PLANAR ANALYTIC NILPOTENT GERMS VIA ANALYTIC FIRST INTEGRALS

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Abstract. We generalize the results of [6] by giving necessary and sufficient conditions for the planar analytic nilpotent germs to have an analytic first integral in $(\mathbb{R}^2, 0)$. The proof of our main result is based on a new method to compute the analytic first integrals of the nilpotent germs, the use of the weight homogeneous polynomials, and the method of characteristic curve for solving linear partial differential equations. Applications of our results to the Kukles-like cubic system are considered.

1. Introduction and main result

The present paper concerns the local classification of real analytic nilpotent germs in $(\mathbb{R}^2, 0)$ which admit analytic first integrals. Consider a planar analytic germ $(P(x,y),\ Q(x,y))$ in $(\mathbb{R}^2, 0)$, i.e., P and Q are analytic functions in x and y having the origin as the singularity. A non-elementary singularity (see Section 2) of the germs is called *nilpotent* if the two eigenvalues at the singularity are both zero but the linear part of the germ at the singularity is non-zero.

It is well known that under an affine change of coordinates and a rescaling of the time variable any real analytic nilpotent germ in $(\mathbb{R}^2, 0)$ can be written in the form

$$\begin{cases} \dot{x} = y + p(x,y), \\ \dot{y} = q(x,y), \end{cases} \tag{1.1}$$

where $p(x,y), q(x,y) = O((|x| + |y|)^2)$ are analytic functions. Moreover, under certain analytic, reversible transformations (see e.g., [15, 22]), system (1.1) can be further transformed into the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = f(x) + yg(x) + y^2 h(x,y), \end{cases} \tag{1.2}$$

where $f(x)$, $g(x)$ and $h(x,y)$ are analytic functions with $f(0) = f'(0) = g(0) = 0$. Write

$$f(x) = ax^k + \dots, \qquad g(x) = bx^l + \dots,$$

where $k \geq 2$, $l \geq 1$, and the dots denote the summation of the terms with degrees higher than k and l, respectively. If $b \neq 0$, we assume without loss of generality that $b > 0$.

We denote by $\mu(F, 0) = m$ the *multiplicity* of an analytic function $F(x)$ at $x = 0$, i.e., m is the largest positive integer such that x^m divides F. If $F \equiv 0$, we define

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$m = \mu(F, 0) = \infty$. With this notation, we have $\mu(f, 0) = k$ and $\mu(g, 0) = l$ in the above.

The local classification of analytic nilpotent germs is a classical problem first considered by Andreev [1] who gave a topological classification of the local phase portraits of the nilpotent germ except for a center or a focus. More precisely, Andreev's classification theorem states as follows.

Andreev's Classification Theorem. ([1]) *Consider* (1.2). *Then the following holds.*

(a) *If $f(x) \equiv 0$, then the line $y = 0$ is a singular line, i.e., it is filled with singularities.*
  (i) *If $g(x) \equiv 0$, then the line $y = 0$ is filled with nilpotent singularities;*
  (ii) *If $g(x) \not\equiv 0$, then the origin is a unique nilpotent singularity.*

(b) *If $f(x) \not\equiv 0$, then the origin is an isolated, nilpotent singularity.*
  (i) *If k is odd and $a > 0$, then the singularity is a nilpotent saddle.*
  (ii) *If k is odd and $a < 0$, then one of the following holds.*
    (ii_1) *If $k < 2l + 1$; or $k = 2l + 1$ and $b^2 + 4a(l + 1) < 0$, then the singularity is a nilpotent center or focus.*
    (ii_2) *If $k = 2l + 1$, $b^2 + 4a(l + 1) \geq 0$ and l is even; or $k > 2l + 1$ and l is even, then the singularity is a nilpotent node.*
    (ii_3) *If $k = 2l + 1$, $b^2 + 4a(l + 1) \geq 0$ and l is odd; or $k > 2l + 1$ and l is odd, then the singularity is formed by one hyperbolic sector and one elliptic sector.*
  (iii) *If k is even and $k < 2l + 1$, then the singularity is a nilpotent cusp.*
  (iv) *If k is even and $k > 2l + 1$, then the singularity is a nilpotent saddle-node.*

We referee the readers to [2] or [22] for more details on the Andreev's classification theorem. Problems concerning nilpotent germs have been considered by various authors. Dumortier [8] and Panazzolo [15] studied the desingularization of the nilpotent germs. Berthier et al [3], Stróżyna [19], Stróżyna and Żoladek [20] gave an analytic or formal orbital classification of the nilpotent germs in the sense of holonomy group for the blowing-up vector fields. To be able to characterize the nilpotent center or focus along the line of the Andreev's classification theorem, Chavarriga *et al* [6] obtained some sufficient conditions for the existence of analytic first integrals in a nilpotent germ, along with some explicit forms of the first integrals.

In this paper, we will extend the results of [6] by giving some sufficient and necessary conditions for the existence of analytic first integrals in a nilpotent germ. More precisely, our main result states as follows.

Main Result. *The analytic nilpotent germ given in* (1.2) *has an analytic first integral in* $(\mathbb{R}^2, 0)$ *if and only if one of the following conditions holds.*

(a) *The nilpotent singularity is non-isolated, i.e., $k = \infty$.*

(b) *The nilpotent singularity is a saddle with either $a > 0$ and $k < 2l + 1$ odd; or $a > 0$, $k = 2l + 1$ and $b/\sqrt{b^2 + 4a(l + 1)}$ being a rational number, and the blowing-up vector field $\widetilde{\mathcal{X}}$ at each of the four saddles on the exceptional divisor is analytically equivalent to*

$$\lambda_1 u(1 + r(z))\frac{\partial}{\partial u} + \lambda_2 v(1 + r(z))\frac{\partial}{\partial v},$$

where r is analytic in z with $r(0)=0$ and $z=u^n v^m$, for $m/n=-\lambda_1/\lambda_2$, $m,n\in\mathbb{N}$ being coprime, and λ_1 and λ_2 being two eigenvalues of the blowing-up vector field at the saddle on the projective lines.

(c) *The nilpotent singularity is a cusp with k even and $k<2l+1$, and the blowing-up vector field $\widetilde{\mathcal{X}}$ at each of the two saddles on the exceptional divisor is analytically orbitally equivalent to its linear part.*

(d) *The nilpotent singularity is an analytic center with $a<0$ and $k<2l+1$ odd.*

Moreover, the following holds.

(i) *In the case* (b) *the nilpotent germ has an analytic first integral of the form*

$$H(x,y)=y^2-\frac{2a}{k+1}x^{k+1}+\ldots, \tag{1.3}$$

if $a>0$ and $k<2l+1$ odd, where the dots denote the summation of the terms of the weight degree higher than $2k+2$ with respect to the weight exponent $(2,k+1)$; or of the form

$$\begin{aligned} H(x,y) \;=\; &\left(2ax^{l+1}+\left(b-\sqrt{b^2+4a(l+1)}\right)y\right)^{n+m}\times\\ &\left(2ax^{l+1}+\left(b+\sqrt{b^2+4a(l+1)}\right)y\right)^{n-m}+\ldots, \end{aligned}$$

if $a>0$ and $k=2l+1$, where $m/n=b/\sqrt{b^2+4a(l+1)}$ with m and n being coprime natural numbers, and the dots denote the sum of the terms with the weight degree higher than $(2l+2)(m+n)$ with respect to the weight exponent $(2,2l+2)$. The separatrices passing through the nilpotent saddle are given by $2ax^{l+1}+\left(b-\sqrt{b^2+4a(l+1)}\right)y+h_1(x,y)=0$ and $2ax^{l+1}+\left(b+\sqrt{b^2+4a(l+1)}\right)y+h_2(x,y)=0$ respectively, where h_1 and h_2 begin with terms of weight degrees higher than $2l+2$.

(ii) *In the cases* (c) *and* (d) *the nilpotent germ has an analytic first integral of the form* (1.3).

The proof of our main result is based on the analysis of the characteristic equation

$$y\frac{\partial H}{\partial x}+\left(f(x)+yg(x)+y^2h(x,y)\right)\frac{\partial H}{\partial x}\equiv 0 \tag{1.4}$$

which a formal first integral H satisfies. Thus if a non-trivial formal power series $H(x,y)$ defined on a planar open domain is convergent and satisfies (1.4), then H becomes an analytic first integral of the nilpotent germ given in (1.2). We will also give a simple, alternative proof of the Andreev's classification theorem and use part of the arguments in the proof of our main result.

One advantage of our method lies in the computation of higher order terms of the analytic first integrals. Following part (b) (ii_2) of the Andreev's classification theorem, part (d) of our main result, and the symmetry of the vector fields under the transformation $(x,y,t)\to(-x,y,-t)$, we easily have the following.

Corollary. *The analytic vector fields*

$$\left(y+\frac{b}{l+1}x^{l+1}+\sum_{i=1}^{n}b_i x^{l+1+2i},\quad -x^{2l+1}+\sum_{j=1}^{m}a_i x^{2l+1+2j}\right),$$

with $m, n \in \mathbb{N}$ arbitrarily, $l \in \mathbb{N}$ odd and $0 < b < 2\sqrt{2}$, has the origin as a center. But there are neither local analytic first integrals at the origin, nor formal ones.

The above corollary is a slight generalization of the Proposition 7 contained in [6] for the case $l = 1$, and $a_i = b_j = 0$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

The paper is organized as follows. In Section 2 we recall some notions and known results to be used later on. In Section 3, we give an alternative proof of the Andreev's classification Theorem. The proof of the main result will be completed in Section 4. In Section 5, we apply our main result to the Kukles-like cubic system and obtain necessary and sufficient conditions for the nilpotent singularity of the system to be an analytic center.

2. Preliminary

A singularity of a planar analytic germ is called *elementary* if at least one of the two eigenvalues of the germ at the singularity is non-zero; otherwise it is called *non-elementary*. An elementary singularity is *degenerate* if one of the eigenvalues is zero. If the real parts of the two eigenvalues are non-zero, the singularity is called *hyperbolic*. A singularity is called a *rational hyperbolic saddle* if it is hyperbolic and the ratio of its eigenvalues is a negative rational number.

At a non-elementary singularity of a planar analytic germ $\mathcal{X}$, the *blowing-up with the weight exponent* α of $(0,0)$ is an analytic surjective map

$$\begin{array}{rccc} \Phi: & \mathbb{S}^1 \times \mathbb{R}^+ & \longrightarrow & \mathbb{R}^2 \\ & (z, \rho) & \longmapsto & (\rho^{\alpha_1} x, \rho^{\alpha_2} y) \end{array}$$

where $\mathbb{S}^1 = \{z = (x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$, $\mathbb{R}^+$ is the set of non-negative real numbers, and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$. The manifold $\widetilde{M} = \mathbb{S}^1 \times \mathbb{R}^+$ is called the *blowing-up* space, and $\mathbb{S}^1 = \Phi^{-1}(0)$ is called the *exceptional divisor* or the *projective line*. The open covers of $\widetilde{M}$, $U_x^+ = \{x > 0\}$, $U_x^- = \{x < 0\}$, $U_y^+ = \{y > 0\}$ and $U_y^- = \{y < 0\}$, are called *local charts* of $\widetilde{M}$. The vector field $\widetilde{\mathcal{X}}$ obtained from $\mathcal{X}$ by the blowing-up change is called the *blowing-up vector fields*. After a sequence of blowing-ups, if all the singularities of the blowing-up vector field on the exceptional divisors are elementary then we call the blowing-up vector field *final*. We denote by Ω the unbounded manifold with the inner boundary formed by the exceptional divisors.

We say that a polynomial $F(x, y)$ is a *weight homogeneous polynomial* if there exist $m \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ such that for any positive real number ρ we have

$$F(\rho^{\alpha_1} x, \rho^{\alpha_2} y) = \rho^m F(x, y).$$

We call α the *weight exponent*, and m the *weight degree* of F with respect to the weight exponent α.

We now recall some known results to be used in the present paper. Consider the planar analytic system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \ldots, \tag{2.1}$$

where the dots denote the summation of the terms of order larger than 1, and the origin is assumed to be a singularity.

Lemma 2.1. *Assume that the origin of system* (2.1) *is a degenerate elementary singularity. Then the system has an analytic first integral in* $(\mathbb{R}^2, 0)$ *if and only if the origin is non-isolated.*

Proof. See [12]. □

Lemma 2.2. *Consider* (2.1)*. Then the following holds.*

(i) *If the singularity at the origin is an elementary saddle, then system* (2.1) *has an analytic first integral in* $(\mathbb{R}^2, 0)$ *if and only if the singularity is a rational hyperbolic saddle and system* (2.1) *in a neighborhood of the origin is analytically equivalent to the following system*

$$\left\{ \begin{array}{l} \dot{u} = \lambda_1 u(1 + r(z)), \\ \dot{v} = \lambda_2 v(1 + r(z)), \end{array} \right. \tag{2.2}$$

where r *is an analytic function in* z *with* $r(0) = 0$*, and* $z = u^n v^m$*, for* $m/n = -\lambda_1/\lambda_2$ *with* $m, n \in \mathbb{N}$ *being coprime and* $\lambda_1,\ \lambda_2$ *being two eigenvalues of system* (2.1) *at the origin.*

(ii) *If the singularity at the origin is non–elementary and its neighborhood is formed by hyperbolic sectors, then system* (2.1) *has an analytic first integral in* $(\mathbb{R}^2, 0)$ *if and only if the following conditions hold:*
   (a) *All the singularities of the final blowing-up vector field* $\widetilde{\mathbf{X}}$ *on* $\partial\Omega$ *are rational hyperbolic saddles;*
   (b) *The vector field* $\widetilde{\mathbf{X}}$ *in some neighborhood of each of these saddles is analytically equivalent to a system of the form* (2.2)*.*

Proof. A complete proof of this lemma can be found in [21]. For the readers' convenience, we sketch the proof in the Appendix. □

Lemma 2.3. *Consider the planar formal differential system*

$$\left\{ \begin{array}{l} \dot{x} = \lambda x + P(x, y), \\ \dot{y} = \mu y + Q(x, y), \end{array} \right.$$

with $x, y \in \mathbb{K}$*, where* $\mathbb{K}$ *is an algebraically closed field of characteristic zero,* $|\lambda| \cdot |\mu| \neq 0$*, and* P *and* Q *are formal series starting with terms of order higher than 1. If* λ *and* μ *are rationally independent or* $\lambda/\mu < 0$*, then there are exactly two solutions through the origin, a linear branch with horizontal tangent and a linear branch with vertical tangent.*

Proof. See [17]. □

Next we recall the method of characteristic curves from, say e.g. [5, Chapter 2], for solving linear partial differential equations. Consider the following first order linear partial differential equation

$$a(x, y)A_x + b(x, y)A_y + c(x, y)A = f(x, y), \tag{2.3}$$

where $A = A(x, y)$, a, b, c and f are C^1 functions.

A curve $(x(t), y(t))$ in the xy–plane is a *characteristic curve* for the partial differential equation (2.3), if at each point (x_0, y_0) on the curve, the vector $(a(x_0, y_0),$

$b(x_0, y_0))$ is tangent to the curve. So, a characteristic curve is a solution of the system

$$\begin{cases} \dfrac{dx}{dt} = a(x(t), y(t)), \\ \dfrac{dy}{dt} = b(x(t), y(t)). \end{cases}$$

If $b(x, y) \neq 0$ then y can be treated as an independent variable and the above system is reduced to the following *characteristic equation*:

$$\frac{dx}{dy} = \frac{a(x, y)}{b(x, y)}. \tag{2.4}$$

Suppose that (2.4) has a solution in the implicit form $g(x, y) = c_1$, where c_1 is an arbitrary constant. We consider the change of variables

$$u = g(x, y), \qquad v = y. \tag{2.5}$$

If (2.5) is invertible, we denote its inverse by $x = p(u, v)$ and $y = q(u, v)$. Then the linear partial differential equation (2.3) becomes the following ordinary differential equation in v (for fixed u)

$$\overline{b}(u, v)\overline{A}_v + \overline{c}(u, v)\overline{A} = \overline{f}(u, v), \tag{2.6}$$

where $\overline{b}$, $\overline{c}$, $\overline{A}$ and $\overline{f}$ are defined by b, c , A and f, respectively, in terms of u and v.

If $\overline{A} = \overline{A}(u, v)$ is a solution of (2.6), then by transformation (2.5)

$$A(x, y) = \overline{A}(g(x, y), y),$$

is a solution of the linear partial differential equation (2.3). Moreover, the general solution of (2.6) becomes the general solution of (2.3) when tracing back the transformation (2.5). This method was first used in [13] to obtain the classification of the invariant algebraic surfaces of the Lorenz system.

3. An alternative proof of the Andreev's classification theorem

The proof of the Statement (a) of Andreev's classification theorem follows easily from the definition of the nilpotent singularity and the form of system (1.2).

We will use the blowing-up technique to prove the statement (b) of the theorem. Let $f(x) \not\equiv 0$. This implies that $2 \leq k < \infty$.

Case 1: $k \leq 2l + 1$. In the local charts $U_x^{\pm}$, by using the weight blowing-up with the weight exponent $(2, k + 1)$:

$$x = \pm X^2, \qquad y = X^{k+1}Y,$$

to system (1.2), where $(X, Y) \in \mathbb{R}^2 \cap \{X \geq 0\}$, we obtain the following blowing-up vector field $\widetilde{\mathcal{X}}$:

$$\begin{cases} \dot{X} = \pm XY/2, \\ \dot{Y} = (\pm 1)^k a + (\pm 1)^l \delta Y \mp \left(\dfrac{k+1}{2}\right) Y^2 + X(\cdots), \end{cases} \tag{3.1}$$

where $\delta = 0$ if $k < 2l + 1$ or $\delta = b$ if $k = 2l + 1$, and the dots denote an analytic function in X and Y. The Y_0 coordinate of the singularity $(0, Y_0)$ on the invariant line $X = 0$ satisfies the equation

$$(\pm 1)^k a + (\pm 1)^l \delta Y_0 \mp \frac{k+1}{2} Y_0^2 = 0.$$

The Jacobian matrix of the vector field $\widetilde{\mathcal{X}}$ at $(0, Y_0)$ is

$$D\widetilde{\mathcal{X}}(0, Y_0) = \begin{pmatrix} \pm Y_0/2 & 0 \\ * & (\pm 1)^l \delta \mp (k+1)Y_0 \end{pmatrix}. \tag{3.2}$$

The condition $k \leq 2l+1$ implies that $a \neq 0$, from which we conclude that $Y_0 \neq 0$. Hence the singularities of the blowing-up vector field $\widetilde{\mathcal{X}}$ are all elementary (if they exist).

In the local charts $U_y^{\pm}$, applying the weight blowing-up with the weight exponent $(2, k+1)$:

$$x = Y^2 X, \qquad y = \pm Y^{k+1},$$

to system (1.2), where $(X, Y) \in \mathbb{R}^2 \cap \{Y \geq 0\}$, we obtain the blowing-up vector field $\widetilde{\mathcal{X}}$:

$$\begin{cases} \dot{X} = \pm 1 + \dfrac{2}{k+1} X \left[\pm a X^k + \delta X^l \pm Y^2(\cdots) \right], \\ \dot{Y} = \dfrac{1}{k+1} \left[\pm a X^k + \delta X^l \pm Y^2(\cdots) \right] Y, \end{cases} \tag{3.3}$$

where $\delta = 0$ if $k < 2l+1$ or $\delta = b$ if $k = 2l+1$, and the dots denote an analytic function in X and Y. This vector field has no interesting singularities on the invariant line $Y = 0$.

Subcase 1: $k < 2l+1$. In this case $\delta = 0$ and $a \neq 0$. If k is odd and $a > 0$, then there are two singularities of $\widetilde{\mathcal{X}}$ in each of U_x^+ and U_x^-. From (3.1) and (3.2) it is easy to see that these four singularities are all rational hyperbolic saddles. After blowing-down we see that the nilpotent singularity of (1.2) is a saddle.

If k is odd and $a < 0$, then the blowing-up vector field $\widetilde{\mathcal{X}}$ has no singularities. Consequently, the nilpotent singularity of (1.2) is either a center or a focus.

If k is even and $a > 0$ (respectively, $a < 0$), then the blowing-up vector field $\widetilde{\mathcal{X}}$ has two singularities (respectively, no singularities) in U_x^+ and no singularities (respectively, two singularities) in U_x^-. In any cases, the two singularities are rational hyperbolic saddles. Hence, the nilpotent singularity of (1.2) is a cusp.

Subcase 2: $k = 2l+1$. In this case $\delta = b$ and $a \neq 0$. The eigenvalues λ_1 and λ_2 of the two singularities (if exist) of the blowing-up vector field $\widetilde{\mathcal{X}}$ in U_x^+ (respectively U_x^-) satisfy

$$\lambda_1 \lambda_2 = \frac{b^2 + 2(k+1)a - b\sqrt{b^2 + 2(k+1)a}}{-2(k+1)},$$

$$\left(\text{respectively} \quad \lambda_1 \lambda_2 = \frac{b^2 + 2(k+1)a - (-1)^l b\sqrt{b^2 + 2(k+1)a}}{-2(k+1)} \right),$$

or

$$\lambda_1 \lambda_2 = \frac{b^2 + 2(k+1)a + b\sqrt{b^2 + 2(k+1)a}}{-2(k+1)}.$$

$$\left(\text{respectively} \quad \lambda_1 \lambda_2 = \frac{b^2 + 2(k+1)a + (-1)^l b\sqrt{b^2 + 2(k+1)a}}{-2(k+1)} \right).$$

If $a > 0$, then the blowing-up vector field $\widetilde{\mathcal{X}}$ has four singularities on the exceptional divisor, which are all saddles. Therefore, the nilpotent singularity of (1.2) is a saddle. Moreover, it follows from (3.1) and (3.2) that the saddles of $\widetilde{\mathbf{X}}$ are rational hyperbolic if and only if $b/\sqrt{b^2 + 4a(l+1)}$ is rational.

If $a<0$ and $b^2+2a(k+1)>0$, then the vector field $\widetilde{\mathcal{X}}$ has a saddle and a node in each of U_x^+ and U_x^-. Moreover, the saddles are separated by the nodes. After blowing-down, we have that the nilpotent singularity is a node.

If $a<0$ and $b^2+2a(k+1)=0$, then the vector field $\widetilde{\mathcal{X}}$ has a saddle-node in each of U_x^+ and U_x^-. Furthermore, the saddle sectors of the two saddle-nodes cannot be consecutive. Hence the nilpotent singularity is also a node.

If $a<0$ and $b^2+2a(k+1)<0$, then the vector field $\widetilde{\mathcal{X}}$ has no singularities on the exceptional divisor. Consequently, the nilpotent singularity is either a center or a focus.

Case 2: $k>2l+1$. In this case $b\neq 0$. In the local charts $U_x^{\pm}$, applying the weight blowing-up with the weight exponent $(1,l+1)$:

$$x=\pm X,\qquad y=X^{l+1}Y,$$

to system (1.2), where $(X,Y)\in\mathbb{R}^2\cap\{X\geq 0\}$, we obtain the blowing-up vector field $\widetilde{\mathcal{X}}$:

$$\left\{\begin{array}{l}\dot X=\pm XY,\\ \dot Y=(\pm1)^l bY\mp(l+1)\,Y^2+X\left[(\pm1)^k aX^{k-2l-2}+Y^2(\cdots)\right],\end{array}\right. \tag{3.4}$$

where the dots denote an analytic function in X and Y. The singularities of $\widetilde{\mathcal{X}}$ on the invariant line $X=0$ are $(0,0)$ with the eigenvalues $\lambda_1=0$, $\lambda_2=(\pm1)^l b$; and $\left(0,(\pm1)^{l+1}\dfrac{b}{l+1}\right)$ with the eigenvalues $\lambda_1=(\pm1)^l\dfrac{b}{l+1}$, $\lambda_2=-(\pm1)^l b$. The latter is a saddle, and the former is degenerate.

In the local charts $U_y^{\pm}$, applying the weight blowing-up change with the weight exponent $(1,l+1)$:

$$x=YX,\qquad y=\pm Y^{l+1},$$

to system (1.2), where $(X,Y)\in\mathbb{R}^2\cap\{Y\geq 0\}$, we obtain the blowing-up vector field $\widetilde{\mathcal{X}}$:

$$\left\{\begin{array}{l}\dot X=\pm1-\dfrac{1}{l+1}X\left[bX^l+Y(\cdots)\right],\\ \dot Y=\dfrac{1}{l+1}Y\left[bX^l+Y(\cdots)\right],\end{array}\right. \tag{3.5}$$

where the dots denote an analytic function in X and Y. It has no interesting singularities on the invariant line $Y=0$.

For the degenerate singularity $(0,0)$ of (3.4), by Theorem 65 of [2] (see also Theorem 7.1 of [22]), we have that if $k-2l$ is odd and $a<0$ (respectively, $a>0$), then it is an unstable node (respectively, a saddle); and if $k-2l$ is even, then it is formed by a parabolic sector and two hyperbolic sectors. Using arguments similar to the proof of Case 1, we have the following local topological constructions of the nilpotent singularity of (1.2):

- If k is odd and $a>0$, then the nilpotent singularity is a saddle.
- If k is odd, $a<0$ and l is even, then the nilpotent singularity is a node.
- If k is odd, $a<0$ and l is odd, then the nilpotent singularity is formed by a hyperbolic sector and an elliptic sector.
- If k is even, then the nilpotent singularity is formed by one parabolic sector and two hyperbolic sectors.

The proof of the theorem is now complete.

4. Proof of the main result

Necessity. Assume that the nilpotent germ has an analytic first integral in $(\mathbb{R}^2, 0)$. Then the nilpotent singularity cannot be a focus, or the one with a parabolic sector or an elliptic sector. For otherwise, the analytic curve passing through the nilpotent singularity would have infinitely many analytic branches, a contradiction to the analyticity. Consequently, from Andreev's local classification theorem we have that the nilpotent singularity is non-isolated, or a saddle, or a cusp, or a center.

Case 1. The nilpotent singularity is non-isolated, i.e. $f(x) \equiv 0$.

In this case, system (1.2) and the system

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = g(x) + yh(x, y), \end{cases}$$

have the same analytic first integrals. Since the origin is a regular point of the last system, it follows from the standard Flow Box Theorem (see e.g., [18]) that it has an analytic first integral in $(\mathbb{R}^2, 0)$. Hence, the nilpotent germ has an analytic first integral in $(\mathbb{R}^2, 0)$. This proves the statement (a).

In the following, we suppose that the nilpotent singularity is isolated.

Case 2. The nilpotent singularity is a saddle.

From statement (i) of (b) in Andreev's local classification theorem we have that k is odd and $a > 0$. The above proof of Case 2 for the Andreev's local classification theorem asserts that if $k > 2l + 1$, then the blowing-up vector field $\widetilde{\mathcal{X}}$ has the degenerate elementary saddle (0, 0) in $U_x^{\pm}$ on the exceptional divisor, which is isolated. It follows from Lemma 2.1 that $\widetilde{\mathcal{X}}$ has no analytic first integrals in any neighborhood of $(0, 0)$. Consequently, we have by Lemma 2.2 that the nilpotent germ (1.2) has no analytic first integrals in $(\mathbb{R}^2,0)$. Therefore, in order for the nilpotent germ to have an analytic first integral we should have $k \leq 2l + 1$.

From statement (ii) of Lemma 2.2 and the above proof of statement (b) in Andreev's local classification theorem under Case 1, we conclude that if the nilpotent germ has an analytic first integral, then either $a > 0$ and $k < 2l + 1$ is odd; or $a > 0$, $k = 2l + 1$ and $b/\sqrt{b^2 + 4a(l + 1)}$ is a rational number. In both situations, the singularities of the blowing-up vector fields on the exceptional divisor are all rational hyperbolic. By Lemma 2.2, around each of these singularities the blowing-up vector field is orbitally equivalent to its linear part.

In what follows we use the weight homogeneous polynomials and the method of characteristic curves for solving linear partial differential equations to compute the analytic or formal first integrals of the nilpotent germs.

Write

$$f(x) = ax^k + a_1x^{k+1} + a_2x^{k+2} + \dots,$$
$$g(x) = bx^l + b_1x^{l+1} + b_2x^{l+2} + \dots$$

We make the following weight change of the variables:

$$x \to \alpha^2 x, \quad y \to \alpha^{k+1} y, \quad t \to \alpha^{-k+1} t.$$

Then system (1.2) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \left(ax^k + a_1\alpha^2x^{k+1} + a_2\alpha^4x^{k+2} + \dots\right) + y\left(b\alpha^{2l-k+1}x^l + \right. \\ \qquad \left. b_1\alpha^{2l-k+3}x^{l+1} + b_2\alpha^{2l-k+5}x^{l+2} + \dots\right) + \alpha^2y^2h(\alpha^2x, \alpha^{k+1}y). \end{cases} \tag{4.1}$$

If $H(x,y)$ is an analytic first integral of system (1.2), then $\alpha^m H(\alpha^2 x, \alpha^{k+1} y)$ with m being any integer is an analytic first integral of system (4.1). Without loss of generality, we assume that the analytic first integral of (4.1) is of the form

$$\widetilde{H}(x,y,\alpha)=\sum_{i=0}^{\infty}\alpha^i H_i(x,y),$$

satisfying $H(x,y)=\widetilde{H}(x,y,\alpha)\Big|_{\alpha=1}$, where $H_0(x,y)$ is a weight homogeneous polynomial of weight degree M with respect to the weight exponent $(2,k+1)$, and $H_i(x,y)$ are weight homogeneous polynomials of weight degree $M+i$ for $i=1,2,\ldots$.

From the definition of analytic first integrals we have

$$\begin{aligned} y\sum_{i=0}^{\infty}\alpha^i\frac{\partial H_i}{\partial x}+\big[\big(ax^k+a_1\alpha^2x^{k+1}+a_2\alpha^4x^{k+2}+\ldots\big)+\\ y\left(b\alpha^{2l-k+1}x^l+b_1\alpha^{2l-k+3}x^{l+1}+b_2\alpha^{2l-k+5}x^{l+2}+\ldots\right)+\\ \alpha^2y^2h(\alpha^2x,\alpha^{k+1}y)\big]\sum_{i=0}^{\infty}\alpha^i\frac{\partial H_i}{\partial y}\equiv 0.\end{aligned} \tag{4.2}$$

Comparing the coefficients of α^0, we have that if $k<2l+1$, then

$$y\frac{\partial H_0}{\partial x}+ax^k\frac{\partial H_0}{\partial y}\equiv 0, \tag{4.3}$$

and if $k=2l+1$, then

$$y\frac{\partial H_0}{\partial x}+\left(ax^{2l+1}+bx^ly\right)\frac{\partial H_0}{\partial y}\equiv 0. \tag{4.4}$$

We now use the method of the characteristic curves for solving linear partial differential equations to construct the weight homogeneous polynomial solutions of (4.3) and (4.4).

Subcase 1: $k=2l+1$. The characteristic equation associated with (4.4) is

$$\frac{dx}{dy}=\frac{y}{ax^{2l+1}+bx^ly}. \tag{4.5}$$

Since $a>0$, $K=b^2+4a(l+1)>0$. Then the last equation has a general solution of the form

$$\left[ax^{2l+2}+bx^{l+1}y-(l+1)y^2\right]\left(\frac{2ax^{l+1}+\left(b-\sqrt{K}\right)y}{2ax^{l+1}+\left(b+\sqrt{K}\right)y}\right)^{\frac{b}{\sqrt{K}}}=c,$$

where c is a constant. Since $\frac{b}{\sqrt{K}}$ is a rational number less than 1, we set $\frac{b}{\sqrt{K}}=\frac{m}{n}$ with $n>m>0$ being relatively prime. Since

$$\begin{aligned}&\left(2ax^{l+1}+\left(b-\sqrt{K}\right)y\right)\left(2ax^{l+1}+\left(b+\sqrt{K}\right)y\right)\\ &=4a\left(ax^{2l+2}+bx^{l+1}y-(l+1)y^2\right),\end{aligned}$$

we can rewrite this last equation as

$$\left(2ax^{l+1}+\left(b-\sqrt{K}\right)y\right)^{n+m}\left(2ax^{l+1}+\left(b+\sqrt{K}\right)y\right)^{n-m}=c'.$$

Consider the change of variables

$$\begin{aligned} u &= \left(2ax^{l+1}+\left(b-\sqrt{K}\right)y\right)^{\frac{n+m}{(n+m,n-m)}}\left(2ax^{l+1}+\left(b+\sqrt{K}\right)y\right)^{\frac{n-m}{(n+m,n-m)}}, \\ v &= x, \end{aligned}$$

where $(n+m, n-m)$ is the great common factor of $n+m$ and $n-m$. Let $x = v$ and $y = \Phi(u, v)$ be its inverse transformation (if the inverse function is not single-valued, we can select an analytic branch of the function such that $y = \Phi(u, v)$ is single-valued). Then equation (4.4) becomes the following ordinary differential equation

$$\Phi(u,v)\frac{d\overline{H}_0}{dv} = 0,$$

where $\overline{H}_0$ is defined by H_0, in terms of u and v.

This last equation has the general solution $\overline{H}_0 = \overline{H}_0(u)$, which is smooth. Hence the general solution $H_0(x, y)$ of (4.4) is of the form

$$\begin{aligned} H_0(x,y) = \overline{H}_0(u) &= \overline{H}_0\Bigg(\left(2ax^{l+1}+\left(b-\sqrt{K}\right)y\right)^{\frac{n+m}{(n+m,n-m)}} \times \\ &\qquad \left(2ax^{l+1}+\left(b+\sqrt{K}\right)y\right)^{\frac{n-m}{(n+m,n-m)}}\Bigg). \end{aligned}$$

In order for $H_0(x, y)$ to be a weight homogeneous polynomial in x, y, $H_0(x, y)$ should be a positive integer power of

$$\left(2ax^{l+1}+\left(b-\sqrt{K}\right)y\right)^{\frac{n+m}{(n+m,n-m)}}\left(2ax^{l+1}+\left(b+\sqrt{K}\right)y\right)^{\frac{n-m}{(n+m,n-m)}}.$$

Since the ratio of eigenvalues of the two singularities of the blowing-up vector given in (3.1) is a negative number, it follows from Lemma 2.3 that the two solutions of the blowing-up vector field passing through the saddle are both linear branches. Since $X = 0$ is invariant to the flow of (3.1), by blowing-down to the vector field $\widetilde{\mathbf{X}}$ and combining the computation of first integrals, we have that the two separatrices of the nilpotent saddle is traced out by the functions:

$$2ax^{l+1}+\left(b-\sqrt{K}\right)y+h_1(x,y)=0, \quad 2ax^{l+1}+\left(b+\sqrt{K}\right)y+h_2(x,y)=0,$$

where h_1 and h_2 denote the terms of weight degree at least $2l + 3$ with respect to the weight exponent $(2, 2l + 2)$. Hence, if the nilpotent germ has analytic first integrals, it should have the analytic first integral of the form

$$\begin{aligned} H(x,y) &= \left(2ax^{l+1}+\left(b-\sqrt{K}\right)y\right)^{n+m} \times \\ &\qquad \left(2ax^{l+1}+\left(b+\sqrt{K}\right)y\right)^{n-m} + \cdots, \end{aligned}$$

where the dots denote the terms of weight degree higher than $2l + 2$. This shows that the analytic first integral of the nilpotent germ has the desired form.

Subcase 2: $k < 2l+1$. Similar to the proof of Subcase 1, we have that the nilpotent germ has an analytic first integral with its first weight homogeneous term being of the form $H_0(x, y) = y^2 - \frac{2a}{k+1}x^{k+1}$. This proves statement (b).

Case 3. The nilpotent singularity is a cusp.

From statement (iii) of (b) in Andreev's local classification theorem, we have that $k < 2l+1$ is even. From (3.2) it is easy to see that the saddles of the blowing-up vector field $\widetilde{\mathcal{X}}$ are all rationally hyperbolic. Suppose that the nilpotent germ has an analytic first integral $H(x,y)$. Using arguments similar to the proof of Case 2, we conclude that the nilpotent germ has an analytic first integral $H(x,y)$ whose first weight homogeneous term is $H_0(x,y) = y^2 - \frac{2a}{k+1}x^{k+1}$. Then statement (c) follows from Lemma 2.2.

Case 4. The nilpotent singularity is a center, and the germ has an analytic first integral.

It follows from (ii_1) of Andreev's local classification theorem that $a < 0$ and $k < 2l+1$ is odd, or $a < 0$, $k = 2l+1$ and $b^2 + 4a(l+1) < 0$. Suppose that $H(x,y)$ is an analytic first integral of the nilpotent germ given in (1.2).

Using arguments similar to the proof of Case 2, we conclude that the first weight homogeneous term $H_0(x,y)$ of $H(x,y)$ satisfies (4.3) if $k < 2l+1$, or (4.4) if $k = 2l+1$.

The general solution of (4.3) is of the form

$$H_0(x,y) = \overline{H}_0(u) = \overline{H}_0\left(ax^{k+1} - \frac{k+1}{2}y^2\right).$$

Since $K = b^2 + 4a(l+1) < 0$, the general solution of (4.4) is of the form

$$\begin{aligned} H_0(x,y) \quad = \quad & \overline{H}_0(u) = \overline{H}_0\left(\left[ax^{2l+2} + bx^{l+1}y - (l+1)y^2\right] \cdot \right. \\ & \left. \exp\left(\frac{2b}{\sqrt{-K}}\arctan\left(\frac{2a}{\sqrt{-K}}\left(\frac{x^{l+1}}{y} + \frac{b}{2a}\right)\right)\right)\right). \end{aligned}$$

Since $ab \neq 0$, $H_0(x,y)$ cannot be a weight homogeneous polynomial with the weight exponent $(2, k+1)$ in x and y. This verifies that if $k = 2l+1$, then the nilpotent germ cannot have analytic or formal first integrals. Consequently, if the nilpotent germ has a center, and an analytic first integral, then we must have $k < 2l+1$.

Treat system (3.1) as a complex planar system. Then it has two pairs of singularities with complex coordinates on the exceptional divisor. The ratio of eigenvalues at both singularities is a negative rational number. It follows from Lemma 2.3 that the blowing-up vector field has exactly two linear branches passing through each of the singularities. Using arguments similar to the proof of Subcase 1 in Case 2, we conclude that there are exactly two linear solutions passing through the nilpotent center. Hence the nilpotent germ has an analytic first integral whose first weight homogeneous term is $H_0(x,y) = y^2 - \frac{2a}{k+1}x^{k+1}$. This proves statement (d).

We have now completed the proof for the necessary part of the theorem.

The sufficient part of the theorem follows easily from the proof for the necessary part.

5. Kukles system

The center-focus problem for the Kukles system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2ya_{12}xy^2 + a_{03}y^3, \end{cases}$$

has been extensively studied (see e.g., [7, 14, 16]). As an application of our method, we consider the following "Kukles-like" cubic system with the origin a nilpotent singularity

$$
\left\{ \begin{array}{l} \dot{x} = y, \\ \dot{y} = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{array} \right. \tag{5.1}
$$

Proposition. *The origin of the Kukles-like cubic system* (5.1) *is an analytic center if and only if* $b_{20} = b_{11} = b_{21} = b_{03} = 0$ *and* $b_{30} < 0$. *Moreover, the following holds.*

a) *If* $b_{12} = b_{02} = 0$ *then the system has the analytic first integral*

$$H(x, y) = y^2 - \frac{b_{30}}{2}x^4;$$

b) *If* $b_{12} = 0$, $b_{02} \neq 0$, *then the system has the analytic first integral*

$$H(x, y) = \left[4b_{02}^4 y^2 + b_{30}\left(4b_{02}^3x^3 + 6b_{02}^2x^2 + 6b_{02}x + 3\right)\right] \exp(-2b_{02}x).$$

c) *If* $b_{12} \neq 0$, *then the system has the analytic first integral*

$$
\begin{aligned}
H(x, y) \quad = \quad & \left[2b_{12}^3y^2 + b_{30}\left(2b_{12}^2x^2 - 2b_{02}b_{12}x + 2b_{02}^2 + 2b_{12}\right)\right] \times \\
& \qquad \exp\left(-2b_{02}x - b_{12}x^2\right) \\
& + \frac{b_{30}}{\sqrt{b_{12}}}\left(2b_{02}^3 + 3b_{02}b_{12}\right)\sqrt{\pi}\exp\left(\frac{b_{02}^2}{b_{12}}\right)\operatorname{erf}\left(\frac{b_{12}x + b_{02}}{\sqrt{b_{12}}}\right).
\end{aligned}
$$

In the above proposition, $\sqrt{b_{12}}$ is a complex number if $b_{12} < 0$, and $\operatorname{erf}(t)$ is the Error function which is entire (for more details, see e.g., [10]).

Proof of Proposition.

The sufficient part can be easily proved by direct calculations.

We now prove the necessary part. By our main result, in order for the origin of (5.1) being an analytic center, we should have $b_{20} = b_{11} = 0$. Applying the weight change of variable

$$x \to \alpha^2 x, \quad y \to \alpha^4 y, \quad t \to \alpha^{-2}t,$$

to (5.1), we obtain the following system

$$
\left\{ \begin{array}{l} \dot{x} = y, \\ \dot{y} = b_{30}x^3 + \alpha^2(b_{02}y^2 + b_{21}x^2y) + b_{12}\alpha^4xy^2 + \alpha^6 b_{03}y^3. \end{array} \right. \tag{5.2}
$$

By the proof of statement (d) of our main result, we can assume that system (5.2) has an analytic first integral of the form $\widetilde{H}(x, y) = \sum\limits_{i=0}^{\infty} \alpha^i H_i(x, y)$ with $H_0(x, y) = y^2 - \frac{b_{30}}{2}x^4$, where, for each $i = 1, \cdots$, $H_i(x, y)$ is a weight homogeneous polynomial of weight degree $8+i$ with respect to the weight exponent $(2, 4)$. Then H_i, $i = 1, 2, \cdots$, satisfies

$$
y\frac{\partial H_i}{\partial x} + b_{30}x^3\frac{\partial H_i}{\partial y} = -(b_{02}y^2 + b_{21}x^2y)\frac{\partial H_{i-2}}{\partial y} - b_{12}xy^2\frac{\partial H_{i-4}}{\partial y} - b_{03}y^3\frac{\partial H_{i-6}}{\partial y}, \tag{5.3}
$$

where $H_i = 0$ for $i < 0$. By induction, we conclude that $H_i = 0$ for $i > 0$ odd.

Applying the change of variables $u = y^2 - \frac{1}{2}b_{30}x^4$ and $v = x$, and its inverse transformation $x = v$ and $y = \sqrt{u + \frac{1}{2}b_{30}v^4}$, to (5.3) with $i = 2$ we have that

$$H_2(x, y) = \overline{H}_2(u, v) = -2b_{02}uv - \frac{1}{5}b_{02}b_{30}v^5 - 2b_{21}\int\left(v^2\sqrt{u + \frac{1}{2}b_{30}v^4}\right)dv.$$

The integral contains an Elliptic function. Hence, in order for H_2 being a weight homogeneous polynomial of weight degree 10, we should have $b_{21} = 0$. Consequently, $H_2 = -2b_{02}xy^2 + \frac{4}{5}b_{02}b_{30}x^5$. Solving equation (5.3) with $i = 4$ yields that $H_4 = \left(2b_{02}^2 - b_{12}\right)x^2y^2 - \frac{1}{3}\left(2b_{02}^2 - b_{12}\right)b_{30}x^6$. For equation (5.3) with $i = 6$ we have that

$$\begin{aligned} H_6(x,y) = \overline{H}_6(u,v) &= -\frac{2}{3}b_{02}\left(2b_{02}^2 - 3b_{12}\right)uv^3 - \frac{1}{7}b_{02}b_{30}\left(2b_{02}^2 - 3b_{12}\right)v^7 \\ &\quad -2b_{03}\int \left(u + \frac{1}{2}b_{30}v^4\right)^{\frac{3}{2}} dv. \end{aligned}$$

The integral also contains an Elliptic function. This implies that b_{03} should be zero.

We have obtained the necessary conditions for the origin of the Kukles-like cubic system to be an analytic nilpotent center. Under these conditions we can check that the analytic functions in the Proposition are first integrals of the system. This completes the proof of the proposition.

APPENDIX: A SKETCH PROOF OF LEMMA 2.2

Proof of statement (i). The proof of the sufficient part is obvious. We now prove the necessary part.

From Lemma 2.1 and the assumptions, it follows that the singularity at the origin is a hyperbolic saddle.

Using the standard stable-unstable manifolds theory, we have that system (2.1) in a neighborhood of the origin is analytically equivalent to

$$(1) \qquad \begin{cases} \dot{u} = \lambda_1 u(1 + g_1(u,v)), \\ \dot{v} = \lambda_2 v(1 + g_2(u,v)), \end{cases}$$

where λ_1 and λ_2 are the eigenvalues of system (2.1) at the origin, and g_1 and g_2 are the analytic functions without constant terms. By applying the characteristic method to compute the analytic first integrals of system (1) in a neighborhood of the hyperbolic saddle, we obtain that λ_1/λ_2 is a negative rational number. This means that the origin is a rational hyperbolic saddle.

With the help of the Poincaré-Dulac Normal Form Theorem we can prove that there is a distinguished transformation of the form

$$(2) \qquad x = u + \phi_1(u,v), \qquad y = v + \phi_2(u,v)$$

with ϕ_1 and ϕ_2 formal series without linear terms and containing the non-resonant terms only, for which system (2.1) is transformed into the following

$$(3) \qquad \begin{cases} \dot{u} = \lambda_1 u(1 + f_1(z)), \\ \dot{v} = \lambda_2 v(1 + f_2(z)), \end{cases}$$

where f_1 and f_2 are formal series in z with $z = u^m v^n$, where m and n are coprime positive integers satisfying $\lambda_1/\lambda_2 = -m/n$. Then using the majorant relations between the series (see e.g., [4]), we prove that the transformation (2) and the system (3) are analytic. This proves the necessary part.

Proof of statement (ii). To prove the necessity, we denote by X the vector field corresponding to system (2.1), and by $\widetilde{X}$ the final blowing-up vector field defined in the unbounded open domain Ω. From the Desingularization Theorem of non-elementary singularities (see e.g., [8, 9]) and the assumptions, it follows that $\widetilde{X}$ has an analytic first integral in a neighborhood of the boundary $\partial\Omega$. Since all singularities on $\partial\Omega$ are elementary, the proof follows from Lemma 2.1 and statement (i) of this lemma.

To prove the sufficiency, we note from the assumptions that there are analytic first integrals in a neighborhood of each of the singularities on $\partial\Omega$. Outside the singularities it follows from [11] that there are analytic first integrals in a neighborhood of $\partial\Omega$. Combining these two types of first integrals, we obtain a global analytic first integral in a neighborhood of $\partial\Omega$. Again using the Desingularization Theorem we have an analytic first integral of system (2.1) in a neighborhood of the origin. This completes the proof of the lemma.

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