \rhd_l^* \varepsilon'$ (resp. $\bar{\varepsilon} \rhd_l^* \bar{\varepsilon}'$) iff it is obtained as follows. This definition is given by induction on the ordered lexicographic pair $\kappa(\varepsilon \rhd^* \varepsilon') = (n, c)$ (resp. $\kappa(\bar{\varepsilon} \rhd^* \bar{\varepsilon}') = (n, c)$) where n is the length of the reduction $\varepsilon \rhd^* \varepsilon'$ (resp. $\bar{\varepsilon} \rhd^* \bar{\varepsilon}'$) and c stands for the complexity of ε (resp. $\bar{\varepsilon}$).*

1. *$\varepsilon \rhd_l^* \varepsilon'$*
   - *If $\varepsilon = \lambda x.u$, then $\varepsilon' = \lambda x.u'$ with $u \rhd_l^* u'$.*
   - *If $\varepsilon = \mu a.u$, then $\varepsilon' = \mu a.u'$ with $u \rhd_l^* u'$.*
   - *If $\varepsilon = \langle u, v\rangle$, then $\varepsilon' = \langle u', v'\rangle$ with $u \rhd_l^* u'$ and $v \rhd_l^* v'$.*
   - *If $\varepsilon = \omega_i u$, then $\varepsilon' = \omega_i u'$ with $u \rhd_l^* u'$.*
   - *If $\varepsilon = (a\,u)$, then $\varepsilon' = (a\,u')$ with $u \rhd_l^* u'$.*
   - *If $\varepsilon = (x\,\bar{\varepsilon})$, then $\varepsilon' = (x\,\bar{\varepsilon}')$ with $\bar{\varepsilon} \rhd_l^* \bar{\varepsilon}'$.*
   - *If $\varepsilon = ((\lambda x.u\,v)\,\bar{\varepsilon})$, then $\varepsilon \rhd_l (u[x := v]\,\bar{\varepsilon}) \rhd_l^* \varepsilon'$.*
   - *If $\varepsilon = ((\mu a.u\;\epsilon)\,\bar{\varepsilon})$, then*
     - *Either $\varepsilon \rhd (\mu a.u[a :=^* \epsilon]\,\bar{\varepsilon}) \rhd_l^* \varepsilon'$.*
     - *Or $\varepsilon \rhd ((\mu a.u\,\theta)\,\bar{s}) \rhd_l^* \varepsilon'$ with $\epsilon \succ \theta$.*
   - *If $\varepsilon = ((\langle u_1, u_2\rangle\;\pi_i)\,\bar{\varepsilon})$, then $\varepsilon \rhd (u_i\,\bar{\varepsilon}) \rhd_l^* \varepsilon'$,*
   - *If $\varepsilon = ((\omega_i u\;[x_1.v_1, x_2.v_2])\,\bar{\varepsilon})$, then $\varepsilon \rhd (v_i[x_i := u]\,\bar{\varepsilon}) \rhd_l^* \varepsilon'$.*
   - *If $\varepsilon = \pi_i$, then $\varepsilon' = \pi_i$.*
   - *If $\varepsilon = [x_1.u_1, x_2.u_2]$, then $\varepsilon' = [x_1.u_1', x_2.u_2']$ with $u_i \rhd_l^* u_i'$.*
2. *$\bar{\varepsilon} \rhd_l^* \bar{\varepsilon}'$*
   - *If $\bar{\varepsilon}$ is not nice, then $\bar{\varepsilon} \succ \bar{s} \rhd_l^* \bar{\varepsilon}'$.*
   - *Else $\bar{\varepsilon} = \varepsilon_1 ... \varepsilon_n$, then $\bar{\varepsilon}' = \varepsilon_1' ... \varepsilon_n'$ with $\varepsilon_i \rhd_l^* \varepsilon_i$ for each $1 \leq i \leq n$.*

We have $\rhd_l^*$ and $\rhd_L^*$ are the same reduction.

6 Finiteness of developments

In the λ-calculus, the finiteness developments theorem stipulates that: for a given term t and a subset R of redexes of t, if we reduce only redexes which belong to R or their *residus*, then any sequence of reductions terminates. Based on this theorem and the local confluence of β-reduction, R. David [5] gave a short proof of the confluence theorem of β-reduction. The notion of *residus* of redex is enough clear in λ-calculus, roughly speaking a *residu* of redex is a redex which is not created, thus if at the beginning in the term t we mark each redex which belongs to R, then we can reduce only these marked redexes. Of course when such redex is copied or transformed by some substitutions, we consider that the new redex is not a created one since it keeps its mark. The same idea will be developed in the next paragraph. First, let us take an example to illustrate this.

Example 6.1 *Let* $t = ((\mu a.u\ [y.p, z.q])\, \varepsilon)$ *and* $R = \{t, (\mu a.u\ [y.p, z.q])\}$, *then* $t \triangleright (\mu a.u\ [y.(p\, \varepsilon), z.(q\, \varepsilon)]) = t_1$ *and* $t \triangleright (\mu a.u[a :=^* [y.p, z.q]]\ \varepsilon) = t_2$. *Observe that if we want to obtain a commun redectum from* t_1 *and* t_2, *we have to reduce some redexes in* t_1 *and* t_2. *Implicitely these redexes are the residus of redexes in* R, *they must not be seen as created redexes. The same term* t *but where the redexes in* R *are marked gives* $t = ((\bar{\mu} a.u\ \lfloor y.p, z.q \rfloor)\, \varepsilon)$.

$t \triangleright (\bar{\mu} a.u\ \lfloor y.(p\, \varepsilon), z.(q\, \varepsilon)\rfloor) \triangleright \bar{\mu} a.u[a :=^* \lfloor y.(p\, \varepsilon), z.(q\, \varepsilon)\rfloor]$

$t \triangleright (\bar{\mu} a.u[a :=^* \lfloor y.p, z.q\rfloor]\, \varepsilon) \triangleright \bar{\mu} a.u[a :=^* \lfloor y.p, z.q \rfloor\ \varepsilon] \triangleright^* \bar{\mu} a.u[a :=^* \lfloor y.(p\, \varepsilon), z.(q\, \varepsilon)\rfloor]$

We mark the commutative (resp classical) redex t *(resp.* $(\mu a.u\ [y.p, z.q])$*) in this way:* $[,]$ *(resp.* μ *) become* $\lfloor , \rfloor$ *(resp.* $\bar{\mu}$*). The commun redecum is the term (without marks)* $\mu a.u[a :=^* [y.(p\, \varepsilon), z.(q\, \varepsilon)]]$.

Therefore for the purpose of this section, we introduce a marked version of the $\lambda\mu^{\wedge\vee}$-calculus.

6.1 The marked terms

Definition 6.1

1. *We extend the syntax of the* $\lambda\mu^{\wedge\vee}$*-calculus by adding new symbols* $\bar{\lambda}$, $\bar{\mu}$, $< . , . >$, Ω_i *and* $\lfloor . , . \rfloor$. *These symbols are called marks. The sets* $\mathbb{T}$ *and* $\mathbb{E}$ *of marked terms are defined by the following grammars:*

$$\mathbb{T} := \mathcal{T} \mid (\bar{\lambda}\mathcal{X}.\mathbb{T}\ \mathbb{T}) \mid (< \mathbb{T}, \mathbb{T} >\ \pi_i) \mid (\Omega_i \mathbb{T}\ [\mathcal{X}.\mathbb{T}, \mathcal{X}.\mathbb{T}]) \mid (\Omega_i \mathbb{T}\ \lfloor \mathcal{X}.\mathbb{T}, \mathcal{X}.\mathbb{T} \rfloor) \mid (\mathbb{T}\ \mathbb{E}) \mid \bar{\mu} a.\mathbb{T}$$

$$\mathbb{E} := \mathcal{E} \mid \mathbb{T} \mid\ \lfloor \mathcal{X}.\mathbb{T}, \mathcal{X}.\mathbb{T} \rfloor$$

2. *The reduction rule* $\blacktriangleright$ *of* $\mathbb{E}$ *consists in the union of the following reduction rules. The meaning of these new reductions is to capture the definition of the finiteness developments, where we reduce only redexes at the beginning or only their residus (which will be exactly the marked redexes).*

   - $(\bar{\lambda} x.t\ u) \blacktriangleright_\beta t[x := u]$
   - $(< t_1, t_2 >\ \pi_i) \blacktriangleright_{\pi_i} t_i$
   - $(\Omega_i t\ [x_1.u_1, x_2.u_2]) \blacktriangleright_D u_i[x_i := t]$
   - $(\Omega_i t\ \lfloor x_1.u_1, x_2.u_2 \rfloor) \blacktriangleright_\Omega u_i[x_i := t]$
   - $((\Omega_i t\ \lfloor x_1.u_1, x_2.u_2 \rfloor)\ \varepsilon) \blacktriangleright_\Delta (\Omega_i t\ \lfloor x_1.(u_1\ \varepsilon), x_2.(u_2\ \varepsilon)\rfloor)$
   - $((t\ \lfloor x_1.u_1, x_2.u_2 \rfloor)\ \varepsilon) \blacktriangleright_\delta (t\ \lfloor x_1.(u_1\ \varepsilon), x_2.(u_2\ \varepsilon)\rfloor)$
   - $(\bar{\mu} a.t\ \varepsilon) \blacktriangleright_\mu \bar{\mu} a.t[a :=^* \varepsilon]$

   We denote by $\blacktriangleright^*$ *the reflexive and transitive closure of* $\blacktriangleright$.

3. *Let* $t \in \mathcal{T}$ *and* r *a redex of* t, *we define* $\hat{r}$ *the marked redex obtained by making* r *as follows:*

   - *If* $r = (\lambda x.u\ v)$, *then* $\hat{r} = (\bar{\lambda} x.u\ v)$
   - *If* $r = (\langle t_1, t_2 \rangle\ \pi_i)$, *then* $\hat{r} = (< t_1, t_2 >\ \pi_i)$

- *If* $r = (\omega_i u\ [x_1.u_1, x_2.u_2])$, *then* $\hat{r} = (\Omega_i u\ [x_1.u_1, x_2.u_2])$
- *If* $r = ((u\ [x_1.u_1, x_2.u_2])\ \varepsilon)$, *then* $\hat{r} = ((u\ \lfloor x_1.u_1, x_2.u_2 \rfloor)\ \varepsilon)$
- *If* $r = (\mu a.t\ \varepsilon)$, *then* $\hat{r} = (\bar{\mu} a.t\ \varepsilon)$

We denote $red(t)$ *the set of all redexes of* t. *Let* R *be a subset of* $red(t)$, *we define* $\hat{t}_R$ *the marked term obtained from* t *by marking each redex of* t *which belongs to* R. *We said that* $\hat{t}_R$ *is the* R*-corresponding marked term to* t.

4. *Let* $\varepsilon \in \mathbb{E}$, *we define the* $\mathcal{E}$*-term* $\check{\varepsilon}$ *by induction on* ε:

- *If* $\varepsilon \in \mathcal{E}$, *then* $\check{\varepsilon} = \varepsilon$
- *If* $\varepsilon = (\bar{\lambda} x.u\ v)$, *then* $\check{\varepsilon} = (\lambda x.\check{u}\ \check{v})$
- *If* $\varepsilon = (< t_1, t_2 >\ \pi_i)$, *then* $\check{\varepsilon} = (\langle \check{t}_1, \check{t}_2 \rangle\ \pi_i)$
- *If* $\varepsilon = (\Omega_i t\ [x.u, y.v])$ *or* $(\Omega_i t\ \lfloor x.u, y.v \rfloor)$, *then* $\check{\varepsilon} = (\omega_i \check{t}\ [x.\check{u}, y.\check{v}])$
- *If* $\varepsilon = (t\ \epsilon)$, *then* $\check{\varepsilon} = (\check{t}\ \check{\epsilon})$
- *If* $\varepsilon = \bar{\mu} a.u$, *then* $\check{\varepsilon} = \mu a.\check{u}$
- *If* $\varepsilon = \lfloor x.u, y.v \rfloor$, *then* $\check{\varepsilon} = [x.\check{u}, y.\check{v}]$

The operation ˇ *quite simply consists in projecting any marked term in the set* $\mathcal{E}$, *i.e, to consider any marked term as any other term without marks.*

Example 6.2 *Let* $t = ((\mu a.u\,[x.p, y.q])\,\varepsilon)$, $R = \{(\mu a.u\,[x.p, y.q])\}$, $S = \{t\}$, $K \subseteq red(p)$, $L \subseteq red(u) \cup red(\varepsilon)$ *and* $T = \{(\mu a.u\,[x.p, y.q]), t\}$, *then*
$\hat{t}_R = ((\bar{\mu} a.u\,[x.p, y.q])\,\varepsilon)$, $\hat{t}_S = ((\mu a.u\ \lfloor x.p, y.q \rfloor)\,\varepsilon)$, $\hat{t}_K = ((\mu a.u\,[x.\hat{p}_K, y.q])\,\varepsilon)$,
$\hat{t}_L = ((\mu a.\hat{u}_L\,[x.p, y.q])\,\hat{\varepsilon}_L)$ *and* $\hat{t}_T = ((\bar{\mu} a.u\ \lfloor x.p, y.q \rfloor)\,\varepsilon)$.

Remark 6.1 1. *Let* t *be a term and* $R \subseteq red(t)$. *It is clear that* $\check{\hat{t}}_R = t$.

2. *When there is only one given set* R *of redexes of a given term* t, *we use the abusive notation* $\hat{t}$ *to denote the* R*-corresponding marked term to* t.

3. *Observe that terms in the forms* $\bar{\lambda} x.t$, $< t_1, t_2 >$ *and* $\Omega_i t$ *are not elements of* $\mathbb{E}$. *Marks can only occur as marks of redexes except in terms of the form* $\bar{\mu} a.t$ *and the form* $(t\ \lfloor x.u, y.v \rfloor)$. *It follows that a* $\blacktriangleright$*-normal form never contains any marks except* $\bar{\mu}$ *or* $\lfloor .\,, . \rfloor$.

4. *It is also clear that any term is a* $\blacktriangleright$*-normal marked term (since* $\blacktriangleright$ *consists only in reducing marked redexes).*

The following lemma shows that the set of marked terms is closed under substitutions and reductions.

Lemma 6.1 1. *If* $t, u, \epsilon \in \mathbb{E}$, *then* $t[x := u], t[a :=^* \epsilon] \in \mathbb{E}$.

2. *If* $t \in \mathbb{E}$ *and* $t \blacktriangleright^* t'$, *then* $t' \in \mathbb{E}$.

3. *If* $t, \varepsilon, \epsilon \in \mathbb{E}$, $t \blacktriangleright t'$, $\varepsilon \blacktriangleright \varepsilon'$ *and* $\epsilon \blacktriangleright^* \epsilon'$, *then* $\varepsilon[x := t] \blacktriangleright^* \varepsilon[x := t']$ *and* $\varepsilon[a :=^* \epsilon] \blacktriangleright^* \varepsilon[a :=^* \epsilon']$.

Proof Easy. □

6.2 Finiteness developments theorem

Definition 6.2 *Let t be a term and R a subset of $red(t)$.*

1. *A sequence of reductions $\hat{t} = t_0 \blacktriangleright t_1 \blacktriangleright t_2 \blacktriangleright ...$ is called R-development of t. It is denoted by $\hat{t} \blacktriangleright^* t'$ if it finishes with the marked term t'. We denote it also by $t \rhd_R^* \check{t'}$ ($t = \check{t_0} \rhd_R \check{t_1} \rhd_R \check{t_2} ... \rhd_R \check{t'}$).*

2. *The term t is said to be R-strongly normalizable iff there are no infinite R-developments of t, i.e, all the R-developments are finite.*

Remark 6.2 *If t is a term and R a set of some redexes of t, then note that $\rhd_R$ and $\blacktriangleright$ reduction are the same, any infinite sequence of $\blacktriangleright$ reductions starting from $\hat{t}$ corresponds to an infinite sequence of $\rhd_R^*$ reductions starting from t. Thus one can be able to identify them, this fact will be implicitly used in the next paragraph, where $\rhd_R$ and $\blacktriangleright$ are confused.*

Lemma 6.2 *Let $\varepsilon, \varepsilon', \epsilon, \epsilon'$ be $\mathcal{E}$-terms and R a set of redexes of ε and ϵ such that $\varepsilon \rhd_R^* \varepsilon'$. If $\epsilon \rhd_R^* \epsilon'$, then, $\varepsilon[a :=^* \epsilon] \rhd_R^* \varepsilon'[a :=^* \epsilon']$.*

Proof By induction on ε. □

Definition 6.3 *Let $\bar{\varepsilon} = \varepsilon_1 ... \varepsilon_m$ be a finite sequence of $\mathcal{E}$-terms.*

1. *A semi-permutative redex of $\bar{\varepsilon}$ is any initial segment in the form $\bar{r}[x.p, y.q]\varepsilon_i$, where $(2 \leq i \leq m)$ and $\bar{r}$ is possibly empty.*

2. *A set of redexes of $\bar{\varepsilon}$ is the union of sets of redexes of each ε_i and any possible semi-permutative redex of $\bar{\varepsilon}$.*

Lemma 6.3 *Let t be a term, $\bar{u} = u_1 ... u_n$ a finite sequence of terms and $\bar{\varepsilon} = \varepsilon_1 ... \varepsilon_m$ a finite sequence of $\mathcal{E}$-terms. Let also $\sigma = [(x_i := u_i)_{1 \leq i \leq n}; (a_j :=^* \varepsilon_j)_{1 \leq j \leq m}]$ and R be a set of redexes of t, $\bar{u}$ and $\bar{\varepsilon}$. If t, $\bar{u}$ and $\bar{\varepsilon}$ are R-strongly normalizable, then $t\sigma$ is R-strongly normalizable.*

Proof First, let $t = (u\,\bar{\varepsilon})$ as in the corollary 3.1. We prove this by induction on the ordered lexicographic pair $(lg(\bar{\varepsilon}), c)$ where $lg(\bar{\varepsilon})$ is the number of $\mathcal{E}$-terms of $\bar{\varepsilon}$ and c denotes the complexity of t. For the simplicity and the clearness of the proof, we prefer to avoid using marked terms.

1. The cases where t is not simple are direct consequences of induction hypothesis (decreasing of c).

2. The case $t = (x_i\,\bar{\epsilon})$: $t\sigma = (u_i\,\bar{\epsilon}\sigma)$, by induction hypothesis (u_i and $\bar{\epsilon}\sigma$) are R-strongly normalizable, we have to examine two cases according to $\bar{\epsilon}$:

   (a) If $\bar{\epsilon}$ is a nice sequence: since we can not reduce the possible created redex $(u_i\ \ \bar{\epsilon}\sigma)$ and for the previous reasons, $t\sigma$ is R-strongly normalizable.

   (b) If $\bar{\epsilon}$ is not a nice sequence: $\epsilon = \epsilon_1 ... \epsilon_{j-2}[x.p, y.q]\epsilon_j ... \epsilon_k$, let's denote $\bar{\epsilon}\sigma$ by $\bar{\epsilon'} = \epsilon'_1 ... \epsilon'_{j-2}[x.p', y.q']\epsilon'_j ... \epsilon'_k$. Suppose that there exists a sequence of infinite R-reductions, starting from $t\sigma$ as follows: $(u_i\ \epsilon'_1\ ... \epsilon'_{j-2})[x.p', y.q']\epsilon'_j ... \epsilon'_k$ $\rhd_R^*\ (u'_i\ \epsilon''_1\ ... \epsilon''_{j-2})[x.p'', y.q'']\epsilon''_j ... \epsilon''_k \rhd_R (u'_i\ \epsilon''_1\ ... \epsilon''_{j-2})[x.(p''\ \epsilon''_j), y.(q''\ \epsilon''_j)] ... \epsilon''_k$ $= t' \rhd_R^* ....$
   By the standardization theorem, $t\sigma \rhd_{st}^* t'$, and this standard reductions is in the form:
   $(u_i\ \epsilon'_1\ ... \epsilon'_{j-2})[x.p', y.q']\epsilon'_j ... \epsilon'_k\ \rhd_R\ (u'_i\ \epsilon'_1\ ... \epsilon'_{j-2})[x.(p'\ \epsilon'_j), y.(q'\ \epsilon'_j)] ... \epsilon'_k \rhd$ $(u'_i\ \epsilon''_1\ ... \epsilon''_{j-2})[x.(p''\ \epsilon''_j), y.(q''\ \epsilon''_j)] ... \epsilon''_k = t' \rhd_R^*$ ....This means that $(u_i\ \ \epsilon'_1\ ... \epsilon'_{j-2})[x.(p'\ \ \epsilon'_j), y.(q'\ \ \epsilon'_j)] ... \epsilon'_k \rhd_R^*$, since we can not reduce the possible created redexes $(p'\ \epsilon'_j)$ and $(q'\ \epsilon'_j)$, thus we have a contradiction with induction hypothesis (decreasing of $lg(\bar{\epsilon})$ from k to $k-1$).

3. The case $t = (\mu a.u\ \epsilon)\ \bar{\epsilon}$, let's denote $u\sigma$, $\epsilon\sigma$ and $\bar{\epsilon}$ by respectively by u', ϵ' and $\bar{\epsilon}'$, which are by induction hypothesis R-strongly normalizable. Therefore, if a sequence of infinite R-reductions exists, it will be in one of the two following forms:

   (a) Either $(\mu a.u'\ \epsilon')\ \bar{\epsilon}' \triangleright_R^* (\mu a.u''\ \epsilon'')\ \bar{\epsilon}'' \triangleright_R (\mu a.u''[a :=^* \epsilon'']\ \bar{\epsilon}'')\ = t' \triangleright_R^* ...$, by the standardization theorem, $t\sigma \triangleright_{st}^* t'$ in following sequence: $(\mu a.u'\ \epsilon')\ \bar{\epsilon}' \triangleright_R (\mu a.u'[a :=^* \epsilon']\ \bar{\epsilon}') \triangleright_R^* (\mu a.u''[a :=^* \epsilon'']\ \bar{\epsilon}'')\ = t' \triangleright_R^* ...$ hence $(\mu a.u'[a :=^* \epsilon']\ \bar{\epsilon}') \triangleright_R^* ...$ and this gives a contradiction with the induction hypothesis (decreasing of $lg(\bar{\epsilon})$ from k to $k-1$).

   (b) Or $\epsilon\bar{\epsilon}$ is not a nice sequence, and we conclude as in the case $2.b$ of head variable x.

□

Theorem 6.1 *Let t be a term and $R \subseteq red(t)$, then t is R-strongly normalizable.*

Proof The proof is similar to the one of the previous lemma. Let $t = (u\,\bar{\varepsilon})$ as in the corollary 3.1. We prove this by induction on the ordered lexicographic pair $(lg(\bar{\varepsilon}), c)$ where c denotes the complexity of t.

1. The cases where t is not simple are direct consequences of induction hypothesis (decreasing of c).

2. The case $t = (x\,\bar{\varepsilon})$

   (a) If $\bar{\varepsilon} = \varepsilon_1...\varepsilon_m$ is nice, then, by induction hypothesis, each ε_i is R-strongly normalizable. Therefore t too.

   (b) Else $t = (x\,\varepsilon_1...\varepsilon_{j-2})[y.p, z.q]\varepsilon_j...\varepsilon_m$. Suppose that there exists a sequence of infinite R-reductions starting from t, then, by induction hypothesis, none of ε_i contains infinite R-reduction. Therefore $t = (x\,\varepsilon_1...\varepsilon_{j-2})[y.p, z.q]\varepsilon_j...\varepsilon_m \triangleright_R^* (x\,\varepsilon'_1...\varepsilon'_{j-2})[y.p', z.q']\varepsilon'_j...\varepsilon'_m \triangleright_R (x\,\varepsilon'_1...\varepsilon'_{j-2})[y.(p'\ \varepsilon'_j), z.(q'\ \varepsilon'_j)]...\varepsilon'_m\ = t'_j \triangleright_R^* ....$ By the standardization theorem $t \triangleright_{st}^* t'_j$, this standard reduction is in the form $t = (x\,\varepsilon_1...\varepsilon_{j-2})[y.p, z.q]\varepsilon_j...\varepsilon_m \triangleright_R (x\,\varepsilon_1...\varepsilon_{j-2})[y.(p\,\varepsilon_j), z.(q\,\varepsilon_j)]...\varepsilon_m \triangleright_R^* (x\,\varepsilon'_1...\varepsilon'_{j-2})[y.(p'\ \varepsilon'_j), z.(q'\ \varepsilon'_j)]...\varepsilon'_m$. This means that $(x\,\varepsilon_1...\varepsilon_{j-2})[y.(p\,\varepsilon_j), z.(q\,\varepsilon_j)]...\varepsilon_m \triangleright_R^* ...$, since we can not reduce the possible created redexes $(p\,\varepsilon_j)$ or $(q\,\varepsilon_j)$, this gives a contradiction with the induction hypothesis (decreasing of $lg(\bar{\varepsilon})$, from m to $m-1$).

3. The case $t = (\mu a.u\,\varepsilon)\bar{\varepsilon}$, this gives by induction hypothesis and the standardization theorem two possibilities to the form of the sequence of the R-infinite reductions:

   (a) $t = (\mu a.u\,\varepsilon)\,\varepsilon_1...\varepsilon_m \triangleright_R (\mu a.u[a :=^* \varepsilon]\,\varepsilon_1)\,\varepsilon_2...\varepsilon_m \triangleright_R^* (\mu a.u'[a :=^* \varepsilon']\,\varepsilon'_1)\,\varepsilon'_2...\varepsilon'_m \triangleright_R^* ....$ This means that $(\mu a.u[a :=^* \varepsilon]\,\varepsilon_1)\,\varepsilon_2...\varepsilon_m \triangleright_R^* ....$ By the lemma 6.3, $u[a :=^* \varepsilon]$ is R-strongly normalizable, therefore this gives a condradiction with the induction hypothesis (decreasing of $lg(\bar{\varepsilon})$, from m to $m-1$).

   (b) $t = (\mu a.u\,\varepsilon)\,\varepsilon_1...[x.p, y.q]\varepsilon_j...\varepsilon_m \triangleright_R (\mu a.u\,\varepsilon)\,\varepsilon_1...[x.(p\,\varepsilon_j), y.(q\,\varepsilon_j)]...\varepsilon_m \triangleright_R^* (\mu a.u'\ \varepsilon')\,\varepsilon'_1...[x.(p'\ \varepsilon'_j), y.(q'\ \varepsilon'_j)]...\varepsilon'_m \triangleright_R^* ....$ This means that $(\mu a.u\,\varepsilon)\,\varepsilon_1...[x.(p\,\varepsilon_j), y.(q\,\varepsilon_j)]...\varepsilon_m \triangleright_R^* ....$ By a similar argument as the previous, this gives a contradiction.

□

Theorem 6.2 (Confluence of $\triangleright_R$) *Let t, t_1, t_2 be terms and $R \subseteq red(t)$.*

1. *If $t \triangleright_R t_1$ and $t \triangleright_R t_2$, then there exists t_3 such that $t_1 \triangleright_R^* t_3$ and $t_2 \triangleright_R^* t_3$.*
2. *If $t \triangleright_R^* t_1$ and $t \triangleright_R^* t_2$, then there exists t_3 such that $t_1 \triangleright_R^* t_3$ and $t_1 \triangleright_R^* t_3$.*

Proof

1. By induction on the complexity of t. Let us check the following cases, the other cases are simple consequences of the induction hypothesis. In this proof we only mark the redexes which will be reduced.
   - $t = (x\,\varepsilon_1)\varepsilon_2 ... \lfloor r.u, s.v \rfloor \lfloor y.p, z.q \rfloor \varepsilon_j ... \varepsilon_m$,
     $t_1 = (x\,\varepsilon_1)\varepsilon_2 ... \lfloor r.(u \lfloor y.p, z.q \rfloor), s.(v \lfloor y.p, z.q \rfloor) \rfloor \varepsilon_j ... \varepsilon_m$
     and $t_2 = (x\,\varepsilon_1)\varepsilon_2 ... \lfloor r.u, s.v \rfloor \lfloor y.(p\,\varepsilon_j), z.(q\,\varepsilon_j) \rfloor ... \varepsilon_m$.
     Therefore check that
     $t_3 = (x\,\varepsilon_1)\varepsilon_2 ... \lfloor r.(u \lfloor y.(p\,\varepsilon_j), z.(q\,\varepsilon_j) \rfloor), s.(v \lfloor y.(p\,\varepsilon_j), z.(q\,\varepsilon_j) \rfloor) \rfloor ... \varepsilon_m$
     is the commun marked redectum obtained of course by R-reductions.
   - $t = ((\mu a.u\, [x.r, y.s])\, \varepsilon\bar{\varepsilon})$.
     - If $t_1 = ((\mu a.u'\, [x.r, y.s])\, \varepsilon\bar{\varepsilon})$ and $t_2 = (\mu a.u\, [a :=^* [x.r, y.s]]\, \varepsilon\bar{\varepsilon})$, then, by the lemma 6.2, we take $t_3 = (\mu a.u'\, [a :=^* [x.r, y.s]]\, \varepsilon\bar{\varepsilon})$.
     - If $t_1 = ((\mu a.u'\, [x.r, y.s])\, \varepsilon\bar{\varepsilon})$ and $t_2 = ((\mu a.u\, [x.(r\ \varepsilon), y.(s\ \varepsilon)])\, \bar{\varepsilon})$, then we take $t_3 = ((\mu a.u'\, [x.(r\ \varepsilon), y.(s\ \varepsilon)])\, \bar{\varepsilon})$.
     - If $t_1 = (\mu a.u[a :=^* [x.r, y.s]]\, \varepsilon\bar{\varepsilon})$ and $t_2 = ((\mu a.u\, [x.(r\ \varepsilon), y.(s\ \varepsilon)])\, \bar{\varepsilon})$, then, both of the commutative and the classical redexes are marked redexes in $\hat{t} = ((\bar{\mu} a.u\ \lfloor x.r, y.s \rfloor)\, \varepsilon\bar{\varepsilon})$, hence $\hat{t_1} = (\bar{\mu} a.u[a :=^* \lfloor x.r, y.s \rfloor]\, \varepsilon\bar{\varepsilon})$ $\triangleright_R (\bar{\mu} a.u[a :=^* \lfloor x.r, y.s \rfloor\ \varepsilon]\, \bar{\varepsilon}) \triangleright_R (\bar{\mu} a.u[a :=^* \lfloor x.(r\ \varepsilon), y.(s\ \varepsilon) \rfloor]\bar{\varepsilon}) = t_3$, and $t_2 = ((\bar{\mu} a.u\ \lfloor x.(r\ \varepsilon), y.(s\ \varepsilon) \rfloor)\, \bar{\varepsilon}) \triangleright_R$ $(\bar{\mu} a.u[a :=^* \lfloor x.(r\ \varepsilon), y.(s\ \varepsilon) \rfloor]\, \bar{\varepsilon}) = t_3$. It is obvious that $\check{t_3} = (\mu a.u[a :=^* [x.(r\ \varepsilon), y.(s\ \varepsilon)]]\bar{\varepsilon})$ is the commun redectum.
2. This is a direct consequence of (1), theorem 6.1 and Newman lemma.

□

Lemma 6.4 *If $t \triangleright^* t'$, then $t = t_0 \triangleright^*_{red(t_0)} t_1 \triangleright^*_{red(t_1)} t_2 \triangleright^*_{red(t_2)} ... \triangleright^*_{red(t_n)} t_{n+1} = t'$.*

Proof Easy. □

Remark 6.3 *It is easy to see that if $u \triangleright^*_{red(u)} v \triangleright^*_{red(u)} w$, then $v \triangleright^*_{red(v)} w$.*

Now the main result of this paper can be easily established.

Theorem 6.3 (Confluence of $\triangleright$) *Let t, t_1, t_2 be terms such that $t \triangleright^* t_1$ and $t \triangleright^* t_2$, then there exists t_3 such that $t_1 \triangleright^* t_3$ and $t_1 \triangleright^* t_3$.*

Proof This is a direct consequence of lemma 6.4 and theorem 6.2.

□

One of the consequences of the standardization theorem is that the normal form if it exists, it can be reached by the leftmost reduction.

Theorem 6.4 *If t' is the normal form of t, then $t \triangleright_l^* t'$.*

Proof Since $t \triangleright^* t'$, then there exists a standard reduction from t to t', i.e, $t \triangleright_{st}^* t'$. We process by induction on $\kappa(t \triangleright_{st}^* t') = (n, c)$. The cases where t is not a simple term are direct consequences of induction hypothesis (decreasing of c). Let us examine the two following cases:

– $t = (x\,\bar\varepsilon)$

- If $\bar\varepsilon = \varepsilon_1...\varepsilon_n$ is nice, then $t' = (x\,\bar\varepsilon')$ where $\bar\varepsilon' = \varepsilon'_1...\varepsilon'_n$ and each $\varepsilon_i \rhd^* \varepsilon'_i$ normal. Therefore the induction hypothesis concludes.
- Else $t = ((x\,\bar r[y.p.z.q])\varepsilon_i\bar k)$ and the standard reduction is in the form $t = ((x\,\bar r)[y.p.z.q]\varepsilon_i\bar k) \rhd ((x\,\bar r)[y.(p\,\varepsilon_i).z.(q\,\varepsilon_i)])\bar k) \rhd^*_{st} t'$. Therefore by induction hypothesis (decreasing of n), we have that $((x\,\bar r)[y.(p\,\varepsilon_i).z.(q\,\varepsilon_i)])\bar k) \rhd^*_l t'$.
  Thus $t = ((x\,\bar r)[y.p.z.q]\varepsilon_i\bar k) \rhd_l ((x\,\bar r)[y.(p\,\varepsilon_i).z.(q\,\varepsilon_i)])\bar k) \rhd^*_l t'$.

- $t = ((\mu a.u\,\varepsilon)\,\bar\varepsilon)$, then the standard reduction from t to the normal form t' is in the following forms: Either $t = ((\mu a.u\,\varepsilon)\,\bar\varepsilon) \rhd ((\mu a.u\,\epsilon)\,\bar r) \rhd^*_{st} t'$ or $t = ((\mu a.u\,\varepsilon)\,\bar\varepsilon) \rhd (\mu a.u[a :=^* \varepsilon]\,\bar\varepsilon) \rhd^*_{st} t'$. Therefore, by induction hypothesis, (decreasing of n) we have that $((\mu a.u\,\epsilon)\,\bar r) \rhd^*_l t'$ and $(\mu a.u[a :=^* \varepsilon]\,\bar\varepsilon) \rhd^*_l t'$, hence $t = ((\mu a.u\,\varepsilon)\,\bar\varepsilon) \rhd ((\mu a.u\,\epsilon)\,\bar r) \rhd^*_l t'$ and $((\mu a.u\,\varepsilon)\,\bar\varepsilon) \rhd_l (\mu a.u[a :=^* \varepsilon]\,\bar\varepsilon) \rhd^*_l t'$.

□

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