Best Basis for Joint Representation: the Median of Marginal Best Bases for Low Cost Information Exchanges in Distributed Signal Representation

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Abstract

The paper addresses the selection of the best representations for distributed and/or dependent signals. Given an indexed tree structured library of bases and a semi-collaborative distribution scheme associated with minimum information exchange (emission and reception of one single index corresponding to a marginal best basis), the paper proposes the median basis computed on a set of best marginal bases for joint representation or fusion of distributed/dependent signals. The paper provides algorithms for computing this median basis with respect to standard tree structured libraries of bases such as wavelet packet bases or cosine trees. These algorithms are effective when an additive information cost is under consideration. Experimental results performed on distributed signal compression confirms worthwhile properties for the median of marginal best bases with respect to the ideal best joint basis, the latter being underdetermined in practice, except when a full collaboration scheme is under consideration.

Keywords: Distributed signals ; Best Joint Basis ; Information fusion ; Wavelets.

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1. Introduction

Among the functional representations one can associate with a *regular* or piecewise regular deterministic signal, low cost information representations are important for many applications involving compression, coding, estimation, dimensionality reduction, *etc.*. In general finding a relevant representation for a whole class of signals is intricate, especially when signals under consideration are impacted by uncertainties/imprecisions inherent to the measurement process and/or specific noise emanating from the acquisition system or external disturbances.

Adaptive and/or fuzzy approaches have shown to be relevant for joint analysis and processing of such class of signals. For instance, in [12] a statistical model associating fuzzy regression, nearest neighbor matching, and neural networks has been proposed for predicting the demand of natural gas by using heterogeneous rooftop unit wireless sensors; in [13] a fuzzy multi-sensor data fusion and a fuzzy Kalman feedback are used for fault detection and effective risk reduction for an integrated vehicle health maintenance system; the analysis of a neuro-fuzzy system involving adaptive wavelet activation that depends on the input signal characteristics is described in [3]; the authors of [10] show that genetic algorithms based on lifting (and thus adaptive) wavelet transforms enables relevant source separation for wide band signals while diminishing different types of noises; in [16] the correlation structure is used to improve the estimation accuracy of highly correlated measurements performed in multi-sensor systems.

In this paper, we analyze a distributed set of signals by using a library of wavelet functions. In contrast to [3] and [10], this paper does not involve adaptive prediction and updating of wavelet coefficients (wavelet lifting). We first derive a finite set of relevant wavelet base for representing a distributed set of signals through a 'lower' and 'upper' wavelet basis, delimiting the set of bases-of-interest (fuzzy functional set). A functional ordering of the bases-of-interest is then proposed for selecting the best joint-basis for distributed compression or coding.

References [17], [5], [6], [7], [11] provide concentration norms and sparsity information costs for best basis selection with respect to one signal observation (*marginal best basis* when considering a distributed set of dependent signals). However, for a distributed acquisition system involving dependent and non-stationary signals, finding a best basis from a joint criterion is not an easy extension of the single signal acquisition case. Given a joint information criterion, the best joint basis cannot be computed without a full collaboration between sensors or gathering of all the data at a central node, whereas both situations are undesirable due to their consequence on sensor architectures and their energy consumptions [9]. In this respect and in order to approach a locally optimal solution, some references such as [1] have investigated semi-collaborative distributions schemes consisting of recursive implementations of the Karhunen-Loève transform. On the other hand, [3], [10], [14, 4, 15] have considered wavelet node splitting subject to prediction and updating stages and conditionally with respect to pre-specified collaboration schemes.

It is worth recalling that the above alternatives are with high computational complexities when the number of sensors/signals is large. In addition, these strategies do not guarantee a convergence to the ideal best joint basis even when the number of recursive operations is significant.

The motivation of the present work is to seek, from the sole knowledge of the marginal best bases (best bases at each individual sensor levels), a basis approximating the joint best basis, which is unknown and undetermined in case of non-collaborative distribution schemes. We show that, on tree structured libraries of functional bases, the median of marginal best bases is a basis with relevant properties for joint representation. In addition, the tree structuring makes the computation of the infimum and supremum bases possible, the latter being useful for evaluating the dispersion of a set of marginal best bases. The paper provides theoretical concepts and algorithmic tools that make the computation of the median, infimum and supremum of a best basis set associated with an additive information cost possible over a tree structured library.

The paper begins by providing the context of best basis selection for distributed signals (Section 2). Then it focuses on defining algebra on tree structured libraries (Section 3). From this algebra, the paper derives algorithms for computing the median, infimum and supremum of a set of bases-of-interests (Section 3). The paper then highlights the relevance of the sample median basis for joint representation when the observation amounts solely to marginal basis consideration (Section 4). The paper finally concludes by discussing some issues concerning the use and interpretation of median, infimum and supremum bases (Section 5).

2. Preliminary notation and issues

2.1. Context

Let us consider distributed compression or distributed fusion of a set of signals delivered by K sensors, *i.e.*, K observations $\mathbf{y_k}$ (partial "views") of a "big" signal s. These observations are available through a model of the form:

$$\mathbf{y_k} = \Theta_k(\mathbf{s}, \xi_\mathbf{k}), \tag{1}$$

where $\Theta_k, \xi_\mathbf{k}$ for $k = 1, 2, \ldots, K$, are respectively operators and noise relating the specificities of the sources/sensors.

Operator Θ_k can be additive (signal s observed in presence of additive noise ξ_1), multiplicative (acquisition systems using coherent radiations), convolutive (transfer function involved in some imaging systems) or masking (missing data inducing a partial loss of information or a partial observation of a whole phenomenon), *etc.*

Figure 1 provides an illustration of the model of Eq. (1) where operator Θ_k is additive with respect to variables s and $\xi_\mathbf{k}$, and masking (has a limited access to the whole signal s):

$$\mathbf{y_k} = \mathbf{s}\mathbb{1}_{\Delta_k} + \xi_\mathbf{k}, \tag{2}$$

where the intervals $(\Delta_k)_{k=1,2,\ldots,K}$ involved in Eq. (2) overlap, yielding dependent observations $(\mathbf{y_k})_{k=1,2,\ldots,K}$. Since multiplicative or convolutive operators can be written in the form of Eq. (2) with appropriate transforms, we will use the distributed system given by the model of Eq. (2) in the following, for the sake of simplicity of presentation. We will moreover use for convenience, the notation $\mathrm{s}_k = \mathbf{s}\mathbb{1}_{\Delta_k}$.

Let us assume that there exists a relevant library of functional bases $\mathcal{B} = \{\mathbf{B}_\ell, \ell = 1, 2, \ldots, L\}$ for representing the signal. In order to avoid any

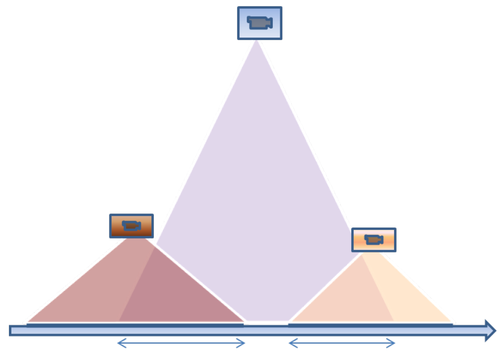

Figure 1: Distributed set of sensors model observation.

confusion when several bases are under consideration, we will denote the representation $\mathbf{B}_\ell[\mathbf{s}_k]$ of signal $\mathbf{s}_k$ on a basis $\mathbf{B}_\ell \in \mathcal{B}$ by $\mathbf{s}_k^{\mathbf{B}_\ell}$.

We evaluate the relevance of representations in different bases $(\mathbf{s}_k^{\mathbf{B}_\ell})_{\ell=1,2,\ldots,L}$ in the particular context of the targeted application, *e.g.*, compression, coding or information fusion. For this purpose, we use an information cost function that is attached to every representation. In the following Section 3, there will be no need to detail the particular information cost function, we will simply assume that an appropriate cost function exists. However, information cost functions will be provided when needed, in Section 4 specifically.

The goal of this paper is to derive a relevant basis $\mathbf{B}^\diamond \in \mathcal{B}$ for the representation of big signal s from its distributed versions $(\mathbf{s}_k)_{\ell=1,2,\ldots,K}$. We emphasize that the scope of the paper is not to derive compression or information fusion methods, but rather the selection of a best representation that guarantees lower information cost with respect to criteria such as sparsity or Shannon information costs (needed for relevant compression/coding/information fusion).

2.2. Problem formulation

With a full collaboration between sensors, or by gathering all the data at a central node, finding a joint best basis can be performed. However, these strategies are with high energy consumption.

Without any collaboration between sensors, one can only compute and use, at sensor levels, the L best marginal bases (one for each sensor); each marginal basis being derived to be relevant for the sole signal s_k representation. However, when considering the whole set of sensors, this strategy is far from optimal.

The above remarks raise the issue of developing semi-collaborative strategies that compromise, in terms of limiting the information exchange and approaching in a certain sense, the best basis for the signal s. Distributed semi-collaborative schemes exchanging correlation structures have been designed in [9], [1], [14] so as to reach, asymptotically, the best joint basis. However, for these methods, convergence is not always guarantee (correlation is an incomplete statistic) and involves high energy consumption cost when it holds true.

In this paper, we propose a semi-collaborative distribution scheme where collaboration simply consists of information exchange regarding best marginal bases. Obviously, it is not possible, from these marginal best bases, to infer the best basis for the joint representation of s due to under determination (semi-collaborative distribution schemes with partial information exchanges do not guarantee convergence to the joint best basis, except for some particular cases).

However, this strategy has 3 main advantages: 1) when library $\mathcal{B}$ is fixed in advance, information sending per sensor reduces to a single index: the index of its best marginal basis (minimal amount of exchange, resulting in very low energy consumption), 2) marginal bases are all relevant bases for s 'partial' representations and the structure of best joint basis is expected to be close to the structure of these best marginal bases and 3) computing best marginal bases is non-recursive.

The problem addressed hereafter concerns deriving a best basis for joint representation conditionally to the knowledge of the best marginal bases. This will be performed by assuming that library $\mathcal{B}$ is constructed with an algebraic order structure which eases the computation of a set of bases with suitable properties, in particular the infimum, the supremum and the median of a given subset of $\mathcal{B}$.

For tree structured libraries with node inclusion properties, such an order structure exists and is derived hereafter from path and terminal node considerations. This order structure will be associated to information variations from parent node to child nodes through a suitable information cost function. Given a subset of bases of $\mathcal{B}$, this order structure makes answering the two questions possible: Question 1): how close are the bases under consideration? Question 2): how to identify the median basis of this subset?

3. Order statistics among a set of bases

In this section, we consider tree based basis libraries generated from a unique root node (the input functional space) and composed with nested orthogonal functional subspaces as tree nodes. For such a tree, a node represents a subspace of the input functional space and the direct sum of subspaces spanned by the bases of child nodes give the subspace spanned by the basis of their father node. One can define an order structure on paths of such a tree (natural order structure relative to precedence of nodes in a path, starting from the root and ending at a terminal node) and one can extend this order structure further to sequence of nodes (subtrees) and the specific collection of nodes composing a basis.

It is noteworthy that an arbitrary tree structured wavelet basis library can be reformulated by using wavelet packet framework. Indeed, in a single tree, this framework makes using a combination of different wavelets and different node splitting schemes (M-band, where M is node variable) possible (multi-wavelets and multivariate band wavelet packet decomposition). We will therefore focus on the library $\mathcal{B}$ of wavelet packet bases in the following. For the sake of simplifying notation, we restrict the presentation to a 2-band wavelet tree splitting scheme. Extensions to 1) an arbitrary M-band wavelet packet library with constant M, 2) multivariate M-band tree structured library or 3) specific tree structured libraries such as the cosine tree, are straightforward.

3.1. Order structures among wavelet packet bases

Let T^* denote a full wavelet packet tree down to a fixed decomposition level J^*. Tree T^* is defined by the collection of nodes (wavelet packet subbands) $T^* = \{\mathbf{W}_{j,n} : 0 \leqslant j \leqslant J^*, n = 0, 1, \ldots, 2^j - 1\}$. A full wavelet packet path P of T^* is defined by the sequence $P = (\mathbf{W}_{j,n_j})_{0 \leqslant j \leqslant J^*}$ where $n_j = n(j) \in \{0, 1, \ldots, 2^j - 1\}$, is recursively defined from $n_0 = 0$ and $n_\ell = 2n_{\ell-1} + \epsilon_\ell$, with $\epsilon_\ell \in \{0, 1\}$ for every $1 \leqslant \ell \leqslant j$. From the above recurrence, any given wavelet packet path, down to a fixed decomposition level $J \leqslant J^*$, can be completely specified by its terminal node $\mathbf{W}_{J,n_J}$ or equivalently by the binary sequence $(\epsilon_\ell)_{1 \leqslant \ell \leqslant J}$ [2]. This path will hereafter be denoted by $P(J, n_J) = (\mathbf{W}_{j,n_j})_{0 \leqslant j \leqslant J}$. For instance, $P(0, 0) = \{\mathbf{W}_{0,0}\}$ is the path consisting solely of the root node.

Consider two arbitrary wavelet packet paths $P(J, n_J)$ and $P(L, p_L)$. Let I be the cardinality of the set $\mathcal{A} = \{n_1, n_2, \ldots, n_J\} \cap \{p_1, p_2, \ldots, p_L\}$ and

$$q_I = \begin{cases} \max \mathcal{A} & \text{if } \mathcal{A} \neq \emptyset, \\ 0 & \text{if } \mathcal{A} = \emptyset. \end{cases}$$

Define an operation $\oplus$ on paths by associating to $P(J, n_J) \oplus P(L, p_L)$, the path $P(I, q_I)$ with terminal node $\mathbf{W}_{I,q_I}$ (smallest wavelet packet space that contains $\mathbf{W}_{J,n_J}$ and $\mathbf{W}_{L,p_L}$).

A subtree T of the wavelet packet tree T^* is a collection $T = (P(J_\ell, n_{J_\ell}))_\ell$ of paths, where $J_\ell \leqslant J^*$ for every ℓ.

Let $\mathcal{T}$ be the set of all wavelet packet subtrees of tree T^* and $T_1 = (P(J_\ell, n_{J_\ell}))_\ell \in \mathcal{T}$, $T_2 = (P(L_k, p_{J_k}))_k \in \mathcal{T}$.

Define an operation $\uplus$ on $\mathcal{T}$ by associating to $T_1 \uplus T_2$, the tree T_0 defined from the convention: $P(I_m, q_{I_m})$ pertains to T_0 if there exists $P(J_\ell, n_{J_\ell}) \in T_1$ and $P(L_k, p_{J_k}) \in T_2$ such that $P(I_m, q_{I_m}) = P(J_\ell, n_{J_\ell}) \oplus P(L_k, p_{J_k})$. It follows that tree T_0 is composed of nodes that are common to T_1 and T_2.

We have: $\uplus$ is a binary operation on $\mathcal{T}$ and,

Theorem 1. *$(\mathcal{T}, \uplus)$ is a commutative monoid with identity element T^*.*

Proof 1. *Commutativity and associativity follow from the properties of the binary operation $\oplus$ on paths. In addition, T^* is the identity since any element of $\mathcal{T}$ is included in T^*.*

Now, let $\mathrm{P}(J, n_J)$ be a wavelet packet path. We have: $\mathrm{P}(J, n_J) \oplus \mathrm{P}(J, n_J) = \mathrm{P}(J, n_J)$. Thus, we can formulate the following proposition.

Proposition 1. *The operation $\oplus$ is idempotent over the set of all wavelet packet paths.*

The idempotence of $\oplus$ induces over wavelet packet paths, an order structure denoted $\preccurlyeq$ and defined by: $\mathrm{P}(J, n_J) \preccurlyeq \mathrm{P}(L, p_L) \Leftrightarrow \mathrm{P}(J, n_J) \oplus \mathrm{P}(L, p_L) = \mathrm{P}(L, p_L)$. This order relation is compatible with the operation $\oplus$. It is the inverse of the natural *set ordering* induced by $\subset$ operation on the wavelet packet subspaces: the largest wavelet space with respect to set inclusion (root node $\mathbf{W}_{0,0}$) is associated with the smallest wavelet packet path.

This order relation, derived from a binary operation on wavelet packet paths, makes subtree ordering possible and eases the derivation of the functional structure of relevant wavelet packet nodes thanks to the computation of lower and upper paths from a given set of subtrees.

The lower (minimal, min) and upper (maximal, max) paths are defined by:

$$\begin{aligned} rcl\mathrm{P}(J, n_J) \preccurlyeq \mathrm{P}(L, p_L) &\iff \mathrm{P}(J, n_J) \oplus \mathrm{P}(L, p_L) = \mathrm{P}(L, p_L) \\ &\iff \min\{\mathrm{P}(J, n_J), \mathrm{P}(L, p_L)\} = \mathrm{P}(J, n_J) \\ &\iff \max\{\mathrm{P}(J, n_J), \mathrm{P}(L, p_L)\} = \mathrm{P}(L, p_L). \end{aligned}$$

Remark 1. *A full wavelet packet tree down to the decomposition level* j *is composed of* 2^j *paths whose terminal nodes are associated with frequency indices* $n_j \in \{0, 1, \dots, 2^j - 1\}$.

Let us consider a re-ordering of these paths obtained from a permutation G *applied on the set* $\{0, 1, \dots, 2^j - 1\}$. *This permutation operates as isotonic with respect to the order defined on wavelet packet paths:*

$$\mathrm{P}(J, n_J) \preccurlyeq \mathrm{P}(L, p_L) \Longrightarrow \mathrm{P}(J, G(n_J)) \preccurlyeq \mathrm{P}(L, G(p_L)).$$

Thus, a re-ordering of the wavelet packet nodes such as the one involved in the Shannon wavelet packet decomposition do not impact on paths/trees ordering.

A subtree being a collection of paths, the collection $\mathcal{T}$ of all subtrees of $\mathtt{T}^*$ associated with $\uplus$ inherits the above path order properties. The extension of these order properties to basis ordering requires the identification of the particular subtrees that are associated with wavelet packet bases (see Section 3.2 below).

3.2. Basis ordering - Extremal bases of a set of bases for the wavelet packet library

A subtree is said to be associated with a basis if the collection of wavelet functions generating its terminal nodes constitute a basis of the input space $\mathbf{W}_{0,0}$. This is equivalent to saying that the direct sum of functional subspaces $\left(\mathbf{W}_{J_\ell, n_{J_\ell}}\right)_\ell$ associated with its terminal nodes equals $\mathbf{W}_{0,0}$. We seek to know whether an arbitrary subtree of $\mathtt{T}^*$ defines a basis. Let us define the following interval:

$$\mathcal{I}_{j,n} = \left[\frac{n}{2^j}, \frac{(n+1)}{2^j}\right[. \tag{3}$$

Then, we can formalize:

Definition 1. *A wavelet packet subtree* $(\mathtt{P}(J_\ell, n_{J_\ell}))_\ell$ *defines a basis if the intervals* $\left(\mathcal{I}_{J_\ell, n_{J_\ell}}\right)_\ell$ *obtained from its terminal nodes form a partition of the interval* $[0, 1[$.

Example 1. *We have:* $\left[\frac{0}{2^1}, \frac{1}{2^1}\right[\cup \left[\frac{2}{2^2}, \frac{3}{2^2}\right[\cup \left[\frac{3}{2^2}, \frac{4}{2^2}\right[= [0, 1[$. *Therefore, the subtree* $(\mathtt{P}(1, 0), \mathtt{P}(2, 2), \mathtt{P}(2, 3))$ *is associated with a basis of the input space. In other words,* $\mathbf{W}_{1,0} \bigoplus \mathbf{W}_{2,2} \bigoplus \mathbf{W}_{2,3} = \mathbf{W}_{0,0}$ *where* $\bigoplus$ *represents the direct sum between functional spaces.*

A basis being a particular collection of subtrees, the collection $\mathcal{B}$ of all wavelet packet bases from $\mathtt{T}^*$ associated with $\uplus$ inherits the properties of $\mathcal{T}$:

Theorem 2. $(\mathcal{B}, \uplus)$ *is a commutative monoid with identity element* $\mathtt{B}^* \equiv \mathtt{T}^*$ *(* $\mathtt{B}^*$ *is the basis corresponding to the terminal nodes of a full wavelet packet tree expansion).*

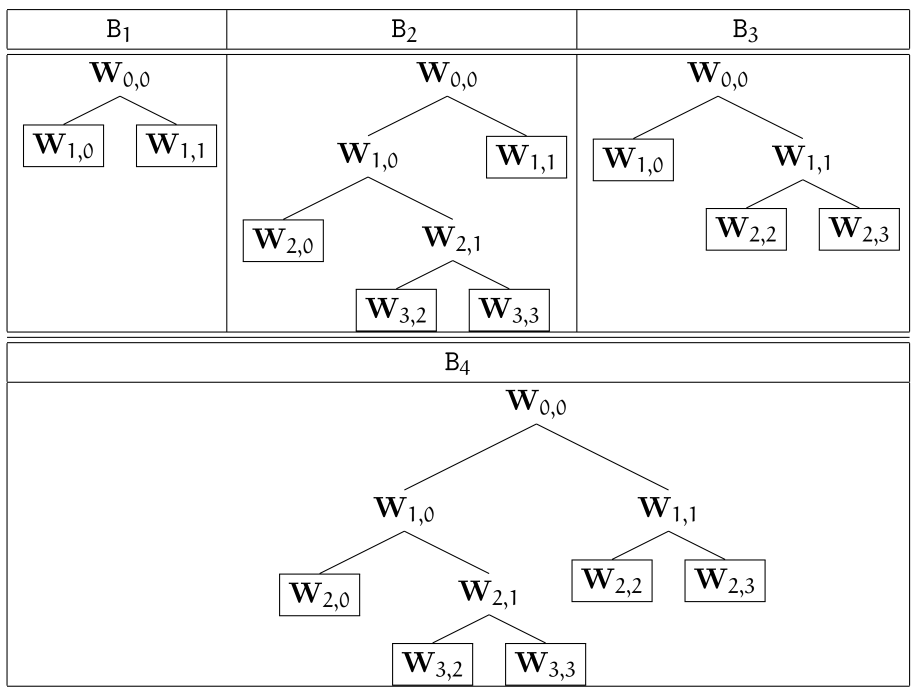

Figure 2: Basis comparison with respect to $\preccurlyeq$. We have: $\min(\mathtt{B}_1, \mathtt{B}_2) = \mathtt{B}_1$, $\inf(\mathtt{B}_2, \mathtt{B}_3) = \mathtt{B}_1$, $\sup(\mathtt{B}_2, \mathtt{B}_3) = \mathtt{B}_4$ and $\max(\mathtt{B}_1, \mathtt{B}_4) = \mathtt{B}_4$

The order structure on $\mathcal{B}$ is such that for two arbitrary wavelet packet bases $B_1 = \bigoplus_\ell \mathbf{W}_{J_\ell, n_{J_\ell}}$ and $B_2 = \bigoplus_k \mathbf{W}_{L_k, p_{L_k}}$ associated respectively with terminal nodes $(P(J_\ell, n_{J_\ell}))_\ell$ and $(P(L_k, p_{L_k}))_k$, we have: $B_1 \preccurlyeq B_2$ if and only if every $\mathcal{I}_{J_\ell, n_{J_\ell}}$ can be written as a partition consisting of elements of $\left(\mathcal{I}_{L_k, p_{L_k}}\right)_k$. As above, this order structure makes basis comparison possible with respect to the lower (min) and upper (max) elements, as well as the greatest lower (infimum, inf) and least upper (supremum, sup) elements. Figure 2 illustrates basis comparison.

Remark 2. *The above order relation between wavelet packet bases/trees may have been defined only by terminal node consideration. But in practice, computing a terminal node assumes that the coefficients associated with every parent node have been computed. In addition, the basis/tree ordering obtained from terminal node consideration is not straightforward. This makes path consideration more convenient for defining the order structure given above.*

By considering the above order structure we derive the infimum and supremum of a set of bases: Tables 1 and 2 provide the algorithms for computing these bases.

Table 1: Computation of the supremum of a subclass of bases.

Set *node unions* to be an empty set
For every query basis from the subclass:
Retrieve all the nodes of this query basis.
Set *node unions* to be the union between the nodes of the query basis and the former *node unions* set.
End
Retrieve the terminal nodes associated with sequence *node unions*. These terminal nodes define the supremum basis among the subclass considered.

Table 2: Computation of the infimum of a subclass of bases.

Retrieve the largest decomposition level J^* involved in the subclass of bases under consideration.
Set *node intersections* to be the sequence of all nodes involved in a full wavelet packet tree down to level J^*.
For every query basis from the subclass:
Retrieve all the nodes of this query basis.
Set *node intersections* to be the intersection between the nodes of the query and the former *node intersections* set.
end
Retrieve the terminal nodes associated with sequence *node intersections*. These terminal nodes define the infimum basis among the subclass of bases considered.

3.3. Median basis

Let $\mathcal{B} = \{\mathbf{B}_k, k = 1, 2, \ldots, K\}$ be a collection of wavelet packet bases. Let J be the maximum decomposition level involved in the tree based decompositions associated with the elements of $\mathcal{B}$.

We consider hereafter, the set of all paths issued from the root node $\mathbf{W}_{0,0}$ and having pairs (J, n_J) as terminal nodes, where $n_J = n_J(\mathtt{P}) \in \{0, 1, \ldots, 2^J - 1\}$. We recall that such a path $\mathtt{P}$ is composed of subbands located at different decomposition levels, $\mathtt{P} = \left(\mathbf{W}_{j,n_j}\right)_{0 \leqslant j \leqslant J^*}$ (see Section 3.1).

Any basis $\mathtt{B}_k$ contributes to path $\mathtt{P}$ in the sense that it admits one of the above subbands $\left\{\mathbf{W}_{j,n_j}, j = 1, 2, \ldots, J\right\}$ as a terminal node.

Let

$$O(\mathbf{W}_{j,n_j}, \mathcal{B}) = \#\left\{\mathtt{B} \in \mathcal{B} : \mathbf{W}_{j,n_j} \subset \mathtt{B}\right\}.$$

Quantity $O(\mathbf{W}_{j,n_j}, \mathcal{B})$ represents the number of occurrences of the subband $\mathbf{W}_{j,n_j}$ in $\mathcal{B}$. Let

$$\mathcal{O}(\mathtt{P}, \mathcal{B}) = \{O(\mathbf{W}_{j,n_j}, \mathcal{B}), j = 0, 1, \ldots, J \text{ and } (j, n_j) \in \mathtt{P}\}.$$

Set $\mathcal{O}(\mathtt{P}, \mathcal{B})$ represents the occurrences in $\mathcal{B}$ of the different subbands pertaining to $\mathtt{P}$.

Then we can define the median subband of the set $\mathcal{B}$ on path $\mathtt{P}$ as

$$\arg_{\mathbf{W}_{j,n_j}} \mathrm{median}\mathcal{O}^*(\mathtt{P}, \mathcal{B}) \tag{4}$$

where $\mathcal{O}^*$ denotes non-null values of $\mathcal{O}$.

The above median subband is unique when K is odd. For even values of K, the sample median is not unique, as in the case of the standard sample median. In the latter case, one may use the middle lower occurrence (decomposition with the shortest path) to define the sample median subband, however, we will consider only odd values of K in the rest of the paper. In addition, the median subband will be sought by describing a subband with a single indexed natural number $\mathbf{W}_{j,n_j} = \mathbf{W}_{N_j}$ with $N = n_j + \sum_{\ell=0}^{j-1} 2^\ell$.

Example 2. *Consider the bases* $\{B_1, B_2, B_3\}$ *given in Figure 2. The largest decomposition level is* $J = 3$. *We thus have 8 full paths corresponding to terminal nodes associated with* $N = 7, 8, \dots, 14$. *Denote* $P_7, P_8, \dots, P_{14}$ *to be the corresponding paths. We have*

$$\begin{aligned}
\mathcal{O}^*(P_7, \{B_1, B_2, B_3\}) &= \{1, 3, 1\} \\
\mathcal{O}^*(P_8, \{B_1, B_2, B_3\}) &= \{1, 3, 1\} \\
\mathcal{O}^*(P_9, \{B_1, B_2, B_3\}) &= \{1, 3, 1\} \\
\mathcal{O}^*(P_{10}, \{B_1, B_2, B_3\}) &= \{1, 10, 1\} \\
\mathcal{O}^*(P_{11}, \{B_1, B_2, B_3\}) &= \{2, 2, 5\} \\
\mathcal{O}^*(P_{12}, \{B_1, B_2, B_3\}) &= \{2, 2, 5\} \\
\mathcal{O}^*(P_{13}, \{B_1, B_2, B_3\}) &= \{2, 2, 6\} \\
\mathcal{O}^*(P_{14}, \{B_1, B_2, B_3\}) &= \{2, 2, 6\}.
\end{aligned} \tag{5}$$

Thus, the median subbands are associated with $N = 1, 2$ *and we can conclude that Median*$\{B_1, B_2, B_3\} = B_1$. *One can remark that, in this example, the median basis is the smallest basis in terms of tree size. This is justified by the fact that* B_2 *and* B_3 *are not comparable.*

Example 3. *Consider now the bases* $\{B_2, B_3, B_4\}$ *given in Figure 2. Then* *we have*

$$\begin{aligned}
\mathcal{O}^*(P_7, \{B_2, B_3, B_4\}) &= \{3, 1, 3\} \\
\mathcal{O}^*(P_8, \{B_2, B_3, B_4\}) &= \{3, 1, 3\} \\
\mathcal{O}^*(P_9, \{B_2, B_3, B_4\}) &= \{9, 1, 9\} \\
\mathcal{O}^*(P_{10}, \{B_2, B_3, B_4\}) &= \{10, 1, 10\} \\
\mathcal{O}^*(P_{11}, \{B_2, B_3, B_4\}) &= \{2, 5, 5\} \\
\mathcal{O}^*(P_{12}, \{B_2, B_3, B_4\}) &= \{2, 5, 5\} \\
\mathcal{O}^*(P_{13}, \{B_2, B_3, B_4\}) &= \{2, 6, 6\} \\
\mathcal{O}^*(P_{14}, \{B_2, B_3, B_4\}) &= \{2, 6, 6\}
\end{aligned} \tag{6}$$

and derive median subbands corresponding to N = *3, 5, 6, 9 and 10, so that Median*$\{B_2, B_3, B_4\} = B_4$.

Theorem 3. *Assume a totally ordered collection of bases*

$$\mathcal{B} = (B_k)_{k=1,2,\ldots,2K+1},$$

with $B_1 \preceq B_2 \preceq \ldots \preceq B_{2K+1}$. *Then from the above defintion of sample median basis, we have Median* $\mathcal{B} = B_K$.

Algorithms given in Tables 1 and 2, as well as the one following from Eq. (4), apply to any tree structure generated from a root node from a recursive node splitting, by using an arbitrary number splits for each node. The algorithms apply even for adaptive node splitting, when the number of bands (subspaces) is computed depending on the signal (adaptive multiband splitting). Note also that many libraries of bases can be structured by using such tree decomposition with the input signal represented at the root node. In this way, these algorithms suit a large number of basis libraries available from the literature.

4. Median of marginal best basis for the joint representation of distributed signals

Section 3 above has emphasized 3 outstanding bases from the algebraic structure of a tree based library of bases. These bases will deserve much interest in this section: under a distributed collaboration scheme associated with a single-index information exchange, the median of the marginal best bases will be associated to the best joint basis whereas extremal bases will provide information on the statistical dispersion of the set of marginal best bases.

The following section provides some convenient information costs with respect to a fast search in tree structured libraries: these information costs, hereafter called *entropies*, are additive in the sense that the corresponding cost splits into a direct sum of functional subspaces, see [5] for details and properties of these information costs.

4.1. Best basis entropies

References [17], [6] and [7] provide different entropy cost functions for the selection of a best basis from a tree structured library such as the wavelet packet tree. In the above references, these cost functions have shown relevancy in compression, coding and denoising problems involving a single signal observation (marginal bases when considering the distributed approach). Among these cost functions, we consider the following sparsity entropies applied to a N-term vector c (set of signal coefficients). For these entropies, lower values relate to higher concentration of the energy of vector c in few coefficients.

Definition 2 (Concentration entropy, [17]). *The* Concentration entropy *of vector* c *is defined as the* ℓ^1*-norm of this vector:*

$$\mathcal{H}^{\textit{1-Norm}}(c) \triangleq \sum_{k=1}^{N} |c_k|. \tag{7}$$

Definition 3 (Sparsity threshold entropy, [5]). *The* sparsity entropy *of vector* c *with respect to the threshold* λ *is defined by:*

$$\mathcal{H}_{\lambda}^{\textit{Thresh}}(c) \triangleq \sum_{k=1}^{N} \mathbb{1}_{]\lambda,+\infty[}(|c_k|). \tag{8}$$

Definition 4 (Sparsity SURE entropy, [6]). *The sparsity* SURE entropy *of vector* c *with respect to the threshold* λ *is defined by:*

$$\mathcal{H}_{\lambda}^{SURE}(c) \triangleq N - 2\sum_{k=1}^{N} \mathbb{1}_{[0,\lambda]}(|c_k|) + \sum_{k=1}^{N} \min(|c_k|^2, \lambda^2). \tag{9}$$

Due to the additivity of these entropies, best basis search for a given observation in a tree structured library is reduced to:

- splitting recursively the tree nodes (starting from the root node),
- comparing the entropies of the parent node with that of the child nodes in order to evaluate the information gain (or loss) when performing the split,
- continue splitting, while the sum of entropies of the child nodes is smaller than that of the parent node.

Let $\mathcal{H}$ be one of the above entropy function. Let $\mathcal{S} = \{s_\ell, \ell = 1, 2, \dots, L\}$ be a sequence of observations. Assume that

- the best bases associated with elements of $\mathcal{S}$ form a totally ordered collection of bases $\mathcal{B} = (B_k)_{k=1,2,\dots,2K+1}$ with $B_1 \preceq B_2 \preceq \dots \preceq B_{2K+1}$, with $2K + 1 \leqslant L$.
- for every $\ell \in \{1, 2, \dots, L\}$, the sequence $\mathcal{E}_\ell$ of entropies of s_ℓ on $\mathcal{B}$ form a convex set $\mathcal{E}_\ell = (\mathcal{H}\{s_\ell, B_k\})_{k=1,2,\dots,2K+1}$, where quantity $\mathcal{H}\{s_\ell, B_k\}$ is the entropy of s_ℓ on basis B_k.

Then, when considering the sum of entropies as a joint criteria for best joint basis selection, the basis Median $\mathcal{B}$ is expected to be the more relevant basis, rather than an arbitrary basis $B \in \mathcal{B}$ for representing the set of observations $\mathcal{S}$.

Remark 3. *The convexity assumption used above for entropies is compatible with the best basis selection: the node splitting continues while entropies are decreasing and the best basis corresponds to the basis below when entropies begin increasing. Note however that*

- *the set of (marginal) best bases $\mathcal{B}$ is not totally ordered in general;*
- *the set of entropies $\mathcal{E}_\ell$ for some $\ell \in \{1, 2, \ldots, L\}$ is not guaranteed to be convex*[4]*.*

Despite these remarks, the following details highlight that the median of the marginal best bases is relevant for practical applications involving joint representations of a distributed set of observations.

More precisely, given the semi-collaborative distribution scheme introduced in Section 2.2 and an entropy cost function, terminal k, for $k \in \{1, 2, \ldots, K\}$, will compute the best basis $\mathbf{B}_k^\diamond \in \mathcal{B}$ for the representation of signal s_k and will send the index of this marginal best basis to other terminals (best marginal basis information exchange). Thus, a sequence $(\mathbf{B}_k^\diamond)_{k=1,2,\ldots,K}$ is available at any terminal so as to make the computation of the median of the best bases possible:

$$\mathbf{B}_{\text{Median}}^\diamond = \text{Median}\{\mathbf{B}_k^\diamond, k = 1, 2, \ldots, K\}.$$

The following experimental results highlight the relevance of $\mathbf{B}_{\text{Median}}^\diamond$ for the representation of the set $(s_k)_{k=1,2,\ldots,K}$.

4.2. Experimental results

Experimental tests concern a simulated 9 sensors based distributed system performing the "acquisition" of permutated versions (different scenarios of a spatial puzzle) of the 3 channels of a color Lena image.

In this system, 1) the Lena image with size 512×512 pixels given in Figure 3-(a) is considered as a big scene, 2) this image (and thus every channel) is split into $K = 9$ overlapping partial point of views of the scene (subimages): a subimage I_k, for $k \in \{1, 2, \ldots, K\}$, represents a squared portion of the scene, where we have considered the same size, 256×256 pixels, for any subimage (see Figure 3-(b)).

[4]The best basis search stops when reaching the first local minimum in order to save decomposition costs, when the set of entropies upon the ordered tree structured bases is not convex.

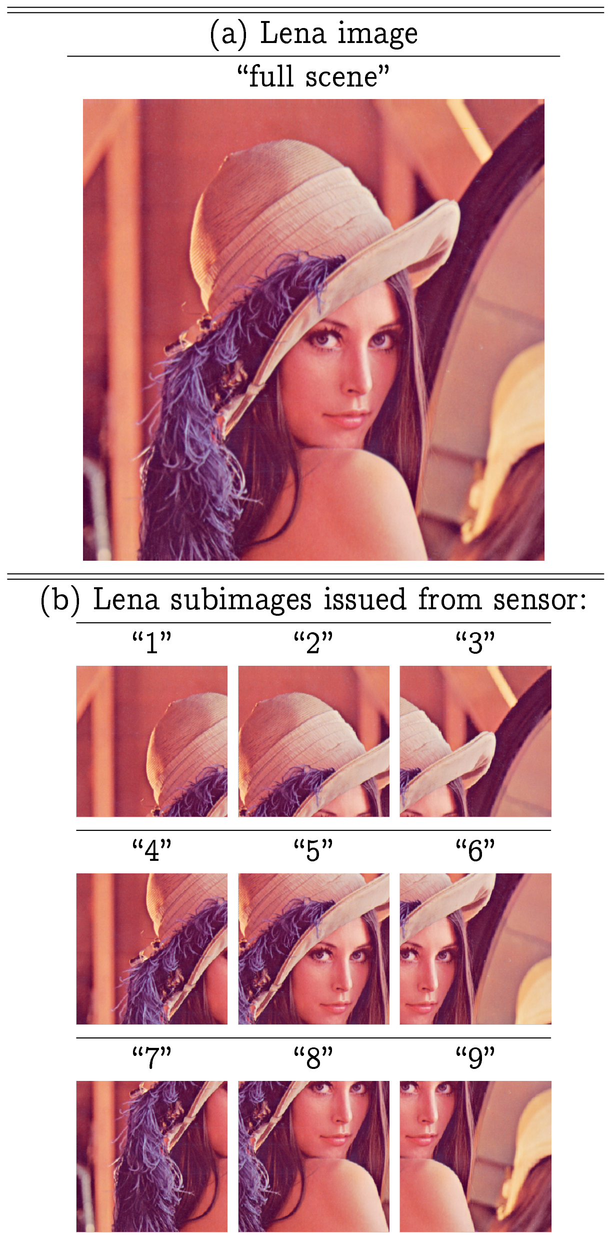

Figure 3: Lena image imaged by a distributed set of sensors.

Table 3: Entropies computed from the experimental setup described in Section 4.2. The entropies of the marginal best bases express the cost of using the basis associated with a partial observation of the bands Red (R), Green (G) and Blue (B) of the Lena image in a distributed acquisition scheme. The mean entropy over the marginal bases represents the expectation of using the best bases associated with an arbitrary terminal in a sequence of distributed acquisitions consisting of random permutations of the 9 subimages composing the scene.

Entropy:	L_1-Norm				Threshold				SURE			
Basis	R	G	B	RGB	R	G	B	RGB	R	V	B	RGB
	$10^7\times$	$10^6\times$	$10^6\times$	$10^6\times$	$10^4\times$	$10^4\times$	$10^4\times$	$10^4\times$	$10^7\times$	$10^7\times$	$10^7\times$	$10^7\times$
$\mathbf{B}_1^\diamond$	1.0914	9.2834	8.9089	19.2837	2.9510	3.9374	3.3207	10.2091	3.5438	4.8832	4.7886	13.2156
$\mathbf{B}_2^\diamond$	1.0877	9.2706	8.8949	19.2532	2.8857	3.9313	3.3240	10.1410	3.4936	4.8822	4.7886	13.1644
$\mathbf{B}_3^\diamond$	1.0894	9.2717	8.8893	19.2504	2.9155	3.9493	3.2823	10.1471	3.5434	4.8832	4.7886	13.2152
$\mathbf{B}_4^\diamond$	1.0896	9.2677	8.8912	19.2485	2.9631	3.9700	3.2853	10.2184	3.4957	4.8499	4.7842	13.1298
$\mathbf{B}_5^\diamond$	1.0886	9.2743	8.8921	19.2550	2.8834	3.9195	3.2872	10.0901	3.4957	4.8499	4.7842	13.1298
$\mathbf{B}_6^\diamond$	1.0876	9.2669	8.8944	19.2489	2.9179	3.9835	3.3326	10.2340	3.4941	4.8662	4.8042	13.1645
$\mathbf{B}_7^\diamond$	1.0894	9.2748	8.8913	19.2555	2.9621	4.0205	3.3639	10.3465	3.5197	4.8830	4.7973	13.2000
$\mathbf{B}_8^\diamond$	1.0875	9.2721	8.8917	19.2513	2.8897	3.9638	3.2987	10.1522	3.5068	4.8467	4.7911	13.1446
$\mathbf{B}_9^\diamond$	1.0881	9.2936	8.9156	19.2973	2.8862	3.9845	3.3468	10.2175	3.4941	4.8652	4.8042	13.1635
$\mathbf{B}_{\mathrm{Med}}^\diamond$	1.0875	9.2707	8.8907	19.2489	2.8866	3.9196	3.2807	10.0869	3.4941	4.8467	4.7842	13.1250
$\mathbf{B}_{\mathrm{Inf}}^\diamond$	1.0895	9.2717	8.8893	19.2505	2.9510	3.9499	3.3207	10.2216	3.5438	4.8832	4.7886	13.2156
$\mathbf{B}_{\mathrm{Sup}}^\diamond$	1.0922	9.3249	8.9322	19.3493	3.0301	4.0966	3.4285	10.5552	3.5197	4.8862	4.8173	13.2232
$\left(\mathbf{B}_k^\diamond\right)_{k=1}^{9}$	1.0888	9.2750	8.8966	19.2604	2.9172	3.9622	3.3157	10.1951	3.5096	4.8677	4.7923	13.1696

Any of the above subimages is considered as an instantaneous observation from a given sensor, the 9 sensors providing the full scene for any experiment, an experiment consisting of the following operations:

- Computation of the marginal best bases $(\mathbf{B}_k^\diamond = \mathbf{B}^\diamond[\mathrm{I}_k])_{1,2,\ldots,K}$ with respect to the entropy functions given in Definitions 2, 3, 4.

- Evaluation of the cost of using the most relevant basis, among the marginal best bases, by computing the entropies of any given subimage $\mathrm{I}_\ell, \ell \in \{k = 1, 2, \ldots, K\}$ on the marginal best bases $(\mathbf{B}_k^\diamond)_{k=1,2,\ldots,K}$. Note that this evaluation assumes that all the data are available at a central node (experimental setup in order to assess performance). We compute this basis, instead of the best joint basis, since computing the latter is combinatorial over a significant number of wavelet packet subtree configurations.

- Computation of

$$\mathbf{B}_{\mathrm{Median}}^\diamond = \mathrm{Median}\{\mathbf{B}_k^\diamond, k = 1, 2, \ldots, K\},$$

as well as the extremal bases:

$$\mathbf{B}^{\diamond}_{\mathrm{Inf}} = \mathrm{Inf}\{\mathbf{B}^{\diamond}_{k}, k = 1, 2, \dots, K\},$$

$$\mathbf{B}^{\diamond}_{\mathrm{Sup}} = \mathrm{Sup}\{\mathbf{B}^{\diamond}_{k}, k = 1, 2, \dots, K\}.$$

Table 3 provides the costs spent in using an arbitrary marginal best basis and the most relevant marginal best basis, as well as the cost of using the median of the marginal best bases. It follows from this table that the entropy of the median basis is more relevant than any marginal best bases. This median basis outperforms the more relevant marginal best basis which is computable only in a fully collaborative distribution scheme.

From the results given in Table 3, the infimum and supremum bases have large entropies in general: this is reasonable since these bases represent extremal trees (smallest and largest) observed from the set of signals. However, these bases provide information on the dispersion of the set of marginal best bases. In particular, when the ℓ^1 concentration norm is used with the Haar wavelet, the infimum and the median bases are very close to each other (they differ by the splitting of a unique node) and this shows that most of the observed bases have approximately the same tree structure.

Table 4: Structural dissimilarity indices $\mathcal{L}(\mathbf{B}) = \sum_{q=1}^{K} \mathcal{M}(\mathbf{B}, \mathbf{B}^{\diamond}_{q})$ where $\mathbf{B}$ is either one among the best bases $\{\mathbf{B}^{\diamond}_1, \mathbf{B}^{\diamond}_2, \dots, \mathbf{B}^{\diamond}_K\}$ or the median/infimum/supremum of this set of best basis. Dissimilarity indices have been computed in the experiment setup of Section 4.2 (see also Table 3).

Entropy:	L_1-Norm				Threshold				SURE			
Basis	R	G	B	RGB	R	G	B	RGB	R	G	B	RGB
$\mathcal{L}(\mathbf{B}^{\diamond}_1)$	588	380	247	1215	318	354	305	977	129	98	55	282
$\mathcal{L}(\mathbf{B}^{\diamond}_2)$	217	271	179	667	262	357	294	913	105	83	55	243
$\mathcal{L}(\mathbf{B}^{\diamond}_3)$	211	210	158	579	408	409	282	1099	134	98	55	287
$\mathcal{L}(\mathbf{B}^{\diamond}_4)$	242	186	140	568	468	403	401	1272	106	79	40	225
$\mathcal{L}(\mathbf{B}^{\diamond}_5)$	184	193	178	555	312	344	272	928	106	79	40	225
$\mathcal{L}(\mathbf{B}^{\diamond}_6)$	196	212	212	620	441	450	372	1263	86	109	65	260
$\mathcal{L}(\mathbf{B}^{\diamond}_7)$	277	211	151	639	648	615	601	1864	201	188	100	489
$\mathcal{L}(\mathbf{B}^{\diamond}_8)$	174	176	143	493	337	450	439	1226	99	80	65	244
$\mathcal{L}(\mathbf{B}^{\diamond}_9)$	227	305	244	776	278	450	482	1210	86	94	65	245
$\mathcal{L}(\mathbf{B}^{\diamond}_{\mathrm{Med}})$	171	173	134	478	279	322	263	864	86	80	40	206
$\mathcal{L}(\mathbf{B}^{\diamond}_{\mathrm{Inf}})$	232	210	158	600	318	380	305	1003	129	98	55	282
$\mathcal{L}(\mathbf{B}^{\diamond}_{\mathrm{Sup}})$	700	506	393	1599	966	841	829	2636	201	187	125	513

Furthermore, if we denote by $\mathcal{N}_{\mathbf{B}}$ the set of terminal node indices of basis $\mathbf{B}$ in tree T:

$$\mathcal{N}_{\mathbf{B}} = \text{Terminal nodes}(\mathbf{B})$$

and define the *structural dissimilarity index* of basis $\mathbf{B}_p$ and basis $\mathbf{B}_q$ by (by counting the nodes that do not pertain to two basis trees):

$$\mathcal{M}(\mathbf{B}_p, \mathbf{B}_q) = \#\mathcal{N}_{\mathbf{B}_p} + \#\mathcal{N}_{\mathbf{B}_q} - 2\#\mathcal{N}_{\mathbf{B}_p} \cap \mathcal{N}_{\mathbf{B}_q},$$

then Table 4 provides cumulative structural dissimilarity indices associated with a basis $\mathbf{B}$ with respect to the set $\{\mathbf{B}_1^\diamond, \mathbf{B}_2^\diamond, \ldots, \mathbf{B}_K^\diamond\}$:

$$\mathcal{L}(\mathbf{B}) = \sum_{q=1}^{K} \mathcal{M}(\mathbf{B}, \mathbf{B}_q^\diamond)$$

These results of Table 4 shows that the median of the best basis is the basis that minimizes almost surely, the structural dissimilarity with respect to the set of observed bases. Experimental results with other images confirm the same remark.

To end this section, one may question the payoff of choosing a system with 1) no-collaboration (coding/compressing independently the signals with respect to the different marginal best bases) 2) a light collaboration (median of marginal best bases discussed in this paper) or 3) a full collaboration (distributed Karhunen-Loève transform for instance, see [9], [1]). Such a choice may strongly depend on the application, since for real-time monitoring by using onboard systems, the sensor architecture is required to be very light, whereas this sensor architecture has no consequence on permanently fixed systems.

5. Conclusion

The paper has presented some algorithmic tools that make basis ordering and order statistics on a set of bases possible. The set of basis under consideration is assumed to be tree structured, with node inclusion properties and an additive information cost is associated with the tree nodes.

The partial ordering associated with such a set of bases facilitates the definition of the median, infimum and supremum of a set of bases. Furthermore, the paper has shown that the median of the marginal best bases is suitable for coding and compression problems since this median basis outperforms the best marginal basis, whereas computing the latter requires more energy than computing the former.

The paper has also provided two particular bases which relate the dispersion of the observed set of marginal acquisitions: the infimum and supremum bases. These bases can be used for example in change detection problems, for instance when seeking an acquisition outlier. In addition, these bases can be helpful in information fusion when mutual information is considered.

Some prospects with respect to the work concern both distributed compression (high dimensional imaging such as 3D and/or video imposes additional constraints) and distributed fusion from compressive acquisitions. For a distributed compressive acquisition, recent prospective works on the topic, in particular [8], have shown that exploiting common sparse supports and correlation structures can lead to a more relevant joint source coding of piecewise regular signals, when the latter are assumed to have Gaussian independent and identically distributed sparse descriptions in some bases. This paper has considered the set of suitable bases as a tree structured library and has provided a (non-parametric) solution for selecting a basis that approaches the performances of the best joint basis, for a given joint information cost (sparsity, concentration, Shannon, *etc.*). Thus, when considering joint sparsity information cost, one can expect that the present paper has also provided a non-parametric answer (in terms of the median of best sparsifying marginal bases) to the selection of a best basis for distributed compressive sensing. The relevancy of the median basis in such a distributed compressive sensing can probably be highlighted theoretically, by analyzing the properties of the median basis with respect to the joint sparsity information and, practically, by simulating several compressive acquisitions associated with different bases, including the median of the best marginal bases.

Other prospects concern establishing the performance of the median of marginal best bases in multiview joint coding and compression problems: the issue regards capturing in this basis, the redundancy (due to the overlapping) of certain types of acquisitions (this overlap can be fixed *a priori* from the sensor location and configurations or estimated by using correlation structures).

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