A Coq-based Implementation of Tarski's Mereogeometry

Richard Dapoigny[1] Patrick Barlatier[1]

[1]LISTIC/Polytech'Savoie
University of Savoie, Annecy, (FRANCE)

Contents

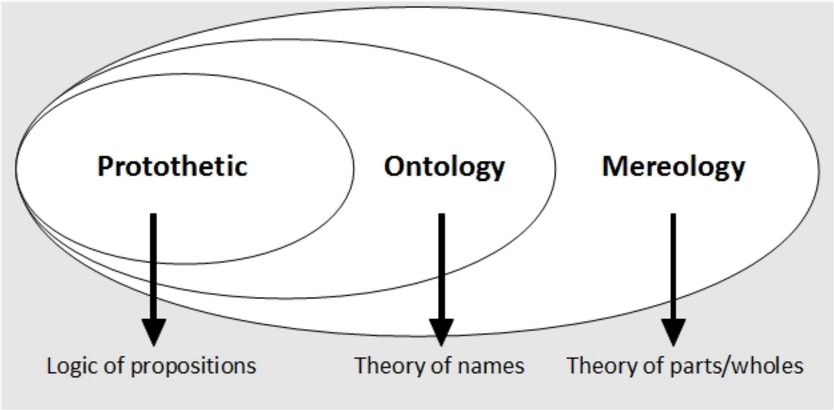
About Mereogeometry

- In Qualitative Spatial Reasoning (QSR), recent papers focus on mereogeometry [Tarski56].
- Mereogeometry seen as an alternative to mereotopology for modeling spatial regions.
- But, Tarki's mereogeometry was lacking a full formalization which would render it more useful for further developments.
- Several author recently suggest a fully formal system based either on First Order Logic [Borgo96] or set theory [Gruszc08], or using parthood together with a sphere predicate [Bennet01]

Some issues

- The mereology that is involved in these papers is a minor part of Leśniewski's systems.
- The logical part is reduced to First-Order Logic and loses its original expressiveness.
- Unlike usual set-based systems which are limited to distributive classes, in Lesniewski's systems collective and distributive classes coexist.
- A part of Leśniewski's systems (mereology) is often mixed with set theory and yields compatibility problems.
- Many approaches to mereogeometry are theoretical and do not provide implemented systems
- The contribution of the present work will address all these issues.

Lesniewski's systems



Lesniewski's systems

1. Protothetic: a higher-order propositional calculus based on an equivalence relation with three axioms.
2. Ontology: based on the "is" (ε) relation whose properties are described with a single axiom, it includes a lot of definitions and theorems.
3. Mereology: the theory of parts and wholes using four axioms (a counterpart of set theory).

Problems

The quasi-totality of papers working on mereology rely on set theory and lose many benefits of Leśniewski's theory.
Protothetic as an exotic formalism is very difficult to assess.

Protothetic

- Introduction of a category for propositions: S.
- Five meta-rules: detachment, substitution, distribution of quantifiers, definitions and the extensionality rule:

$$\forall p\, q\, f, (p \equiv q) \;\equiv\; (f(p) \equiv f(q))$$

- All usual connectives (implication, conjunction, disjunction) defined from equivalence and negation (Tarski's thesis).
- Protothetic has an algebraic correspondence with groupoids.
- Using the Coq theorem prover we are able to:
  - the Coq setoid will account to the groupoid (able to express extensionality, the counterpart of the algebraic composition law),
  - mapping propositions to the sort *Prop*,
  - express meta-rules of protothetic in a more usable way.

Ontology

- Introduction of a category for names: N
- Two subcategories, singular names which denote objects and plurals which correspond to sets of objects (e.g., $C_1 \varepsilon \mathit{circle}$)[1]
- A single axiom introducing names.
- An extended definition structure:

$$\forall A v_1 \ldots v_n, (A \,\varepsilon\, \psi(v_1, \ldots v_n) \equiv A \,\varepsilon\, A \wedge \theta)$$

[1] The left argument of ε is always a singular.

Ontology

- A set of definitions is provided (strong equality, weak equality, weak inclusion, strong inclusion, ...)
- No classes are defined in the ontology, but there are plurals (abstraction).
- Is there a way to get subsumption? Yes, weak (or strong inclusion)

$$\forall A\, a\, b, (weak\_inclusion(a, b) \equiv A \,\varepsilon\, a \supset A \,\varepsilon\, b)$$

means: every *a* is (a)*b*

Ontology

- Strong inclusion:

$$\forall a\, b, (strong\_inclusion(a, b) \;\equiv \sim \forall A, \sim (A \,\varepsilon\, a) \;\wedge\; \forall C, \;(C \,\varepsilon\, a \;\supset\; C \,\varepsilon\, b))$$

- Since *a* and *b* are plurals, they describe abstract names.
- If *a* and *b* denote respectively square and quadrilateral, then the above definition assumes that for all, "one of the square" and "one of the quadrilateral", square is strongly included in quadrilateral is equivalent to: for some object o_i, o_i is one of the quadrilaterals and for all o_i, if o_i is one of square then, o_i is one of the quadrilaterals.

Mereology

- Introduces three functors *pt* (part-of[1]), *el* (element-of[2]) and *Kl* (class in the collective sense).
- Two axioms over *pt*, stand resp. for asymmetry and transitivity

$$\forall A\, B,\ A\,\varepsilon\,(pt\ B) \quad \supset \quad B\,\varepsilon\,distinct(pt\ A)$$
$$\forall A\, B\, C,\ A\,\varepsilon\ \wedge\ (pt\ B)\ B\,\varepsilon\,(pt\ C) \quad \supset \quad A\,\varepsilon\,(pt\ C)$$

- A definition for the *el* functor:

$$\forall A\, B,\ A\,\varepsilon\,(el\ B)\ \equiv\ A\,\varepsilon\,A\ \wedge\ singular\_equality A\, B\ \vee\ A\,\varepsilon\,(pt\ B)$$

[1] a.k.a. proper-part-of in usual ontologies
[2] a.k.a. part-of in usual ontologies

Mereology

- A definition for the *Kl* functor:

$$\begin{aligned}\forall A\, a, (A \,\varepsilon\, (Kl\; a) \;\equiv A \,\varepsilon\, A \wedge \sim \forall B, \sim (B \,\varepsilon\, a) \wedge \\ \forall B, (B \,\varepsilon\, a \;\supset\; B \,\varepsilon\, (el\; A)) \wedge \\ \forall B, (B \,\varepsilon\, (el\; A)) \;\supset \sim \forall CD, \sim (C \,\varepsilon\, a \;\wedge\; D \,\varepsilon\, (el\; C) \wedge \\ D \,\varepsilon\, (el\; B))\end{aligned}$$

- Two axioms for the *Kl* functor, resp. uniqueness and existence:

$$\forall A\, B\, a,\; (A \,\varepsilon\, (Kl\; a) \;\wedge\; B \,\varepsilon\, (Kl\; a) \quad\supset\quad singular\_equality\; A\; B)$$
$$\forall A\, a,\; (A \,\varepsilon\, a \quad\supset\quad \sim \forall B\; \sim (B \,\varepsilon\, (Kl\; a)))$$

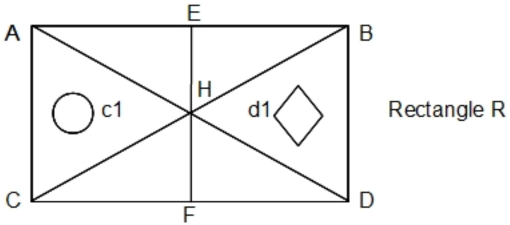
Mereology

Consequences:

- The theory is sound and complete [Clay68, Lejewski69]
- Many theorems are defined on the basis of the previous axioms and definitions.
- Mereology can be extended with appropriate definitions without compromising its soundness and completeness, e.g., with the *col* functor (collection):

$$\forall A\, a, \quad (A\,\varepsilon\,(col\ a) \;\equiv\; A\,\varepsilon\,A \;\wedge \forall B,\; B\,\varepsilon\,(el\ A) \;\supset \sim \forall CD, \\ \sim (C\,\varepsilon\,a \;\wedge\; D\,\varepsilon\,(el\ C) \;\wedge\; D\,\varepsilon\,(el\ B) \;\wedge\; C\,\varepsilon\,(el\ A)))$$

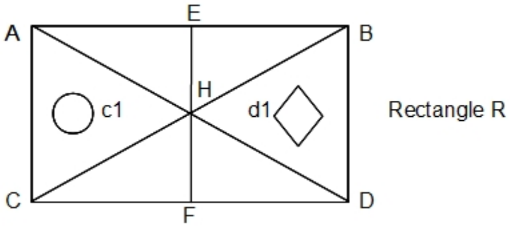
Example



$$\forall Aa,(A\varepsilon(Kl\,a)\equiv A\varepsilon A\wedge\sim\forall B,\sim(B\varepsilon a)\wedge\forall B,(B\varepsilon a\supset B\varepsilon(el\,A))\wedge\forall B,(B\varepsilon(el\,A))\supset\sim\forall CD,\sim(C\varepsilon a\wedge D\varepsilon(el\,C)\wedge D\varepsilon(el\,B))$$

- A rectangle labeled *R* includes geometric parts where *a* describes the squares of *R*: $A\,\varepsilon\,Kl(\textit{square of }R)$
- The name *A* denotes an object, i.e., rectangle *R*.
- For some *B*, $B\,\varepsilon$ (*square of R*) (e.g., *AEFC*).
- Any square of *R* is an element of *A* e.g., $EBDF\,\varepsilon\,el\,A$.

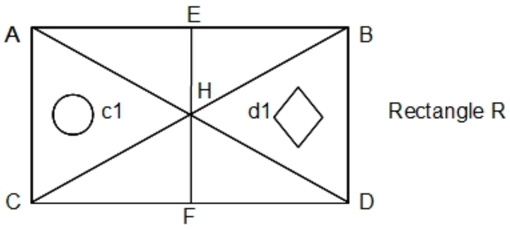
Example



$$\forall Aa,(A\varepsilon(Kl a)\equiv A\varepsilon A\wedge\sim\forall B,\sim(B\varepsilon a)\wedge\forall B,(B\varepsilon a\supset B\varepsilon(el A))\wedge\forall B,(B\varepsilon(el A))\supset\sim\forall CD,\sim(C\varepsilon a\wedge D\varepsilon(el C)\wedge D\varepsilon(el B))$$

- for any element B of R, e.g., the triangle BHD, an object C which is a square of R should exists ($EBFD$), while another object D should also exist as both an element of C and an element of B. If C denotes $EBFD$ and D, the diamond $d1$, then we easily show that $d1$ is both element of $EBFD$ and BHD.

Example



$\forall Aa,(A\varepsilon(Kla)\equiv A\varepsilon A\wedge\sim\forall B,\sim(B\varepsilon a)\wedge\forall B,(B\varepsilon a\supset B\varepsilon(elA))\wedge\forall B,(B\varepsilon(elA))\supset\sim\forall CD,\sim(C\varepsilon a\wedge D\varepsilon(elC)\wedge D\varepsilon(elB))$

- Let us now define *B* as the collective object including *c*1 (a circle) and *BHD*. Then, there exists *C* which is a square of *R* e.g., *EBFD* and *D* which is both element of *C* and *B*, e.g., the diamond *d*1.

Motivations for Mereogeometry

- The expressive power of mereogeometries exceeds that of mereotopological theories which are limited to describe topological properties [Borgo08].
- Ability of mereogeometry to reconstruct points from region-based primitives.
- A number of papers devoted to this area has regularly increased since the 90's (e.g.,[Bennet01, Gruszc08, Borgo96]).
- Mereogeometry allows in many cases a direct mapping from empirical entities and laws to theoretical entities and formulas.

Tarski's Mereogeometry

Many mereogeometries have been investigated, but they are shown to be sub-systems of Tarski's mereogeometry [Borgo08]. Following a suggestion of Leśniewski, Tarski has represented geometry without points in a theory called mereogeometry. Assumptions:

1. Mereogeometry relies on Leśniewski's mereology.
2. A single primitive notion: *sphere* a.k.a. *ball*.
3. Nine definitions including *point* and *solid*.
4. Four axioms.

It is shown that mereogeometry has a model in ordinary three-dimensional Euclidian geometry (Theorem B).

Clarifying the framework

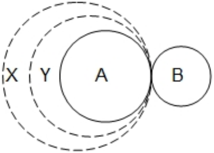
Assumptions:

- Mereogeometry is an application of Leśniewski's mereology.
- *ball* is a primitive which has the status of a plural.
- *sum* (in definition 8) refers to the concept of collection (*col*).
- The term class is not understood as the Leśniewski's class (*Kl*) but rather as a class in the Whitehead and Russell style (i.e., a plural).
- A series of notions is expressed as definitions in the Leśniewskian's style.

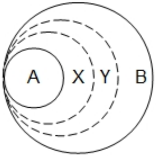
Definitions

- Externally Tangent (ET)
- Internally Tangent (IT)
- Externally Diametrically Tangent (EDT)
- Internally Diametrically Tangent (IDT)
- Concentric (CON)
- Point (POINT)
- Equidistant (EQUID)
- Solid (TarskiD8)
- Interior Point (IPOINT)

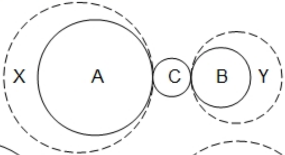
Definitions



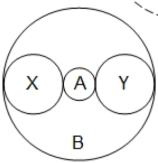
Ball *A* externally tangent to ball *B*



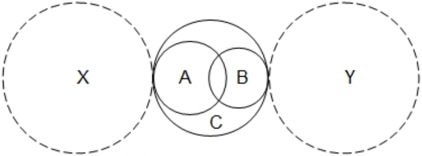
Ball *A* internally tangent to ball *B*



Balls *A* and *B* externally diametrically tangent to ball *C*



Ball *A* is concentric with ball *B*



Ball *A* and *B* internally diametrically tangent to ball *C*

Using the Coq Language

- Coq is used as a meta-language for expressing theories.
- It has a large active user base.
- An expressive higher-order type system.
- Types in Coq naturally fit all categories of mereology ($Prop \to Prop \to Prop$ for equivalence, $N \to N$ for *pt*, $N \to N \to Prop$ for ε, etc.).
- Ability to express setoids (set + equivalence) with rewriting tactics.
- Semi-automated reasoning with tactics and *Ltac*.
- Each term is typed: a formula has type *Prop* and a term having as type this formula is a proof for it.

Typing the Universe of Discourse in Coq

It consists first in assigning types to functors:

`Parameter` *epsilon* : $N \rightarrow N \rightarrow Prop$.
`Parameter` *coll* : $N \rightarrow N$.
...
`Parameter` *balls* : N.
`Parameter` *solids* : N.
`Parameter` *et* : $N \rightarrow N$.
`Parameter` *it* : $N \rightarrow N$.
`Parameter` *edt* : $N \rightarrow N \rightarrow N$.
`Parameter` *idt* : $N \rightarrow N \rightarrow N$.
`Parameter` *con* : $N \rightarrow N$.
`Parameter` *point* : $N \rightarrow N$.
`Parameter` *equid* : $N \rightarrow N \rightarrow N$.
`Parameter` *ipoint* : $N \rightarrow N$.

Expressing Definitions in Coq

$\texttt{Parameter}\ ET : \forall\ A\ B,\ A\ \varepsilon\ et\ B \equiv\equiv ((A\ \varepsilon\ balls) \wedge (B\ \varepsilon\ balls) \wedge (A\ \varepsilon\ ext\ B) \wedge$
$\forall\ X\ Y, ((X\ \varepsilon\ balls) \wedge (Y\ \varepsilon\ balls) \wedge (A \leqslant X \wedge X\ \varepsilon\ ext\ B) \wedge$
$(A \leqslant Y \wedge Y\ \varepsilon\ ext\ B)) \supset (X \leqslant Y \vee Y \leqslant X)).$

$\texttt{Parameter}\ IT : \forall\ A\ B,\ A\ \varepsilon\ it\ B \equiv\equiv ((A\ \varepsilon\ balls) \wedge (B\ \varepsilon\ balls) \wedge (A < B) \wedge$
$\forall\ X\ Y, ((X\ \varepsilon\ balls) \wedge (Y\ \varepsilon\ balls) \wedge (A \leqslant X \wedge X \leqslant B) \wedge$
$(A \leqslant Y \wedge Y \leqslant B)) \supset (X \leqslant Y \vee Y \leqslant X)).$

$\texttt{Parameter}\ EDT : \forall\ A\ B\ C,\ A\ \varepsilon\ edt\ B\ C \equiv\equiv ((A\ \varepsilon\ balls) \wedge (B\ \varepsilon\ balls) \wedge$
$(C\ \varepsilon\ balls) \wedge (B\ \varepsilon\ et\ A) \wedge (C\ \varepsilon\ et\ A) \wedge \forall\ X\ Y, ((X\ \varepsilon\ balls) \wedge$
$(Y\ \varepsilon\ balls) \wedge (B \leqslant X \wedge X\ \varepsilon\ ext\ A) \wedge (C \leqslant Y \wedge Y\ \varepsilon\ ext\ A)) \supset (X\ \varepsilon\ ext\ Y)).$

$\texttt{Parameter}\ IDT : \forall\ A\ B\ C,\ A\ \varepsilon idt\ B\ C \equiv\equiv ((A\ \varepsilon\ balls) \wedge (B\ \varepsilon\ balls) \wedge (C\ \varepsilon\ balls)$
$\wedge (B\ \varepsilon\ it\ A) \wedge (C\ \varepsilon\ it\ A) \wedge \forall\ X\ Y, ((((\ X\ \varepsilon\ balls) \wedge (Y\ \varepsilon\ balls)$
$\wedge (X\ \varepsilon\ ext\ A) \wedge (Y\ \varepsilon\ ext\ A) \wedge (B\ \varepsilon\ ext\ X) \wedge (C\ \varepsilon\ ext\ Y)) \supset (X\ \varepsilon\ ext\ Y)).$

Expressing Definitions in Coq

`Parameter` $CON : \forall\ A\ B,\ A\ \varepsilon\ con\ B \equiv\equiv ((A\ \varepsilon\ balls) \wedge (B\ \varepsilon\ balls)) \wedge singular\_equality\ A\ B \vee (A < B \wedge \forall\ X\ Y, ((X\ \varepsilon\ balls) \wedge (Y\ \varepsilon\ balls) \wedge (A\ \varepsilon\ edt\ X\ Y) \wedge (X\ \varepsilon\ it\ B) \wedge (Y\ \varepsilon\ it\ B)) \rightarrow (B\ \varepsilon\ idt\ X\ Y)) \vee (B < A \wedge \forall\ X\ Y, (X\ \varepsilon\ balls) \wedge (Y\ \varepsilon\ balls) \wedge (B\ \varepsilon\ edt\ X\ Y) \wedge (X\ \varepsilon\ it\ A) \wedge (Y\ \varepsilon\ it\ A) \supset (A\ \varepsilon\ idt\ X\ Y))).$

`Parameter` $POINT : \forall\ P\ B,\ P\ \varepsilon\ (point\ B) \equiv\equiv ((P\ \varepsilon\ P) \wedge (B\ \varepsilon\ balls) \wedge \forall\ B', (B'\ \varepsilon\ balls) \wedge B'\ con\ B).$

`Parameter` $EQUID : \forall\ A\ B\ C,\ A\ \varepsilon\ equid\ B\ C \equiv\equiv ((A\ \varepsilon\ balls) \wedge (B\ \varepsilon\ balls) \wedge (C\ \varepsilon\ balls) \wedge \neg\forall\ X, \neg((X\ \varepsilon\ balls) \wedge (X\ \varepsilon\ con\ A) \wedge \forall\ Y, \neg((Y\ \varepsilon\ balls) \wedge Y\ \varepsilon\ (union\ B\ C) \wedge (Y \leqslant X) \vee (Y\ \varepsilon\ ext\ X)))).$

`Parameter` $TarskiD8 : \forall\ A,\ A\ \varepsilon\ solids \equiv\equiv \neg\forall\ B, \neg(B\ \varepsilon\ B \wedge (B\ \varepsilon\ coll\ balls) \wedge (A\ \varepsilon\ subcoll\ B)).$

`Parameter` $IPOINT : \forall\ P\ X\ C,\ P\ \varepsilon\ (ipoint\ X) \equiv\equiv (X\ \varepsilon\ solids \wedge P\ \varepsilon\ (point\ C) \wedge \neg\forall\ A', \neg((A'\ \varepsilon\ balls) \wedge (A'\ \varepsilon\ P) \wedge (A' \leqslant X))).$

The Axiom System

Axioms broadly separated into external (to the theory) and internal axioms:

i External axioms which include:
   - an axiom stating the existence of a correspondence between notions of the geometry of solids and notions of ordinary point geometry,
   - two axioms establishing a correspondence between notions of the geometry of solids and topology.

ii Internal axioms that are derivable from Leśniewski's mereology.

The Axiom System

External axioms:

Axiom 1

The notions of point and equidistance of two points to a third satisfy all axioms of ordinary Euclidean geometry of three dimensions.

- $\Rightarrow$ points as they are introduced in definition 6 (*POINT*) correspond to points of an ordinary point-based geometry
- $\Rightarrow$ the relation *EQUID* corresponds to an ordinary equidistance relation

The Axiom System

Axiom 2

If *A* is a solid, the class α of all interior points of *A* is a non-empty regular open set.

Axiom 3

If the class α of points is a non-empty regular open set, there exists a solid *A* such that α is the class of all its interior points.

Consistency of these axioms has been analyzed in [Gruszc08].

Internal axioms

Axiom 4

If *A* is a ball and *B* a part of *A*, there exists a ball *C* which is a part of *B*.

Axiom 5

If *A* is a solid and *B* a part of *A*, then *B* is also a solid.

Using Lemmas from Leśniewski's ontology and from mereology, these axioms can be turned to theorems.

Coq demonstrations with tactics

```
Theorem TA4 : ∀ A B, (A ε balls ∧ B ε el A) ⊃ ¬∀ C, ¬(C ε balls ⊃ C ε el B).
Proof.
    intros A B;elimP.
    apply_Lem_in_goal (intro_ex_m2 B).
    decompose_and H.
    assert (H1:=H0);apply_Lem_in_hyp (OntoT5 A balls) H0.
    apply_Lem_in_hyp (MereoT49 A balls ) H1.
    apply_Lem_in_hyp (MereoT43 A B) H.
    assert (H3:(B ε solids)); [
        elim_def1 TarskiD8 B;apply_Lem_in_goal (intro_ex_m1 B);exists A;splitP;assumption |
        clear H H0 H1;assert (H1:=H3);apply_Lem_in_hyp (OntoT5 B solids) H1;apply_def_in_hyp
                                                                        (TarskiD8 B) H3;
        apply_Lem_in_hyp (elim_ex_m1 B) H;elim_for_some H C;decompose_and H;
        apply_def_in_hyp (MD3 C balls) H0;decompose_and H4;clear H2;apply_Lem_in_hyp
                                                                        (MereoT42 C B) H;
        specialize (H4 B);apply_Lem_in_hyp H4 H;clear H4;apply_Lem_in_hyp (elim_ex_coll2 balls C
                                                                                   B) H;
        elim H;intros E H4;clear H;elim H4;intros F H2; decompose_and H2;exists F;elimP;
        assumption
    ].
Qed.
```

Coq demonstrations with tactics

```
Theorem TA4' : ∀ A B, (A ε solids ∧ B ε el A) ⊃ B ε solids.
Proof.
   intros A B;elimP.
   decompose_and H.
   apply_def_in_hyp (TarskiD8 A) H0.
   apply_Lem_in_hyp (elim_ex_m1 A) H1.
   elim_for_some H1 C.
   decompose_and H2.
   apply_Lem_in_hyp (MereoT43 A B) H.
   elim_def1 TarskiD8 B.
   apply_Lem_in_goal (intro_ex_m1 B).
   exists C.
   decompose_and H1.
   splitP;[ assumption |
      assumption |
      apply_Lem_in_goal (Transitive_subcoll B AC);splitP;assumption
   ].
Qed.
```

There is a sixth axiom not detailed here, which is merely an alternative of axioms 4 and 5.

What do we gain?

- We have provided a more solid conceptual framework for qualitative spatial reasoning by moving some axioms into theorems.
- It is fully formalized and computer verified.
- We have developed a library (including a hundred of lemmas) with which users can more easily gain access to mereogeometry.
- Leśniewski's theory is an expressive alternative for supporting spatial reasoning.
- As an object language, it may also have potential for new treatments of constancy and change over time.

Future works

- Extend the number of definitions and lemmas.
- Express a request as a lemma and solve it with Coq (like higher-order unification).
- We are working on a tactic *autoMereogeo* able to prove a goal with a decision algorithm.
- Further objective: write an inference algorithm for more automatization in spatial reasoning.
- Much remains to do!

Thanks for your attention.
Any question?

[Bennet01] B. Bennett, *A categorical axiomatisation of region-based geometry*. Fundamenta Informaticae 46(1-2), 145–158, (2001)

[Borgo96] S. Borgo, N. Guarino and C. Masolo, *A pointless theory of space based on strong congruence and connection.* Procs. of 5th Int. Conf. on Principle of Knowledge Representation and Reasoning (KR'96), Morgan Kaufmann, 220–229, (1996)

[Borgo08] S. Borgo and C. Masolo, Full Mereogeometries, the review of symbolic logic, 1(1), 1–46, (2008)

[Clay68] R.E. Clay, *The consistency of Leśniewski's mereology relative to the real number system*. The Journal of Symbolic Logic, 33, 251–257, (1968)

[Gruszc08] R. Gruszczyǹski and R. Pietruszczak, *Full Development of Tarski's Geometry of Solids*. The Bulletin of Symbolic Logic 14(4), 481–540, (2008) [Henkin63] L. Henkin, A theory of propositional types, Fundamenta Mathematicae 52, 323–334 (1963).

[Lejewski69] C. Lejewski, *Consistency of Leśniewski's Mereology*. Journal of Symbolic Logic, 34 (3), 321–328, (1969)

[Tarski56] A. Tarski, *Foundations of the geometry of solids* in Logics, Semantics, Metamathematics. Papers from 1923-1938 by Alfred Tarski, Clarendon Press, (1956)