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A combinatorial discussion on finite dimensional Leavitt path algebras

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Abstract

Any finite dimensional semisimple algebra A over a field K is isomorphic to a direct sum of finite dimensional full matrix rings over suitable division rings. We shall consider the direct sum of finite dimensional full matrix rings over a field K. All such finite dimensional semisimple algebras arise as finite dimensional Leavitt path algebras. For this specific finite dimensional semisimple algebra A over a field K, we define a uniquely determined specific graph - called a truncated tree associated with A - whose Leavitt path algebra is isomorphic to A. We define an algebraic invariant $\kappa(A)$ for A and count the number of isomorphism classes of Leavitt path algebras with the same fixed value of $\kappa(A)$. Moreover, we find the maximum and the minimum K-dimensions of the Leavitt path algebras of possible trees with a given number of vertices and we also determine the number of distinct Leavitt path algebras of line graphs with a given number of vertices.

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1. Introduction

By the well-known Wedderburn-Artin Theorem [4], any finite dimensional semisimple algebra A over a field K is isomorphic to a direct sum of finite dimensional full matrix rings over suitable division rings. We shall consider the direct sum of finite dimensional full matrix rings over a field K. All such finite dimensional semisimple algebras arise as finite dimensional Leavitt path algebras as studied in [2]. The Leavitt path algebras are introduced independently by Abrams-Aranda Pino in [1] and by Ara-Moreno-Pardo in [3] via different approaches.

In general, the Leavitt path algebra $L_K(E_1)$ can be isomorphic to the Leavitt path algebra $L_K(E_2)$ for non-isomorphic graphs E_1 and E_2. In this paper, we introduce a class of specific graphs which we call the class of truncated trees, denoted by $\mathcal{T}$, and prove that for any finite acyclic graph E there exists a unique element F in $\mathcal{T}$ such that $L_K(E)$ is isomorphic to $L_K(F)$. Furthermore, for any two acyclic graphs E_1 and E_2 and their corresponding truncated trees F_1 and F_2 we have

$$L_K(E_1) \cong L_K(E_2) \text{ if and only if } F_1 \cong F_2.$$

For a given finite dimensional Leavitt path algebra $A = \bigoplus_{i=1}^{s} M_{n_i}(K)$ with $2 \leq n_1 \leq n_2 \leq \ldots \leq n_s = N$, the number s is the number of minimal ideals of A and N^2 is the maximum of the dimensions of the minimal ideals. Therefore, the integer $s + N - 1$ is an algebraic invariant of A which we denote by $\kappa(A)$.

Then, we prove that the number of isomorphism classes of finite dimensional Leavitt path algebras A, with the invariant $\kappa(A) > 1$, having no ideals isomorphic to K is equal to the number of distinct truncated trees with $\kappa(A)$ vertices. The number of distinct truncated trees with m vertices is computed in Proposition 3.4.

We also compute the best upper and lower bounds of the K-dimension of possible trees on m vertices, as a function of m and the number of sinks.

In the last section, we calculated the number of isomorphism classes of Leavitt path algebras of line graphs with m vertices as a function of m.

2. Preliminaries

We start by recalling the definitions of a path algebra and a Leavitt path algebra. For a more detailed discussion see [1]. A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements E^0 and E^1 are called *vertices* and *edges*, respectively. For each $e \in E^0$, $s(e)$ is the source of e and $r(e)$ is the range of e. If $s(e) = v$ and $r(e) = w$, then v is said to emit e and w is said to receive e. A vertex which does not receive any edges is called a *source,* and a vertex which emits no edges is called a *sink*. An *isolated* vertex is both a sink and a source. A graph is *row-finite* if $s^{-1}(v)$ is a finite set for each vertex v. A row-finite graph is *finite* if E^0 is a finite set.

A *path* in a graph E is a sequence of edges $\mu = e_1 \ldots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. The *source* of μ and the *range* of μ are defined as $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$ respectively. The number of edges in a path μ is called the *length* of μ, denoted by $l(\mu)$. If $s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle*. A graph E is called *acyclic* if E does not have any cycles.

The *total-degree* of the vertex v is the number of edges that either have v as its source or as its range, that is, $totdeg(v) = \left|s^{-1}(v) \cup r^{-1}(v)\right|$. A finite graph E is a *line graph* if it is connected, acyclic and $totdeg(v) \leq 2$ for every $v \in E^0$. A line graph E is called an *m-line graph* if E has m vertices.

For $n \geq 2$, define E^n to be the set of paths of length n, and $E^* = \bigcup_{n\geq 0} E^n$ the set of all paths. Given a vertex v in a graph, the number of all paths ending at v is denoted by $n(v)$.

The path K-algebra over E, KE, is defined as the free K-algebra $K[E^0 \cup E^1]$ with the relations:

(1) $v_i v_j = \delta_{ij} v_i$ for every $v_i, v_j \in E^0$,
(2) $e_i = e_i r(e_i) = s(e_i) e_i$ for every $e_i \in E^1$.

Given a graph E, define the extended graph of E as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$ where $(E^1)^* = \{e_i^* \mid e_i \in E^1\}$ is a set with the same cardinality as E and disjoint from E so that the map assigning e* to e is a one-to-one correspondence; and the functions r' and s' are defined as

$$r'|_{E^1} = r, \quad s'|_{E^1} = s, \quad r'(e_i^*) = s(e_i) \quad \text{and} \quad s'(e_i^*) = r(e_i).$$

The Leavitt path algebra of E, $L_K(E)$, with coefficients in K is defined as the path algebra over the extended graph $\widehat{E}$, which satisfies the additional relations:

(CK1) $e_i^* e_j = \delta_{ij} r(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$,
(CK2) $v_i = \sum_{\{e_j \in E^1 \mid s(e_j) = v_i\}} e_j e_j^*$ for every $v_i \in E^0$ which is not a sink, and emits only finitely many edges.

The conditions (CK1) and (CK2) are called the Cuntz-Krieger relations. Note that the condition of row-finiteness is needed in order to define the equation (CK2).

Finite dimensional Leavitt path algebras are studied in [2] by Abrams, Aranda Pino and Siles Molina. The authors characterize the structure theorems for finite dimensional Leavitt path algebras. Their results are summarized in the following proposition:

2.1. Proposition. (1) *The Leavitt path algebra $L_K(E)$ is a finite-dimensional K-algebra if and only if E is a finite and acyclic graph.*

(2) *If $A = \bigoplus_{i=1}^{s} M_{n_i}(K)$, then $A \cong L_K(E)$ for a graph E having s connected components each of which is an oriented line graph with n_i vertices, $i = 1, 2, \cdots, s$.*

(3) *A finite dimensional K-algebra A arises as a $L_K(E)$ for a graph E if and only if $A = \bigoplus_{i=1}^{s} M_{n_i}(K)$.*

(4) *If $A = \bigoplus_{i=1}^{s} M_{n_i}(K)$ and $A \cong L_K(E)$ for a finite, acyclic graph E, then the number of sinks of E is equal to s, and each sink v_i $(i = 1, 2, \cdots, s)$ has $n(v_i) = n_i$ with a suitable indexing of the sinks.*

3. Truncated Trees

For a finite dimensional Leavitt path algebra $L_K(E)$ of a graph E, we construct a distinguished graph F having the Leavitt path algebra isomorphic to $L_K(E)$ as follows:

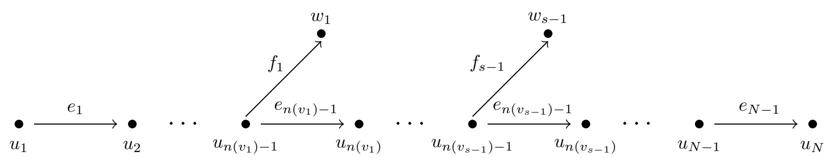
3.1. Theorem. *Let E be a finite, acyclic graph with no isolated vertices. Let $s = |S(E)|$ where $S(E)$ is the set of sinks of E and $N = \max\{n(v) \mid v \in S(E)\}$. Then there exists a unique (up to isomorphism) tree F with exactly one source and $s + N - 1$ vertices such that $L_K(E) \cong L_K(F)$.*

Proof. Let the sinks $v_1, v_2, \ldots, v_s$ of E be indexed such that

$$2 \leq n(v_1) \leq n(v_2) \leq \ldots \leq n(v_s) = N.$$

Define a graph $F = (F^0, F^1, r, s)$ as follows:

$$\begin{aligned} F^0 &= \{u_1, u_2, \ldots, u_N, w_1, w_2, \ldots w_{s-1}\} \\ F^1 &= \{e_1, e_2, \ldots, e_{N-1}, f_1, f_2, \ldots, f_{s-1}\} \\ s(e_i) &= u_i \quad \text{and} \quad r(e_i) = u_{i+1} \qquad i = 1, \ldots, N-1 \\ s(f_i) &= u_{n(v_i)-1} \quad \text{and} \quad r(f_i) = w_i \qquad i = 1, \ldots, s-1. \end{aligned}$$



Clearly, F is a directed tree with unique source u_1 and $s + N - 1$ vertices. The graph F has exactly s sinks, namely $u_N, w_1, w_2, \ldots w_{s-1}$ with $n(u_N) = N$, $n(w_i) = n(v_i)$, $i = 1, \ldots, s-1$. Therefore, $L_K(E) \cong L_K(F)$ by Proposition 2.1.

For the uniqueness part, take a tree T with exactly one source and $s+N-1$ vertices such that $L_K(E) \cong L_K(T)$. Now $N = \max\{n(v) \mid v \in S(E)\}$ is equal to the square root of the maximum of the K-dimensions of the minimal ideals of $L_K(E)$ and also of $L_K(T)$. So there exists a sink v in T with $|\{\mu_i \in T^* \mid r(\mu_i) = v\}| = N$. Since, any vertex in T is connected to the unique source by a uniquely determined path, the unique path joining v to the source must contain exactly N vertices, say $a_1, ..., a_{N-1}, v$ where a_1 is the unique source and the length of the path joining a_k to a_1 being equal to $k-1$ for any $k = 1, 2, ..., N-1$. As $L_K(E) = \bigoplus_{i=1}^{s} M_{n_i}(K)$ with s summands, all the remaining $s-1$ vertices, say $b_1, ..., b_{s-1}$, must be sinks by Proposition 2.1(4). For any vertex a different from the unique source, clearly $n(a) > 1$. Also, there exists an edge g_i with $r(g_i) = b_i$ for each $i = 1, \ldots, s-1$. Since $s(g_i)$ is not a sink, it follows that $s(g_i) \in \{a_1, a_2, ..., a_{N-1}\}$, more precisely $s(g_i) = a_{n(b_i)-1}$ for $i = 1, 2, ..., s-1$. Thus T is isomorphic to F. □

We name the graph F constructed in Theorem 3.1 as the *truncated tree associated with* E.

3.2. Proposition. *With the above definition of* F*, there is no tree* T *with* $|T^0| < |F^0|$ *such that* $L_K(T) \cong L_K(F)$.

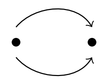
Proof. Notice that since T is a tree, any vertex contributing to a sink represents a unique path ending at that sink.

Assume on the contrary there exists a tree T with n vertices and $L_K(T) \cong A = \bigoplus_{i=1}^{s} M_{n_i}(K)$ such that $n < s + N - 1$. Since N is the maximum of n_i's there exists a sink v with $n(v) = N$. But in T the number $n - s$ of vertices which are not sinks is less than $N - 1$. Hence the maximum contribution to any sink can be at most $n - s + 1$ which is strictly less than N. This is the desired contradiction. □

Remark that the above proposition does not state that it is impossible to find a graph G with smaller number of vertices having $L_K(G)$ isomorphic to $L_K(E)$. The next example illustrates this point.

3.3. Example. Consider the graphs G and F.

Both $L_K(G) \cong M_3(K) \cong L_K(F)$ and $|G^0| = 2$ where as $|F^0| = 3$.



G

F

Given any graphs G_1 and G_2, $L_K(G_1) \cong L_K(G_2)$ does not necessarily imply $G_1 \cong G_2$. However, for truncated trees F_1, F_2 we have $F_1 \cong F_2$ if and only if $L_K(F_1) \cong L_K(F_2)$. So there is a one-to-one correspondence between the Leavitt path algebras and the truncated trees.

Consider a finite dimensional Leavitt path algebra $A = \bigoplus_{i=1}^{s} M_{n_i}(K)$ with $2 \leq n_1 \leq n_2 \leq \ldots \leq n_s = N$. Here, the number s is the number of minimal ideals of A and N^2 is the maximum of the dimensions of the minimal ideals. Therefore, the integer $s+N-1$ is an algebraic invariant of A which is denoted by $\kappa(A)$. Notice that the number of isomorphism classes of finite dimensional Leavitt path algebras A, with the invariant $\kappa(A) > 1$, having no ideals isomorphic to K is equal to the number of distinct truncated trees with $\kappa(A)$ vertices by the previous paragraph. The next proposition computes this number.

3.4. Proposition. *The number of distinct truncated trees with m vertices is 2^{m-2}.*

Proof. In a truncated tree, $n(v_1) \neq n(v_2)$ for any two distinct non-sinks v_1 and v_2. For every sink v, there is a unique non-sink w so that there exists an edge e with $s(e) = w$ and $r(e) = v$. Namely the non-sink w is with $n(w) = n(v) - 1$. This w is denoted by $b(v)$.

Now, define $d(u) = |\{v : n(v) \leq n(u)\}|$ for any $u \in E^0$. Clearly, $d(u)$ is equal to the sum of $n(u)$ and the number of sinks v with $n(b(v)) < n(u)$ for any $u \in E^0$. Assign an m-tuple $\alpha(E) = (\alpha_1, \alpha_2, ..., \alpha_m) \in \{0,1\}^m$ to a truncated tree E with m vertices by letting $\alpha_j = 1$ if and only if $j = d(v)$ for some vertex v which is not a sink. Clearly, there is just one vertex v with $n(v) = 1$, namely the unique source of E and that vertex is not a sink, so $\alpha_1 = 1$. Since there cannot be any non-sink v with $d(v) = m$, it follows that $\alpha_m = 0$.

Conversely, for $\beta = (\beta_1, \beta_2, ..., \beta_m) \in \{0,1\}^m$ with $\beta_1 = 1$ and $\beta_m = 0$ there exists a unique truncated tree E with m vertices such that $\alpha(E) = \beta$: If $\beta_i = 1$, then assign a non-sink v to E with $n(v) = |\{k : 1 \leq k < i \text{ and } \beta_k = 1\}|$. If $\beta_i = 0$ and $j = |\{k : 1 \leq k < i \text{ and } \beta_k = 1\}|$ then construct a sink which is joined to the non-sink v with $n(v) = j$. Clearly, the graph E is a truncated tree with m vertices and $\alpha(E) = \beta$.

Hence the number of distinct truncated trees with m vertices is equal to 2^{m-2} which is the number of all elements of $\{0,1\}^m$ with the first component 1 and the last component 0. $\square$

Hence, we have the following corollary.

3.5. Corollary. *Given $n \geq 2$, the number of isomorphism classes of finite dimensional Leavitt path algebras A with $\kappa(A) = n$ and which do not have any ideals isomorphic to K is 2^{n-2}.*

4. Bounds on the K-Dimension of finite dimensional Leavitt Path Algebras

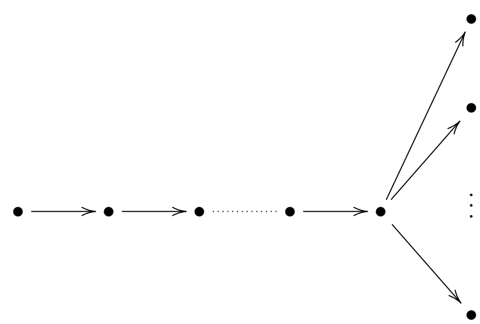
For a tree F with m vertices, the K-dimension of $L_K(F)$ is not uniquely determined by the number of vertices only. However, we can compute the maximum and the minimum K-dimensions of $L_K(F)$ where F ranges over all possible trees with m vertices.

4.1. Lemma. *The maximum K-dimension of $L_K(E)$ where E ranges over all possible trees with m vertices and s sinks is attained at a tree in which $n(v) = m - s + 1$ for each sink v. In this case, the value of the dimension is $s(m - s + 1)^2$.*

Proof. Assume E is a tree with m vertices. Then $L_K(E) \cong \bigoplus_{i=1}^{s} M_{n_i}(K)$, by Proposition 2.1 (3) where s is the number of sinks in E and $n_i \leq m - s + 1$ for all $i = 1, \ldots s$. Hence

$$\dim L_K(E) = \sum_{i=1}^{s} n_i^2 \leq s(m - s + 1)^2.$$

Notice that there exists a tree E as sketched below



with m vertices and s sinks such that $\dim L_K(E) = s(m - s + 1)^2$. $\square$

4.2. Theorem. *The maximum K-dimension of $L_K(E)$ where E ranges over all possible trees with m vertices is given by $f(m)$ where*

$$f(m) = \begin{cases} \dfrac{m(2m+3)^2}{27} & if \quad m \equiv 0 (mod\ 3) \\ \dfrac{1}{27}(m+2)(2m+1)^2 & if \quad m \equiv 1 (mod\ 3) \\ \dfrac{4}{27}(m+1)^3 & if \quad m \equiv 2 (mod\ 3) \end{cases}$$

Proof. Assume E is a tree with m vertices. Then $L_K(E) \cong \bigoplus_{i=1}^{s} M_{n_i}$ where s is the number of sinks in E. Now, to find the maximum dimension of $L_K(E)$, determine the maximum value of the function $f(s) = s(m - s + 1)^2$ for $s = 1, 2, \ldots, m - 1$. Extending the domain of $f(s)$ to real numbers $1 \leq s \leq m - 1$ f becomes a continuous function, hence its maximum value can be computed.

$$f(s) = s(m - s + 1)^2 \Rightarrow \frac{d}{ds}\left(s(m - s + 1)^2\right) = (m - 3s + 1)(m - s + 1)$$

Then $s = \dfrac{m+1}{3}$ is the only critical point in the interval $[1, m-1]$ and since $\dfrac{d^2 f}{ds^2}(\dfrac{m+1}{3}) < 0$, it is a local maximum. In particular f is increasing on the interval $\left[1, \dfrac{m+1}{3}\right]$ and decreasing on $\left[\dfrac{m+1}{3}, m-1\right]$. There are three cases:

Case 1: $m \equiv 2 \pmod 3$. In this case $s = \dfrac{m+1}{3}$ is an integer and maximum K-dimension of $L_K(E)$ is $f\left(\dfrac{m+1}{3}\right) = \dfrac{4}{27}(m+1)^3$ and $n_i = \dfrac{2(m+1)}{3}$, for each $i = 1, 2, \ldots, s$.

Case 2: $m \equiv 0 \pmod 3$. Then: $\dfrac{m}{3} = t < t + \dfrac{1}{3} = s < t+1$ and

$$f\left(\frac{m}{3}\right) = \frac{(2m+3)^2 m}{27} = \alpha_1 \text{ and } f\left(\frac{m}{3}+1\right) = \frac{4m^2(m+3)}{27} = \alpha_2.$$

Note that, $\alpha_1 > \alpha_2$. So α_1 is maximum K -dimension of $L_K(E)$ and $n_i = \dfrac{2}{3}m + 1$, for each $i = 1, 2, \ldots, s$.

Case 3: $m \equiv 1 \pmod 3$. Then $\dfrac{m-1}{3} = t < t + \dfrac{2}{3} = s < t+1$ and

$$f\left(\frac{m-1}{3}\right) = \frac{4}{27}(m+2)^2(m-1) = \beta_1$$

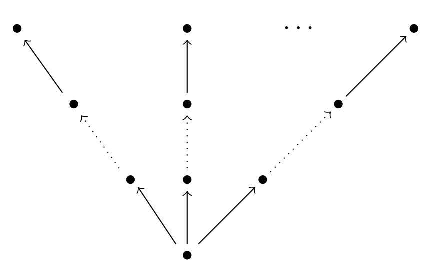
and

$$f\left(\frac{m+2}{3}\right) = \frac{1}{27}(2m+1)^2(m+2) = \beta_2.$$

In this case $\beta_2 > \beta_1$ and so β_2 gives the maximum K-dimension of $L_K(E)$ and $n_i = \dfrac{2m+1}{3}$, for each $i = 1, 2, \ldots, s$. □

4.3. Theorem. *The minimum K-dimension of $L_K(E)$ where E ranges over all possible trees with m vertices and s sinks is equal to $r(q+2)^2 + (s-r)(q+1)^2$, where $m - 1 = qs + r, \quad 0 \le r < s$.*

Proof. We call a graph a *bunch tree* if it is obtained by identifying the unique sources of the finitely many disjoint oriented finite line graphs as seen in the figure.



Let $\mathcal{E}(m, s)$ be the set of all bunch trees with m vertices and s sinks. Every element of $\mathcal{E}(m, s)$ can be uniquely represented by an s -tuple $(t_1, t_2, ..., t_s)$ where each t_i is the number of vertices different from the source contributing to the i^{th} sink, with $1 \le t_1 \le t_2 \le ... \le t_s$ and $t_1 + t_2 + ... + t_s = m - 1$.

Let $E \in \mathcal{E}(m,s)$ with $t_s - t_1 \le 1$. This E is represented by the s-tuple $(q, \dots, q, q+1, \dots, q+1)$ where $m - 1 = sq + r$, $0 \le r < s$.

Now, claim that the dimension of E is the minimum of the set

$$\{\dim L_K(F) : F \text{ tree with } s \text{ sinks and } m \text{ vertices}\}.$$

If we represent $U \in \mathcal{E}(m,s)$ by the s-tuple $(u_1, u_2, ..., u_s)$ then $E \neq U$ implies that $u_s - u_1 \ge 2$.

Consider the s-tuple $(t_1, t_2, ..., t_s)$ where $(t_1, t_2, ..., t_s)$ is obtained from $(u_1 + 1, u_2, ..., u_{s-1}, u_s - 1)$ by reordering the components in increasing order.

In this case, the dimension d_U of U is

$$d_U = (u_1+1)^2 + \ldots + (u_s+1)^2.$$

Similarly, the dimension d_T of the bunch graph T represented by the s-tuple $(t_1, t_2, ..., t_s)$, is

$$d_T = (t_1+1)^2 + \ldots + (t_s+1)^2 = (u_1+2)^2 + \ldots + (u_{s-1}+1)^2 + u_s^2.$$

Hence

$$d_U - d_T = 2(u_s - u_1) - 2 > 0.$$

Repeating this process sufficiently many times, the process has to end at the exceptional bunch tree E showing that its dimension is the smallest among the dimensions of all elements of $\mathcal{E}(m,s)$.

Now let F be an arbitrary tree with m vertices and s sinks. As above assign to F the s-tuple $(n_1, n_2, ..., n_s)$ with $n_i = n(v_i) - 1$ where the sinks v_i, $i = 1, 2, \dots, s$ are indexed in such a way that $n_i \le n_{i+1}$, $i = 1, \dots, s-1$. Observe that $n_1 + n_2 + \cdots + n_s \ge m - 1$. Let $\beta = \sum_{i=1}^{s} n_i - (m-1)$. Since $s \le m - 1$, $\beta \le \sum_{i=1}^{s}(n_i - 1)$. Either $n_1 - 1 \ge \beta$ or there exists a unique $k \in \{2, \dots, s\}$ such that $\sum_{i=1}^{k-1}(n_i - 1) < \beta \le \sum_{i=1}^{k}(n_i - 1)$. If $n_1 - 1 \ge \beta$, then let

$$m_i = \begin{cases} n_1 - \beta & , \quad i = 1 \\ n_i & , \quad i > 1 \end{cases}.$$

Otherwise, let

$$m_i = \begin{cases} 1 & , \quad i \le k-1 \\ n_k - \left(\beta - \sum_{i=1}^{k-1}(n_i - 1)\right) & , \quad i = k \\ n_i & , \quad i \ge k+1 \end{cases}.$$

In both cases, the s-tuple $(m_1, m_2, \dots, m_s)$ that satisfies $1 \le m_i \le n_i$, $m_1 \le m_2 \le \cdots \le m_s$ and $m_1 + m_2 + \cdots + m_s = m - 1$ is obtained. So, there exists a bunch tree M namely the one corresponding uniquely to $(m_1, m_2, \dots, m_s)$ which has dimension $d_M \le d_F$. This implies that $d_F \ge d_E$.

Hence the result follows. □

4.4. Lemma. *The minimum K-dimension of $L_K(E)$ where E ranges over all possible trees with m vertices occurs when the number of sinks is $m-1$ and is equal to $4(m-1)$.*

Proof. By the previous theorem observe that

$$\dim L_K(E) \ge r(q+2)^2 + (s-r)(q+1)^2$$

where $m - 1 = qs + r$, $0 \le r < s$. Then

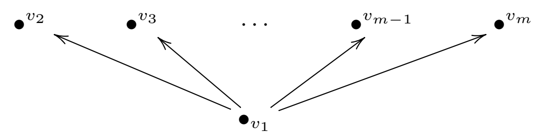
$$r(q+2)^2 + (s-r)(q+1)^2 = (m-1)(q+2) + qr + r + s.$$

Thus

$$(m-1)(q+2)+qr+r+s-4(m-1) = (m-1)(q-2)+qr+r+s \geq 0 \quad if \quad q \geq 2.$$

If $q = 1$, then $-(m-1)+2r+s = -(m-1)+r+(m-1) = r \geq 0$. Hence $\dim L_K(E) \geq 4(m-1)$.

Notice that there exists a truncated tree E with m vertices and $\dim L_K(E) = 4(m-1)$ as sketched below :



□

5. Line Graphs

In [2], the Proposition 5.7 shows that a semisimple finite dimensional algebra $A = \bigoplus_{i=1}^{s} M_{n_i}(K)$ over the field K can be described as a Leavitt path algebra $L_K(E)$ defined by a line graph E, if and only if A has no ideals of K-dimension 1 and the number of minimal ideals of A of K-dimension 2^2 is at most 2. On the other hand, if $A \cong L_K(E)$ for some m-line graph E then $m-1 = \sum_{i=1}^{s}(n_i - 1)$, that is, m is an algebraic invariant of A.

Therefore the following proposition answers a reasonable question.

5.1. Proposition. *The number A_m of isomorphism classes of Leavitt path algebras defined by line graphs having exactly m vertices is*

$$A_m = P(m-1) - P(m-4)$$

where $P(t)$ is the number of partitions of the natural number t.

Proof. Any m-line graph has $m-1$ edges. In a line graph, for any edge e there exists a unique sink v so that there exists a path from $s(e)$ to v. In this case we say that e is directed towards v. The number of edges directed towards v is clearly equal to $n(v)-1$. Let E and F be two m -line graphs. Then $L_K(E) \cong L_K(F)$ if and only if there exists a bijection $\phi : S(E) \to S(F)$ such that for each v in $S(E)$, $n(v) = n(\phi(v))$. Therefore the number of isomorphism classes of Leavitt path algebras determined by m-line graphs is the number of partitions of $m-1$ edges in which the number of parts having exactly one edge is at most two. Since the number of partitions of k objects having at least three parts each of which containing exactly one element is $P(k-3)$, the result $A_m = P(m-1) - P(m-4)$ follows. □

References

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra 293 (2) (2005), 319 - 334.

[2] G. Abrams, G. Aranda Pino, M. Siles Molina, Finite-dimensional Leavitt path algebras, J. Pure Appl. Algebra 209 (2007) 753 - 762.

[3] P. Ara, M.A. Moreno, and E. Pardo, Nonstable K-theory for graph algebras, Alg. Rep. Thy. 10 (2007), 157–178.

[4] T.Y. Lam, A First Course In Noncommutative Rings, Springer-Verlag 2001.