Topics on Gravity Outside of Four Dimensions

Adel Bouchareb

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Adel BOUCHAREB

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DIRECTEURS DE THESE : Karim Ait Moussa Professeur Université de Constantine
Gérard Clément D.R. LAPTh (Annecy, France)

DEVANT LE JURY

PRESIDENT :

Reda Attallah	Professeur	Université d'Annaba

EXAMENATEURS:

Habib Aissaoui	M.C.A	Université de Constantine
Azeddine Boudine	M.C.A	Université de Oum El Bouaghi
Ahcène Djoudi	Professeur	Université d'Annaba

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General Introduction

Gravity is one of the most beautiful theories of modern physics. It is based on an elegant mathematical structure-the differential geometry of curved spacetime-and supported by all available experimental tests. Gravity has a special status, as it is intimately connected with the geometry of space and time. The basic idea in Einstein's formulation of gravity-General Relativity (GR)- is that gravity is (a manifestation of the curvature of the) geometry of spacetime.

The spacetime has no *a priori* fixed shape and it is dynamical. The structure of the spacetime is governed by the local distribution of the matter that it contains and the motion of the matter depends on the curvature of the spacetime.

The theory is based on two principles, the first one is that the physics remains unaltered under a general coordinate transformation (diffeomorphisms). The second principle, known as the (Einstein) equivalence principle, generalizes the accurately experimentally verified equality between the gravitational and inertial masses. It can be stated in many forms, one of them is the following: *Local gravitational effects can be got rid of by choosing an inertial frame.* The dynamics of the theory is governed by the Einstein equation, which is a set of coupled partial differential equations of the metric of the spacetime. The metric is a mathematical object (a tensor) that tells us how to measure distances on the spacetime. Theses differential equations are highly nonlinear which make them almost impossible to solve by direct methods. The only way to find exact solutions of these equations is making some assumptions about the form of the metric (ansatzë).

An outstanding problem in modern physics is reconciling Quantum Field (QFT) Theory with General Relativity. So far all attempts to quantize gravity failed.

The central objects in GR (and the main purpose of the present thesis), and which may shed light on the problem of Quantum Gravity are Black Holes. They are the physical systems where gravity, statistical physics and quantum mechanics meet. The idea of the existence of such objects traces back to 1789, when reverend John Michell and later in 1795 (but independently) Simon Pierre de La Place. Both predicted the existence of stars that are so massive that even light can not escape from them. Their prediction was based on the corpuscular nature of light, that it is, the idea of "dark stars" was abandoned after the Young two-slit interference experiment showing that light has wavy nature in 1801.

The idea was left dormant for many years until the advent of the General Relativity by Albert Einstein in 1916. Few months after the publication of the final form of Einstein's theory, a German mathematician and astronomer, Karl Schwarzschild succeeded to find the first exact solution of the complicated Einstein equation. He assumed a spherically symmetric shape of the spacetime. The Schwarzschild solution shows a "singularity" at the very same radius of the Michell and Laplace "dark stars". It was realized that at this radius, the light undergoes an infinite redshift in such a way that it never reaches an outside observer. On the other side it was shown the geometry at Schwarzschild radius is perfectly smooth. An observer crossing this area will notice nothing special. The aforementioned singularity turns to be only an artifact of the coordinate system.

Astrophysical black holes may form at the final stage of stellar evolution, the stars of masses of few solar masses may collapse into black holes.

Currently about 20 binary stars are known to exist in our milky way. They are believed to contain black holes of few solar masses. Supermassive black holes are believed to exist at the centers of galaxies including ours.

The purpose of this thesis is not discuss astrophysical black holes but to consider black holes from the theoretical point of view. It is believed that black hole physics are may shed light on one of the fundamental problems of contemporary physics, namely to find new ideas to reconcile dynamical spacetime geometry and quantum mechanics. Though the final goal is still far from reach, considerable progress on some aspects of black holes has been made during the last couple of decades.

The most interesting feature of black holes form the theoretical perspective is that they are thermodynamical objects.

Many of the black hole problems may be addressed in $(2 + 1)$-dimensions.

A more ambitious theory is string theory. This theory claims providing a unified description of all interactions including quantum gravity. The basic idea of string theory is to replace point elementary particles of the conventional Quantum Field Theory by one dimensional extended object called strings of which the vibrational modes correspond to the elementary particles. The spectrum of the string contain a massless spin-2 particle which can be identified with the graviton (the mediator of the gravitational interactions). The theory is free of tachyons (particles with speed greater than light's speed) if one introduces a new symmetry, called supersymmetry (SUSY), between bosons (integer spin particles) and fermions (half-integer spin particles). String theory with such a symmetry is known as superstring theory. The consistency of this theory requires the dimension of spacetime to be ten. The problem of superstring theory is that it is not unique, there are five distinct string theories. The main success of the "second string theory" is the discovery that all the five string theories are related to each other. Moreover all these theories arise as different limits of a unique theory coined "M-theory". This theory is not yet known. However, it is approximated by eleven dimensional gravity at low energies. Supergravity is the local version of supersymmetry. Superstring theory is not a theory of strings only but it contains other extended objects called "D-branes". A D-brane is a hypersurface where open strings are attached. Yang-Mills fields are thought of to live on D-brane. A breakthrough was provided by the study of the nonperturbative aspects of D-branes. That is the microscopic derivation of the black hole entropy. Using a special case in five dimensions, Strominger and Vafa [**?**] succeed to count the microstates associated with D-branes (which are linked to black holes). The extremal charged black holes which solve low energy limit supergravity.

This Thesis

The thesis is divided into two loosely connected parts: the first one is concerned with three dimensional Topologically massive gravity and the other is devoted to generating solutions of black objects within five minimal dimensional supergravity theory.

The First Part

The first chapter is meant to introduce a few selected issues about three dimensional gravity and defining Topologically Massive Gravity (TMG).

The second chapter is aimed to first describe in some details the formalism based on dimensional reduction of $(2+1)$-dimensional gravity to $(1+0)$-dimensions. This formalism is used to construct a solution describing a black hole family in TMG with cosmological constant.

The third chapter constitutes the core of the first part. After introducing the method of Abbot-Deser-Tekin for calculating conserved quantities in gravity theories, we give the detailed calculation of the Killing charges in Topologically Massive Gravity. Using the formalism presented in the previous chapter, we greatly simplify the formula of the conserved charges in the case of stationary axially symmetric spacetimes. The fourth chapter deals with thermodynamics of Topologically Massive Gravity. The relevant thermodynamical quantities entering in the first law are computed. Amongst them is the entropy, which is computed using the method of conical singularities. The different quantities together with the mass and the angular momentum, computed in the third chapter, all fit nicely in the first law of the black hole thermodynamics.

The Second Part

The chapter five is devoted to preliminary notions such as nolinear sigma model and coset manifold. In the sixth chapter we present the dimensionl reduction of five dimensional minimal supergravity down to (3+0)-dimensions to obtain, after dualizing Kaluza-Klein gauge vectors to scalars, a nolinear sigma model coupled to (3+0)-dimensional gravity. The next task is to make manifest the hidden symmetries of the theory. We show that the the isotropy group of the scalar manifold is the split real form of the exceptional group G_2, then the scalar matrix is given. The sixth chapter is devoted to the generating technique of black objects. Especially transforming neutral black hole and neutral black ring into charged ones is presented.

Part I

GRAVITY IN (2+1) DIMENSIONS

1
(2+1) dimensional gravity in a nutshell

$(2+1)-$dimensional gravity (2 space dimensions and one time dimension)(with or without cosmological constant) is essentially a gedanken laboratory to study issues about quantum gravity. This theory provides us with a simpler picture of the more realistic but complicated $(3+1)$-dimensional gravity. At first sight, gravity in $(2+1)$ dimensions looks trivial and may be unattractive. This is mainly due to the lack of local degrees of freedom and the absence of Newtonian limit. The geometry of the spacetime is flat outside the source of matter (or has a constant curvature if one considers a cosmological constant). However, local distribution of matter affects globally the geometry of the spacetime. For instance, the spacetime around a point particle is conical with a deficit angle proportional to the particle's mass.

The most dramatic turn in (2+1)-dimensional gravity, was the discovery of a solution with almost the usual features of black holes. This solution named BTZ after Bañados, Teitelboim and Zanelli [2].

One can remedy the problem of absence of degrees of freedom by adding to the Hilbert-Einstein action an $SO(1,2)$ gravitational Chern-Simons term. The theory with such a modification is called Topologically Massive Gravity (TMG). It is a consistent theory of gravity with a massive graviton having one degree of freedom [1] .

1.1 The action

$(2+1)$-dimensional gravity (with cosmological constant) is the theory of gravity described by the Einstein-Hilbert action

$$I = \frac{1}{2\kappa}\int_{\mathcal{M}} (R - 2\Lambda) + I_{matter}, \tag{1.1}$$

in one dimension of time and two dimensions of space. As usual the integration is hold on the manifold M equipped with a metric$g$, R is the scalar curvature, Λ is the cosmological constant, and $\kappa \equiv 8\pi G$ whit G is the Newton constant. I_{matter} is the matter action.

The resulting Euler-Lagrange equations are the Einstein equations

$$\mathcal{G}_{\mu\nu} \equiv \kappa T_{\mu\nu}, \tag{1.2}$$

where

$$\mathcal{G}_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu},$$

$G_{\mu\nu}$being the Einstein tensor defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}.$$

Even this theory has the very same form of the $(3+1)-$dimensional General Relativity, it is very peculiar. Indeed this theory does not contain any dynamical degrees of freedom.

1.2 The Peculiarity of $(2+1)$-dimensional General Relativity

In any dimension the local geometric information about the spacetime is encoded in the Riemann tensor. The latter can be decomposed into two parts as

$$R_{\mu\nu}{}^{\rho\sigma} = g_{[\mu}{}^{[\rho} S_{\nu]}{}^{\sigma]} + C_{\mu\nu}{}^{\rho\sigma}$$

where the tensor $S_{\mu\nu}$

$$S_{\mu\nu} \equiv \frac{4}{D-1} R_{\mu\nu} - \frac{4}{(D-1)(D-2)} R g_{\mu\nu},$$

is a combination of the traces of the Riemann tensor. $C_{\mu\nu\rho\sigma}$is a traceless and conforrmally invariant tensor (vanishes if the metric is a factor times the flat metric). This tensor is called Weyl tensor. The 'true' gravitational degrees of freedom are contained in the Weyl. While the Ricci tensor and scalar curvature can be identified with the matter degrees of freedom, as it is suggests by Einstein equations. The peculiarity of $(2+1)-$dimensional general relativity comes from the fact that the Riemann tensor has the same number of the independent entries as the Ricci tensor implying that Weyl tensor vanishes identically in three dimensions. This has dramatic consequences: every solution of (1.2) with $\Lambda = 0$ is flat, $\Lambda < 0$ is Ads and $\Lambda > 0$ is dS.

This means that the curvature is concentrated at the location of the matter. In other words there is no propagating degrees of freedom, no gravitational waves at the classical level and no graviton at the quantum level.

1.2.1 Counting the degrees of freedom

The absence of the degrees of freedom can be further checked on by a naive counting. The dynamical variables of general relativity in $D-$dimensions

are constant time hypersurface metric and its time derivative. The phase space of the theory is then $2 \times \frac{1}{2} D(D-1)$, but there are D of the Einstein equations which are not dynamical but rather constraints, besides, we have D gauge freedom. We are left then with

$$2 \times \frac{1}{2} D(D-1) - D - D = 2 \times \frac{1}{2} D(D-3)$$

In conclusion there are $\frac{1}{2}D(D-3)$ degrees of freedom per point. In $D=4$, on gets the two polarizations of the gravitons, but in $D=3$ there are no degrees of freedom.

1.2.2 The Newtonian limit

The $(2+1)-$dimensional gravity has no Newtonian limit. In the weak field approximation of $d > 2$ dimensional GR

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ ,$$

Einstein equations read (we drop the cosmological constant for the time being)

$$\Box h_{\mu\nu} = 2\kappa T_{\mu\nu}$$

one recovers the Newtonian limit (namely Poisson's law) $\nabla^2 \phi = 4\pi G \rho$ by the identifications $h_{00} = \phi$ and $T_{00} = 4\pi G\rho$. The geodesic equation

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + 2\frac{D-3}{D-2}\partial^i \phi = 0$$

Obviously for $D=3$, a test particle does feel any interaction.

1.3 The gravitational field of a point mass in (2+1)-dimensions

We will show that spacetime is flat outside the matter source. The mass is reflected in an angle deficit in the space and the spin has the effect of rendering the time coordinate helical.

1.3.1 Static solution

Let us consider a spinless point particle of mass M at rest at rest of the coordinates system. The energy-momentum tensor is given by

$$\sqrt{|g|} T^{00} = -M \delta^{(2)}(\mathbf{x}), \quad T^{0i} = T^{ij} = 0$$

We assume the ansatz for the metric

$$g_{ij} = e^{\phi(\mathbf{x})}, \quad g_{i0} = 0, \quad g_{00} = -N^2(\mathbf{x}).$$

The equations of motion be reduce to

$$N(\mathbf{x}) = \mathrm{cons\,tan\,t},$$

which can be set to $N(\mathbf{x}) = 1$ by rescaling the time, and

$$\nabla^2 \phi(\mathbf{x}) = -2\kappa M \delta^{(2)}(\mathbf{x}).$$

This is the Green's function in two dimensions. Its solution is

$$\phi(\mathbf{x}) = -\frac{\kappa}{\pi} M \ln \|\mathbf{x}\| \,.$$

The line element is given then by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \|\mathbf{x}\|^{2(\alpha-1)}\,\mathrm{d}\mathbf{x}^2, \quad \alpha \equiv 1 - \frac{\kappa}{2\pi} M,$$

Let us focus on the space line

$$\begin{aligned}\mathrm{d}l^2 &= \|\mathbf{x}\|^{2(\alpha-1)}\,\mathrm{d}\mathbf{x}^2 \\ &= \|\mathbf{x}\|^{2(\alpha-1)}\,(\mathrm{d}\,\|\mathbf{x}\|^2 + \|\mathbf{x}\|^2\,\mathrm{d}\varphi^2)\end{aligned}$$

If we make the change of variables

$$r = \frac{\|\mathbf{x}\|^{\alpha}}{\alpha} \text{ and } \phi = \alpha\varphi,$$

we obtain the line element of flat space in usual polar coordinates

$$\mathrm{d}l^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2,$$

however the range of the angular variable is unusual, *viz.* $0 \le \phi \le 2\pi\alpha$. Which means that a wedge of angle $2\pi(1-\alpha)$ is cut out from the flat space and the edges are identified. This defines a cone.

1.3.2 Spinning point mass

We return now to the spinning point particle with angular momentum J The energy-momentum tensor is given by

$$\begin{aligned}\sqrt{|g|}T^{00} &= -M\delta^{(2)}(\mathbf{x}), \quad T^{0i} = 0 \\ \sqrt{|g|}T^{ij} &= \mathrm{J}\varepsilon^{ij}\partial_j\delta^{(2)}(\mathbf{x})\end{aligned}$$

The line element is given then by

$$\mathrm{d}s^2 = -(\mathrm{d}t - \frac{\kappa}{\pi}\mathrm{J}\mathrm{d}\phi)^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2,$$

If we define a new time τ

$$\tau = t - \frac{\kappa}{\pi}\mathrm{J}\phi, \tag{3.3}$$

The line element becomes that of a flat conical spacetime

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2,$$

nevertheless, the transformation (3.3)is singular at $\mathbf{x} = \mathbf{0}$, and the time has a unusual structure as it is helical. At constant t, the new time τ is identified as

$$\tau \equiv \tau - \frac{2\kappa}{\alpha}\mathrm{J}n$$

since $\phi \equiv \phi + 2\pi n\alpha$.

1.4 BTZ Black Holes

The BTZ black hole[2] is obtained from the universal covering AdS_3 metric

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} - 1\right)\mathrm{d}t^2 + \left(\frac{r^2}{\ell^2} - 1\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 \tag{4.4}$$

if we identify $\varphi \sim \varphi + 2\pi$ (this can be done since ∂_φ is a Killing vector). (4.4) will be a black hole. For constant t, $r < \ell$ are trapped surfaces. We know from section (1.3.1) that identifying $\varphi \sim \varphi + 2\pi\sqrt{M}$ has the effect of adding a mass then

$$\mathrm{d}s^2 = -\left(\frac{r^2}{\ell^2} - M\right)\mathrm{d}t^2 + \left(\frac{r^2}{\ell^2} - M\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 \tag{4.5}$$

where the following rescaling was made

$$\varphi \to \frac{\varphi}{\sqrt{M}}, \qquad t \to \frac{t}{\sqrt{M}}, \qquad r \to \sqrt{M}r$$

One can further add an angular momentum too the black hole. This can be done by the identifications:

$$\begin{aligned}\hat{t} &= \frac{r_+}{\ell}t - r_-\varphi \\ \hat{\varphi} &= \frac{r_+}{\ell}\varphi - \frac{r_-}{\ell^2}t \\ \hat{r}^2 &= \ell^2\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\end{aligned}$$

1.5 Topologically Massive Gravity

As it was seen in the previous section, the $(2+1)-$dimensional gravity theory is trivial due to the fact that the traceless part of the Riemann tensor (namely the Weyl tensor) vanishes identically in three dimensions. However, there exists a tensor which can play the role of the Weyl tensor in three dimension, that is the conformal Cotton tensor

$$C^{\mu\nu} = \frac{1}{\sqrt{|g|}}\epsilon^{(\mu\alpha\beta}\mathrm{D}_\alpha G^{\nu)}{}_\beta\,.$$

This has the same symmetries as the Einstein tensor, and one can hope to restore some of the degrees of freedom contained in Weyl tensor if the Cotton tensor is added to the Einstein tensor in the equation of motion.

Since $C^{\mu\nu}$ is one derivative higher order than Einstein tensor, it must be multiplied by a factor $\frac{1}{\mu}$ of inverse mass dimension. The equation of motion then reads

$$\mathcal{E}_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = \kappa T_{\mu\nu}. \tag{5.6}$$

In the same manner that Einstein term can be derived from the Einstein-Hilbert action (1.1), the Cotton term can be derived from the following Chern-Simons action

$$I_{CS} = \frac{1}{2}\int_{\mathcal{M}}\mathrm{d}^3x\sqrt{-g}\varepsilon^{\lambda\mu\nu}\Gamma^r_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma_{r\nu} + \frac{2}{3}\Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\nu r}\right). \tag{5.7}$$

This action is called topological term since it does not depend on the metric but only on the connections.

The theory described with Einstein-Hilbert action plus the Chern-Simons term is called Topologically massive gravity (TMG) [1] . Adding the topological term modifies the theory in a non trivial way. This can be seen by the linearized equation of motion about Minkowski vacuum, this yields

$$(\Box + \mu^2)\phi = 0,$$

with

$$\phi \equiv (\delta_{ij} + \hat{\partial}_i \hat{\partial}_j) h^{ij}, \quad \hat{\partial}_i \equiv \partial_i (-\nabla^2)^{-\frac{1}{2}}.$$

This indicates the presence of propagating of one massive degree of freedom, hence the name massive gravity.

Bibliography

[1] S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* **48** (1982) 975; *Ann. Phys., NY* **140** (1982) 372.

[2] M. Bañados, C. Teitelboim and J. Zanelli, *Phys. Rev. Lett.* **69** (1992) 1849; M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Phys. Rev.* D **48** (1993) 1506.

2
TMG Black Holes

TMG black hole solutions was found in [1]. The purpose of this chapter is to extend this solution to include cosmological constant.

2.1 The sationnary axial symmetric spacetimes in (2+1)-dimensional gravity:

We consider three dimensional spacetime with two commuting $U(1)$ isometries. The two $U(1)$ isometries are generated by two non-commuting Killing vectors ∂_t and ∂_φ. We are interested in axially symmetric solutions, so we will take φ coordinate periodic *i.e* $\varphi \sim \varphi + 2\pi$ (the orbits of ∂_φ are closed).

2.1.1 Dimensional Reduction:

Upon dimensional reduction, along ∂_t and ∂_φ isometries, from $(2+1)$ to $(1+0)$, three dimensional metric can be then cast into the form

$$\mathrm{d}s^2 = \lambda_{ab}(\mathrm{d}x^a + B^a\mathrm{d}\rho)(\mathrm{d}x^b + B^b\mathrm{d}\rho) + \frac{h_{\rho\rho}(\rho)}{|\det\lambda|}\,\mathrm{d}\rho^2\,, \quad a = 1,2 \tag{1.1}$$

$$x^0 = t, \qquad x^1 = \varphi, \qquad x^2 = \rho \tag{1.2}$$

where λ is the 2×2 matrix, $h_{\rho\rho}(\rho)$ is the metric on the ρ direction.B^a is the Kaluza-Klein gauge vector which can be removed through the coordinate transformations $x^a \to x^a + F^a(\rho)$.we denote by e the einbein in the ρ direction.

The metric (1.1) then takes the form

$$ds^2 = \lambda_{ab}(\rho)\,dx^a dx^b + \frac{e^2(\rho)}{|\det\lambda|}\,d\rho^2\,, \tag{1.3}$$

2.1.2 Parametrization of the metric

The metric (1.3) is $SL(2,\mathbb{R})$ invariant. This fact leads us to define a useful parametrization of the matrix λ.

There is a one-to-one mapping between vectors of $\mathbb{R}^{2,1}$ (The three dimensional Minkowski space) and 2×2 real symmetrical matrices (this due to the isomorphism $SL(2,\mathbb{R}) \approx SO(2,1)$).

Let's first, define the Lorentzian invariant dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ belonging ,

$$\mathbf{A}\cdot\mathbf{B} = \eta_{ij}A^i B^j$$

and their Lorentz invariant cross product by

$$(\mathbf{A}\times\mathbf{B})^i = \eta^{ij}\epsilon_{jkl}A^k B^l\,, \tag{1.4}$$

where ϵ_{ijk} is the Levi-Cevita symbol, with $\epsilon_{012} = +1$ (thus $\epsilon^{012} = -1$).

Notice that some of the usual cross product identities may change due to the Lorentzian signature of the metric η_{ij}.

For instance, the identity $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})$ has the opposite sign of the usual one.

We will define the "light cone " components A^+ and A^- and the "transverse" component A^Y of a vector $\mathbf{A} = (A^0, A^1, A^2)$

$$A^{\pm} \equiv A^0 \pm A^1\,, \tag{1.5}$$

$$A^Y \equiv A^2 \tag{1.6}$$

Recall that the $SL(2,\mathbb{R})$ algebra is given by

$$\left[\tau^i, \tau^j\right] = \epsilon^{ijk}\tau_k$$

where the generators are the following Pauli matrices.

$$\tau^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,,\ \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\ \tau^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{1.7}$$

To each vector $\mathbf{V} = (V^0, V^1, V^2) \in \mathbb{R}^{1,2}$ one can associate a real symmetrical matrix

$$M = \tau^0\boldsymbol{\tau}.\mathbf{V} = \begin{pmatrix} V^+ & V^Y \\ V^Y & V^- \end{pmatrix}.$$

and the determinant of M is given by

$$\begin{aligned} \det M &= (V^0)^2 - (V^1)^2 - (V^2)^2 \\ &= \eta_{ij}V^i V^j \\ &= -\left\|\boldsymbol{V}\right\|^2 \end{aligned}$$

Conversely, for every real symmetric 2×2 matrix M, there is a vector $\boldsymbol{V} \in \mathbb{R}^{1,2}$ defined by

$$\boldsymbol{V} = -\frac{1}{2}Tr(\boldsymbol{\tau}\tau^0 M)$$

$$\begin{aligned} V^0 &= \frac{1}{2}(M_{11} + M_{22}) \\ V^1 &= \frac{1}{2}(M_{11} - M_{22}) \\ V^2 &= M_{12} \end{aligned}$$

In particular, there is a vector $\mathbf{X} = (X^0, X^1, X^2)$ associated with λ such that

$$\lambda = \begin{pmatrix} X^+ & Y \\ Y & X^- \end{pmatrix}, \tag{1.8}$$

$$\begin{aligned} \mathcal{R}^2 &\equiv |\det \lambda| \\ &= \mathbf{X}^2 = \eta_{ij} X^i X^j = -T^2 + X^2 + Y^2 \,. \end{aligned} \tag{1.9}$$

It will be useful to associate to each vector $\mathbf{A} \in \mathbb{R}^{1,2}$, a traceless matrix noted $\mathbf{A}$ or a such that

$$a \equiv \mathbf{A} \equiv \boldsymbol{\tau} \cdot \boldsymbol{A} = \begin{pmatrix} -A^Y & -A^- \\ A^+ & A^Y \end{pmatrix} \tag{1.10}$$

Pauli matrices satisfy

$$\tau^i \tau^j = \eta^{ij} + \epsilon^{ijk} \tau_k \,, \qquad \tau^{iT} = \tau^0 \tau^i \tau^0 = \tau_i \,. \tag{1.11}$$

It follows immediately from the above two properties that for any two$R^{1,2}$ vectors $\mathbf{A}$, $\mathbf{B}$ one has

$$\mathbf{AB} = \mathbf{A} \cdot \mathbf{B}\,\mathbb{I} + \mathbf{A} \times \mathbf{B} \,, \tag{1.12}$$

where 1 is the 2×2 unit matrix. Also, the application of (**??**) to the vector $\mathbf{X}$ yields

$$\lambda = \tau^0 x \,, \quad \lambda^{-1} = -\frac{1}{\mathcal{R}^2} x \tau^0 \,. \tag{1.13}$$

From the background metric (1.3), we compute the Christoffel symbols

$$\Gamma^a_{2b} = \frac{1}{2} \left(\lambda^{-1} \lambda'\right)^a{}_b \,, \quad \Gamma^2_{ab} = -\frac{1}{2e^2} \mathcal{R}^2 \, \lambda'_{ab} \,, \quad \Gamma^2_{22} = -\mathcal{R}^{-1} \mathcal{R}' \,, \tag{1.14}$$

where the prime stands for the derivative $d/d\rho$, and the corresponding Ricci tensor

$$R^a{}_b = -\frac{1}{2e} \Big((e^{-1} \mathcal{R}\mathcal{R}')' \mathbb{I} + (e^{-1} l)' \Big)^a{}_b \,, \quad R^2{}_2 = \frac{1}{e} \Big(-(e^{-1} \mathcal{R}\mathcal{R}')' + e^{-1} \frac{1}{2} \mathbf{X}'^2 \Big) \,, \tag{1.15}$$

where l is the matrix associated with the vector

$$\mathbf{L} \equiv \mathbf{X} \times \mathbf{X}' \,. \tag{1.16}$$

The Ricci scalar

$$R = \frac{1}{e} \Big(-2(e^{-1} \mathcal{R}\mathcal{R}')' + e^{-1} \frac{1}{2} \mathbf{X}'^2 \Big) \,,$$

2.1.3 (2+1)-Dimensional Gravity as Particle Mechanics

The parametrization (1.3) reduces the action (1.1) to the form

$$I = \frac{1}{2\kappa_3} \int \mathrm{d}^2 x \int \mathrm{d}\rho \, L \,,$$

The effective action which governs the dynamics is given by

$$L = \tfrac{1}{2} \left[e^{-1} \, \mathbf{X}'^2 - 4e \, \Lambda + \frac{1}{\mu} e^{-2} \mathbf{X} \cdot (\mathbf{X}' \times \mathbf{X}'') \right] \tag{1.17}$$

The variable $x^1$and $x^2$are toroidal coordinates, then dynamically irrelevant, they can be integrated out. The effective action which governs the dynamics is given by

$$I_E = \frac{1}{2\kappa_1} \int \mathrm{d}\rho \, L \,, \tag{1.18}$$

where

$$\frac{1}{\kappa_1} \equiv \frac{V_2}{\kappa_3} \text{ and } V_2 \equiv \int_{\mathbb{T}^{1,1}} \mathrm{d}^2 x$$

To make transparent the interpretation of the Lagrangian (1.17), let us turn off the Chern-Simons action. The remaining Einstein-Hilbert action describes Lorentz in-

variant mechanics of a particle. The particle's world-line (geodesic) is parametrized by ρ. The "position" of the particle is $\mathbf{X}(\rho)$ (the embedding of the world-line into the an auxiliary Minkowski target space $\mathbb{R}^{1,2}$) . The variable $e(\rho)$ is an einbein on the world-line. The equations of motion are derived by externalizing the action (1.18) with respect to $\mathbf{X}$, this yields

$$\mathbf{X}'' = 0.$$

The variation of the action with respect to the field e gives the constraint

$$\mathbf{X}'^2 + 4e^2\,\Lambda = 0. \tag{1.19}$$

One can solve the previous equation in e and replace back it in the action (1.17), this can be always done because (1.19) is not dynamical. The result is the more familiar action

$$\begin{aligned} I_E &= -\int \mathrm{d}\rho \, \sqrt{|\Lambda \mathbf{X}'^2|} \\ &= -\sqrt{|\Lambda|} \int \sqrt{\mathrm{d}\tau^2}, \end{aligned}$$

where τ is the proper time of the particle

$$d\tau^2 = -\eta_{ij} \mathrm{d}X^i \mathrm{d}X^j = -\mathbf{X}'^2 \mathrm{d}\rho^2.$$

To recover the non-relativistic limit, we choose the gauge $X^0 \equiv T = \tau$, then

$$I_E = -\sqrt{|\Lambda|} \int \mathrm{d}T \sqrt{1 - \vec{U}^2}, \quad \vec{U} \equiv \begin{pmatrix} \dfrac{\mathrm{d}X}{\mathrm{d}T} \\ \\ \dfrac{\mathrm{d}Y}{\mathrm{d}T} \end{pmatrix}.$$

In the limit of velocities small with respect to the velocity of light

$$I_E = \int \mathrm{d}T \left(-\sqrt{|\Lambda|} + \frac{1}{2} \sqrt{|\Lambda|} \vec{U}^2 + ... \right).$$

One can identify the particle's mass, being $\sqrt{|\Lambda|}$.

2.1.4 TMG as Particle Mechanics

Let us now turn on the Chern-Simons term. Using (1.14), one has

$$I_{C-S} = \frac{1}{2\kappa_1\mu}\int \mathrm{d}\rho\, e^{-2}\mathbf{X}\cdot(\mathbf{X}'\times\mathbf{X}'') \tag{1.20}$$

This adds to the free particle action (1.18), an "interaction" potential. The dynamics is governed now by the sum of the free particle action (1.18) and the interaction (1.20). As aside note, the full action is second order in ρ, the extremization of such an action needs either to add boundary conditions on the velocities $\mathbf{X}'(\rho_i) = \mathbf{X}'(\rho_f)$ or to add of suitable a surface term. The equations of motion is then

$$\mathbf{X}'' = \frac{1}{2\mu e}\left[3(\mathbf{X}'\times\mathbf{X}'') + 2(\mathbf{X}\times\mathbf{X}''')\right] \tag{1.21}$$

And the variation with respect to the Lagrange multiplier e yields constraint

$$\mathbf{X}'^2 - \frac{2}{e\mu}\mathbf{X}.(\mathbf{X}'\times\mathbf{X}'') + 4e^{-2}\,\Lambda = 0. \tag{1.22}$$

These equations can be integrated once. This can be performed thanks to the conserved angular momentum $\mathbf{J}$ associated to Lorentz transformations on the auxiliary Minkowski space $\mathbb{R}^{1,2}$. The conserved angular momentum $\mathbf{J}$ is found to be

$$\mathbf{J} = \mathbf{L} + \mathbf{S},$$

$$\text{with}\ \ \mathbf{L} \equiv \mathbf{X}\times\mathbf{X}' \ \ \text{and}\ \ \mathbf{S} \equiv -\frac{1}{2\mu e}[\mathbf{X}'\times(\mathbf{X}\times\mathbf{X}') - 2\mathbf{X}\times(\mathbf{X}\times\mathbf{X}'')]. \tag{1.23}$$

The angular momentum $\mathbf{L}$ is associated the pure Einstein-Hilbert action, while $\mathbf{S}$ is correspond to Chern-Simons term. From (1.21) , one can rewrite(1.23) as

$$2\mathbf{X}^2\mathbf{X}'' = 2e\mu\mathbf{J} - 2e\mu\mathbf{X}\times\mathbf{X}' + (\mathbf{X}\cdot\mathbf{X}')\mathbf{X}' - (6e^2\Lambda + \frac{5}{2}\mathbf{X}'^2)\mathbf{X}, \tag{1.24}$$

and the constraint reduces to

$$(\mathbf{X}'^2 + 4e^2\Lambda) = -\frac{2}{\mu e}\mathbf{X}\cdot(\mathbf{X}'\times\mathbf{X}'') = -\frac{4}{3}\mathbf{X}\cdot\mathbf{X}'' \tag{1.25}$$

2.1.5 Solving the equations of motion

In this section we will solve the equations of motion (1.24). We will also assume that $X(\rho)$ is an analytic function of at $\rho = 0$, so it can be expanded in Taylor series

$$\mathbf{X}(\rho) = \sum_{n=0}^{\infty}\frac{1}{n!}\rho^n\boldsymbol{\alpha}_n$$

The coefficients $\boldsymbol{\alpha}_n$ are given

$$\boldsymbol{\alpha}_n = \left.\frac{\mathrm{d}^n\mathbf{X}(\rho)}{\mathrm{d}\rho^n}\right|_{\rho=0}$$

Let us now expand equation (1.24) order by order in ρ. The order ρ^0 term just fixes the angular momentum vector

$$\mathbf{J} = 2\boldsymbol{\alpha}_0\times\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_1(\boldsymbol{\alpha}_0\cdot\boldsymbol{\alpha}_1) - \boldsymbol{\alpha}_0(6e^2\Lambda - \frac{5}{2}\boldsymbol{\alpha}_1) \tag{1.26}$$

Let us further make an educated guess and suppose that $\boldsymbol{\alpha}_2$ is null and by a suitable translation of coordinates setting $\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 = 0$. It follows that

$$\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 = \pm \left\|\boldsymbol{\alpha}_1\right\| \boldsymbol{\alpha}_0 \tag{1.27}$$

Expanding the Hamiltonian constraint to order ρ^0 gives

$$\boldsymbol{\alpha}_1 + 4e^2\Lambda = -2\frac{1}{e\mu}\boldsymbol{\alpha}_0 \cdot (\boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2) = \mp 2\frac{1}{e\mu}\left\|\boldsymbol{\alpha}_1\right\| \boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_2 \tag{1.28}$$

Comparing to the second form of the Hamiltonian constraint

$$\boldsymbol{\alpha}_1 + 4e^2\Lambda = -\frac{4}{3}\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_2 \tag{1.29}$$

we see that either $\alpha_1^2 = \dfrac{4}{9}\dfrac{\mu^2}{\zeta^2}$ and $\alpha_0 \cdot \alpha_2 \neq 0$ or $\alpha_1^2 = -4e^2\Lambda$ and $\alpha_0 \cdot \alpha_2 = 0$.

If $\alpha_1^2 = -4e^2\Lambda$ then we can show that all higher order terms in the Taylor expansion vanish, so the solution is just the BTZ black hole. Finally, let us consider the case where $\boldsymbol{\alpha}_1^2 = \dfrac{4}{9}\dfrac{\mu^2}{\zeta^2}$. In this case the Hamiltonian constraint to the lowest order reduces to

$$\boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_0 = -\frac{1}{3}\mu^2 e^{-2}\left(1 + 9\frac{\Lambda}{\mu^2}\right) \tag{1.30}$$

Expanding the constraint equation (1.25) to the linear order ρ^1 we discover that $\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_3 = 0$ then to ρ^{m-2},ρ^{m-1} and ρ^{m-1}

$$\boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_m = \boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_m = \boldsymbol{\alpha}_2 \cdot \boldsymbol{\alpha}_m = 0 \tag{1.31}$$

for all $m \geq 3$. The equations of motion to order ρ

$$2\boldsymbol{\alpha}_2^2\boldsymbol{\alpha}_3 + 2e\mu\boldsymbol{\alpha}_0 \times \boldsymbol{\alpha}_2 - \frac{1}{2}e^2\left(\mu^2 + 9\Lambda\right)\boldsymbol{\alpha}_1.$$

This can be solved by

$$\boldsymbol{\alpha}_3 = 0 \quad \text{and} \quad \boldsymbol{\alpha}_0 \times \boldsymbol{\alpha}_2 = \frac{1}{2}e^2\left(\mu^2 + 9\Lambda\right)\boldsymbol{\alpha}_1.$$

Proceeding in the same manner we find that the equation of motion to the ρ^2 is solved by

$$\boldsymbol{\alpha}_4 = 0 \quad \text{and} \quad \boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 = -\frac{2}{3}e\mu\boldsymbol{\alpha}_2.$$

Now we will see that all the higher order terms in Taylor expansion vanish. Let us suppose that the cubic and higher terms up to some m vanish

$$\mathbf{X}(\rho) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\rho + \frac{1}{2}\boldsymbol{\alpha}_2\rho^2 + \sum_{k=m}^{\infty}\frac{1}{k!}\rho^k\boldsymbol{\alpha}_k \tag{1.32}$$

Using the Hamiltonian constraints (1.25), the equations of motion to the order ρ^{m-2} reduces to

$$\frac{2}{(m-2)!}\boldsymbol{\alpha}_2^2\boldsymbol{\alpha}_m = 0$$

We have just proven by induction that the higher terms vanish Indeed, we have checked that $\boldsymbol{\alpha}_3 = 0$ We have assumed then that the higher terms up to m vanish and proven that in this case the higher order terms up to $m+1$ must vanish.

The conclusion is

$$\mathbf{X}(\rho) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1\rho + \frac{1}{2}\boldsymbol{\alpha}_2\rho^2 \tag{1.33}$$

with

$$\boldsymbol{\alpha}_2^2 = \boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 = 0, \quad \boldsymbol{\alpha}_1^2 = \nu^2, \quad \boldsymbol{\alpha}_0 \cdot \boldsymbol{\alpha}_2 = -z,$$
$$\boldsymbol{\alpha}_0 \times \boldsymbol{\alpha}_2 = \frac{z}{\nu}\boldsymbol{\alpha}_1, \quad \boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2 = -\frac{2}{3}e\mu\boldsymbol{\alpha}_2. \tag{1.34}$$

with

$$z \equiv \nu^2(1-\beta^2)$$
$$\nu \equiv \frac{2}{3}e\mu, \quad \beta^2 \equiv \frac{1}{4}\left(1 - \frac{27\Lambda}{\mu^2}\right) \tag{1.35}$$

2.1.6 Parametrization

We now proceed to the parametrization of the constant vectors subject to the Eqns (1.34).

Let us start with the null vector $\boldsymbol{\alpha}_2$. A generic null vector can always be put in the form

$$\boldsymbol{\alpha}_2 = (a, a\cos\theta, a\sin\theta)., \tag{1.36}$$

with a real and $0 \leq \theta < 2\pi$. From (1.1) and (1.8),

At large ρ, one has

$$g_{\phi\phi} \sim \tfrac{1}{2}\alpha_2^- \rho^2 = a\rho^2 \sin^2 \tfrac{1}{2}\theta \tag{1.37}$$

This should be nonnegative (to avoid CTC's at infinity) thus either 1) $\theta \neq 0$ and $a > 0$, or 2) $\theta = 0$.

In the first case ($\theta \neq 0$ and $a > 0$), making the change to a rotating frame $d\varphi \to d\hat{\varphi} = d\varphi - \cot(\theta/2)dt$ transforms $\boldsymbol{\alpha}_2$ to

$$\hat{\boldsymbol{\alpha}}_2 = (\hat{a}, \hat{a}\cos\hat{\theta}, \hat{a}\sin\hat{\theta}).,\tag{1.38}$$

but with $\hat{\alpha}^Y = 0$t leading to $\hat{\theta} = \pi$. $\hat{a}$ can be set equal to 1 by a scale transformation $\rho \to \dfrac{\hat{\rho}}{\hat{a}}$ and $t \to \dfrac{\hat{t}}{\hat{a}}$, thus

$$\boldsymbol{\alpha}_2 = (1, -1, 0). \tag{1.39}$$

In the other case, the magnitude of c can be set equal to 1 but the sign remains unconstrained

$$\boldsymbol{\alpha}_2 = (\pm 1, \pm 1, 0). \tag{1.40}$$

It was seen in [?] that this choice corresponds either to a non-black hole solution or to the precedent solution.

Let us note

$$\boldsymbol{\alpha}_0^2 \equiv -\beta^2\rho_0^2, \tag{1.41}$$

The equations (1.34) are fulfilled by the following parametrization:

$$\boldsymbol{\alpha}_2 = (1, -1, 0)\,, \quad \boldsymbol{\alpha}_1 = (\omega, -\omega, -\nu)\,, \quad \boldsymbol{\alpha}_0 = \frac{1}{2}(z+u, z-u, -\frac{2\omega z}{\nu})\,, \tag{1.42}$$

where

$$u \equiv \frac{\omega^2 z}{\nu^2} + \frac{\rho_0 \beta^2}{z}.$$

It is to notice that in this parametrization, there are only two free parameters, namely ρ_0 and ω.

2.1.7 The Metric

Now we are ready to write the final form of the metric.

With the help of (1.8) and (1.9), the metric (1.3) may be put into the form

$$ds^2 = -\frac{\mathcal{R}^2}{X^-}\mathrm{d}t^2 + X^-\left(\mathrm{d}\varphi + \frac{Y}{X^-}\mathrm{d}t\right)^2 + \frac{e^2}{\mathcal{R}^2}\,\mathrm{d}\rho^2 \tag{1.43}$$

Using (1.33),(1.41) and (1.42), we have

$$X^+ = \nu^2\left(1-\beta^2\right)$$

$$X^- \equiv (\rho+\omega)^2 + \beta^2\left(\frac{\rho_0^2}{1-\beta^2} - \omega^2\right),$$

$$\mathcal{R}^2 \equiv \mathbf{X}^2(\rho) = \beta^2(\rho^2 - \rho_0^2)$$

$$Y = -\nu\left[\rho + \omega\left(1-\beta^2\right)\right]$$

recall that

$$\nu \equiv \frac{2}{3}e\mu, \quad \beta^2 \equiv \frac{1}{4}\left(1 - \frac{27\Lambda}{\mu^2}\right)$$

By replacing these in (1.43), we get

$$\begin{aligned} \mathrm{d}s^2 = &-\beta^2\frac{\rho^2-\rho_0^2}{r^2}\,\mathrm{d}t^2 + r^2\left[\mathrm{d}\varphi - \nu\frac{\rho+(1-\beta^2)\omega}{r^2}\,\mathrm{d}t\right]^2 \\ &+ \frac{e^2}{\beta^2}\frac{\mathrm{d}\rho^2}{\rho^2-\rho_0^2}\,, \end{aligned} \tag{1.44}$$

with $r^2 \equiv X^-$which can be written as

$$r^2 = \rho^2 + 2\omega\rho + \omega^2(1-\beta^2) + \frac{\beta^2\rho_0^2}{1-\beta^2}\,, \tag{1.45}$$

The natural background for the black hole family (1.44) is the extreme black hole $\rho_0 = 0$ with $\omega = 0$,

$$\mathrm{d}\bar{s}^2 = -\beta^2\,\mathrm{d}t^2 + \rho^2\left[\mathrm{d}\varphi - \frac{1}{\rho}\,\mathrm{d}t\right]^2 + \frac{1}{\zeta^2\beta^2}\frac{\mathrm{d}\rho^2}{\rho^2}\,. \tag{1.46}$$

The vector X associated with (1.44) is of the form[1] From the wedge products

$$\boldsymbol{\alpha}\times\boldsymbol{\alpha} = -\boldsymbol{\alpha}\,, \quad \boldsymbol{\alpha}\times\boldsymbol{\alpha} = -z\boldsymbol{\alpha}\,, \quad \boldsymbol{\alpha}\times\boldsymbol{\alpha} = \frac{\beta^2\rho^2}{z}\boldsymbol{\alpha} - \boldsymbol{\alpha}, \tag{1.47}$$

[1] In this section we return to the convention introduced in Eq. (1.3), i.e. background geometrical quantities are overlined.

we obtain

$$\mathbf{L} = \alpha\,\rho^2 + 2\beta\,z\rho - \beta \times \gamma\,, \tag{1.48}$$

$$\mathbf{S} = -\alpha\,\rho^2 - 2\beta\,z\rho - \frac{1}{3}\gamma + \frac{4}{3}z\beta \times \gamma\,, \tag{1.49}$$

leading to the constant super angular momentum

$$\mathbf{J} = \frac{2\beta^2}{3}\left[-\frac{1+2\beta^2}{1-\beta^2}\rho_0^2\boldsymbol{\alpha} + \boldsymbol{\gamma}\right]. \tag{1.50}$$

2.2 Global Structure

This black hole is regular for all $\rho \neq \pm\rho_0$ (geodisically complete). It may be extended through the horizons$\pm\rho_0$

$$R = \frac{1-4\beta^2}{2}\nu^2 \ = 6\Lambda,$$
$$R_{\mu\nu}R^{\mu\nu} = \frac{3-8\beta^2+8\beta^4}{4}\nu^4\,, \tag{2.1}$$

There is no curvature singularity (at least for $\Lambda \neq 0$).

However the orbits of $m = \dfrac{\partial}{\partial\phi}$ should be nontimelike for the absence of CTC's $i.e. m^2 \geq 0$. Therefore r^2 must be nonnegative.

CT's are absent if

$$\beta^2 < 1 \quad \text{and} \quad \omega^2 < {\rho_0}^2/(1-\beta^2)\,. \tag{2.2}$$

For $\beta^2 < 1$ and $\omega^2 > \rho_0^2/(1-\beta^2)$,and $\omega > 0$ $\rho \in [\rho_-, \rho_+]$,$(\rho_- < \rho_+ < -\rho_0)$ where ρ_- and ρ_+ are the roots of

$$r^2 = \rho^2 + 2\omega\rho + \omega^2(1-\beta^2) + \frac{\beta^2\rho_0^2}{1-\beta^2}\,, \tag{2.3}$$

$r^2 = 0$

$$\rho_\pm = -\omega \pm \beta\sqrt{\omega^2 - \frac{{\rho_0}^2}{1-\beta^2}}\,. \tag{2.4}$$

3

Conserved Quantities for TMG with arbitrary background

3.1 Preliminaries

The definition of conserved charges is related to the mass-energy tensor, but the definition of the latter is a subtle subject in general relativity. It might come as surprise for non-specialist that a covariant mass-energy tensor of gravitational interaction does not exist. To understand this fact one has to return to the basics of general relativity. The principle of equivalence states that "One can get rid of all the local physical effects of the gravitational field by jumping in an inertial (free falling) frame ". One can then cancel the energy-momentum tensor at every point. The consequence of this is the non-localizability of the gravitational conserved charges. Only the global charges of the whole spacetime are conserved. An other approach is that quasi-local charges..... Since the conserved quantities are in general associated with the symmetries of the background (vacuum) not with the full theory, one can look for a pseudotensor (a tensor that is covariant only under a subset of the general coordinate transformations). Following [3, 4, 5] we will show how to build a pseudotensor and compute the conserved charges. Consider a gravitation theory(a theory which invariant under diffeomorphisms), with a generalized Einstein tensor (GET) $\mathcal{E}_{\mu\nu}\left[g\right]$ and matter mass-energy tensor $T_{\mu\nu}$. The equation of motion

$$\mathcal{E}_{\mu\nu}\left[g\right] = \kappa T_{\mu\nu} \tag{1.1}$$

where $E_{\mu\nu}\left[g\right]$ is a Generalized Einstein Tensor (GET) of a gravitation theory, $T_{\mu\nu}$ is the matter energy-momentum tensor, and $\kappa = 8\pi G$ is the Einstein gravitational constant.

Given a background metric $\bar{g}_{\mu\nu}$solving the vacuum field equations

$$\mathcal{E}_{\mu\nu}\left[\bar{g}\right] = 0\,. \tag{1.2}$$

The metric $g_{\mu\nu}$, can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\ , \tag{1.3}$$

where $h_{\mu\nu}$ represents deviation from the background solution ($h_{\mu\nu}$ vanishes at infinity, however it is not assumed to be small everywhere).

In the following we will refer by background fields to the fields constructed from the background metric. The fields built from $h_{\mu\nu}$ will be called linearized fields. Furthermore we will adopt the convention that indices on linearized tensors (together with background tensors of course) are raised and lowered with $\bar{g}$.

By formally expanding $\mathcal{E}_{\mu\nu}[g]$ around $h_{\mu\nu}$

$$\mathcal{E}_{\mu\nu}[g] = \mathcal{E}_{\mu\nu}[\bar{g}] + \left.\frac{\delta\mathcal{E}_{\mu\nu}}{\delta g_{\alpha\beta}}\right|_{\bar{g}} h^{\alpha\beta} + \mathcal{O}(h^2) \tag{1.4}$$

Noting $\delta\mathcal{E}_{\mu\nu} \equiv \left.\frac{\delta\mathcal{E}_{\mu\nu}}{\delta g_{\alpha\beta}}\right|_{\bar{g}} h^{\alpha\beta}$, the exact Field equations (**??**) can then be written as

$$\delta\mathcal{E}_{\mu\nu} = \kappa(T_{\mu\nu} + t_{\mu\nu}), \tag{1.5}$$

where

$$t_{\mu\nu} \equiv \frac{1}{\kappa}(\mathcal{E}_{\mu\nu} - \delta\mathcal{E}_{\mu\nu})$$

$t_{\mu\nu}$ is interpreted as the energy-momentum pseudotensor of the gravitational field (generalization of the flat background to an arbitrary (curved) background). The quantity

$$\tau_{\mu\nu} \equiv T_{\mu\nu} + t_{\mu\nu}$$

is then the total energy-momentum pseudotensor of matter and gravitation.

We will construct from $\tau_{\mu\nu}$ a locally conserved quantity.

For this, let's notice first that the linearized GET inherits Bianchi identity from the exact theory, indeed, we have

$$\begin{aligned} \mathrm{D}_\nu\mathcal{E}^{\mu\nu} &= \bar{\mathrm{D}}_\nu\bar{\mathcal{E}}^{\mu\nu} + \bar{\mathrm{D}}_\nu\delta\mathcal{E}^{\mu\nu} + \mathcal{O}(h^2) \\ &= \bar{\mathrm{D}}_\nu\bar{\mathcal{E}}^{\mu\nu} + \delta\Gamma^{\mu}_{\mu\rho}\bar{\mathcal{E}}^{\rho\nu} + \delta\Gamma^{\nu}_{\mu\rho}\bar{\mathcal{E}}^{\mu\rho} + \bar{\mathrm{D}}_\nu\delta\mathcal{E}^{\mu\nu} + \mathcal{O}(h^2), \end{aligned}$$

Using Bianchi identity for the exact theory

$$\mathrm{D}_\nu\mathcal{E}^{\mu\nu} = 0. \tag{1.6}$$

and that of the background

$$\bar{\mathrm{D}}_\nu\bar{\mathcal{E}}^{\mu\nu} = 0. \tag{1.7}$$

to together with (1.2), one has the linearized Bianchi identity

$$\bar{\mathrm{D}}_\nu\delta\mathcal{E}^{\mu\nu} = 0\,, \tag{1.8}$$

This implies that

$$\bar{\mathrm{D}}_\nu\tau^{\mu\nu} = 0 \tag{1.9}$$

with d the background covariant derivative. Now if the background admits an isometry ξ_μ, i.e.

$$\bar{\mathrm{D}}_{(\mu}\xi_{\nu)} = 0 \tag{1.10}$$

then

$$\bar{\mathrm{D}}_\mu(\tau^{\mu\nu}\xi_\nu) = 0\,. \tag{1.11}$$

is covariantly conserved (as a consequence of (1.9) and (1.10)). But the quantity between parenthesis is a vector then

$$\bar{\mathrm{D}}_\mu(\tau^{\mu\nu}\xi_\nu) = \frac{1}{\sqrt{|\bar{g}|}}\partial_\mu\left(\sqrt{|\bar{g}|}\tau^{\mu\nu}\xi_\nu\right) = 0\,. \tag{1.12}$$

implying that is $\sqrt{|\bar{g}|}\tau^{\mu\nu}\xi_\nu$ is locally conserved.

and that the charge

$$Q^\mu(\xi) = \int_{\mathcal{M}} \sqrt{|\bar{g}|}\tau^{\mu\nu}\xi_\nu \tag{1.13}$$

is conserved.

In practice, it is not an easy to calculate $t^{\mu\nu}$ (in order to get $\tau^{\mu\nu}$) for specific physical problems. It is convenient to use the L-H side of (1.5)

$$\tau^{\mu\nu}\xi_\nu = \frac{1}{\kappa}\delta\mathcal{E}^{\mu\nu}\xi_\nu$$

then

$$Q^\mu(\xi) = \frac{1}{\kappa}\int_{\mathcal{M}} \sqrt{|\bar{g}|}\delta\mathcal{E}^{\mu\nu}\xi_\nu \tag{1.14}$$

We define the "superpotential"

$$\mathcal{K}^\mu \equiv \delta\mathcal{E}^{\mu\nu}\xi_\nu = \partial_\nu\mathcal{F}^{\mu\nu} = \frac{1}{\sqrt{|\bar{g}|}}\partial_\nu\left(\sqrt{|\bar{g}|}\mathcal{F}^{\mu\nu}\right), \tag{1.15}$$

Using Stokes theorem, one gets

$$Q^\mu(\xi) = \frac{1}{\kappa}\int_{\mathcal{M}}\partial_\nu\left(\sqrt{|\bar{g}|}\mathcal{F}^{\mu\nu}\right) = \frac{1}{\kappa}\int_{\partial\mathcal{M}}\sqrt{|\bar{g}|}\mathcal{F}^{\mu\nu}\mathrm{d}S_\nu. \tag{1.16}$$

3.2 Conserved Quantities For TMG Solutions

The ADT procedure explained above will be applied to TMG with cosmological constant.

3.2.1 The linearized Theory

Generalized Einstein tensor is defined by

$$\mathcal{E}_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu}\,.$$

where the Einstein tensor is given by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}\,, \qquad \mathcal{G}_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu}\,. \tag{2.17}$$

and the Cotton tensor is

$$C^{\mu\nu} = \frac{1}{\sqrt{|g|}}\epsilon^{(\mu\alpha\beta}\mathrm{D}_\alpha G^{\nu)}{}_\beta\,, \tag{2.18}$$

with $\epsilon^{\mu\alpha\beta}$ the antisymmetric symbol.

In what follows $\delta\mathcal{T}$ indicates the part linear in $h_{\mu\nu}$ of the field $\mathcal{T}$ which reduced to $\bar{\mathcal{T}}$ on the background. Notice that $\delta\mathcal{T}^{\mu\nu} \neq g^{\mu\alpha}g^{\nu\beta}\delta\mathcal{T}_{\alpha\beta}$.

One can find the relation between $\delta T^{\mu\nu}$ and $\delta T_{\alpha\beta}$ as follows

$$\begin{aligned}\delta\mathcal{T}^{\mu\nu} &= \delta(g^{\mu\alpha}g^{\nu\beta}\delta\mathcal{T}_{\alpha\beta})\\ &= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\delta\mathcal{T}_{\alpha\beta} + \delta g^{\mu\alpha}g^{\nu\beta}\bar{\mathcal{T}}_{\alpha\beta} + g^{\mu\alpha}\delta g^{\nu\beta}\bar{\mathcal{T}}_{\alpha\beta}\\ &= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\delta\mathcal{T}_{\alpha\beta} - h^{\mu\alpha}\bar{\mathcal{T}}_{\alpha}{}^{\nu} - h^{\nu\beta}\bar{\mathcal{T}}^{\mu}{}_{\beta}\\ &= \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\delta T_{\alpha\beta} - h^{\mu}{}_{\alpha}\bar{\mathcal{T}}^{\alpha\nu} - h^{\nu}{}_{\alpha}\bar{\mathcal{T}}^{\mu\alpha}\end{aligned}$$

where the linearized metric

$$\delta g_{\mu\nu} \equiv h_{\mu\nu}, \quad \delta g^{\mu\nu} = -h^{\mu\nu} \tag{2.19}$$

The trace of the linearized metric is $h \equiv g^{\mu\nu}h_{\mu\nu}$, and using the identity for any matrix M, $\delta \det M/\det M = \mathrm{Tr}\delta M$ one can compute the linearized determinant $\delta g/g = h$, and we have also

$$\delta\frac{1}{\sqrt{|g|}} = -\frac{1}{2}\frac{1}{\sqrt{|g|}}h\,.$$

The linearized Christoffel symbols are found to be

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\bigg(\mathrm{D}_{\mu}h^{\rho}{}_{\nu} + \mathrm{D}_{\nu}h^{\rho}{}_{\mu} - \mathrm{D}^{\rho}h_{\mu\nu}\bigg)\,. \tag{2.20}$$

One can deduce the linearized Ricci tensor

$$\begin{aligned}\delta R_{\mu\nu} &= \mathrm{D}_{\rho}\delta\Gamma^{\rho}_{\mu\nu} - \mathrm{D}_{\nu}\delta\Gamma^{\rho}_{\mu\rho}\\ &= \tfrac{1}{2}\bigg(\mathrm{D}^{\lambda}\mathrm{D}_{\nu}h_{\lambda\mu} + \mathrm{D}^{\lambda}\mathrm{D}_{\mu}h_{\lambda\nu} - \mathrm{D}^{\lambda}\mathrm{D}_{\lambda}h_{\mu\nu} - \mathrm{D}_{\mu}\mathrm{D}_{\nu}h\bigg)\,,\end{aligned} \tag{2.21}$$

The linearized Ricci scalar reads

$$\delta R \equiv \delta R_{\mu\nu}\,g^{\mu\nu} - h^{\mu\nu}R_{\mu\nu} = -\mathrm{D}^{\lambda}\mathrm{D}_{\lambda}h + \mathrm{D}_{\mu}\mathrm{D}_{\nu}\,h^{\mu\nu} - h^{\mu\nu}R_{\mu\nu}, \tag{2.22}$$

The linearized cosmological Einstein tensor is

$$\delta\mathcal{G}_{\mu\nu} \equiv (\delta G_{\mu\nu} + \Lambda h_{\mu\nu}) = \delta R_{\mu\nu} - \tfrac{1}{2}\,g_{\mu\nu}\,\delta R - \tfrac{1}{2}(R-2\Lambda)\,h_{\mu\nu}\,,\,.$$

Replacing (2.21) (2.22), one ends up with

$$\begin{aligned}\delta\mathcal{G}_{\mu\nu} &= \tfrac{1}{2}\bigg(\mathrm{D}^{\lambda}\mathrm{D}_{\nu}h_{\lambda\mu} + \mathrm{D}^{\lambda}\mathrm{D}_{\mu}h_{\lambda\nu} - \mathrm{D}^{\lambda}\mathrm{D}_{\lambda}h_{\mu\nu} - \mathrm{D}_{\mu}\mathrm{D}_{\nu}h\bigg)\\ &\quad - \tfrac{1}{2}g_{\mu\nu}\bigg(- \mathrm{D}^{\lambda}\mathrm{D}_{\lambda}h + \mathrm{D}_{\mu}\mathrm{D}_{\nu}\,h^{\mu\nu}\bigg) - \tfrac{1}{2}\bigg((R-2\Lambda)\,h_{\mu\nu}\bigg)\\ &\quad + \tfrac{1}{2}g_{\mu\nu}R_{\lambda\rho}h^{\lambda\rho},\end{aligned} \tag{2.23}$$

The linearized Cotton tensor is found to be

$$\delta C^{\mu\nu} = \frac{1}{\sqrt{|g|}}\epsilon^{(\mu\alpha\beta}\bigg(\mathrm{D}_{\alpha}\delta G^{\nu)}{}_{\beta} + \delta\Gamma^{\nu)}_{\alpha\lambda}G^{\lambda}{}_{\beta}\bigg) - \tfrac{1}{2}hC^{\mu\nu}\,. \tag{2.24}$$

3.2.2 The Superpotentials

Let us start with the superpotental corresponding to Einstein tensor,

$$
\begin{aligned}
\mathcal{K}_E^{\mu} \equiv \xi_\nu \delta G^{\mu\nu} &= \tfrac{1}{2}\xi_\nu\Big(\mathrm{D}_\lambda \mathrm{D}^\nu h^{\lambda\mu} + \mathrm{D}_\lambda \mathrm{D}^\mu h^{\lambda\nu} - \mathrm{D}^\lambda \mathrm{D}_\lambda h^{\mu\nu} - \mathrm{D}^\mu \mathrm{D}^\nu h\Big) \\
&+ \tfrac{1}{2}\xi^\mu\Big(\mathrm{D}_\lambda \mathrm{D}^\lambda h - \mathrm{D}_\lambda \mathrm{D}_\nu h^{\lambda\nu}\Big) + \tfrac{1}{2}\xi_\nu\Big(-4R^{(\mu\lambda}h_\lambda{}^{\nu)} + (R-2\Lambda)h^{\mu\nu}\Big) \\
&+ \tfrac{1}{2}\xi^\mu R_{\lambda\rho}h^{\lambda\rho},
\end{aligned} \tag{2.25}
$$

After integration by parts

$$
\begin{aligned}
\mathcal{K}_E^{\mu} &= \tfrac{1}{2}\mathrm{D}_\lambda\Big(\xi^\lambda \mathrm{D}_\nu h^{\mu\nu} - \xi^\mu \mathrm{D}_\nu h^{\lambda\nu} + \xi_\nu \mathrm{D}^\mu h^{\lambda\nu} - \xi_\nu \mathrm{D}^\lambda h^{\mu\nu} + \xi^\mu \mathrm{D}^\lambda h - \xi^\lambda \mathrm{D}^\mu h\Big) \\
&+ \tfrac{1}{2}\left(\mathrm{D}_\lambda \xi_\nu \mathrm{D}^\lambda h^{\mu\nu} + \mathrm{D}_\lambda \xi^\mu \mathrm{D}_\nu h^{\lambda\nu} - \mathrm{D}_\lambda \xi^\mu \mathrm{D}^\lambda h\right) \\
&+ \tfrac{1}{2}\xi^\nu\Big(-2R^{\mu\lambda}h_{\lambda\nu} - R_{\nu\lambda}h^{\lambda\mu} + R^\mu{}_{\rho\lambda\nu}h^{\lambda\rho} + (R-2\Lambda)h^{\mu\nu}\Big) \\
&+ \tfrac{1}{2}\xi^\mu R_{\lambda\rho}h^{\lambda\rho}
\end{aligned}
$$

and we have used the identity valid for Killing vectors

$$
\mathrm{D}_\mu \mathrm{D}_\nu \xi_\lambda = R^\rho{}_{\mu\nu\lambda}\xi_\rho\,. \tag{2.26}
$$

Finally

$$
\mathcal{K}_E^{\mu} = \mathrm{D}_\lambda \mathcal{F}_E^{\mu\lambda}(\xi) - \xi^\nu \mathcal{G}^{\mu\lambda}h_{\lambda\nu} + \frac{1}{2}\xi^\mu \mathcal{G}^{\lambda\rho}h_{\lambda\rho} - \frac{1}{2}\xi^\nu \mathcal{G}^\mu{}_\nu h\,, \tag{2.27}
$$

with the Einstein superpotential:

$$
\begin{aligned}
\mathcal{F}_E^{\mu\nu}(\xi) = \tfrac{1}{2}\big(&\xi^\nu \mathrm{D}_\lambda h^{\lambda\mu} - \xi^\mu \mathrm{D}_\lambda h^{\lambda\nu} + \xi_\lambda \mathrm{D}^\mu h^{\lambda\nu} - \xi_\lambda \mathrm{D}^\nu h^{\lambda\mu} + \xi^\mu \mathrm{D}^\nu h - \xi^\nu \mathrm{D}^\mu h \\
&+ h^{\nu\lambda}\mathrm{D}_\lambda \xi^\mu - h^{\mu\lambda}\mathrm{D}_\lambda \xi^\nu + h\mathrm{D}^{[\mu}\xi^{\nu]}\big)\,,
\end{aligned} \tag{2.28}
$$

Notice the that (2.27) is a total divergence for pure GR , since the extra terms vanish on-shell.

Now we shall turn to the Cotton term

$$
\mathcal{K}_C^{\mu} \equiv \xi_\nu \delta C^{\mu\nu} \tag{2.29}
$$

$$
\begin{aligned}
&= \frac{1}{2\sqrt{|g|}}\Bigg[\mathrm{D}_\alpha\Big(\epsilon^{\mu\alpha\beta}\xi_\nu \delta G^\nu{}_\beta + \epsilon^{\nu\alpha\beta}\xi_\nu \delta G^\mu{}_\beta + \epsilon^{\mu\nu\beta}\xi_\nu \delta G^\alpha{}_\beta\Big) \\
&- \tfrac{1}{2}\epsilon^{\nu\alpha\beta}\mathrm{D}_\alpha \xi_\nu \delta G^\mu{}_\beta - \tfrac{1}{2}\epsilon^{\mu\nu\beta}\xi_\nu \mathrm{D}_\alpha \delta G^\alpha{}_\beta \\
&+\epsilon^{(\nu\alpha\beta}\xi_\nu \delta\Gamma^{\mu)}_{\alpha\lambda}G^\lambda{}_\beta - \tfrac{1}{2}\sqrt{|g|}\xi_\nu h C^{\mu\nu}\Bigg]\,.
\end{aligned} \tag{2.30}
$$

Using

$$
\mathrm{D}_\alpha \delta G^\alpha{}_\beta = -\delta\Gamma^\alpha_{\alpha\lambda}G^\lambda{}_\beta + \delta\Gamma^\lambda_{\alpha\beta}G^\alpha{}_\lambda\,, \tag{2.31}
$$

and the identity

$$
\epsilon^{\mu\alpha\beta}\xi_\nu \equiv \delta^\mu_\nu \epsilon^{\alpha\beta\rho}\xi_\rho + \delta^\alpha_\nu \epsilon^{\beta\mu\rho}\xi_\rho + \delta^\beta_\nu \epsilon^{\mu\alpha\rho}\xi_\rho\,, \tag{2.32}
$$

we shall cast $\mathcal{K}_C^{\mu}$ into the same form as(2.27), namely a total derivative term plus the extra terms where the Einstein tensor being replaced by the Cotton one. The aim is that the Einstein part and the Cotton part of each term will combine and vanish on-shell.

$$\mathcal{K}^{\mu}_{C} \equiv \frac{1}{2\sqrt{|g|}} T^{\mu} - \tfrac{1}{2}\Big(2\xi^{\nu} C^{\lambda\mu} h_{\lambda\nu} - \xi^{\mu} C^{\lambda\rho} h_{\lambda\rho} + \xi^{\nu} C^{\mu}{}_{\nu} h \Big)$$

where

$$\begin{aligned} T^{\mu} &= \mathrm{D}_{\alpha}\Big(2\epsilon^{\mu\alpha\beta}\xi_{\nu}\delta G^{\nu}{}_{\beta} - \epsilon^{\mu\alpha\nu}\xi_{\nu}\delta G \Big) - \epsilon^{\nu\alpha\beta}\mathrm{D}_{\alpha}\xi_{\nu}\delta G^{\mu}{}_{\beta} \\ &+ 2\Big(\epsilon^{\alpha\beta\nu}\delta\Gamma^{\mu}_{\alpha\lambda} + \epsilon^{\mu\alpha\nu}\delta\Gamma^{\beta}_{\alpha\lambda} + \epsilon^{\beta\mu\nu}\delta\Gamma^{\alpha}_{\alpha\lambda} \Big)\xi_{\nu} G^{\lambda}{}_{\beta} \\ &+ \Big(\epsilon^{\lambda\alpha\beta}\xi_{\nu}\mathrm{D}_{\alpha}G^{\mu}{}_{\beta} + \epsilon^{\mu\alpha\beta}\xi_{\nu}\mathrm{D}_{\alpha}G^{\lambda}{}_{\beta} - \epsilon^{\lambda\alpha\beta}\xi^{\mu}\mathrm{D}_{\alpha}G_{\nu\beta} \Big) h_{\lambda}{}^{\nu} \,. \end{aligned} \tag{2.33}$$

T^{μ} is expected to be a total derivative, thus we shall integrate the rhs of (2.33) part. To this end we define the vector

$$\eta^{\nu} \equiv \frac{1}{2\sqrt{|g|}} \epsilon^{\nu\rho\sigma}\mathrm{D}_{\rho}\xi_{\sigma} \,, \tag{2.34}$$

and express the 3-dimensional Riemann tensor in terms of the Einstein tensor,

$$R^{\rho\sigma}_{\mu\nu} = \epsilon_{\mu\nu\alpha}\epsilon^{\rho\sigma\beta} G^{\alpha}{}_{\beta} \tag{2.35}$$

we have from (2.26) and (2.32)

$$\mathrm{D}^{\mu}\eta^{\nu} = \frac{1}{\sqrt{|g|}} \epsilon^{\mu\rho\lambda}\xi_{\rho} G^{\nu}{}_{\lambda} \,. \tag{2.36}$$

Using the definition (2.34), the second term of (2.33) can be rewritten as

$$\begin{aligned} &- \frac{1}{2}\epsilon^{\nu\alpha\beta}\mathrm{D}_{\alpha}\xi_{\nu}\delta G^{\mu}{}_{\beta} = \sqrt{|g|}\eta^{\beta}\delta G^{\mu}{}_{\beta} \\ &= \frac{1}{2}\sqrt{|g|}\mathrm{D}_{\lambda}\Big(\eta^{\lambda}\mathrm{D}_{\nu}h^{\mu\nu} - \eta^{\mu}\mathrm{D}_{\nu}h^{\lambda\nu} + \eta_{\nu}\mathrm{D}^{\mu}h^{\lambda\nu} - \eta_{\nu}\mathrm{D}^{\lambda}h^{\mu\nu} + \eta^{\mu}\mathrm{D}^{\lambda}h - \eta^{\lambda}\mathrm{D}^{\mu}h \Big) \\ &+ \frac{1}{2}\sqrt{|g|}\Big(\mathrm{D}_{\lambda}\eta^{\mu}\mathrm{D}_{\nu}h^{\lambda\nu} - \mathrm{D}_{\lambda}\eta_{\nu}\mathrm{D}^{\mu}h^{\lambda\nu} + \mathrm{D}_{\lambda}\eta_{\nu}\mathrm{D}^{\lambda}h^{\mu\nu} - \mathrm{D}_{\lambda}\eta^{\mu}\mathrm{D}^{\lambda}h \Big) \\ &+ \frac{1}{2}\sqrt{|g|}\eta^{\beta}\Big(h^{\lambda}{}_{\beta}G^{\mu}{}_{\lambda} - hG^{\mu}{}_{\beta} \Big) \,, \end{aligned} \tag{2.37}$$

while the third term can be rewritten as

$$\begin{aligned} &\Big(\epsilon^{\alpha\beta\nu}\delta\Gamma^{\mu}_{\alpha\lambda} + \epsilon^{\mu\alpha\nu}\delta\Gamma^{\beta}_{\alpha\lambda} + \epsilon^{\beta\mu\nu}\delta\Gamma^{\alpha}_{\alpha\lambda} \Big)\xi_{\nu}G^{\lambda}{}_{\beta} = \\ &= \frac{1}{2}\sqrt{|g|}\mathrm{D}^{\alpha}\eta^{\lambda}\Big(- \mathrm{D}_{\alpha}h^{\mu}{}_{\lambda} - \mathrm{D}_{\lambda}h^{\mu}{}_{\alpha} + \mathrm{D}^{\mu}h_{\alpha\lambda} + \delta^{\mu}_{\alpha}\mathrm{D}_{\lambda}h \Big) \\ &+ \frac{1}{2}\epsilon^{\mu\alpha\nu}\xi_{\nu}G^{\lambda}{}_{\beta}\mathrm{D}_{\alpha}h^{\beta}{}_{\lambda} \,. \end{aligned} \tag{2.38}$$

Collecting these, and again integrating by parts, we arrive at

$$\begin{aligned}T^\mu = {}& \mathrm{D}_\lambda\Bigg[\epsilon^{\mu\lambda\beta}\xi_\nu\delta G^\nu{}_\beta - \frac{1}{2}\epsilon^{\mu\lambda\nu}\xi_\nu\delta G + \frac{1}{2}\sqrt{|g|}\Big(\eta^\lambda\mathrm{D}_\nu h^{\mu\nu} - \eta^\mu\mathrm{D}_\nu h^{\lambda\nu} \\ & + \eta_\nu\mathrm{D}^\mu h^{\lambda\nu} - \eta_\nu\mathrm{D}^\lambda h^{\mu\nu} + \eta^\mu\mathrm{D}^\lambda h - \eta^\lambda\mathrm{D}^\mu h + \mathrm{D}_\nu\eta^\mu h^{\lambda\nu} \\ & - \mathrm{D}_\nu\eta^\mu h^{\lambda\nu} + \mathrm{D}^\mu\eta^\lambda h - \mathrm{D}^\lambda\eta^\mu h\Big) + \frac{1}{2}\epsilon^{\mu\lambda\nu}\xi_\nu G^\rho{}_\beta h^\beta{}_\rho\Bigg] \\ & - \sqrt{|g|}\Big(\mathrm{D}_\lambda\mathrm{D}_\nu\eta^{[\mu}h^{\lambda]\nu} + \mathrm{D}_\lambda\mathrm{D}^{[\mu}\eta^{\lambda]}h\Big) \\ & - \frac{1}{2}\epsilon^{\mu\lambda\nu}\mathrm{D}_\lambda\Big(\xi_\nu G^\rho{}_\beta\Big)h^\beta{}_\rho + \frac{1}{2}\sqrt{|g|}\eta^\beta\Big(h^\lambda{}_\beta G^\mu{}_\lambda - hG^\mu{}_\beta\Big) \\ & + \frac{1}{2}\Big(\epsilon^{\lambda\alpha\beta}\xi_\nu\mathrm{D}_\alpha G^\mu{}_\beta + \epsilon^{\mu\alpha\beta}\xi_\nu\mathrm{D}_\alpha G^\lambda{}_\beta - \epsilon^{\lambda\alpha\beta}\xi^\mu\mathrm{D}_\alpha G_{\nu\beta}\Big)h_\lambda{}^\nu\,. \end{aligned} \tag{2.39}$$

Using (2.34) and (2.36), and the identity (obtained [13] by computing $[\mathrm{D}^\lambda, \mathrm{D}^\nu]\eta^\mu$, first from (2.35), then from (2.36))

$$\epsilon^{\rho\nu\sigma}\xi_\sigma\mathrm{D}^\lambda G^\mu{}_\rho - \epsilon^{\rho\lambda\sigma}\xi_\sigma\mathrm{D}^\nu G^\mu{}_\rho = \sqrt{|g|}\Big(g^{\mu\lambda}G^{\tau\nu}\eta_\tau - g^{\mu\nu}G^{\tau\lambda}\eta_\tau + 2G^{\mu\lambda}\eta^\nu - 2G^{\mu\nu}\eta^\lambda\Big)\,, \tag{2.40}$$

we finally obtain

$$\mathcal{K}^\mu_C = \mathrm{D}_\lambda\mathcal{F}^{\mu\lambda}_C(\xi) - \xi^\nu C^{\lambda\mu}h_{\lambda\nu} + \frac{1}{2}\xi^\mu C^{\lambda\rho}h_{\lambda\rho} - \frac{1}{2}\xi^\nu C^\mu{}_\nu h\,, \tag{2.41}$$

with the Cotton superpotential:

$$\begin{aligned}\mathcal{F}^{\mu\nu}_C(\xi) \equiv \mathcal{F}^{\mu\nu}_E(\eta) + \frac{1}{2\sqrt{|g|}}\epsilon^{\mu\nu\rho}\Bigg[\xi_\rho h^\lambda_\sigma G^\sigma_\lambda + \frac{1}{\sqrt{|g|}}\xi_\lambda\Big(2\epsilon^{\mu\nu\rho}\delta G^\lambda{}_\rho - \epsilon^{\mu\nu\lambda}\delta G\Big) \\ + \frac{1}{2}h\Big(\xi_\sigma G^\sigma_\rho + \frac{1}{2}\xi_\rho R\Big)\Bigg]\,.\end{aligned}$$

The total current

$$\mathcal{K}^\mu = \mathrm{D}_\lambda\mathcal{F}^{\mu\lambda}(\xi)$$

and superpotential

$$\mathcal{F}^{\mu\lambda}(\xi) \equiv \mathcal{F}^{\mu\nu}_E(\xi) + \frac{1}{\mu}\mathcal{F}^{\mu\nu}_C(\xi)$$

The mass

$$\mathcal{M} = Q^0(\xi^{(t)}) \equiv \frac{1}{\kappa}\int_{\partial M}\mathcal{F}^{0i}dS_i$$

and the angular momentum

$$\mathcal{J} = Q^0(\xi^{(\varphi)}) \equiv \frac{1}{\kappa}\int_{\partial M}\mathcal{F}^{0i}dS_i$$

3.2.3 Application To Stationary Axially Symmetric Solutions

Now we specialize to stationary axisymmetric spacetimes, and compute the conserved quantities associated with the two commuting isometries ∂_t and ∂_φ.

The linearized metric components are

$$h_{ab} = \delta\lambda_{ab}\,, \quad h_{22} = -2e^2\frac{\delta\mathcal{R}}{\mathcal{R}^3}\,. \tag{2.42}$$

We will also need the mixed components

$$h^a{}_b = (\lambda^{-1}\delta\lambda)^a{}_b = \left(\frac{\delta\mathcal{R}}{\mathcal{R}}\mathbb{I} + \frac{1}{\mathcal{R}^2}\sigma\right)^a{}_b\,, \quad h^2{}_2 = -2\frac{\delta\mathcal{R}}{\mathcal{R}}\,, \tag{2.43}$$

where σ is the matrix associated with the vector

$$\mathbf{\Sigma} \equiv \mathbf{X} \times \delta\boldsymbol{X}\,. \tag{2.44}$$

and we have used (1.12) and (1.13). Note that (2.43) implies $h = 0$.

Let us apply this formalism to the computation of the ADT charges (**??**). Choosing the boundary ∂M to be a circle, we need only to compute the (02) superpotential components. We begin by the computation of $F_E^{02}(V)$ for an arbitrary spacetime vector V^a ($V^2 = 0$). It is convenient to first compute the covariant components

$$\begin{aligned}
\mathcal{F}_{Ea2}(V) &= \frac{1}{2}(\lambda V)_a\bigg(- \partial_2 h^2{}_2 + \Gamma^2_{22}h^2{}_2 + \Gamma^b_{2c}h^c{}_b\bigg) \\
&\quad - \frac{1}{2}(\lambda V)_b\partial_2 h^b{}_a + \frac{1}{2}h^2{}_2\partial_2(\lambda V)_a \\
&= \frac{1}{2}(\lambda V)_a\bigg[2\left(\frac{\delta\mathcal{R}}{\mathcal{R}}\right)' + 2\frac{\mathcal{R}'\delta\mathcal{R}}{\mathcal{R}^2} + \frac{1}{2}Tr(\lambda^{-1}\lambda'\lambda^{-1}\delta\lambda)\bigg] \\
&\quad - \frac{1}{2}(\lambda V)_b\big(\lambda^{-1}\delta\lambda\big)'^b{}_a - \frac{\delta\mathcal{R}}{\mathcal{R}}(\lambda V)'_a\,.
\end{aligned} \tag{2.45}$$

This leads to

$$\begin{aligned}
\mathcal{F}_E^{02}(V) &= \frac{1}{2}\zeta^2\bigg\{V\bigg(\mathbf{X}\cdot\delta\mathbf{X}' + 2\mathcal{R}'\delta\mathcal{R}\bigg) \\
&\quad + V\lambda\bigg(- \sigma' + 2\frac{\mathcal{R}'}{\mathcal{R}}\sigma - 2\mathcal{R}\delta\mathcal{R}\lambda^{-1}\lambda'\bigg)\lambda^{-1} - 2\mathcal{R}\delta\mathcal{R}V'\bigg\}^0\,.
\end{aligned} \tag{2.46}$$

After some algebra with use of (1.12) , (1.13) and (1.11) one finds

$$\lambda(-\sigma' + 2\frac{\mathcal{R}'}{\mathcal{R}}\sigma - 2\mathcal{R}\delta\mathcal{R}\lambda^{-1}\lambda')\lambda^{-1} = \tau^0(2\mathcal{R}'\delta\mathcal{R}\mathbb{I} - \delta l)\tau^0 = (-2\mathcal{R}'\delta\mathcal{R}\mathbb{I} - \delta l^T)\,, \tag{2.47}$$

and finally

$$\mathcal{F}_E^{02}(V) = \frac{1}{2}e^{-2}\left[\bigg(\mathbf{X}\cdot\delta\mathbf{X}'\,\mathbb{I} - \delta l\bigg)V - 2\mathcal{R}\delta\mathcal{R}V'\right]^0\,. \tag{2.48}$$

Now we move on to the computation of the Cotton part of the superpotential. Using $\xi'^a = 0$, the vector defined in (2.34) takes the form[1]

$$\eta^a = \frac{1}{2e}\epsilon^{ab}\xi'_b = -\frac{1}{2e}\left(x'\xi\right)^a\,. \tag{2.49}$$

[1] Our convention $\epsilon^{012} = -1$ implies $\epsilon^{ab2} = -\epsilon^{ab}$ with $\epsilon^{01} = +1$.

Replacing in (2.48), and using again (1.12) one gets

$$\mathcal{F}_E^{02}(\eta) = \frac{1}{4}e^{-3}\left\{\left(\delta\mathbf{L}\cdot\mathbf{X}'\,\mathbb{1} - \delta\left(\mathbf{X}'\times\mathbf{L}\right) + \frac{1}{2}\delta(\mathbf{X}'^2)\mathbf{X} - \delta(\mathbf{X}^2)\mathbf{X}''\right)\xi\right\}^0 . \tag{2.50}$$

The next term in (**??**) contributes

$$\begin{aligned}\frac{1}{\sqrt{|g|}}(\lambda\xi)_c\left(\epsilon^{0b}\delta R^c{}_b - \frac{1}{4}\epsilon^{0c}\delta R\right) &= \frac{1}{2}e^{-3}\left[\xi\lambda\left(\delta l' + \frac{1}{4}\delta(\mathbf{X}'^2)\mathbb{1}\right)\tau^0\right]^0 \\ &= \frac{1}{2}e^{-3}\left[\left(\mathbf{X}\times\delta\mathbf{L}' + \tfrac{1}{4}\delta(\mathbf{X}'^2)\mathbf{X} - \mathbf{X}\cdot\delta\mathbf{L}'\mathbb{1}\right)\xi\right]^0 \\ &= \frac{1}{2}e^{-3}\left[\left(\delta\left(\mathbf{X}\times\mathbf{L}'\right) - \frac{1}{2}\delta(\mathbf{X}^2)\mathbf{X}'' + \left(\mathbf{X}''\cdot\delta\mathbf{X}\right)\mathbf{X}\right.\right. \\ &\quad \left.\left. + \frac{1}{4}\delta(\mathbf{X}'^2)\mathbf{X} - \mathbf{X}\cdot\delta\mathbf{L}'\mathbb{1}\right)\xi\right]^0 .\end{aligned} \tag{2.51}$$

Finally, the last term in (**??**) contributes

$$\begin{aligned}\frac{1}{2\sqrt{|g|}}\epsilon^{0a}(\lambda\xi)_a\left(h^b{}_c R^c{}_b + h^2{}_2 R^2{}_2\right) &= \frac{1}{2}e^{-3}\left(\frac{1}{\mathcal{R}^2}\mathbf{\Sigma}\cdot\mathbf{L}' + \frac{\delta\mathcal{R}}{\mathcal{R}}[\mathbf{X}'^2 - (\mathcal{R}\mathcal{R}')']\right)(x\xi)^0 \\ &= -\frac{1}{2}e^{-3}\mathbf{X}''\cdot\delta\mathbf{X}(x\xi)^0 ,\end{aligned} \tag{2.52}$$

where we have used

$$\mathbf{\Sigma}\cdot\mathbf{L}' = (\mathbf{X}\cdot\mathbf{X}'')\mathcal{R}\delta\mathcal{R} - \mathcal{R}^2(\mathbf{X}''\cdot\delta\mathbf{X}) . \tag{2.53}$$

Collecting(2.50), (2.51) and(2.52), we end up with a simple expression for the Cotton superpotential

$$\mathcal{F}_C^{02}(\xi) = \frac{1}{2}e^{-3}\left\{\left(\delta\mathbf{X}\times\mathbf{L}' - \frac{1}{2}\delta\mathbf{X}'\times\mathbf{L} + \frac{1}{2}\mathbf{X}'\cdot\delta\mathbf{L}\mathbb{1} - \mathbf{X}\cdot\delta\mathbf{L}'\mathbb{1}\right)\xi\right\}^0 . \tag{2.54}$$

The net ADT charge (**??**) is a linear combination of the Einstein superpotential (2.48) (for $V = \xi$) and the Cotton superpotential (2.54). To make contact with the SAM approach of [**?**], we define the spin super angular momentum

$$\mathbf{S} \equiv \frac{1}{e\mu}\left[\tfrac{1}{2}\mathbf{X}'\times\mathbf{L} - \mathbf{X}\times\mathbf{L}'\right] , \tag{2.55}$$

and the total conserved super angular momentum [9]

$$\mathbf{J} = \mathbf{L} + \mathbf{S} , \tag{2.56}$$

in terms of which the Killing charge is given by

$$Q(\xi) = -\frac{\pi}{\kappa e}\left\{\left(\delta\mathbf{J} - \mathbf{X}\cdot\delta\mathbf{X}'\mathbb{1} + \frac{1}{e\mu}(\mathbf{X}\cdot\delta\mathbf{L}' - \frac{1}{2}\mathbf{X}'\cdot\delta\mathbf{L})\mathbb{1}\right)\xi\right\}^0 . \tag{2.57}$$

Choosing ξ to be one of the two Killing vectors of (1.3), $\xi_{(t)} = (-1,\,0)$ and $\xi_{(\varphi)} = (0,\,1)$, we finally obtain the mass and angular momentum of the field configuration $X + \delta X$ relative to the background X:

$$\mathcal{M} = -\frac{\pi}{\kappa e}\left[\delta J^Y + \mathbf{X}\cdot\delta\mathbf{X}' + \frac{1}{e\mu}\left(\frac{1}{2}\mathbf{X}'\cdot\delta\mathbf{L} - \mathbf{X}\cdot\delta\mathbf{L}'\right)\right] , \tag{2.58}$$

$$\mathcal{J} = \frac{\pi}{\kappa e}\delta J^- . \tag{2.59}$$

3.2.4 TMG BH Conserved Charges

The mass and angular momentum for the solution (1.44) can be easily computed, the result is

$$\mathcal{M} = \frac{16}{27}\frac{\pi}{\kappa}\frac{\mu^2}{\zeta}\frac{\beta^2\left(1-\beta^2\right)}{c}\omega$$

$$J = \frac{4}{9}\frac{\pi}{\kappa}\mu\beta^2\left[\frac{1-\beta^2}{c}\omega^2 - \frac{1+\beta^2}{1-\beta^2}c\rho_0^2\right]$$

4
TMG Black Holes Thermodynamics

4.0.5 Chern-Simons Entropy

The first ingredient is the black hole entropy S. This is the sum

$$S = S_E + S_C \tag{0.1}$$

of an Einstein and a Cotton (or Chern-Simons) contribution. The Einstein entropy is as usual proportional to the horizon "area" (perimeter in the present case),

$$S_E = \frac{4\pi^2}{\kappa} r_h \,. \tag{0.2}$$

The general formula for the Chern-Simons contribution to the entropy was first given by Solodukhin [11],

$$S_{CS} = -\frac{2\pi}{\kappa\mu} \int_\Sigma \omega_{ab,\varphi} e^a_\mu e^b_\nu \hat{\epsilon}^{\mu\nu} \quad .$$

where

$$\hat{\epsilon}^{\alpha\beta} \equiv \frac{\epsilon^{\mu\nu\varphi}}{\sqrt{|g|}} (n^\alpha n_\mu)(n^\beta n_\nu)$$

$$(n^\alpha n_\mu) \equiv n^\alpha_1 n_{1\mu} + n^\alpha_2 n_{2\mu},$$

$\omega_{ab} = \omega_{ab,\mu} dx^\mu$ is the spin connection and n_1 and n_2 is a pair of vectors normal to the horizon hypersurface Σ and orthogonal to each other.

The orthonormal basis $e^a = e^a_\mu dx^\mu$ for the metric (1.43) is
evaluated on the horizon, where $\omega_{ab} = \omega_{ab,\mu} dx^\mu$ is the spin connection.
From the black hole metric (1.44), which is already in the ADM form

$$\mathrm{d}s^2 = -N^2 \,\mathrm{d}t^2 + r^2(\mathrm{d}\varphi + N^\varphi \,\mathrm{d}t)^2 + \frac{1}{(\zeta r N)^2} \,\mathrm{d}\rho^2 \,, \tag{0.3}$$

The dreibein e^a for the metric (0.3) is

$$e^0 = N \,\mathrm{d}t \,, \quad e^1 = r(\mathrm{d}\varphi + N^\varphi \mathrm{d}t) \,, \quad e^2 = \frac{1}{\zeta r N} \,\mathrm{d}\rho \,. \tag{0.4}$$

The corresponding spin connections are

$$\begin{aligned} {\omega^0}_2 &= \zeta r[N' e^0 + \frac{1}{2} r (N^\varphi)' e^1] \,, \\ {\omega^0}_1 &= \zeta r \frac{1}{2} r (N^\varphi)' e^2 \,, \\ {\omega^1}_2 &= \zeta r \left[\frac{1}{2} r (N^\varphi)' e^0 + N \frac{r'}{r} e^1 \right] \,, \end{aligned} \tag{0.5}$$

$$S_{CS} = -\frac{2\pi}{\kappa\mu} \int_0^{2\pi} \omega_{02,\varphi} \, \mathrm{d}\varphi \,, \tag{0.6}$$

leading to

$$S_{CS} = -\frac{2\pi^2}{\kappa\mu} \zeta r_h^3 (N^\varphi)'(\rho_0) = -\frac{2\pi^2}{\kappa} \frac{1}{e\mu} \frac{L^-}{\sqrt{X^-}} \Bigg| \tag{0.7}$$

$$= -\frac{4\pi^2}{3\kappa} \sqrt{\frac{c}{1-\beta^2}} \left[(1-2\beta^2)\rho_0 + (1-\beta^2)\frac{\omega}{c} \right] . \tag{0.8}$$

The total entropy is

$$S = \frac{8\pi^2}{3\kappa\sqrt{1-\beta^2}} \left[(1+\beta^2)\rho_0 + (1-\beta^2)\omega \right] \, . \tag{0.9}$$

4.0.6 The First Law of Black Hole Thermodynamics

We read the Hawking temperature

$$T_H = \frac{1}{2\pi} n^\rho \, \partial_\rho N|_{\rho=\rho_0} = \frac{\mu\beta^2}{3\pi} \frac{\rho_0}{r_h} \,, \tag{0.10}$$

where $n^\rho = \sqrt{|g|} = \zeta r N$, and

$$r_h = r(\rho_0) = \frac{1}{\sqrt{1-\beta^2}} [\rho_0 + (1-\beta^2)\omega] \tag{0.11}$$

is the horizon areal radius. The horizon angular velocity is

$$\Omega_h = -N^\varphi(\rho_0) = \frac{\sqrt{1-\beta^2}}{r_h} \,. \tag{0.12}$$

Putting (**??**), (**??**) and (0.9) together, it is easy to check that the first law is satisfied for independent variations of the black hole parameters ρ_0 and ω.

$$\mathrm{d}\mathcal{M} = T_H \mathrm{d}S + \Omega_h \mathrm{d}\mathcal{J} \, . \tag{0.13}$$

Conclusion

We have shown that black holes exist in topologically massive gravity with cosmological constant. They are free of causal singularity for a given range parameters. We have discussed the problem of the conserved charges in gravity and given their values in any background. Finally, it was shown that TMG Black Holes do obey to the first law of Black Hole thermodynamics.

Bibliography

[1] K. Ait Moussa, G. Clément and C. Leygnac, *Class. Quantum Grav.* **20** (2003) L277.

[2] G. Clément, *Phys. Rev.* D **68** (2003) 024032.

[3] L.F. Abbott and S. Deser, *Nucl. Phys.* B **195** (1982) 76.

[4] S. Deser and B. Tekin, *Phys. Rev. Lett.* **89** (2002) 101101; *Phys. Rev.* D **67** (2003) 084009.

[5] S. Deser and B. Tekin, *Class. Quantum Grav.* **20** (2003) L259.

[6] S. Deser, I. Kanik and B. Tekin, *Class. Quantum Grav.* **22** (2005) 3383.

[7] S. Ölmez, Ö. Sarioğlu and B. Tekin, *Class. Quantum Grav.* **22** (2005) 4355.

[8] G. Clément, *Phys. Rev.* D **49** (1994) 5131.

[9] G. Clément, *Class. Quantum Grav.* **11** (1994) L115.

[10] K. Ait Moussa, G. Clément, H. Guennoune and C. Leygnac, in preparation.

[11] S.N. Solodukhin, Phys. Rev. D **74** (2006) 024015.

[12] Y. Tachikawa, *Class. Quantum Grav.* **24** (2007) 737.

[13] S. Weinberg, "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity" (Wiley, New York, 1972).

Part II

GRAVITY IN FIVE DIMENSIONS

Introduction

Let us consider a five-dimensional (super)gravity action. The solution of the corresponding equations of motion (the metric $\hat{g}_{\mu\nu}$and the graviphoton potential $\hat{A}_{\mu}$). We assume that all the fields are independent of two coordinates (one timelike and the other spacelike). That is the solution is stationary and axisymmetric. This allows to reduce the five-dimensional (super)gravity action down to three dimensions. We mean by reducing just dropping the dependence on the mentioned coordinates.

In three dimensions, it is always possible to trade vectors for scalars. These scalars together with those stemming directly upon dimensional reduction, will constitute the "matter" part of a three-dimensional gravitational action. Moreover they arrange themselves in such a way they form a non-linear σ-model coupled to three-dimensional gravity. The experience shows that such non-linear σ-models obtained upon reduction of (super)gravity theories (and dualisation of vector fields) are based on interesting structures called coset spaces. This allow to pack the scalars (that are the all degrees of freedom of the theory) into a (Hermitian) matrix which is in a representation of the group G of the isometries of the scalar manifold. We will refer to the former matrix by the scalar matrix. Once achieving to do this, one can cast the reduced action into a nice form which has the feature of being G symmetry manifest. Acting on the scalar matrix globally by the G elements doesn't alter the three dimensional action (neither the three-dimensional metric). The invariance property property provides us with powerful generating technique (constructing new (super)gravity field equations exact solutions from already known ones).

That is, starting from a solution having two (commuting) killing vectors $(\hat{g}_{\mu\nu}, \hat{A}_{\mu})$ that we call the "seed" solution. One can obtain the corresponding scalar matrix $\mathcal{M}$ by following the steps explained above. All computations done, one has $(h_{ij}, \mathcal{M})$. By acting with suitable elements of G on $\mathcal{M}$, one obtain a new scalar matrix $\mathcal{M}'$, h_{ij} being inert , one has the transformed couple $(h_{ij}, \mathcal{M}')$. The next move is to unpack the scalars fields from the new scalar matrix. The scalars initially coming from dualising vector fields should be dedualised. All what remains to be done is to lift the solution from three dimensions to five.

The central problem dealt with in this part consists to applying this program to the minimal supergravity in five dimensions. One start by reducing the action of the bosonic part of the theory (which is all what we need for our purposes). The action is composed by the Einstein-Hilbert gravitational term, the Maxwell term and a $U(1)$ Chern-Simons term. By simple counting degrees of freedom one can see that this theory has 8 degrees of freedom. It comes as a no surprise that the reduction and the dualisation of the graviphoton potential together with the Kaluza-Klein vector field will give 8 scalars. These scalars span an eight-dimensional scalar manifold (the target space of the nonlinear sigma models). The isometry group of the former manifold is the group $G_{2(2)}$ a non-compact real form of the exceptional group G_2. The dimension of this group is 14 and since the isotropy group is $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$

which has 8 dimensions one has a coset space of dimension $14 - 6 = 8$ as it should be.

Exponentiating a suitable set of the generators of the algebra that spans the $G_{2(2)}$ group (this set is called the solvable algebra), one can get the representative that will give the scalar matrix. The scalar matrix is the central tool for the generating technique, once it was found, all what remains is to apply procedure explained above.

5

Dimensional Reduction, Non-Linear σ-Models, Cosets And All That

5.1 Non-linear σ-model coupled to gravity

Let $\mathcal{L}$ be a Lagrangian defined on a n-dimensional spacetime manifold $\mathbb{M}_n$ with coordinates $\{x^i\}$ where i runs from 1 to n, and a metric h_{ij}. The Lagrangian depends on (scalar) fields ϕ^A taking their values in a target space $\mathcal{T}$ with coordinates $\{\phi^A(x)\}, A = 1, ..., \dim \mathcal{T}$ and a metric G_{AB}.

The scalar fields ϕ^A map the the space time to the target space, that is

$$\begin{aligned} \phi^A : & \mathbb{M}_n \longrightarrow \mathcal{T} \\ & x \longmapsto \phi^A(x) \end{aligned}$$

The target space is sometimes called the scalar manifold. The action of the non-linear σ-nonlinear model coupled to gravity is given by

$$I = \int d^n x \sqrt{|h|}[R + h^{ij}\partial_i \phi^A(x) \partial_j \phi^B(x) G_{AB}] \tag{1.1}$$

R being the Ricci scalar with respect to the metric h.

The variation of the above action yields the field equations:

$$\begin{aligned} R_{ij} - \partial_i \phi^A(x) \partial_j \phi^A(x) G_{AB} = 0 \\ D^i \partial_i \phi^A(x) = 0 \end{aligned}$$

with R_{ij} and D denote respectively the Ricci tensor and the covariant derivative with respect to the spacetime metric. The fields that are solutions of these equations are called the harmonic maps.

An interesting situation occurred when the target space happens to be a (pseudo)Riemannian homogeneous space (or coset space) G/H with G is the isometry group of the target space, and H is a subgroup of G.

In the following section we will describe nonlinear σ-models based on Coset spaces.

5.2 Coset spaces

Let us first define what a coset G/H means, and start with an example. Consider the n-dimensional sphere S^n. The sphere is invariant under $G = SO(n+1)$, the group of the $(n+1)$-dimensional rotations. G is called the *isometry group* and its elements are *isometries*. If G *acts transitively* on the space, that is, given any two points in the manifold they can be connected by a transformation, such a space is *homogeneous*. Obviously the sphere is a Homogeneous space as every two points on S^n are connected by an $SO(n+1)$ isometry. However the rotation connecting these two points is not unique as every point on S^n is invariant under $H = SO(n)$ subgroup. H is known as the *isotropy group*. It is clear that if a transformation $\tilde{g} \in G$ maps some point onto another point, then the composition of the two transformations $g = \tilde{g}.h$, with $h \in H$, will do the same. Therefore every point on the sphere can be associated with a class of group elements $g \in G$ that are equivalent up to multiplication by elements $h \in H$ (from the right). Such equivalent classes are called *cosets* and noted G/H. The sphere is an $SO(n+1)/SO(n)$ coset.

6

The $G_{2(2)}$ σ-model for five-dimensional minimal supergravity

Introduction

6.1 Five-to-three dimensional reduction

The bosonic sector of five-dimensional minimal supergravity is defined by the Einstein-Maxwell-Chern-Simons action

$$\hat{I}_5 := \frac{1}{2{\kappa_5}^2}\int d^5x\left[\sqrt{|\hat{g}|}\left(-\hat{R}-\frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}\right)\right. \tag{1.1}$$

$$\left.-\frac{1}{12\sqrt{3}}\hat{\epsilon}^{\mu\nu\rho\sigma\lambda}\hat{F}_{\mu\nu}\hat{F}_{\rho\sigma}\hat{A}_\lambda\right], \tag{1.2}$$

where $\hat{F}_{\mu\nu} = 2\partial_{[\mu}\hat{A}_{\nu]}$, $\mu, \nu, \cdots = 1, \cdots, 5$, and $\hat{\epsilon}^{\mu\nu\rho\sigma\lambda}$ is the five-dimensional antisymmetric symbol.

We starting with the 5-dimensional Einstein action

$$I_5 = \frac{1}{2{\kappa_5}^2}\int d^5x\sqrt{-\hat{g}}\hat{R} \tag{1.3}$$

which, up to a surface term, may be written

$$I_5 = \frac{1}{2{\kappa_5}^2}\int d^Dx\sqrt{-\hat{g}}(\Omega_{\bar{\mu}\bar{\nu}\bar{\rho}}\Omega^{\bar{\mu}\bar{\nu}\bar{\rho}} - 2\Omega_{\bar{\mu}\bar{\nu}\bar{\rho}}\Omega^{\bar{\rho}\bar{\mu}\bar{\nu}} - 4\Omega_{\bar{\rho}\bar{\mu}}{}^{\bar{\mu}}\Omega^{\bar{\rho}}{}_{\bar{\nu}}{}^{\bar{\nu}}) \tag{1.4}$$

where $\Omega_{\bar{\mu}\bar{\nu}\bar{\rho}}$ are the anholonomy coefficients.

We split the vielbein (actually the fnbein) and the vector field as

$$\hat{E}_\mu{}^{\bar{\mu}} = \begin{pmatrix} e_i{}^{\bar{\imath}} & e_i{}^{\bar{a}} \\ e_a{}^{\bar{\imath}} & e_a{}^{\bar{a}} \end{pmatrix}, \hat{A} = (A_a, A_i) \tag{1.5}$$

The flat Lorentz indices $\bar{\mu}, \bar{\nu}, \bar{\rho} \ldots = 0, \ldots, 4$ are split into $\bar{\imath}, \bar{\jmath}, \bar{k} \ldots = 1, 2, 3$ and $\bar{a}, \bar{b}, \ldots = 0, 1$.

The metric takes the form

$$\hat{g}_{\mu\nu} = \begin{pmatrix} h_{ij} + \lambda_{ab} B_i{}^a B_j{}^b & \lambda_{ab} B_i{}^a \\ \lambda_{ab} B_j{}^b & \lambda_{ab} \end{pmatrix} \tag{1.6}$$

The indices $i, j, \ldots$ are raised by the metric $h_{ij} \equiv \eta_{\bar{\imath}\bar{\jmath}} e_i{}^{\bar{\imath}} e_j{}^{\bar{\jmath}}$, $\eta_{\bar{\imath}\bar{\jmath}} = \mathrm{diag}[+1, +1, +1]$, $\lambda_{ab} \equiv \eta_{\bar{a}\bar{b}} e_a{}^{\bar{a}} e_b{}^{\bar{b}}$, $\eta_{\bar{a}\bar{b}} = \mathrm{diag}[-1, +1]$; $h = \det(h_{ij})$ and $\tau = |\lambda_{ab}|$, respectively; $B_i{}^a = e^a{}_{\bar{a}} e_i{}^{\bar{a}}$ is the Kaluza-Klein gauge vector field strength.

If we choose a gauge $e_a{}^{\bar{\imath}} = 0$, the only non-vanishing components are

$$\Omega_{\bar{\imath}\bar{\jmath}\bar{k}}$$

,

$$\Omega_{\overline{\jmath}\overline{\imath}\bar{c}} = e_{a\bar{c}} e^i{}_{\bar{\imath}} e^j_{\bar{\jmath}} G_{ij}{}^a$$

$$\Omega_{\bar{\imath}\bar{b}\bar{c}} = -\Omega_{\bar{b}\bar{\imath}\bar{c}} = -e^a{}_{\bar{b}} \tilde{e}^i_{\bar{\imath}} \partial_i e_{a\bar{c}} \tag{1.7}$$

where

$$G_{ij}{}^a = 2\partial_{[i} B_{j]}{}^a \tag{1.8}$$

is the Kaluza-Klein strength field.

Here we have assumed that all fields are independent of the extra coordinates z^a. This yields the 3-dimensional action

$$I_3 = \frac{1}{2\kappa_3{}^2} \int d^3x \sqrt{h} \tau [R - \frac{1}{4} \lambda_{ab} G_{ij}{}^a G_{kl}{}^b h^{ij} h^{kl} + \frac{1}{4} h^{ij} (\lambda^{ab} \lambda^{cd} - \lambda^{ad} \lambda^{cb}) \partial_i \lambda_{ab} \partial_j \lambda_{cd}] \tag{1.9}$$

or

$$I_3 = \frac{1}{2\kappa_3{}^2} \int d^3x \sqrt{h} \tau [R - \frac{1}{4} \tau \lambda_{ab} G_{ij}{}^a G_{kl}{}^b h^{ij} h^{kl} - \frac{1}{4} h^{ij} \partial_i \ln \tau \partial_j \ln \tau + \frac{1}{4} h^{ij} \partial_i \lambda_{ab} \partial_j \lambda^{ab}] \tag{1.10}$$

We can eliminate τ by a Weyl rescaling

$$e_i{}^{\bar{\imath}} \to \tau^{-\frac{1}{2}} e_i{}^{\bar{\imath}} \tag{1.11}$$

or equivalently

$$h_{ij} \to \tau h_{ij}$$

to obtain

$$I_3 = \frac{1}{2\kappa_3{}^2} \int d^3x \sqrt{h} [R - \frac{1}{4} \tau \lambda_{ab} G_{ij}{}^a G_{kl}{}^b h^{ij} h^{kl} + \frac{1}{4} h^{ij} \partial_i \lambda_{ab} \partial_j \lambda^{ab} - \frac{1}{4} h^{ij} \partial_i \ln \tau \partial_j \ln \tau] \tag{1.12}$$

Let move on to the Maxwell term

$$\mathcal{L}_M \equiv \frac{1}{4} \sqrt{|\hat{g}|} \hat{F}^{\mu\nu} \hat{F}_{\mu\nu}$$

In flat indices

$$\hat{F}_{\bar{\mu}\bar{\nu}} = 2\hat{E}_{[\bar{\mu}}{}^{\mu} \partial_{\mu} \hat{A}_{\bar{\nu}]} - \Omega_{\bar{\mu}\bar{\nu}}{}^{\bar{\rho}} \hat{A}_{\bar{\rho}}$$

and

$$\hat{A}_{\bar{\mu}} = \hat{E}_{\bar{\mu}}{}^{\mu} \hat{A}_{\mu}$$

giving

$$A_{\bar{\imath}} = \sqrt{\tau} e_{\bar{\imath}}{}^{i} A'_{i},$$

where

$$A'_i \equiv A_i - B_i^a A_a$$

and
Finally

$$A_{\bar{a}} = e_{\bar{a}}^{a} A_a.$$

$$F_{\bar{\imath}\bar{\jmath}} = \tau e_{\bar{\imath}}{}^{i} e_{\bar{\jmath}}{}^{j} \left[2\partial_{[i} A'_{j]} + G^a_{ij} A_a \right]$$

$$F_{\bar{\imath}\bar{a}} = \tau e_{\bar{\imath}}{}^{i} e^a_{\bar{a}} \partial_i A_a$$

The reduced Maxwell lagrangian reads

$$\mathcal{L}_M = \hat{E}\hat{F}^{\bar{\mu}\bar{\nu}} \hat{F}_{\bar{\mu}\bar{\nu}} = \sqrt{h}(\tau F'^{ij} F'_{ij} + 2\lambda^{ab} h^{ij} \partial_i A_a \partial_j A_b)$$

Where

$$F'_{ij} \equiv 2\partial_{[i} A'_{j]} + G^a_{ij} A_a$$

Chern-Simons now

$$\mathcal{L}_{CS} = \frac{1}{12\sqrt{3}} \hat{\epsilon}^{\mu\nu\rho\sigma\lambda} \hat{F}_{\mu\nu} \hat{F}_{\rho\sigma} \hat{A}_{\lambda} = \frac{1}{3\sqrt{3}} \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} \epsilon^{\bar{a}\bar{b}} \left[F_{\bar{\imath}\bar{\jmath}} F_{\bar{k}\bar{a}} A_{\bar{b}} - F_{\bar{\imath}\bar{a}} F_{\bar{\jmath}\bar{b}} A_{\bar{k}} \right]$$

After some algebra, switching to to curved indices and discarding a surface term, one has:

$$\mathcal{L}_{CS} = -\frac{1}{6\sqrt{3}} \epsilon^{ijk} \epsilon^{ab} A_a \partial_k A_b \left[3F'_{ij} - G^c_{ij} A_c \right]$$

6.2 Dualization

Dualisation can be done by enforcing Bianchi identity, this can be achieved by means of Lagrange multipliers.

Since F'_{ij} doesn't obey Bianchi identity, we introduce an auxiliary field $\tilde{F}_{ij}$ defined by

$$\begin{aligned} \tilde{F}_{ij} &\equiv F'_{ij} - G^c_{ij} A_c \\ &= 2\partial_{[i} A'_{j]} \end{aligned}$$

Which obviously fulfills Bianchi identity.
We add the following term to the Lagrangian

$$\mathcal{L}_{lag.} \equiv -\frac{\sqrt{3}}{2} \epsilon^{ijk} \mu \partial_k \tilde{F}_{ij} - \frac{\sqrt{3}}{2} \epsilon^{ijk} \omega_a \partial_k G^a_{ij}$$

The factors were chosen to give simple normalization.
The variation $\frac{\delta \mathcal{L}}{\delta \tilde{F}_{ij}} = 0$ gives

$$F'^{ij} = \frac{\sqrt{3}}{\tau\sqrt{h}}\epsilon^{ijk}\eta_k\,, \qquad \eta_k = \partial_k\mu + \frac{1}{3}\epsilon^{ab}A_a\partial_k A_b\,. \tag{2.13}$$

and $\frac{\delta\mathcal{L}}{\delta G^a_{ij}} = 0$ yields

$$\lambda_{ab}G^{bij} = \frac{1}{\tau\sqrt{h}}\epsilon^{ijk}V_{ak}\,, \qquad V_{ak} = \partial_k\omega_a - \sqrt{3}A_a\left(\partial_k\mu + \frac{1}{9}\epsilon^{bc}A_b\partial_k A_c\right)\,, \tag{2.14}$$

The expressions of F'^{ij} and G^{bij} are pure algebraic (not differential) equations, then can be replaced in the Lagrangian.

The result is:

$$\begin{aligned}\mathcal{L} = \sqrt{h}\Big\{R - \frac{1}{2}h^{ij}\Big[&\frac{1}{2}\partial_i\lambda_{ab}\partial_j\lambda^{ab} + \frac{1}{2}\partial_i\ln\tau\partial_j\ln\tau - \tau^{-1}V_{ai}\lambda^{ab}V_{bj}\\ &+3\left(\partial_i\psi_a\lambda^{ab}\partial_j\psi_b - \tau^{-1}\eta_i\eta_j\right)\Big]\Big\}\,.\end{aligned} \tag{2.15}$$

where we have made the replacements:

$$A_i \to \sqrt{3}A_i, A_a \to \sqrt{3}\psi_a$$

6.3 The Hidden Symmetries of the mSUGRA5

If we note the scalars $\Phi^A(A = 1, 2, \ldots, 8)$ the scalar (with respect to the 3-dimensional space) λ_{ab},ω_a,ψ_a and μ, the above Lagrangian describes 3-dimensional gravity coupled to a nonlinear $\sigma-$model:

$$\mathcal{L} = \sqrt{h}(R - G_{AB}\frac{\partial\Phi^A}{\partial x^i}\frac{\partial\Phi^B}{\partial x^j}h^{ij}) \tag{3.16}$$

The metric of the target space is given by

$$dS^2 = \frac{1}{2}Tr(\lambda^{-1}d\lambda\lambda^{-1}d\lambda) + \frac{1}{2}\tau^{-2}d\tau^2 - \tau^{-1}V^T\lambda^{-1}V + 3\left(d\psi^T\lambda^{-1}d\psi - \tau^{-1}\eta^2\right)\,. \tag{3.17}$$

The dualized action (and the scalar manifold) is invariant under isometries of the target space:

$$\Phi'^A = \Phi^A + \varepsilon X^A(\Phi)$$

X^A's are Killing vectors, which can be found by solving the Killing equation:

$$X_{(A;B)} = 0$$

Semi-colon denotes G_{AB} covariant derivative. Instead of solving Killing equation directly (which is very hard), one can find the isometry group, let us note it G by indirect methods.

There are obvious symmetries, the remnant of $GL(5,\mathbb{R})$ upon reduction: $GL(3,\mathbb{R})\ltimes\mathbb{R}^3$, the generators of this group are

$$M_a{}^b = 2\lambda_{ac}\frac{\partial}{\partial\lambda_{cb}} + \omega_a\frac{\partial}{\partial\omega_b} + \delta^b_a\omega_c\frac{\partial}{\partial\omega_c} + \psi_a\frac{\partial}{\partial\psi_b} + \delta^b_a\mu\frac{\partial}{\partial\mu} \tag{3.18}$$

$$N^a = \frac{\partial}{\partial \omega_a}\,, \tag{3.19}$$

$$Q = \frac{\partial}{\partial \mu} \tag{3.20}$$

and the remnant of the gauge symmetry

$$R^a = \frac{\partial}{\partial \psi_a} + 3\mu\frac{\partial}{\partial \omega_a} - \epsilon^{ab}\psi_b\left(\frac{\partial}{\partial \mu} + \psi_c\frac{\partial}{\partial \omega_c}\right) \tag{3.21}$$

where a runs from 1to2.

Now, since $SL(3,\mathbb{R}) \subset G$ (five dimensional gravity is a consistent truncation of minimal five dimensional supergravity), one must add two generators say L_a to ${M_a}^b$and N^a to form an $\mathfrak{sl}(3,\mathbb{R})$ algebra,

$$\left[{M_a}^b, L_c\right] = (\delta^b_c L_a + \delta^b_a L_c)\,, \tag{3.22}$$

$$[N^a, L_b] = {M_b}^a\,, \tag{3.23}$$

$$[L_a, L_b] = 0\,. \tag{3.24}$$

The commutation of L_a with Q requires the introduction of two generators P_a such that

$$[Q, L_a] = P_a\,, \tag{3.25}$$

Finally commutation of L_a with the R^a requires four more generators, a traceless tensor ${A_a}^b$ and a scalar T,

$$[R^a, L_b] = {A_b}^a + \delta^a_b T\,. \tag{3.26}$$

We make the assumption that the algebra $\mathfrak{g}$ is minimal and closes with a single scalar generator T (leading to vanishing ${A_a}^b$), thus

$$[R^a, L_b] = \delta^a_b T\,. \tag{3.27}$$

The five generators L_a, P_a, and T are determined by solving the Lie brackets, up to a single integration constant, this is determined by demanding that T is an isometry of the metric (3.17). The result is

$$\begin{aligned} T = {} & \left[2\mu\lambda_{bc} + 6\epsilon^{de}\lambda_{bd}\psi_c\psi_e\right]\frac{\partial}{\partial \lambda_{bc}} \\ & + \left[3\mu\omega_b + 3\tau\psi_b - \epsilon^{cd}\omega_c\psi_b\psi_d + 4\tau\lambda^{cd}\psi_b\psi_c\psi_d\right]\frac{\partial}{\partial \omega_b} \\ & + \left[\omega_b + \mu\psi_b + 2\epsilon^{cd}\lambda_{bd}\psi_c\right]\frac{\partial}{\partial \psi_b} \\ & + \left[\mu^2 + \tau - \epsilon^{bc}\omega_b\psi_c + 2\tau\lambda^{bc}\psi_b\psi_c\right]\frac{\partial}{\partial \mu}\,, \end{aligned} \tag{3.28}$$

The L_a, $P_a$can be fully determined from $[R^a, T] = 2\epsilon^{ab} P_b$ and $[P_a, T] = 3L_a$.

$$\begin{aligned}
L_a = & \left[2\omega_b\lambda_{ac} + 2\mu\left(\lambda_{bc}\psi_a - 3\lambda_{ab}\psi_c\right) + 2\epsilon^{de}\lambda_{bd}\psi_a\psi_c\psi_e\right]\frac{\partial}{\partial\lambda_{bc}} \\
& + \left[\omega_a\omega_b + \tau\lambda_{ab} - \mu^3\epsilon_{ab} - \mu^2\psi_a\psi_b\right. \\
& \left. -2\mu\epsilon^{cd}\lambda_{ad}\psi_b\psi_c + 6\tau\psi_a\psi_b + 2\tau\lambda^{cd}\psi_a\psi_b\psi_c\psi_d\right]\frac{\partial}{\partial\omega_b} \\
& + \left[-\mu^2\epsilon_{ab} + \mu\lambda_{ab} + \omega_b\psi_a\right. \\
& \left. -\mu\psi_a\psi_b + \epsilon^{cd}\lambda_{bd}\psi_a\psi_c\right]\frac{\partial}{\partial\psi_b} \\
& + \left[\mu\omega_a - \mu^2\psi_a - \mu\epsilon^{cd}\lambda_{ad}\psi_c + \tau\psi_a + \tau\lambda^{bc}\psi_a\psi_b\psi_c\right]\frac{\partial}{\partial\mu},
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
P_a = & \left[2\lambda_{bc}\psi_a - 6\lambda_{ab}\psi_c\right]\frac{\partial}{\partial\lambda_{bc}} \\
& + \left[-3\mu^2\epsilon_{ab} - 2\mu\psi_a\psi_b - 2\epsilon^{cd}\lambda_{ad}\psi_b\psi_c\right]\frac{\partial}{\partial\omega_b} \\
& + \left[-2\mu\epsilon_{ab} + \lambda_{ab} - \psi_a\psi_b\right]\frac{\partial}{\partial\psi_b} \\
& + \left[\omega_a - 2\mu\psi_a - \epsilon^{cd}\lambda_{ad}\psi_c\right]\frac{\partial}{\partial\mu}.
\end{aligned} \tag{3.30}$$

The generators obey to the following algebra:

$$\left[M_a{}^b, M_c{}^d\right] = \delta_c^b M_a{}^d - \delta_a^d M_c{}^b\,. \tag{3.31}$$

$$\left[M_a{}^b, N^c\right] = -(\delta_a^c N^b + \delta_a^b N^c)\,, \tag{3.32}$$
$$\left[M_a{}^b, Q\right] = -\delta_a^b Q\,, \tag{3.33}$$
$$\left[N^a, N^b\right] = 0\,, \tag{3.34}$$
$$\left[Q, N^a\right] = 0\,. \tag{3.35}$$

$$\left[M_a{}^b, R^c\right] = -\delta_a^c R^b\,, \tag{3.36}$$
$$\left[N^a, R^b\right] = 0\,, \tag{3.37}$$
$$\left[Q, R^a\right] = 3N^a\,, \tag{3.38}$$
$$\left[R^a, R^b\right] = 2\epsilon^{ab} Q\,. \tag{3.39}$$

$$\begin{aligned}
\left[M_a{}^b, P_c\right] &= \delta_c^b P_a\,, && (3.40)\\
\left[M_a{}^b, T\right] &= \delta_a^b T\,, && (3.41)\\
[N^a, P_b] &= \delta_b^a Q\,, && (3.42)\\
[N^a, T] &= R^a\,, && (3.43)\\
[Q, P_a] &= -2\epsilon_{ab} R^b\,, && (3.44)\\
[Q, T] &= Tr(M)\,, && (3.45)\\
[R^a, P_b] &= -3M_b{}^a + \delta_b^a Tr(M)\,, && (3.46)\\
[R^a, T] &= 2\epsilon^{ab} P_b\,, && (3.47)\\
[L_a, P_b] &= 0\,, && (3.48)\\
[L_a, T] &= 0\,, && (3.49)\\
[P_a, P_b] &= 2\epsilon_{ab} T\,, && (3.50)\\
[P_a, T] &= 3L_a\,. && (3.51)
\end{aligned}$$

$$[R^a, L_b] = \delta_b^a T\,. \tag{3.52}$$

This is a rank 2 algebra which can be put in the Cartan form, with

$$\begin{array}{ll}
H_1 = -\dfrac{M_0{}^0 + M_1{}^1}{\sqrt{6}}\,, & H_2 = -\dfrac{M_0{}^0 - M_1{}^1}{\sqrt{2}}\,,\\
E_1 = -M_1{}^0\,, & E_{-1} = -M_0{}^1\,,\\
E_2 = \frac{1}{\sqrt{3}}R^0\,, & E_{-2} = \frac{1}{\sqrt{3}}P_0\,,\\
E_3 = \frac{1}{\sqrt{3}}R^1\,, & E_{-3} = \frac{1}{\sqrt{3}}P_1\,,\\
E_4 = \frac{1}{\sqrt{3}}Q\,, & E_{-4} = \frac{1}{\sqrt{3}}T\,,\\
E_5 = -N^0\,, & E_{-5} = L_0\,,\\
E_6 = -N^1\,, & E_{-6} = L_1\,,
\end{array} \tag{3.53}$$

Figure 6.1: The root diagram for g_2.

The roots are given by

$$\begin{aligned}
\boldsymbol{\alpha}_1 &= (0, -\sqrt{2})\,, \\
\boldsymbol{\alpha}_2 &= (\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}})\,, \\
\boldsymbol{\alpha}_3 &= (\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{2}})\,, \\
\boldsymbol{\alpha}_4 &= (\frac{\sqrt{2}}{\sqrt{3}}, 0)\,, \\
\boldsymbol{\alpha}_5 &= (\frac{\sqrt{3}}{\sqrt{2}}, \frac{1}{\sqrt{2}})\,, \\
\boldsymbol{\alpha}_6 &= (\frac{\sqrt{3}}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\,,
\end{aligned}$$

$\boldsymbol{\alpha}_1 + \ \boldsymbol{\alpha}_2$ being simple roots, one has

$$\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1 + \ \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_4 = \boldsymbol{\alpha}_1 + 2\ \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_5 = \boldsymbol{\alpha}_1 + 3\ \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_6 = 2\boldsymbol{\alpha}_1 + \ \mathbf{3}\boldsymbol{\alpha}_2.$$

The root space diagram is that of the 14-dimensional algebra $\mathfrak{g}_{2(2)}$ which is the real form of $\mathfrak{g}_2$.

6.4 Building $G_{2(2)}/[SL(2,\mathbb{R})\times SL(2,\mathbb{R})]$ coset

An 8×8 matrix representation of $\mathfrak{g}_{2(2)}$ may be found in [15] (one has to do a Weyl trick namely multiplying by $-i$ the Z matrices there) Then one has generic block decomposition:

j_M $(M=1,\cdots,14)$

$$j=\begin{pmatrix} S & \tilde{V} & \sqrt{2}U \\ -\tilde{U} & -S^T & \sqrt{2}V \\ \sqrt{2}V^T & \sqrt{2}U^T & 0 \end{pmatrix}, \tag{4.54}$$

where S is a 3×3 matrix, U and V are 3-component column matrices, U^T and V^T the corresponding transposed row matrices, and $\tilde{U}$, $\tilde{V}$ are the 3×3 dual matrices $\tilde{U}_{ij}=\epsilon_{ijk}U_k$. The matrices ${m_a}^b$, n^a and ℓ_a generating $SL(3,R)$ are of type S, the corresponding 3×3 blocks being

$$\begin{aligned} S_{{m_0}^0}&=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; S_{{m_0}^1}=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ S_{{m_1}^0}&=\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; S_{{m_1}^1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ S_{n^0}&=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; S_{n^1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\ S_{\ell_0}&=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; S_{\ell_1}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.55}$$

The matrices p_a and q are of type U, the corresponding 1×3 blocks being

$$U_{p_0}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; U_{p_1}=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; U_q=\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{4.56}$$

The matrices r^a and t are of type V, the corresponding 1×3 blocks being

$$V_{r^0}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; V_{r^1}=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; V_t=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.57}$$

Due to the form of (4.54), the transposed matrices j_A^T are related to the original matrices j_A by

$$j_A^T=-Kj_AK\,, \tag{4.58}$$

where K has the block structure

$$K=\begin{pmatrix} 0 & \mathbb{I} & 0 \\ \mathbb{I} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.59}$$

It follows from (4.54) and (**??**) that the matrix M has the symmetrical block structure

$$M = \begin{pmatrix} A & B & \sqrt{2}U \\ B^T & C & \sqrt{2}V \\ \sqrt{2}U^T & \sqrt{2}V^T & S \end{pmatrix}, \tag{4.60}$$

where A and C are symmetrical 3×3 matrices, B is a 3×3 matrix, U and V are 3-component column matrices, and S a scalar. It also follows from (4.58) that the inverse matrix is given by

$$M^{-1} = KMK = \begin{pmatrix} C & B^T & -\sqrt{2}V \\ B & A & -\sqrt{2}U \\ -\sqrt{2}V^T & -\sqrt{2}U^T & S \end{pmatrix}, \tag{4.61}$$

Computation of the product (**??**), with the matrices (**??**), (**??**) and (**??**) gives the coset matrix M in the form (4.60), with

$$\begin{aligned}
A &= \begin{pmatrix} \begin{matrix}\big[(1-y)\lambda + (2+x)\psi\psi^T - \tau^{-1}\tilde{\omega}\tilde{\omega}^T \\ +\mu(\psi\psi^T\lambda^{-1}J - J\lambda^{-1}\psi\psi^T)\big]\end{matrix} & \tau^{-1}\tilde{\omega} \\ \tau^{-1}\tilde{\omega}^T & -\tau^{-1} \end{pmatrix}, \\
B &= \begin{pmatrix} (\psi\psi^T - \mu J)\lambda^{-1} - \tau^{-1}\tilde{\omega}\psi^T J & \begin{matrix}\big[(-(1+y)\lambda J - (2+x)\mu + \psi^T\lambda^{-1}\tilde{\omega})\psi \\ +(z - \mu J\lambda^{-1})\tilde{\omega}\big]\end{matrix} \\ \tau^{-1}\psi^T J & -z \end{pmatrix}, \\
C &= \begin{pmatrix} (1+x)\lambda^{-1} - \lambda^{-1}\psi\psi^T\lambda^{-1} & \lambda^{-1}\tilde{\omega} - J(z - \mu J\lambda^{-1})\psi \\ \tilde{\omega}^T\lambda^{-1} + \psi^T(z + \mu\lambda^{-1}J)J & \begin{matrix}\big[\tilde{\omega}^T\lambda^{-1}\tilde{\omega} - 2\mu\psi^T\lambda^{-1}\tilde{\omega} \\ -\tau(1+x-2y-xy+z^2)\big]\end{matrix} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
U &= \begin{pmatrix} (1+x-\mu J\lambda^{-1})\psi - \mu\tau^{-1}\tilde{\omega} \\ \mu\tau^{-1} \end{pmatrix}, \\
V &= \begin{pmatrix} (\lambda^{-1} + \mu\tau^{-1}J)\psi \\ \psi^T\lambda^{-1}\tilde{\omega} - \mu(1+x-z) \end{pmatrix}, \\
S &= 1 + 2(x-y),
\end{aligned} \tag{4.62}$$

with

$$\tilde{\omega} = \omega - \mu\psi . \quad x = \psi^T\lambda^{-1}\psi , \quad y = \tau^{-1}\mu^2 , \quad z = y - \tau^{-1}\psi^T J\tilde{\omega} . \tag{4.63}$$

Bibliography

[1] E. Cremmer, B. Julia, H. Lü and C.N. Pope, "Higher-dimensional origin of $d = 3$ coset symmetries", arXiv:hep-th/9909099.

[2] G. Neugebauer and D. Kramer, Ann. Phys. (Leipzig) **24** (1969) 62.

[3] W. Kinnersley, Journ. Math. Phys. **14** (1973) 651.

[4] G. Clément, Gen. Rel. Grav. **18** (1986) 861; G. Clment and D. Gal'tsov, Phys. Rev. D **54** (1996) 265.

[5] B. Julia, "Group disintegrations", in *Superspace and supergravity*, eds. S.W. Hawking and M. Roček (Cambridge University Press, Cambridge 1981) 331.

[6] N. Marcus and J.H. Schwarz, Nucl. Phys. B **228** (1983) 145.

[7] E. Cremmer, "Supergravities in 5 dimensions", in *Superspace and supergravity*, eds. S.W. Hawking and M. Roček (Cambridge University Press, Cambridge 1981) 267.

[8] A.H. Chamseddine and H. Nicolai, Phys. Lett. **96**B (1980) 89.

[9] S. Mizoguchi and N. Ohta, Phys. Lett. B **441** (1998) 123.

[10] S. Mizoguchi and G. Schroeder, Class. Quantum Grav, **17** (2000) 835.

[11] M. Possel and S. Silva, Phys. Lett. B **580** (2004) 273.

[12] N.G. Scherbluk, "Hidden symmetries in five-dimensional supergravity", MS Thesis, Moscow State University (2006) (unpublished).

[13] A. Bouchareb, G. Clment, C-M. Chen, D. V. Gal'tsov, N. G. Scherbluk and T. Wolf, Phys. Rev. D **76** (2007) 104032.

[14] D. Maison, Gen. Rel. Grav. **10** (1979) 717.

[15] M. Gunaydin and F. Grsey, J. Math. Phys. **14** (1973) 1651..

[16] Y. Morisawa, S. Tomisawa and Y. Yasui, Phys. Rev. D **77** (2008) 064019.

[17] M. Gunaydin, A. Neitzke, O. Pavlyk and B. Pioline, "Quasi-conformal actions, quaternionic discrete series and twistors: $SU(2,1)$ and $G_{2(2)}$", arXiv:0707.1669.

[18] M. Berkooz and B. Pioline, "5D black holes and non-linear sigma models", arXiv:0802.1659.

[19] D. Gaiotto, W. Li and M. Padi, JHEP 0712:093 (2007); W. Li, "Non-supersymmetric attractors in symmetric coset spaces" arXiv:0801.2536.

[20] G. Clément, Phys. Rev. D **57** (1998) 4885; G. Clément, Grav. Cosmol. **5** (1999) 281.

[21] A. Bouchareb and G. Clément, work in progress.

7

Solution Generating Technique in five-dimensional minimal supergravity

7.1 $G_{2(2)}/[SL(2,\mathbb{R}) \times SL(2,\mathbb{R})]$ Generating Technique

7.1.1 The subgroup preserving asymptotic flatness:

In five dimensions, there are two possible boundary conditions: asymptotically flat M_5, or asymptotically Kaluza-Klein $M_4 \times S^1$spacetime.

The metric of the former is given by

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_3 \tag{1.1}$$

where

$$d\Omega_3 \equiv d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2$$

is the volume element of the three sphere.

and the latter can be obtained by adding to the 4-dimensional Minkowski space M_4 a fifth dimension (wrapped on the circle), the metric of such a space reads

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega + dz^2 \tag{1.2}$$

$$d\Omega \equiv d\theta^2 + \sin^2\theta d\phi^2$$

For both of the above asymptotic spacetimes one can derive the rigid $G_{2(2)}$ transformation(s) preserving them (*i.e.* their isotropy group).

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2) \tag{1.3}$$

The Boundary conditions for $\mathbb{M}_5$

Consider the metric (1.1) and take $z^0 = t$ and $z^1 = \ell\psi$ one has :

$$\lambda_{00} = -1, \quad \lambda_{11} = \frac{r^2}{\ell^2}\cos\theta^2, \quad \lambda_{10} = 0$$
$$\tau = \frac{r^2}{\ell^2}\cos^2\theta, \quad \omega_0 = 0, \quad \omega_1 = 0$$
$$\psi_0 = 0\,, \quad \psi_1 = 0, \quad \mu = 0,$$

the coset representative :

$$\mathcal{M} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{r^2}{\ell^2}\cos^2\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\ell^2}{r^2\cos^2\theta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\ell^2}{r^2\cos^2\theta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{r^2}{\ell^2}\cos^2\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Only the combination $G_+ \equiv P_1 - T$ preserve the above nonconstant matrix (at spatial infinity). This turns to be charge-generating transformation, and will be used to generate electrically charged solution from neutral seed solution.

But also one can exploit the coset model to its full potentialities, this can be achieved by reducing (1.1) with respect to $z^0 = t$ and $z^1 = \ell(\psi + \phi)$, (this a generalization of Giusto and Saxena [?] remarks to the minimal SUGRA), it gives

$$\lambda_{00} = -1, \quad \lambda_{11} = \frac{r^2}{\ell^2}, \quad \lambda_{10} = 0, \quad \tau = \frac{r^2}{\ell^2},$$

Kaluza- Klein vectors

$$B_0^\eta = 0, \quad B_1^\eta = \ell\cos 2\theta$$

The 3-dimensional metric

$$h_{ij}dx^i dx^j = \frac{r^2}{4\ell^2}\left[dr^2 + r^2 d\theta^2 + \sin\theta^2\cos\theta^2 d\eta^2\right]$$

The dualization yields

$$\omega_0 = 0, \quad \omega_1 = \frac{r^2}{\ell^2}$$

and obviously

$$\psi_0 = 0\,, \quad \psi_1 = 0, \quad \mu = 0,$$

The coset representative in this case

$$\mathcal{M}=\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & -\dfrac{4\ell^2}{r^2} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \dfrac{4\ell^2}{r^2} & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

At spatial infinity,

$$\mathcal{M}_\infty=\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

There are six transformations leaving invariant the matrix $\mathcal{M}_\infty$:

$$\begin{aligned} &F_+ = -{M_1}^0 - L_0\,, \quad F_0 = {M_1}^1\,, \quad F_- = -{M_0}^1 + N^0\,,\\ &G_+ = P_1 - T\,, \quad G_0 = P_0 + R^0\,, \quad G_- = R^1 + Q\,. \end{aligned} \tag{1.4}$$

This can be seen ,just by verifying that the product of each one of the above generators with $\mathcal{M}_\infty$ is antisymmetric

$$(\mathcal{M}_\infty J_\alpha)^T = -(\mathcal{M}_\infty J_\alpha)$$

These generators form an $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$ algebra

$$\begin{aligned} &[J_0, J_\pm] = \pm J_\pm\,, \quad [J_+, J_-] = J_0\,,\\ &\left[\bar{J}_0, \bar{J}_\pm\right] = \pm \bar{J}_\pm\,, \quad \left[\bar{J}_+, \bar{J}_-\right] = \bar{J}_0\,,\\ &\left[J_\alpha, \bar{J}_\beta\right] = 0\,, \end{aligned}$$

with

$$J_\alpha = \frac{1}{4}(F_\alpha - G_\alpha)\,, \quad \bar{J}_\alpha = \frac{1}{4}(3F_\alpha + G_\alpha)\,.$$

The three transformations F_0, F_-, and G_- are pure gauge transformations (general coordinates transformations), and thus doesn't affect physics. It remains three physical transformations preserving asymptotic flatness are the spin-generating transformation F_+ , G_+ the charging transformation already found, and the transformation G_0.

Their matrix representations can be found ind the Appendix.

The Boundary conditions for $\mathbb{M}_4 \times S^1$:

Choosing $z^0 = t$ and $z^1 = z$, the data from the metric (1.2) gives the asymptotic coset representative :

$$\mathcal{M}_\infty^{KK} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

There are 6 transformations that preserve this configuration:

$$\begin{aligned} &J_1 = {M_0}^1 + {M_1}^0\,, \quad J_2 = N^0 + L^0\,, \quad J_3 = N^1 - L^1\,, \\ &J_4 = Q - T\,, \quad J_5 = P_0 + R^0\,, \quad J_6 = R^1 + P_1. \end{aligned} \tag{1.5}$$

It is worthy to note that one can go from $\mathcal{M}_\infty^{KK}$ to $\mathcal{M}_\infty$ by through the expression

$$\mathcal{M}_\infty = P_{SH}^T \mathcal{M}_\infty^{KK} P_{SH}$$

where

$$P_{SH} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The P_{SH} quiet generally can transform a black string into a black hole in 5-dimensions.

7.1.2 Charging Neutral Seeds

Consider the charging transformation

$$P_C = \begin{pmatrix} c^2 & 0 & 0 & s^2 & 0 & 0 & \sqrt{2}sc \\ 0 & c & 0 & 0 & 0 & s & 0 \\ 0 & 0 & c & 0 & -s & 0 & 0 \\ s^2 & 0 & 0 & c^2 & 0 & 0 & \sqrt{2}sc \\ 0 & 0 & -s & 0 & c & 0 & 0 \\ 0 & s & 0 & 0 & 0 & c & 0 \\ \sqrt{2}sc & 0 & 0 & \sqrt{2}sc & 0 & 0 & c^2+s^2 \end{pmatrix}, \tag{1.6}$$

where $c \equiv \cosh\alpha$, $s \equiv \sinh\alpha$.

The transformation (1.6) acting on a neutral ($\psi_a = \mu = 0$) seed solution (λ_{ab}, ω_a) leads to the target space following data, extracted from the transformed matrix representative $\mathcal{M}'$,

$$\tau' = D^{-1}\tau\,, \tag{1.7}$$
$$\lambda'_{11} = D^{-2}\lambda_{11}\,, \tag{1.8}$$
$$\lambda'_{12} = D^{-2}[c^3\lambda_{12} + s^3\lambda_{11}\omega_1]\,, \tag{1.9}$$
$$\omega'_1 = D^{-2}[c^3(c^2+s^2+2s^2\lambda_{11})\omega_1 - s^3(2c^2+(c^2+s^2)\lambda_{11})\lambda_{12}]\,, \tag{1.10}$$
$$\omega'_2 = \omega_2 + D^{-2}s^3[-c^3\lambda_{12}^2 + s(2c^2-\lambda_{11})\lambda_{12}\omega_1 - c^3\omega_1^2]\,, \tag{1.11}$$
$$\psi'_1 = \sqrt{3}D^{-1}sc(1+\lambda_{11})\,, \tag{1.12}$$
$$\psi'_2 = \sqrt{3}D^{-1}sc(c\lambda_{12} - s\omega_1)\,, \tag{1.13}$$
$$\mu' = \sqrt{3}D^{-1}sc(c\omega_1 - s\lambda_{12})\,, \tag{1.14}$$

with $D = c^2 + s^2\lambda_{11} = 1 + s^2(1+\lambda_{11})$.

7.2 The Seeds

7.2.1 Black Holes:

The metric for five-dimensional rotating black hole, in the Boyer-Lindquist coordinates, is ([?]):

$$\begin{aligned} ds^2 = {} & \frac{\rho^2}{4\Delta}\,dx^2 + \rho^2\,d\theta^2 \; - dt^2 + (x+a^2)\,\sin^2\theta\,d\phi^2 + (x+b^2)\,\cos^2\theta\,d\psi^2 \\ & + \frac{r_0^2}{\rho^2}\left[dt + a\,\sin^2\theta\,d\phi + b\,\cos^2\theta\,d\psi\right]^2\,. \end{aligned} \tag{2.15}$$

Here,

$$\rho^2 = x + a^2\,\cos^2\theta + b^2\,\sin^2\theta\,, \tag{2.16}$$
$$\Delta = (x+a^2)(x+b^2) - r_0^2\,x\,. \tag{2.17}$$

The angles ϕ and ψ lie in the interval $[0, 2\pi]$, while the angle θ belongs to $[0, \pi/2]$.

The black hole horizon is located at $x = x_+$ where

$$x_\pm = \frac{1}{2}\left[r_0^2 - a^2 - b^2 \pm \sqrt{(r_0^2-a^2-b^2)^2 - 4a^2b^2}\right]\,. \tag{2.18}$$

The determinant of this metric is given by

$$\sqrt{-g} = \frac{1}{2}\sin\theta\cos\theta\,\rho^2\,. \tag{2.19}$$

The metric (2.15) is invariant under the following transformation

$$a \leftrightarrow b\,, \quad \theta \leftrightarrow \left(\frac{\pi}{2} - \theta\right)\,, \quad \phi \leftrightarrow \psi\,. \tag{2.20}$$

It possesses 3 Killing vectors, ∂_t, ∂_ϕ and ∂_ψ.

7.2.2 Black Rings

Black rings with one angular momentum

$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}d\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right], \tag{2.21}$$

where

$$F(\xi) = 1 + \lambda\xi, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi)\,, \tag{2.22}$$

and

$$C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,. \tag{2.23}$$

The dimensionless parameters λ and ν are in the range

$$0 < \nu \le \lambda < 1\,. \tag{2.24}$$

with $-\infty \le y \le -1$ and $-1 \le x \le 1$, spatial asymptotic infinity is recovered when $x \to -1, y \to -1$. The axis of rotation around the ψ direction is at $y = -1$, and the axis of rotation around ϕ is divided into two pieces: $x = 1$ is the disk bounded by the ring, and $x = -1$ is its complement from the ring to infinity. The horizon is located at $y = -1/\nu$. Outside it, at $y = -1/\lambda$, lies an ergosurface.

The angular variables must be identified with periodicity

$$\Delta\psi = \Delta\phi = 4\pi\frac{\sqrt{F(-1)}}{|G'(-1)|} = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu} \tag{2.25}$$

and the two parameters λ, ν must satisfy

$$\lambda = \frac{2\nu}{1+\nu^2}\,. \tag{2.26}$$

Black rings with two angular momenta

An exact solution for a black ring with both rotations been achieved by Pomeransky and Sen'kov in [?] They have furthermore managed to present it in a fairly compact form:

$$ds^2 = -\frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 - \frac{F(x,y)}{H(y,x)}d\psi^2 - 2\frac{J(x,y)}{H(y,x)}d\psi d\phi + \frac{F(y,x)}{H(y,x)}d\phi^2 - \frac{2k^2H(x,y)}{(x-y)^2(1-\nu)^2}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)}\right)\,. \tag{2.27}$$

Here we follow the notation introduced in [?], except that we have chosen mostly plus signature, and exchanged $\phi \leftrightarrow \psi$ to conform to the notation in (2.21). It worth to mention that the angles ϕ and ψ have been rescaled here to have canonical periodicity 2π.

The one-form Ω characterizing the rotation is

$$\Omega = -\frac{2k\lambda\sqrt{(1+\nu)^2-\lambda^2}}{H(y,x)}\Big[(1-x^2)y\sqrt{\nu}d\phi + \frac{1+y}{1-\lambda+\nu}\left(1+\lambda-\nu+x^2y\nu(1-\lambda-\nu)+2\nu x(1-y)\right)\,d\psi\Big], \tag{2.28}$$

and the functions G, H, J, F become

$$\begin{aligned}
G(x) &= (1-x^2)\left(1+\lambda x+\nu x^2\right)\,,\\
H(x,y) &= 1+\lambda^2-\nu^2+2\lambda\nu(1-x^2)y+2x\lambda(1-y^2\nu^2)+x^2y^2\nu(1-\lambda^2-\nu^2)\,,\\
J(x,y) &= \frac{2k^2(1-x^2)(1-y^2)\lambda\sqrt{\nu}}{(x-y)(1-\nu)^2}\left(1+\lambda^2-\nu^2+2(x+y)\lambda\nu-xy\nu(1-\lambda^2-\nu^2)\right)\,,
\end{aligned} \tag{2.29}$$

$$\begin{aligned}
F(x,y) = \frac{2k^2}{(x-y)^2(1-\nu)^2}\Big[&G(x)(1-y^2)\left[\left((1-\nu)^2-\lambda^2\right)(1+\nu)+y\lambda(1-\lambda^2+2\nu-3\nu^2)\right]\\
&+G(y)\big[2\lambda^2+x\lambda((1-\nu)^2+\lambda^2)+x^2\left((1-\nu)^2-\lambda^2\right)(1+\nu)+x^3\lambda(1-\lambda^2-3\nu^2+2\nu^3)\\
&-x^4(1-\nu)\nu(-1+\lambda^2+\nu^2)\big]\Big].
\end{aligned}$$

When $\lambda=0$ one finds flat spacetime. In order to recover the metric (2.21) one must take $\nu\to 0$, identify $R^2=2k^2(1+\lambda)^2$ and rename $\lambda\to\nu$.

The parameters λ and ν are restricted to

$$0\le\nu<1\,,\qquad 2\sqrt{\nu}\le\lambda<1+\nu \tag{2.30}$$

for the existence of regular black hole horizons. The bound $\lambda\ge 2\sqrt{\nu}$ is actually a Kerr-like bound on the rotation of the S^2. To see this, consider the equation for vanishing $G(y)$,

$$1+\lambda y+\nu y^2=0\,, \tag{2.31}$$

Electrifying Myers and Perry

Starting with the five-dimensional Myers and Perry [?]:

$$ds^2=-dt^2+\frac{\rho^2}{4\Delta}dx^2+\rho^2d\theta^2+(x+a^2)\sin^2\theta d\phi^2+(x+b^2)\cos^2\theta d\psi^2+\frac{r_0^2}{\rho^2}\left(dt+a\sin^2\theta d\phi+b\cos^2\theta d\psi\right)^2, \tag{2.32}$$

where

$$\rho^2=x+a^2\cos^2\theta+b^2\sin^2\theta,\qquad \Delta=(x+a^2)(x+b^2)-r_0^2x. \tag{2.33}$$

Choosing $z^0=t,\quad z^1=\psi$, we find :

$$\lambda_{00}=-1+\frac{r_0^2}{\rho^2},\quad \lambda_{01}=\frac{r_0^2}{\rho^2}b\cos^2\theta,\quad \lambda_{11}=(x+b^2)\cos^2\theta+\frac{r_0^2}{\rho^2}b^2\cos^4\theta,$$

$$\tau=\left(x+b^2-r_0^2+\frac{r_0^2}{\rho^2}a^2\cos^2\theta\right)\cos^2\theta, \tag{2.34}$$

$$a^0_\phi=-\tau^{-1}\frac{r_0^2}{\rho^2}(x+b^2)a\sin^2\theta\cos^2\theta,\qquad a^1_\phi=\tau^{-1}\frac{r_0^2}{\rho^2}ab\sin^2\theta\cos^2\theta, \tag{2.35}$$

and the three-metric

$$h_{ij}dx^i dx^j = \tau\left(\frac{\rho^2}{4\Delta}dx^2 + \rho^2 d\theta^2 + \frac{\Delta}{\tau}\sin^2\theta\cos^2\theta d\phi^2\right), \qquad \sqrt{h} = \frac{1}{2}\tau\rho^2\sin\theta\cos\theta. \tag{2.36}$$

The dualization of the vector fields gives

$$\omega_0 = -\frac{r_0^2}{\rho^2}a\cos^2\theta, \qquad \omega_1 = -\frac{r_0^2}{\rho^2}ab\cos^4\theta. \tag{2.37}$$

The action of our charge-generating transformation with parameter α ($c = \cosh\alpha$, $s = \sinh\alpha$) on this neutral seed leads to the transformed fields according to (1.7). Performing the inverse dualization, we obtain the charged black hole solution

$$ds'^2 = -D^{-2}\left(1 - \frac{r_0^2}{\rho^2}\right)(dt + \Omega')^2 + D\left[\frac{\rho^2 dx^2}{4\Delta} + \rho^2 d\theta^2 + \left(x + a^2 + \frac{r_0^2 a^2}{\rho^2 - r_0^2}\sin^2\theta\right)\sin^2\theta d\phi^2\right.$$

$$\left. + 2\frac{r_0^2 ab}{\rho^2 - r_0^2}\sin^2\theta\cos^2\theta d\phi d\psi + \left(x + b^2 + \frac{r_0^2 b^2}{\rho^2 - r_0^2}\cos^2\theta\right)\cos^2\theta d\psi^2\right], \tag{2.38}$$

$$\Omega' = -r_0^2\left[\left(\frac{c^3 a}{\rho^2 - r_0^2} + \frac{s^3 b}{\rho^2}\right)\sin^2\theta d\phi + \left(\frac{c^3 b}{\rho^2 - r_0^2} + \frac{s^3 a}{\rho^2}\right)\cos^2\theta d\psi\right], \tag{2.39}$$

$$A' = \sqrt{3}scD^{-1}\frac{r_0^2}{\rho^2}\left[dt + (ca + sb)\sin^2\theta d\phi + (cb + sa)\cos^2\theta d\psi\right], \tag{2.40}$$

with

$$D = 1 + s^2\frac{r_0^2}{\rho^2}.$$

The same solution is obtained if reduction is carried out with respect to the angular variable ϕ instead of ψ. Note that it is regular on the polar axis $\sin\theta = 0$.

This solution could be put in the following form

$$d\bar{s}^2 = -dt^2 - \frac{2\bar{q}}{\bar{\rho}^2}\bar{\nu}(dt - \bar{\omega}) + \frac{\bar{f}}{\bar{\rho}^4}(dt - \bar{\omega})^2 + \frac{\bar{\rho}^2 r^2}{\bar{\Delta}}dr^2 + \bar{\rho}^2 d\theta^2 + (r^2 + \bar{a}^2)\sin^2\theta d\phi^2 \tag{2.41}$$

$$+ (r^2 + \bar{b}^2)\cos^2\theta d\psi^2, \tag{2.42}$$

$$\bar{A} = \frac{\sqrt{3}\bar{q}}{\bar{\rho}^2}(dt - \bar{\omega}), \tag{2.43}$$

where

$$\bar{\nu} = \bar{b}\sin^2\theta d\phi + \bar{a}\cos^2\theta d\psi, \qquad \bar{\omega} = \bar{a}\sin^2\theta d\phi + \bar{b}\cos^2\theta d\psi, \qquad \bar{f} = 2\bar{m}\bar{\rho}^2 - \bar{q}^2,$$
$$\bar{\Delta} = (r^2 + \bar{a}^2)(r^2 + \bar{b}^2) + \bar{q}^2 + 2\bar{a}\bar{b}\bar{q} - 2\bar{m}r^2, \qquad \bar{\rho}^2 = r^2 + \bar{a}^2\cos^2\theta + \bar{b}^2\sin^2\theta.$$

The metrics ds'^2 and $d\bar{s}^2$ are related by the following coordinate and parameter transformation:

$$r^2 = x + s^2(r_0^2 - a^2 - b^2) - 2absc, \qquad 2\bar{m} = (1 + 2s^2)r_0^2,$$
$$\bar{q} = -scr_0^2, \qquad \bar{a} = -ca - sb, \qquad \bar{b} = -cb - sa,$$

implying

$$\bar{\rho}^2 = D\rho^2 = \rho^2 + s^2 r_0^2, \qquad \bar{\Delta} = \Delta.$$

Comparing then the electromagnetic potentials, we find $\bar{A} = -A'$, so the two solutions are identical under charge conjugation (or a simultaneous sign change of t, ϕ and ψ

7.2.3 Forging a charged doubly spinning black rings

One can either choose $dx^1 = dt$ and $dx^2 = d\psi$ or $dx^1 = dt$ and $dx^2 = d\phi$. Switching between the two choices is achieved by making the exchange $F(x,y) \leftrightarrow F(y,x)$ and $\Omega_\phi \leftrightarrow \Omega_\psi$. For the first choice ($dx^2 = d\psi$), the seed λ_{ab} can be read off from Eq. (1) of PS,

$$\tau = \frac{F(y,x)}{H(x,y)}, \quad \lambda_{00} = -\frac{H(y,x)}{H(x,y)}, \quad \lambda_{10} = \Omega_\psi \lambda_{00}\,, \tag{2.44}$$

while the seed ω_a must be obtained by dualizing the B^a_ϕ,

$$\tau \lambda_{ab} G^{bij} = \frac{1}{\sqrt{h}} \epsilon^{ijk} \left[\partial_k \omega_a - \psi_a (\partial_k \mu + \frac{1}{3\sqrt{3}} \epsilon^{bc} \psi_b \partial_k \psi_c) \right] \tag{2.45}$$

with $G^a_{ij} = \partial_i B^a_j - \partial_j B^a_i$, and

$$B'^1_\phi = \Omega_\phi - \Omega_\psi B^1_\phi\,, \quad B^1_\phi = -\frac{J(x,y)}{F(y,x)}, \tag{2.46}$$

where $\Omega_\phi = \Omega_\phi(x,y)$ and $\Omega_\psi = \Omega_\psi(x,y)$ are given in Eq. (2) of PS. Inspection of relations (1.7) shows that it is not necessary to compute ω_2, while the computation of ω_1 yields simply

$$\omega_1(x,y) = -\Omega_\phi(y,x)\,. \tag{2.47}$$

Similarly, the ω_1 corresponding to the second choice ($dx^2 = d\phi$) is

$$\hat{\omega}_1(x,y) = \Omega_\psi(y,x)\,. \tag{2.48}$$

To write down the charged solution, there remains to dualize back the ω'_a and μ' to the a'^a_ϕ and A'_ϕ. It is easy to show (without explicit dualization) from the above relations, that $a'^1_\phi = a^1_\phi$, while

$$G'^{0i\phi} = c^3 G^{1i\phi} + s^3 \left[-\omega_1 G^{2i\phi} + \frac{\lambda_{11}^2}{\tau\sqrt{h}} \epsilon^{ij} \partial_j \Omega_\psi \right], \tag{2.49}$$

$$F'^{i\phi} = B'^{1\phi} \partial^i \psi'_1 + B'^{2\phi} \partial^i \psi'_2 - \frac{\sqrt{3}sc}{D\tau\sqrt{h}} \epsilon^{ij} (c\partial_j \omega_1 + s\lambda_{11}^2 \partial_j \Omega_\psi)\,. \tag{2.50}$$

We have solved the first duality equation (2.49) for the second choice ($dx^2 = d\psi$), with the result

$$\hat{B}'^1_\psi(x,y) = c^3 \hat{B}^1_\psi(x,y) - s^3 \hat{B}^1_\phi(y,x)\,, \tag{2.51}$$

leading to

$$\hat{\Omega}'_\psi(x,y) = c^3 \hat{\Omega}_\psi(x,y) - s^3 \hat{\Omega}_\phi(y,x) = \Omega'_\psi(x,y)\,. \tag{2.52}$$

This shows that the charged solution does not depend on the choice of the second Killing vector (∂_ψ or ∂_ϕ).

The final charged black ring metric is

$$\begin{aligned} ds'^2 = -D^{-2} \frac{H(y,x)}{H(x,y)} (dt + \Omega')^2 + D \Bigg[- \frac{F(x,y)}{H(y,x)} d\phi^2 - 2\frac{J(x,y)}{H(y,x)} d\phi d\psi \\ + \frac{F(y,x)}{H(y,x)} d\psi^2 + \frac{2k^2 H(x,y)}{(1-\nu)^2 (x-y)^2} \left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} \right) \Bigg], \end{aligned} \tag{2.53}$$

$$\Omega' = (c^3\Omega_\psi(x,y) - s^3\Omega_\phi(y,x))d\psi + (c^3\Omega_\phi(x,y) + s^3\Omega_\psi(y,x))d\phi\,, \tag{2.54}$$

$$\begin{aligned} A' = \sqrt{3}scD^{-1}\Bigg[&\frac{2\lambda(1-\nu)(x-y)(1-\nu xy)}{H(x,y)}dt \\ &+\left(-c\frac{H(y,x)}{H(x,y)}\Omega_\psi(x,y) + s\Omega_\phi(y,x)\right)d\psi \\ &+\left(-c\frac{H(y,x)}{H(x,y)}\Omega_\phi(x,y) - s\Omega_\psi(y,x)\right)d\phi\Bigg]\,, \end{aligned} \tag{2.55}$$

with

$$D = 1 + s^2\frac{2\lambda(1-\nu)(x-y)(1-\nu xy)}{H(x,y)}\,. \tag{2.56}$$

This is to be compared with the charged black ring given in [**?**], Sect. 4 (exchange ψ and ϕ!). A difference is that Elvang et al. start with a seed having an extra parameter (dipole charge), which can be fine tuned so that Dirac-Misner strings are absent. Such string singularities arise if the orbits of ∂_ψ (our ψ, their ϕ) do not close off at $x = 1$. In the present case it is clear that both $\Omega'_\psi(1,y)$ and $A'_\psi(1,y)$ are proportional to $\Omega_\phi(y,1)$, which does not vanish, so that string singularities are unavoidable. Specifically,

$$\Omega'_\psi(1,y) = -s^3\frac{4k\lambda}{\sqrt{(1+\nu)^2 - \lambda^2}}\,. \tag{2.57}$$

The vector field Ω'_ψ can be made regular by a translation, leading to a NUT charge which can in principle be cancelled by a NUT-generating transformation.

Bibliography

[1] A. A. Pomeransky and R. A. Sen'kov, "Black ring with two angular momenta", arXiv:hep-th/0612005

[2] S. Giusto and A. Saxena, Class. Quantum Grav. **24** (2007) 4269.

[3] J. Ford, S. Giusto, A. Peet and A. Saxena, Class. Quantum Grav. **25** (2008) 075014.

Conclusion

We developed a solution generating technique in minimal five dimensional supergravity based on the hidden dualities of this theory. It was found that the nonlinear sigma model which arises upon dimensional reduction and dualization the vector fields of this theory has a coset structure $G_{2(2)}/[SL(2,\mathbb{R}) \times SL(2,\mathbb{R})]$, where the isometry group $G_{2(2)}$ is a compact form of the rank 2 exceptional group G_2. One of the transformations which conserve asymptotic flatness is identified as a transformation charging electrically neutral solutions. This transformation is applied to a neutral rotating black hole solution to obtain a charged rotating black hole in five dimensions. This charging procedure is then used to get a charged doubly spinning black ring from a neutral. The resulting solution suffers from a singularity known as Dirac-Misner string.