Rehabilitating space-times with NUTs

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Abstract

We revisit the Taub-NUT solution of the Einstein equations without time periodicity condition, showing that the Misner string is still fully transparent for geodesics. In this case, analytic continuation can be carried out through both horizons leading to a Hausdorff spacetime without a central singularity, and thus geodesically complete. Furthermore, we show that, in spite of the presence of a region containing closed time-like curves, there are no closed causal *geodesics*. Thus, some longstanding obstructions to accept the Taub-NUT solution as physically relevant are removed.

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I. INTRODUCTION

The Taub-NUT solution of the vacuum Einstein equations with Lorentzian signature [1–4] remains one of the most puzzling results of General Relativity. It realizes the idea of gravitational electric-magnetic duality and is often interpreted, by analogy with the Dirac magnetic monopole, as the field of a gravitational dyon with the usual mass m and a "magnetic" mass n [5]. Astrophysicists gave tribute to this solution suggesting to explore its signatures in the microlensing data [6]. At the same time, most theorists consider it as unphysical because of the presence of a Misner string singularity on the polar axis (the gravitational analogue of the Dirac string for the magnetic monopole), with still debated features, and regions containing closed timelike curves (CTCs).

To make the string unobservable, Misner suggested to impose periodicity on the time coordinate [3] which entails, however, further serious problems. First, the space-time then contains closed time-like curves everywhere. Second, with this periodicity condition the analytically extended Taub-NUT spacetime is either geodesically incomplete [3, 4] (extension can be carried out through only one of the two horizons), or can be maximally extended to a geodesically complete but non-Hausdorff spacetime [7, 8].

Another option is to preserve causality in the large by abandoning the time periodicity condition, thereby retaining the Misner string as an unremovable singularity. It was suggested by Bonnor and others [9], that the Misner string should be interpreted as a singular material source of angular momentum. On the other hand, Miller et al. [10] have shown that the vacuum Taub-NUT spacetime (without the time periodicity condition) can be maximally extended à la Kruskal through both horizons. Because the extended spacetime presents a coordinate singularity on the the polar axis, they considered it to be geodesically incomplete [10, 11].

In this Letter, we consider motion in the Taub-NUT space without time periodicity in greater detail and show that the Misner string is fully transparent for geodesics hitting it. Since with this interpretation the analytical continuation at the horizon is not problematic and there is no central singularity, the whole extended Taub-NUT space-time turns out to be *geodesically complete*, removing the major obstruction to give it physical significance.

The other problem with the Taub-NUT solution consists in the presence of a region surrounding the Misner string containing CTCs, which are generally considered to violate causality [4]. We show that for certain values of the parameter C which was previously used to fix the location of the Misner string, these CTCs are not *causal geodesics*, and thus do not lead to causality violations for a freely falling observer.

II. THE SETUP

We start with the family of Taub-NUT spacetimes

$$ds^2 = -f(dt - 2n(\cos\theta + C)\, d\varphi)^2 + f^{-1}dr^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \qquad \rho^2 = r^2 + n^2 \quad (2.1)$$

with $f = (r^2 - 2rm - n^2)/\rho^2$, where m is the ordinary mass, n is the "magnetic" mass or NUT parameter and C is an additional parameter related to the "large" coordinate transformation $t \to t + C\varphi$. Note that C should be considered as physical rather than pure gauge parameter, since it changes the asymptotic behavior of the metric. Its introduction was often used to modify the position of the Misner string: for $C = -1$ it lies at the southern hemisphere, for $C = 1$ — at the northern, for $C = 0$ at both of them. Note also that the areal radius $(r^2 + n^2)^{1/2}$ is always finite, so the space-time has no central singularity. This is possible because the maximally analytically extended Taub-NUT spacetime [10] has two distinct regions at spacelike infinity $r \to \pm\infty$.

The metric is symmetric [3, 5, 12, 13] under time translations, generated by the Killing vector $K_t = \partial_t$, and $so(3)$ local rotations associated with $K_{(i)}$, $i = x, y, z$, which can be compactly presented as

$$\begin{aligned} K_{(\pm)} &= K_{(x)} \pm iK_{(y)} = \mathrm{e}^{\pm i\varphi}\left(\pm i\partial_\theta - \cot\theta\,\partial_\varphi - \frac{2n(1 + C\cos\theta)}{\sin\theta}\,\partial_t\right)\,, \\ K_{(z)} &= \partial_\varphi + 2nC\,\partial_t\,, \end{aligned} \quad (2.2)$$

The associated four first integrals of geodesic equations $K_{(a)\mu}\dot{x}^\mu$ ($\dot{x}^\mu = dx^\mu/d\tau$) read:

$$E = \left(\dot{t} - 2n(\cos\theta + C)\dot{\varphi}\right) f\,, \quad (2.3)$$

$$J_\pm = J_x \pm iJ_y = \left(2nE\sin\theta - \rho^2(i\dot{\theta} - \sin\theta\cos\theta\dot{\varphi})\right)\mathrm{e}^{\pm i\varphi}\,, \quad (2.4)$$

$$J_z = 2nE\cos\theta + \rho^2\sin^2\theta\dot{\varphi}\,, \quad (2.5)$$

with J_x, J_y, J_z forming a Cartesian vector $\vec{J}$. This can be decomposed into the mutually orthogonal orbital and "spin" parts [13]

$$\vec{L} + \vec{S} = \vec{J}\,, \qquad \vec{L} = \rho^2\,\hat{r}\wedge\dot{\hat{r}}\,, \qquad \vec{S} = 2nE\hat{r}\,, \quad (2.6)$$

where $\hat{r} = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$ is a unit vector normal to the two-sphere. It follows from the orthogonality of $\vec{L}$ and $\vec{S}$ that

$$\vec{J}\cdot\hat{r} = 2nE\,. \tag{2.7}$$

In the magnetic monopole case, such a first integral means that the the trajectory of the charged particle lies on the surface of a cone with axis $\vec{J}$ originating from the magnetic monopole source $r = 0$. However the Taub-NUT gravitational field has no apex for the "cone". Rather this means that the geodesic intersects all the two-spheres of radius r on the same small circle, or parallel, $\mathcal{C}$ with polar axis $\vec{J}$. Squaring (2.6) leads to

$$\vec{J}^2 = \vec{L}^2 + 4n^2E^2\,, \tag{2.8}$$

which can be rewritten as

$$\rho^4[\dot\theta^2 + \sin^2\theta\,\dot\varphi^2] = l^2\,, \tag{2.9}$$

with $l^2 = J^2 - 4n^2E^2$ (denoting $J^2 = \vec{J}^2$). Inserting this into the normalization condition $\dot{x}^\mu\dot{x}^\nu g_{\mu\nu} = \varepsilon$ ($= -1$ for timelike and 0 for null geodesics) leads to the effective radial equation

$$\dot{r}^2 + f(r)\left[\frac{l^2}{\rho^2} - \varepsilon\right] = E^2\,, \tag{2.10}$$

which is identical to the equation for radial motion in the equatorial plane for the metric (2.1) without the term $-2n\cos\theta d\varphi$.

III. MISNER STRING CROSSING

Passing to the new parameter λ on the geodesic defined by $d\tau = \rho^2 d\lambda$, and putting $\xi = \cos\theta$ we obtain

$$\left(\frac{d\xi}{d\lambda}\right)^2 = -J^2\,\xi^2 + 4nEJ_z\,\xi + (l^2 - J_z^2)\,. \tag{3.11}$$

Assuming $J^2 \neq 0$ ($J^2 = 0$ implies from (2.8) $E = 0$ and $l = 0$), Eq. (3.11) is solved (up to an additive constant to λ) by [11]

$$\cos\theta = J^{-2}\left[2nEJ_z + lJ_\perp\cos(J\lambda)\right] = \cos\psi\cos\eta + \sin\psi\sin\eta\cos(J\lambda)\,, \tag{3.12}$$

where $J_\perp^2 = J^2 - J_z^2\,, \tan\eta = l/2nE,\ \tan\psi = J_\perp/J_z$. Eq. (3.11) has two turning points $\theta_\pm$ such that

$$\cos\theta_\pm = J^{-2}\,(2nEJ_z \pm lJ_\perp) = \cos(\psi \mp \eta)\,. \tag{3.13}$$

It follows that the trajectory crosses periodically the Misner string, $\cos\theta_\pm = \pm 1$ only if

$$J_z = 2nE\,, \quad \text{or} \quad J_z = -2nE\,. \tag{3.14}$$

The only geodesics which can cross both components of the Misner string are those with $\eta = \pi/2$ ($E = J_z = 0$), leading to $\dot{t} = \dot{\varphi} = 0$; according to (2.10), in the stationary sector ($f(r) > 0$) these can only be spacelike geodesics. The trajectory can also stay on the Misner string component $\theta = 0$ or π if (3.14)) is satisfied with $2nE = \pm J$.

The differential equation (2.5) for φ can be rewritten as

$$\frac{d\varphi}{d\lambda} = \frac{1}{2}\left[\frac{J_z - 2nE}{1-\cos\theta(\lambda)} + \frac{J_z + 2nE}{1+\cos\theta(\lambda)}\right], \tag{3.15}$$

with $\cos\theta(\lambda)$ given by (3.12). This is solved by [11]

$$\varphi - \varphi_0 = \arctan\left[\frac{\cos\psi - \cos\eta}{1-\cos(\psi-\eta)}\tan\frac{J\lambda}{2}\right] + \arctan\left[\frac{\cos\psi + \cos\eta}{1+\cos(\psi-\eta)}\tan\frac{J\lambda}{2}\right]. \tag{3.16}$$

For trajectories crossing the North Misner string, with $J_z = 2nE$, this reduces to

$$\varphi - \varphi_1 = \arctan\left(\cos\eta\,\tan\left(\frac{J\lambda}{2}\right)\right), \tag{3.17}$$

with $\eta = \arcsin l/J$, $\varphi_1 = \varphi_0 - \mathrm{sgn}(\tan(J\lambda/2))\pi/2$. A similar formula applies in the case of the South Misner string, with η replaced by $\pi - \eta$ and $J\lambda$ replaced by $J\lambda - \pi$ (note that according to (3.12) the North Misner string is crossed for $\lambda = 2k\pi/J$, while the South Misner string is crossed for $\lambda = (2k+1)\pi/J$, k integer). In the case e.g. of the North Misner string, this gives on account of (3.12),

$$\cos(\varphi - \varphi_1) = \frac{J_z}{J_\perp}\tan\left(\frac{\theta}{2}\right), \tag{3.18}$$

consistent with (2.7) (the choice $\varphi_1 = 0$ in (3.18) corresponds to the choice $\vec{J} = (J_\perp, 0, J_z)$ in (2.7)). Clearly the Misner string is completely transparent to the geodesic motion!

When the parameter λ varies over a period, e.g. $\lambda \in [-\pi/J,\, \pi/J]$, the argument of the first or second arctan in (3.16) varies from $-\infty$ to $+\infty$ for $J_z \mp 2nE > 0$, and from $+\infty$ to $-\infty$ for $J_z \mp 2nE < 0$. It is identically zero for $J_z \mp 2nE = 0$. Accordingly, the variation of φ over a period is

$$\Delta\varphi = \pi\left[\mathrm{sgn}(J_z - 2nE) + \mathrm{sgn}(J_z + 2nE)\right]. \tag{3.19}$$

This means that for $J_z^2 > 4n^2E^2$ ($|\Delta\varphi| = 2\pi$) the parallel $\mathcal{C}$ circles the North-South polar axis, i.e. the Misner string. For $J_z^2 < 4n^2E^2$, ($|\Delta\varphi| = 0$) $\mathcal{C}$ does not circle the Misner string. And for $J_z = \pm 2nE$ ($|\Delta\varphi| = \pi$), $\mathcal{C}$ goes through the North or South pole, as discussed above.

IV. ABSENCE OF CLOSED CAUSAL GEODESICS

The ADM form of the metric (2.1) is

$$ds^2 = -\frac{f\rho^2 \sin^2\theta}{\Sigma}\, dt^2 + f^{-1}dr^2 + \rho^2 d\theta^2 + \Sigma\left(d\varphi + \frac{2nf(\cos\theta + C)}{\Sigma}\, dt\right)^2 , \qquad (4.20)$$

with $\Sigma(r,\theta) = \rho^2\sin^2\theta - 4n^2 f(\cos\theta + C)^2$. For $f(r) < 0$, Σ is positive definite, while for $f(r) > 0$ (outside the horizon), which which assume further, Σ becomes negative, and closed timelike curves (CTCs) appear, in a neighborhood of the Misner string given by $\Sigma(r,\theta) < 0$. The surface $\Sigma(r,\theta) = 0$ bounding this CTC neighborhood is a causal singularity of the spacetime, where the signature of the spacetime changes from $(-+++)$ outside to $(+++-)$ inside. This singularity is, just as the Misner string itself, completely transparent to geodesic motion. Nevertheless, the occurrence of CTCs in a spacetime is usually considered to violate causality [4]. An observer travelling around such a CTC would eventually return to his original spacetime position after a finite proper time lapse, thus opening the possibility for time travel. However, unless this observer is freely falling, such a CTC travel would necessarily involve accelerations generated e.g. by rocket engines. One can argue that the back-reaction of these matter accelerations on the spacetime geometry would deform it in such a way that chronology would ultimately be preserved. If this reasoning is correct, causality violation can only occur in spacetimes with closed timelike geodesics (CTGs), or possibly closed null geodesics (CNGs). We now show that there are no closed timelike or null geodesics in the Taub-NUT spacetime with $|C| \le 1$.

Combining the Eqs. (2.3,2.5) and passing to λ-parametrization one is led to split $t(\lambda) = t_r(\lambda) + t_\theta(\lambda)$ satisfying

$$\frac{dt_\theta}{d\lambda} = 4n^2E + n\left[\frac{(C+1)(J_z - 2nE)}{1-\cos\theta(\lambda)} + \frac{(C-1)(J_z + 2nE)}{1+\cos\theta(\lambda)}\right] , \qquad (4.21)$$

$$\frac{dt_r}{d\lambda} = E\frac{\rho^2}{f(r)} , \qquad (4.22)$$

with $\cos\theta(\lambda)$ given by (3.12). The explicit solution to equation (4.21) is [11]

$$t_\theta(\lambda) = 4n^2E\lambda + 2n(C+1)\arctan\left[\frac{\cos\psi - \cos\eta}{1-\cos(\psi-\eta)}\tan\frac{J\lambda}{2}\right] + 2n(C-1)\arctan\left[\frac{\cos\psi + \cos\eta}{1+\cos(\psi-\eta)}\tan\frac{J\lambda}{2}\right] , \qquad (4.23)$$

in the interval $-\pi/J < \lambda < \pi/J$. The resulting variation of t_θ over a period $2\pi/J$ of λ is

$$\Delta t_\theta = 2\pi n \left[\frac{4nE}{J} + (C+1)\mathrm{sgn}(J_z - 2nE) + (C-1)\mathrm{sgn}(J_z + 2nE)\right] . \tag{4.24}$$

During a period $2\pi/J$ of the angular motion,

$$\Delta t_\theta \geq 4\pi n \left(\frac{2nE}{J} - 1\right) \tag{4.25}$$

for $|C| \leq 1$. Also, from (4.22) and (2.10),

$$\frac{dt_r}{d\lambda} = \frac{E\rho^2}{f(r)} \geq E^{-1}[l^2 - \varepsilon\rho^2] \geq E^{-1}[l^2 - \varepsilon n^2] \tag{4.26}$$

in the stationary sector $f(r) > 0$. This leads to

$$\Delta t_r \geq \frac{2\pi}{EJ}\left[l^2 - \varepsilon n^2\right] , \tag{4.27}$$

over the same period. Adding the two together, we obtain

$$\Delta t = \Delta t_r + \Delta t_\theta \geq \frac{2\pi}{E}\left[J - 2nE - \varepsilon n^2/J\right] . \tag{4.28}$$

For $\varepsilon = -1$ this is clearly positive definite. For $\varepsilon = 0$, this can vanish only for $2nE = J$ ($l = 0$). But in this case $\Delta t_\theta \geq 0$, while $dt_r/d\lambda$, and thus also Δt_r, is positive definite. Thus, for $|C| \leq 1$ all timelike or null geodesics which stay in the stationary sectors $r > r_h$ are causal (future directed).

The above reasoning fails for $|C| > 1$, in which case the lower bound (4.25) is replaced by $\Delta t_\theta \geq 4\pi n(2nE/J - |C|)$. One can show [14] that, for any parameter set (m, n), one can find a value of C such that there are CNGs (and, presumably, CTGs for larger values of $|C|$). For instance, for $m = 0$ and $C = -\sqrt{3}$, the circle $t =$ const., $r = \sqrt{3}n$, $\theta = \arctan\sqrt{2}$ is a null geodesic.

V. CONCLUSION

We have shown that, contrary to longstanding prejudice (for a recent discussion see [11]), Taub-NUT space-time without periodic identification of time is geodesically complete. This is valid for the whole family of metrics with arbitrary C, and presumably can be extended to other spacetimes with NUT parameter. Moreover, in spite of the presence of a region where the azimuthal coordinate is timelike and the temporal coordinate spacelike, there are for $|C| \leq 1$ no closed timelike or null *geodesics* which could violate causality.

We realize that our results do not remove all objections against physical attribution of the metric with NUTs to the real world, in particular, we do not consider quantum effects [5, 12]. Still, we hope that our findings remove some important obstructions to recognition of these spacetimes as physically relevant and will stimulate further work in this direction.

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