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THE BROUWER FIXED POINT THEOREM WITH EQUIVALENCES, EXTENSIONS, AND APPLICATIONS

A THESIS

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To Bonni

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TABLE OF CONTENTS

INTRODUCTION

The purpose of this paper is to present a complete proof of the Brouwer Fixed Point Theorem, equivalent statements to the theorem, extensions, and applications to all of these.

Brouwer's theorem deals with certain continuous mappings whose domain is a subset of some Euclidean n-space, which we shall denote by E^n. That part of the domain of the mapping in which we are interested is called a Euclidean n-cell, or merely an n-cell. In this paper we define an n-cell as any set which is homeomorphic to the closed unit ball in E^n, $B = \{x: x \in E^n, \|x\| \leq 1\}$. Observe that an n-cell is not necessarily a set in E^n.

The Brouwer Fixed Point Theorem states that every Euclidean n-cell has the fixed point property. To say that a set A has the fixed point property, we mean that whenever a continuous function f maps A into itself, there is at least one point x in A such that $f(x) = x$. Then x is called a fixed point of f.

Before proceeding to the discussion of Brouwer's theorem, we need to mention a few pertinent facts that will be used in this paper. The first is known as Tietze's extension theorem, the proof of which can be found in Hocking and Young [7].

Tietze's Extension Theorem. Let X be a normal space, and let $f: C \rightarrow I^n$ be a continuous mapping of the closed subset C of X into the unit cube I^n in E^n, where $I^n = \{x: x=(x_1, x_2, \ldots, x_n), 0 \leq x_i \leq 1\}$. Then there is a continuous mapping $f^*: X \rightarrow I^n$ such that $f^*(x) = f(x)$ for all $x \in C$.

In the generalizations of Brouwer's theorem we encounter a linear topological space. This is a linear space (vector space) with a topology which makes the vector operations of addition and scalar multiplication continuous (as functions of two variables). If X and Y are two linear topological spaces, we say X and Y are linearly homeomorphic, if there exists a homeomorphism h of X onto Y such that

$$h(\alpha x+\beta y) = \alpha h(x) + \beta h(y)$$

for all $x,y \in X$, all scalars α,β.

The following theorem is extremely important in the study of normed linear spaces. The proof can be found in Wilansky [13].

Theorem. Every finite dimensional real normed linear space is linearly homeomorphic to some Euclidean n-space.

It is easily shown that if h is a linear homeomorphism, then h^{-1} is also a linear homeomorphism. It is also straightforward to show that if X is a complete metric space, then any image of X under a linear homeomorphism is complete. Two important consequences of these statements and the theorem above are that every finite dimensional real normed linear space X is complete, and every closed, bounded set in X is compact.

In this paper we adopt some notation which may not be universal. We use the symbol $\square$ to indicate that a proof has been completed. We let $\bar{A}$ denote the closure of the set A relative to the topology of the space in which A lies. For two sets A and B, the notation A-B means $A \cap B^c$, where B^c is the complement of B relative to the space which contains B. When speaking of the composition of two mappings f and g, we define $f \circ g(x) = f(g(x))$. Other notation which is used will be defined when it is used.

CHAPTER I

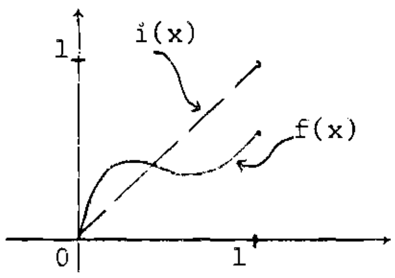
THE BROUWER FIXED POINT THEOREM

Before proceeding to the discussion of Brouwer's Fixed Point Theorem in E^n, we first prove the theorem for E^1. The proof for this case is quite easy and does not require the extensive structure necessary in the general case.

In E^1 let I be the closed unit interval [0,1]. Brouwer's theorem can be stated:

Theorem 1.1. If f is a continuous mapping from I into I, then there exists an $x_o \varepsilon I$ so that $f(x_o) = x_o$.

Remark. Geometrically, this theorem says that the graphs of f(x) and the identity mapping i(x) = x must intersect at some point in I. The diagram indicates what is happening.



Proof. Consider the function g(x) = f(x)-x. Note that g is continuous on [0,1]. If g(x) = 0 for some x ε I, then f(x) = x. So assume g(0) ≠ 0 and g(1) ≠ 0. Otherwise, we are finished. Since for each $x \in I, 0 \leq f(x) \leq 1$, we must have $g(0) = f(0)-0>0$ and $g(1) = f(1)-1<0$. By the Intermediate Value Theorem, there is a point x_o between 0 and 1 such that $g(x_o) = 0$. Thus, $f(x_o) = x_o$. ▯

In order to prove the Brouwer Fixed Point Theorem, we first need to develop the concept of an n-simplex and some of its basic properties. Intuitively, an n-simplex in E^n is a generalization of a triangle in E^2.

Definition. The points $x_o, x_1, \cdots, x_n$ in $E^m (m \geq n)$ are said to be *in general position* if the vectors $\overline{x_1 - x_o}$, $\overline{x_2 - x_o}$, $\cdots$, $\overline{x_n - x_o}$ are linearly independent. The choice of x_o as the "origin" is arbitrary in determining their linear independence.

Definition. Let $x_o, x_1, \cdots, x_n$ be in general position in $E^m (m \geq n)$. The *n-simplex* S associated with these points is

$$S = \{x: x = \sum_{i=o}^{n} t_i x_i, \text{ where each } t_i \geq 0 \text{ and } \sum_{i=o}^{n} t_i = 1\}.$$

The points $x_o, x_1, \cdots, x_n$ are called the vertices of S, and we denote this n-simplex by $S = |x_o, x_1, \cdots, x_n|$.

Note that if $x_o, x_1, \cdots, x_n$ are in general position in E^n, then every x in E^n can be uniquely expressed as a linear combination of these points, say $x = \sum_{i=o}^{n} t_i x_i$, with the stipulation that $\sum_{i=o}^{n} t_i = 1$. In such a representation, the t_i's may be negative.

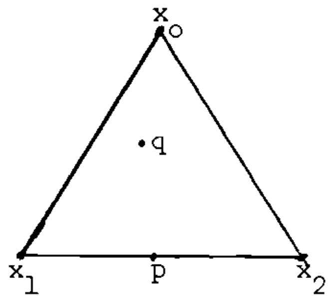
Recall that a point x is an interior point of a set A in E^m, if there exists an open set U in E^m such that $x \in U \subset A$. If $x \in A$ is not an interior of A, then x is a boundary point of A. Observe that if $S = |x_o, x_1, \cdots, x_n|$ is an n-simplex in E^m, where $m > n$, there are no interior points of S, by this definition. Instead, we shall define "inner" points of S as points which are interior points of S when S is viewed as a subset of E^n, rather than E^m. Then it can be shown that x is an inner point of S, where $x = \sum_{i=o}^{n} t_i x_i$ and $\sum_{i=o}^{n} t_i = 1$ if, and only if, each $t_i > 0$. We call x a boundary point of S if, and only if, at least one $t_i = 0$ and each $t_i \geq 0$. A thorough discussion of this can be found in Bers [1]. One further step is to decompose the boundary of S into faces (or sides).

Definition. A k-side (k-face) of an n-simplex $S = |x_o, x_1, \cdots, x_n|$ is a k-simplex whose vertices are a subset of $\{x_o, x_1, \cdots, x_n\}$. A k-side is said to have dimension k.

Notice that if $k<n$, the k-side is a subset of the boundary of S. Also, if x is a boundary point of S, then $x = \sum_{i=o}^{n} t_i x_i$ where $\sum_{i=o}^{n} t_i = 1$, each $t_i \geq 0$, and at least one $t_i = 0$. Without loss of generality, assume $t_i = 0$ for $k<i\leq n$, and for $0\leq i\leq k$, $t_i > 0$. If this is not the case, reorder the x_i's to do this. Then $x \in |x_o, x_1, \cdots, x_k|$, which is a k-side of S. By construction, this is the side of S of least dimension which contains x. We make the following definition.

Definition. Let $x \in S = |x_o, x_1, \cdots, x_n|$. The carrier side of S for the point x is the side of least dimension containing x.

For example, in the 2-simplex $S = |x_o, x_1, x_2|$ shown below, the carrier side of x_o is the 0-simplex $S_o = \{x_o\}$. The carrier side of p is the 1-simplex $S_1 = \{tx_1 + (1-t)x_2 \colon 0\leq t\leq 1\}$; that is, the side joining x_1 and x_2. The carrier side of q is S.



Note that if x is an inner point of S, the carrier side of S for x is all S and has dimension n. For a boundary point x of S, the dimension of the carrier side of S for x is less than n.

Another concept we need is that of a simplical subdivision of an n-simplex S. We are interested in a particular type of subdivision called a "barycentric subdivision." We first consider the notion of a barycenter of a finite collection of points.

Definition. Let $p_o, p_1, \cdots, p_k (k \geq 0)$ be points in E^n, where p_j has the Cartesian coordinate representation $p_j = (a_{j1}, a_{j2}, \cdots, a_{jn})$ for $j = 0, 1, \cdots, k$. The *barycenter* $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ of the points $p_o, p_1, \cdots, p_k$ has coordinates

$$\bar{x}_i = \frac{1}{k+1} \sum_{j=o}^{k} a_{ji}.$$

The barycenter $\bar{x}$ of $p_o, p_1, \cdots, p_k$ is denoted by $\bar{x} = B(p_o, p_1, \cdots, p_k)$. This definition is suggestive of the center of gravity of particles of mass 1 located at the points $p_o, p_1, \cdots, p_k$.

Now consider an n-simplex $S = |x_o, x_1, \cdots, x_n|$. We list all possible barycenters of all non-empty subsets of $\{x_o, x_1, \cdots, x_n\}$ as follows:

Stage 1: (using subsets consisting of one point)

$B(x_o), B(x_1), \cdots, B(x_n)$ (Note that $B(x_i) = \{x_i\}$)

Stage 2: (using subsets consisting of two distinct points)

$B(x_o,x_1)$, $B(x_o,x_2)$, $\cdots$, $B(x_o,x_n)$, $B(x_1,x_2)$, $B(x_1,x_3)$, $\cdots$, $B(x_1,x_n)$, $\cdots$, $B(x_{n-1},x_n)$.

⋮

Stage k: (using subsets consisting of k distinct points)

$B(x_o,x_1,\cdots,x_{k-1})$, $\cdots$

⋮

Stage n: (using subsets consisting of n distinct points)

$B(x_o,x_1,\cdots,x_{n-1})$, $B(x_o,x_1,\cdots,x_{n-2},x_n)$, $\cdots$, $B(x_o,x_2,x_3,\cdots,x_n)$, $B(x_1,x_2,\cdots,x_n)$

Stage n+1: (using subsets consisting of n+1 distinct points)

$B(x_o,x_1,\cdots,x_n)$

Using these barycenters as vertices, we form a collection of n-simplexes by choosing a barycenter from each stage, starting with stage 1 and proceeding successively through stage n+1, so that the points which are used to determine the barycenter in stage k are common to the set of points used to determine the barycenter chosen from stage k+1. We give an example to illustrate. The 4-simplex $|B(x_o)$, $B(x_o,x_1)$, $B(x_o,x_1,x_4)$, $B(x_o,x_1,x_3,x_4)$, $B(x_o,x_1,x_2,x_3,x_4)|$ is subsimplex of the type described for the 4-simplex $S = |x_o,x_1,x_2,x_3,x_4|$. The

4-simplex $|B(x_o), B(x_o,x_1), B(x_1,x_2,x_3), B(x_1,x_2,x_3,x_4),$ $B(x_o,x_1,x_2,x_3,x_4)|$, however, is not one of the desired type, since x_o is not common to the set of points used to determine the third barycenter.

Definition. Let $S = |x_o,x_1,\cdots,x_n|$. The *first barycentric subdivision of S* is the collection of n-subsimplexes as described above.

It is easily seen that there are exactly (n+1)! subsimplexes in the first barycentric subdivision of S. Before proceeding further we illustrate this process with a concrete example in E^2.

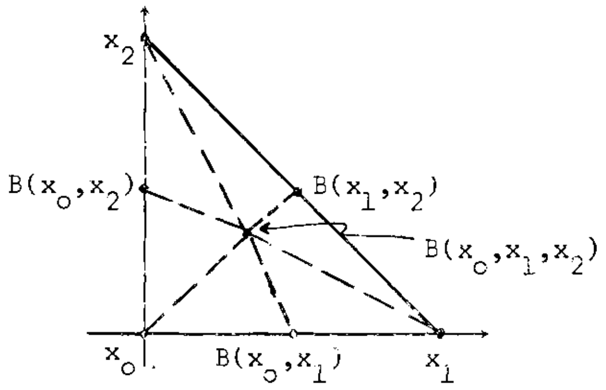
Let $x_o = (0,0)$, $x_1 = (1,0)$, and $x_2 = (0,1)$. Let $S = |x_o,x_1,x_2|$ be a 2-simplex in E^2. (See diagram below.)

Stage 1: $B(x_o) = (0,0)$, $B(x_1) = (1,0)$, $B(x_2) = (0,1)$

Stage 2: $B(x_o,x_1) = (\frac{1}{2},0)$, $B(x_o,x_2) = (0,\frac{1}{2})$, $B(x_1,x_2) = (\frac{1}{2},\frac{1}{2})$

Stage 3: $B(x_o,x_1,x_2) = (\frac{1}{3},\frac{1}{3})$

Choosing vertices in the prescribed manner, we get 6 subsimplexes, namely, $|B(x_o), B(x_o,x_1), B(x_o,x_1,x_2)|$, $|B(x_o), B(x_o,x_2), B(x_o,x_1,x_2)|$, $|B(x_1), B(x_o,x_1), B(x_o,x_1,x_2)|$, $|B(x_1), B(x_1,x_2), B(x_o,x_1,x_2)|$, $|B(x_2), B(x_o,x_2), B(x_o,x_1,x_2)|$, and $|B(x_2), B(x_1,x_2), B(x_o,x_1,x_2)|$. Geometrically, we have S subdivided as follows:



For the second barycentric subdivision of an n-simplex S, we take a first barycentric subdivision of each subsimplex obtained in the first barycentric subdivision of S. Likewise, we may define the kth barycentric subdivision of S in a similar manner. In Bers [1] it is shown that the diameter of the n-simplex $S = |x_o, x_1, \cdots, x_n|$, which is defined by $d(S) = \sup\{\|x-y\| : x,y \in S\}$, is precisely $d(S) = \max\{\|x_i - x_j\| : i,j = 0,1,\cdots,n\}$. Intuitively, it is the length of the longest 2-side. It is easily seen that in the kth barycentric subdivision of S, for any subsimplex T, $d(T) \leq \left(\frac{n-1}{n}\right)^k d(S)$. Thus, by taking enough barycentric subdivisions of S, we create a grid in S of n-subsimplexes so that the diameter of any subsimplex is as small as desired. Henceforth, we shall refer to any kth barycentric subdivision of S merely as a simplical subdivision of S, unless we need a specific barycentric subdivision.

The following lemma will be useful in the proof of Brouwer's theorem.

Lemma 1. Let A be a side of dimension n-1 of an n-subsimplex in some simplical subdivision K of $S = |x_o, x_1, \cdots, x_n|$. Then A is shared by exactly two n-subsimplexes in K, if A is not on the boundary of S; and A is common to exactly one n-subsimplex, if it is on the boundary of S.

Proof. We prove this by induction on the number of barycentric subdivisions. Let K_1 be the first barycentric subdivision of S. Let A be an (n-1)-side of S_1 in K_1, where $S_1 = |B(x_{i_0}), B(x_{i_0},x_{i_1}), \cdots, B(x_0,x_1,\cdots,x_n)|$ as described earlier. Without loss of generality, we may assume $x_{i_j} = x_j$; otherwise, renumber. For simplicity of notation, let $V_j = B(x_0,x_1,\cdots,x_{j-1})$, i.e., the vertex of S_1 chosen from stage j. Then $A = |v_1,v_2,\cdots,v_{j-1},v_{j+1},\cdots,v_{n+1}|$ for some $1\le j\le n+1$. First assume $j = n+1$. Then $A = |v_1,v_2,\cdots,v_n|$ and the points in A are convex linear combinations of the n points $x_0,x_1,\cdots,x_{n-1}$. By previous comments, A must be contained in the boundary of S. Since the (n+1)-st vertex for every subsimplex is $B(x_0,x_1,\cdots,x_n)$, there can be no n-simplex other than S_1 which has A as an (n-1)-side.

Now assume $j \neq n+1$. Then A is missing the vertex $v_j = B(x_0,x_1,\cdots,x_{j-1})$. We want to determine exactly how many n-simplexes have A as an (n-1)-side. Every n-simplex which contains A must be of the form

$$T = |B(x_0), B(x_0,x_1), \cdots, B(x_0,x_1,\cdots,x_{j-2}), \tilde{v}_j,$$

$$B(x_0,x_1,\cdots,x_{j-1},x_j), \cdots, B(x_0,x_1,\cdots,x_n)|.$$

By the method of constructing T, $\tilde{v}_j$ must involve all the previous vertices $x_0,x_1,\cdots,x_{j-2}$. Also v_{j+1} must involve all the vertices of $\tilde{v}_j$. Thus, there are exactly two choices for $\tilde{v}_j$, namely, $B(x_o,x_1,\cdots,x_{j-2},x_{j-1})$ or $B(x_o,x_1,\cdots,x_{j-2},x_j)$. Moreover, since A contains v_{n+1}, there are points in A which are convex linear combinations of all n+1 points $x_o,x_1,\cdots,x_n$ with all non-zero coefficients. One, for example, is $v_{n+1} = \sum_{i=o}^{n} \frac{1}{n+1} x_i$. Thus, A is not on the boundary of S. We shall then say A is interior to S, and A is shared by exactly two n-subsimplexes in K_1. We have proven the lemma for K_1.

Now assume the lemma is true for the kth barycentric subdivision of S, call it K_k. We want to show it is true for K_{k+1}. Recall that K_{k+1} is just the first barycentric subdivision of K_k. Another way of viewing K_{k+1} is the following: Let S_k be an n-simplex in K_k. Take the first barycentric subdivision of S_k. We get (n+1)! subsimplexes of S_k. The collection of all subsimplexes for each S_k in K_k is the set of n-subsimplexes in K_{k+1}.

Let A be an (n-1)-side in K_{k+1}. Then A is an (n-1)-side for an n-simplex T which is a subsimplex of some n-simplex S_k in K_k. If A is interior to S_k, then A is an (n-1)-side for exactly two n-subsimplexes of S_k and is, therefore, common to exactly two n-subsimplexes of S. Also, A is interior to S. If A is on the boundary of S_k, we must consider two possibilities. If A is on a boundary of S_k which is interior to S, then A is common to exactly one subsimplex in S_k. However, that (n-1)-side of S_k which contains A is shared by one other n-simplex S_k' in K_k. The same argument holds for A in S_k'. Thus, if A is interior to S, it follows that A is shared by exactly two n-subsimplexes in

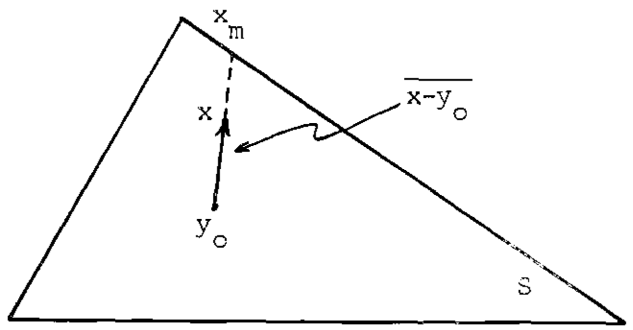
K_{k+1}. If A is on a boundary of S_k which is a boundary of S, then A is common to exactly one n-simplex in S_k, and the (n-1)-side of S_k containing A is common to S_k only in K_k. Thus, A is common to exactly one n-subsimplex in K_{k+1} if it is on the boundary of S. This proves the inductive step. The lemma is, therefore, true for any simplical (barycentric) subdivision of S. ▯

We digress from our discussion of simplexes for a moment to mention some other points which are necessary for the background to the proof of Brouwer's theorem. Recall that the theorem says every Euclidean n-cell has the fixed point property. We shall prove the theorem is true for every n-simplex in E^n. In order to establish the result for any n-cell, we need to show that an n-cell is homeomorphic to an n-simplex, and that under homeomorphisms, the fixed point property is invariant.

Lemma 2. A Euclidean n-cell is homeomorphic to an n-simplex.

Proof. All that is necessary to show is that every n-simplex is homeomorphic to the unit ball $B = \{x: \|x\| \le 1\}$ in E^n. Since every Euclidean n-cell is homeomorphic to B and the composition of homeomorphisms is a homeomorphism, we shall have established our lemma.

Let $S = |x_o, x_1, \cdots, x_n|$ be an n-simplex in E^n. Let y_o be an inner point of S. We are going to define a mapping on S which shrinks S to a unit ball around y_o. The diagram below will illustrate the idea. Let $x \in S$. Consider the vector $\overline{x - y_o}$. Geometrically, this vector emanates from the point y_o in the direction $\overline{x - y_o}$ (see diagram). Assume $x \neq y_o$. Then $\|x-y_o\| > 0$.



Now consider $\alpha(\overline{x-y_o})$ where $\alpha > 0$. This is a vector in the same direction as $(x-y_o)$. There exists an $x_m \in S$ such that $x_m - y_o = \alpha_m(x-y_o)$ with $\alpha_m \geq 1$, and such that $\|x_m-y_o\| = \max\{\|\alpha(x-y_o)\|: \alpha \geq 0$ and $\alpha(x-y_o) + y_o \in S\}$. This says there exists a furthest point $x_m \in S$ from y_o along the positive directed segment $\overline{x - y_o}$. Such a point exists since S is closed and bounded. Define $h: S \rightarrow B$ as follows:

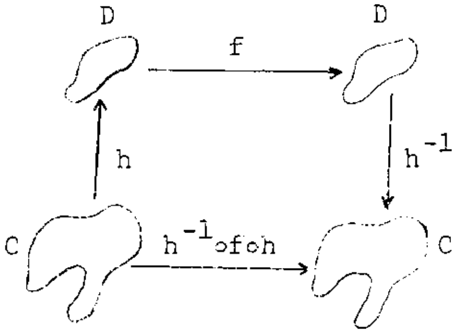
$$h(x) = \begin{cases} 0 & \text{for } x = y_o \\ \dfrac{x-y_o}{\|x_m-y_o\|} & \text{for } x \in S,\ x \neq y_o \end{cases}$$

Observe that $\|x-y_o\| \leq \alpha_m\|x-y_o\| = \|\alpha_m(x-y_o)\| = \|x_m-y_o\|$. So $\|h(x)\| \leq 1$. It is easy to check that h continuous on S, that h is one-to-one and onto, and that h^{-1} is continuous. Thus, S is homeomorphic to $B = \{x: \|x\| \leq 1\}$. □

We now show that if a set C has the fixed point property, then every homeomorphic image of C has that property, too.

Lemma 3. The fixed point property is a topological invariant; i.e., is preserved under homeomorphisms. This proof is essentially that of Whyburn [14].

Proof. Let C be a set with the fixed point property. Suppose C is homeomorphic to D. Let h: C⟶D be a homeomorphism. (The diagram below may help in visualizing these statements.) Let f: D⟶D be a continuous mapping. Then the mapping $h^{-1} \circ f \circ h$ is a continuous mapping of C into C, and, hence, has a fixed point, say x_o. That is, $h^{-1} \circ f \circ h\ (x_o) = x_o$.



Let $h(x_o) = y_o$ where $y_o \varepsilon D$. We claim that $f(y_o) = y_o$. Observe that since $h^{-1} \circ f \circ h(x_o) = x_o$, we have $f \circ h(x_o) = h(x_o)$, or $f(h(x_o)) = h(x_o)$. Thus, $f(y_o) = y_o$. □

We have proven these previous lemmas in order that we need only prove the Brouwer Fixed Point Theorem for an n-simplex in E^n. We now prove two more lemmas which will aid in the proof for an n-simplex. These lemmas and the proof of the theorem for an n-simplex are essentially those presented by Whyburn [14].

Lemma 4. Let K by any simplical subdivision of the n-simplex $S = |x_o, x_1, \cdots, x_n|$, and let v(e) be a mapping of the vertices of K into the vertices of S such that for any vertex e of K, v(e) is a vertex of the carrier side of S for e. Then there exist an odd number of n-simplexes in K whose vertices map in a one-to-one fashion onto the vertices of S.

Proof. Each n-simplex in K whose vertices map in a one-to-one fashion onto those of S shall be called an R-simplex. Also, call an (n-1)-simplex in K an R-side provided its vertices map in a one-to-one fashion onto the points $x_o, x_1, \cdots, x_{n-1}$ under v.

The proof will be by induction on the dimensionality n of S. For n=0, we have $S = |x_o| = \{x_o\}$. There is exactly one n-simplex in any simplical subdivision of S. That simplex is S itself, and the lemma is trivially true. So assume the lemma is true for dimension n-1. We adopt the following notation:

N = the number of R-simplexes in K.

$\alpha(T)$ = the number of R-sides on an n-simplex T in K.

α = the number of R-sides of K lying on the boundary of S.

Let T be an n-simplex of K with vertices $y_o, y_1, \cdots, y_n$. Then T has n+1 sides of dimension n-1. Denote the jth (n-1)-side of T by

$$T_j = |y_o, y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_n|.$$

If T is an R-simplex, then one, and only one, vertex y_j maps onto x_n. All (n-1)-sides of T containing y_j are not R-sides. Only T_j is an R-side. Thus $\alpha(T) = 1$.

Assume that T is not an R-simplex. Then either the vertices of T map onto the set $\{x_o, x_1, \cdots, x_{n-1}\}$, or they do not. If they do not, then no (n-1)-side of T maps its vertices onto the set, and $\alpha(T) = 0$. If they do map onto this set, we claim that $\alpha(T) = 2$. We must have exactly two of the vertices of T mapping onto some x_j where $0 \le j \le n-1$. The remaining vertices of T map onto the remaining x_i for $i \ne j$. By relabeling the vertices of T, if necessary, we have that $v(y_o) = v(y_1) = x_j$. Thus only the sides T_o and T_1 are R-sides of T.

Restating these results, we have that $\alpha(T) = 1$ if T is an R-simplex and $\alpha(T) = 0$ or $\alpha(T) = 2$ if T is not an R-simplex. Thus,

$$N \equiv \sum_{T \epsilon K} \alpha(T) \pmod 2.$$

We now need to know how many times an R-side of K is counted in the sum $\sum_{T \epsilon K} \alpha(T)$. Referring back to Lemma 1, we see that an R-side is counted only once if it lies on the boundary of S and twice if it is interior to S. So

$$\alpha \equiv \sum_{T \epsilon K} \alpha(T) \pmod 2.$$

Therefore,

$$N \equiv \alpha \pmod 2. \qquad (*)$$

According to (*) we need to count the number of R-sides lying on the boundary of S. Let W be an (n-1)-simplex in K (i.e., an (n-1)-side). If W lies on the boundary of S, say $W \subset |x_o, x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n|$, then the mapping v takes the vertices of W into the set $\{x_i: i \neq j\}$. Thus, for W to be an R-side, it must be that $W \subset |x_o, x_1, \cdots, x_{n-1}|$. Assume W is an R-side.

Setting $S^{n-1} = |x_o, x_1, \cdots, x_{n-1}|$, we notice that $K^{n-1} = K \cap S^{n-1}$ is a simplical subdivision of S^{n-1}. Furthermore, W is an (n-1)-simplex in S^{n-1} whose vertices map in a one-to-one fashion onto the vertices of S^{n-1} under v. Note that v restricted to S^{n-1} still fulfills the requirement that v(e) is a vertex of the carrier side of S^{n-1} for any vertex e in K^{n-1}. This makes W an R-simplex in K^{n-1}. Thus, W is an R-side in K if and only if W is an R-simplex in K^{n-1}. By the induction hypothesis, there exist an odd number of R-simplexes in K^{n-1}. So α is an odd number, and by (*) we have that N is odd. $\square$

We use this lemma to prove the following lemma.

Lemma 5. If $A_o, A_1, \cdots, A_n$ are non-empty closed sets of $S = |x_o, x_1, \cdots, x_n|$ such that for each set of distinct integers $i_o, i_1, \cdots, i_k$ $(0 \leq i_j \leq n)$, the side $|x_{io}, x_{i_1}, \cdots, x_{i_k}|$ is contained in $A_{i_o} \cup A_{i_1} \cup \cdots \cup A_{i_k}$, then $\bigcap_{j=o}^{n} A_j$ is non-empty.

Proof. Let $\varepsilon > 0$ be given. Let K_ε be a simplical subdivision of S such that each simplex in K_ε has diameter less than ε.

Let e be a vertex of K_ε. Denote the carrier side of S for e by $S(e) = |x_{i_o}, x_{i_1}, \cdots, x_{i_k}|$. By hypothesis, $S(e) \subset \bigcup_{j=o}^{k} A_{i_j}$, and, hence, $e \in A_{i_j}$ for at least one j, $0 \leq j \leq k$. Pick one such set.

Define $v(e) = x_{i_j}$, the vertex corresponding to the set A_{i_j} chosen above. Having done this for each vertex $e \in K_\varepsilon$, we have a mapping of the sort described in Lemma 4. Consequently, the number of R-simplexes in K_ε is odd, and there is at least one n-simplex T in K_ε whose vertices map in a one-to-one fashion onto the vertices of S. In order that a vertex e of T map into x_j, it is necessary that $e \in A_j$. Thus, $T \cap A_j$ is non-empty for $j = 0,1,\cdots,n$. Recall that the diameter of T is less than ε.

Consider a sequence $\{K(m)\}$ of simplical subdivisions of S where the diameter of all simplexes in K(m) is less than $\frac{1}{m}$, $m=1,2,\cdots$. Then there exists a sequence of n-simplexes $\{T_m\}$ in S with the diameter of T_m less than $\frac{1}{m}$ and $T_m \cap A_j$ non-empty for $j = 0,1,\cdots,n$. For each m, let p_m be a point in T_m. Then the sequence $\{p_m\}$ is contained in the compact set S. So there is a subsequence $\{p_{m_i}\}$ which converges to a point $p \in S$. Using the notation d(y,A) to mean the distance from the point y to the set A, and recalling that for each j, there is a point $a_j \in T_m \cap A_j$, we have that

$$d(p_m,A_j) \le d(p_m,a_j) \le \text{diameter of } T_m < \frac{1}{m}.$$

For a fixed set A, d(y,A) is a continuous function defined on E^n. It then follows that

$$\lim_{i\to\infty} d(p_{m_i},A_j) = d(p,A_j) \qquad j=0,1,\cdots,n.$$

Combining these results, we have that

$$0 \le \lim_{i\to\infty} d(p_{m_i},A_j) \le \lim_{i\to\infty} \frac{1}{m_i} = 0.$$

Hence, $d(p,A_j) = 0$ for $j=0,1,\cdots,n$. Since each A_j is closed, $p \varepsilon A_j$, and, so $p \varepsilon \bigcap_{j=o}^{n} A_j$. □

We are now ready to prove the Brouwer Fixed Point Theorem.

Theorem 1.2. Every Euclidean n-cell has the fixed point property.

Proof. Let $S = |x_o,x_1,\cdots,x_n|$ be an n-simplex. Let f be a continuous mapping from S into S. For $x \varepsilon S$, x has a unique representation as $x = \sum_{i=o}^{n} \alpha_i x_i$ where each $\alpha_i \geq 0$ and $\sum_{i=o}^{n} \alpha_i = 1$. Denote x by

$$x = (\alpha_o,\alpha_1,\cdots,\alpha_n) \tag{1}$$

and f(x) by

$$f(x) = (\alpha_o',\alpha_1',\cdots,\alpha_n'). \tag{2}$$

Define the following sets in S:

$$A_j = \{x\colon\ x \varepsilon S,\ \alpha_j \geq \alpha_j'\} \qquad j=0,1,\cdots,n$$

We need to show that each A_j is non-empty and closed in S. For each $j=0,1,\cdots,n$, define the projection mapping P_j by

$$P_j(x) = \alpha_j.$$

Then each P_j is continuous on S, and the mapping Q_j defined by

$$Q_j(x) = P_j(x) - P_j(f(x))$$

is continuous for each j. Then

$$A_j = \{x\colon x \in S,\ Q_j(x) \geq 0\} = Q_j^{-1}([0,\infty)).$$

Being the inverse image of a closed set, A_j is closed, since Q_j is continuous. Clearly, $x_j \in A_j$. So A_j is non-empty.

Now let $S^k = |x_{i_o}, x_{i_1}, \cdots, x_{i_k}|$. We want to show that $S^k \subset \bigcup_{j=0}^{k} A_{i_j}$, in order to apply Lemma 5. In (1) and (2), each $\alpha_i \geq 0$ and each $\alpha_i' \geq 0$, and $\sum_{i=o}^{n} \alpha_i = \sum_{i=o}^{n} \alpha_i' = 1$. Let $x \in S^k$, and let $I = \{i_o, i_1, \cdots, i_k\}$. Letting $x = (\alpha_o, \alpha_1, \cdots, \alpha_n)$, if $j \notin I$, $\alpha_j = 0$. Thus,

$$\sum_{i \in I} \alpha_i = 1$$

Assume that for each $i \in I$, $\alpha_i < \alpha_i'$, i.e., $x \notin A_i$ for any $i \in I$. Then

$$1 = \sum_{i \in I} \alpha_i < \sum_{i \in I} \alpha_i' \leq \sum_{i=o}^{n} \alpha_i'.$$

This is a contradiction. Thus $x \in A_i$ for at least one $i \in I$, and $S^k \subset \bigcup_{j=o}^{k} A_{i_j}$. Applying Lemma 5, it follows that there exists a point $x_o \in S$ such that $x_o \in \bigcap_{j=o}^{n} A_j$. For $x_o = (\beta_o, \beta_1, \cdots, \beta_n)$ we have that $\beta_i > \beta_i'$ for $i=0,1,\cdots,n$. Since $\sum_{i=o}^{n} \beta_i = \sum_{i=o}^{n} \beta_i' = 1$, we get that $\beta_i = \beta_i'$ for each i. So $f(x_o) = x_o$.

Reiterating our previous comments, by showing an n-simplex has the fixed point property, we have that every n-cell has the fixed point property. □

CHAPTER II

EQUIVALENCES TO BROUWER'S FIXED POINT THEOREM

We now present three statements which are equivalent to Brouwer's Fixed Point Theorem and prove their equivalence. Often an application of the theorem is more convenient if one of these alternate forms is used. These equivalences are stated by Hurewicz and Wallman [8].

Theorem 2.1. The following four statements are equivalent:

I. A continuous mapping from an n-cell in E^n into itself has a fixed point. (Brouwer's theorem.)

II. Let B_n be a closed convex ball in E^n, say $B_n = \{x: \|x\| \leq 1\}$. Let S_{n-1} be the (n-1)-sphere associated with B_n; in this case, $S_{n-1} = \{x: \|x\| = 1\}$. Then there exists no continuous function f which maps B_n into S_{n-1} and keeps such point of S_{n-1} fixed.

III. Let S_{n-1} be as described in II. Then there exists no function f(t,x) from $[0,1] \times S_{n-1}$ into S_{n-1} which is continuous in the pair (t,x) and which has the boundary conditions

$$f(0,x) = x_o \quad (\text{where } x_o \in S_{n-1})$$

$$f(1,x) = x$$

for each $x \in S_{n-1}$.

IV. Let I_n be a cube in E^n, say $I_n = \{x: |x_i| \leq 1, i=1,2,\cdots,n\}$. Let C_i and C_i' be the faces of I_n, defined by

$$C_i = \{x: x \in I_n, x_i=1\}$$

and

$$C_i' = \{x: x \in I_n, x_i=-1\}$$

Let K_i be a closed set separating C_i and C_i'; i.e., $I_n - K_i = U_i \cup U_i'$ where $U_i \cap U_i'$ is empty, U_i and U_i' are open relative to I_n, and $C_i \subset U_i$, $C_i' \subset U_i'$. Then $\bigcap_{i=1}^{n} K_i$ is non-empty.

Remark. We could prove this in a cyclic fashion. However, the proofs are interesting enough to prove their equivalence in the order IV $\Leftrightarrow$ I $\Leftrightarrow$ II $\Leftrightarrow$ III.

Proof. IV $\Rightarrow$ I. Let $B = \{x: x \in E_n, \|x\| \leq \frac{1}{2}\}$. Let $f: B \to B$ be continuous, and let g be a continuous mapping of I_n onto B which leaves each $x \in B$ fixed. Then $f \circ g: I_n \to B$ is a continuous mapping. For $x = (x_1, x_2, \cdots, x_n)$, define the projection P_i by $P_i(x) = x_i$. Then each P_i is continuous on E^n. Moreover, $P_i(f \circ g)$ is continuous on I_n. Now define the continuous mapping Q_i by $Q_i(x) = P_i(x) - P_i(f \circ g(x))$, for $x \in I_n$, $i=1,2,\cdots,n$. Consider the sets

$$K_i = \{x: x \in I_n, Q_i(x) = 0\} \quad \text{for } i=1,2,\cdots,n.$$

Clearly $K_i = Q_i^{-1}[0]$. Since the set $\{0\}$ is closed and Q_i is continuous, K_i is closed.

Let $U_i = \{x: x \in I_n, Q_i(x)>0\}$ and $U_i' = \{x: x \in I_n, Q_i(x)<0\}$. Obviously $U_i \cap U_i'$ is empty, and $I_n - K_i = U_i \cup U_i'$. Since $U_i = Q_i^{-1}[(0,\infty)]$ and $U_i' = Q_i^{-1}[(-\infty,0)]$, by continuity of Q_i, both U_i and U_i' are open in I_n. This is true for each $i=1,2,\cdots,n$.

We now need to show that $C_i \subset U_i$ and $C_i' \subset U_i'$. If $x \in C_i$, then $x_i = 1$, and $\|f\circ g(x)\| \leq \frac{1}{2}$. So $|P_i(f\circ g(x))| < 1$, and $Q_i(x) > 0$. Hence, $C_i \subset U_i$. Likewise, $C_i' \subset U_i'$. Thus, by IV there is an $x_o \in \bigcap_{i=1}^{n} K_i$. Hence, $f \circ g(x_o) = x_o$. Since $f \circ g: I_n \rightarrow B$, we must have $x_o \in B$. So, $g(x_o) = x_o$, and it follows that $f(x_o) = x_o$. Since every n-cell in E^n is homeomorphic to B, we have that IV implies I. $\square$

I $\Rightarrow$ IV. Let K_i be a closed set which separates C_i and C_i', for $i=1,2,\cdots,n$. Let U_i and U_i' be open sets in I_n such that $I_n - K_i = U_i \cup U_i'$, $U_i \cap U_i'$ is empty, and $C_i \subset U_i$, $C_i' \subset U_i'$. For $x \in I_n$, let $v(x)$ be the mapping whose i-th component, $v_i(x)$, is defined by

$$v_i(x) = \begin{cases} -d(x,K_i) & \text{if } x \in U_i, \\ d(x,K_i) & \text{if } x \in U_i' \\ 0 & \text{if } x \in K_i \end{cases}$$

where $d(x,K_i) = \inf\{\|x-k\|: k \in K_i\}$. Now define

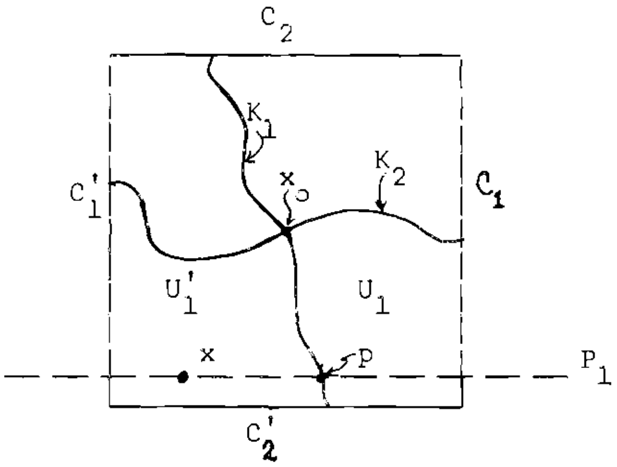
$$f(x) = x + v(x) \qquad \text{for } x \in I_n.$$

We want to show that $f: I_n \rightarrow I_n$ and that f is continuous.

Let $x \in I_n$. We first show $|x_i + v_i(x)| \leq 1$ for $i=1,2,\cdots,n$. We illustrate this with the diagram below. Fix i. Consider the hyperplane

$$P_i = \{y: y \in E^n, \ y_j = x_j \ \text{ for } j \neq i\}$$

Since K_i separates C_i and C_i', $P_i \cap K_i$ must be nonempty. Otherwise, C_i and C_i' can be connected by P_i and, hence, are not separated by K_i. Thus, there is a point $p \in K_i$ with $p_j = x_j$ for $j \neq i$, and $d(x,K_i) \leq d(x,p) = |x_i - p_i|$.



Case 1. Suppose $x \in U_i$. Then $x_i > p_i$ and $d(x,K_i) \leq x_i - p_i$. So, $p_i = x_i - (x_i - p_i) \leq x_i - d(x,K_i) = x_i + v_i(x) \leq x_i$. That is,

$$-1 \leq p_i \leq x_i + v_i(x) \leq x_i \leq 1.$$

Hence, $|x_i + v_i(x)| \leq 1$.

Case 2. Suppose $x \in U_i'$. Then $p_i > x_i$ and $d(x,K_i) \le p_i - x_i$. So, $x_i \le x_i + d(x,K_i) = x_i + v_i(x) \le x_i + (p_i - x_i) = p_i$. That is,

$$-1 \le x_i \le x_i + v_i(x) \le p_i \le 1.$$

Hence, $|x_i + v_i(x)| \le 1$.

Case 3. Suppose $x \in K_i$. Then $x_i + v_i(x) = x_i$, and $|x_i + v_i(x)| \le 1$. Thus $f\colon I_n \to I_n$

We now need to show that f is continuous. Note that f(x) is continuous if and only if each $v_i(x)$ is continuous. We assume it is known that for a fixed non-empty set A in E^n, d(x,A) is a continuous function defined on E^n.

Case 1. Let $x \in U_i$. Let $\{x^j\}$ be a sequence in I_n with $x^j \to x$. For some integer N, for all $k \ge N$, $x^j \in U_i$, since U_i is open in I_n. Thus, for $k \ge N$

$$v_i(x^j) = -d(x^j,K_i) \longrightarrow -d(x,K_i) = v_i(x).$$

Case 2. Let $x \in U_i'$. This case is similar to Case 1.

Case 3. Let $x \in K_i$. Let $\varepsilon > 0$ be given. Let $V = \{y\colon y \in I_n, d(x,y) < \varepsilon\}$, and let $y \in V$. Since $x \in K_i$, $d(y,K_i) \le d(y,x) < \varepsilon$, and

$$|v_i(y) - v_i(x)| = |\pm d(y,K_i) - 0| = d(y,K_i) < \varepsilon.$$

Thus, each $v_i(x)$ is continuous, and, hence, f(x) is continuous.

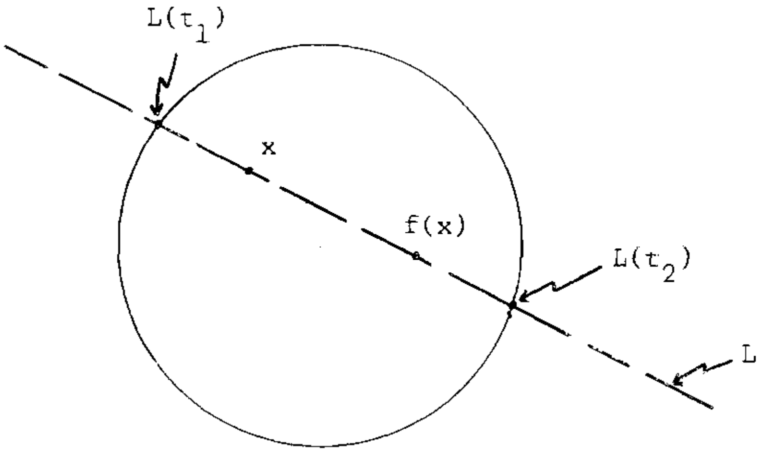
By I, there exists an $x_o \in I_n$ such that $f(x_o) = x_o$. Thus, $v(x_o) = 0$, and $x_o \in K_i$ for $i=1,2,\cdots,n$. Hence, $x_o \in \bigcap_{i=1}^{n} K_i$. ▯

I $\Longrightarrow$ II: Suppose there exists a continuous mapping $f: B_n \rightarrow S_{n-1}$ such that

$$f(x) = x \qquad \text{for each } x \in S_{n-1}.$$

Then let $g(x) = -x$ be defined on S_{n-1}. Note that g is continuous on S_{n-1}, so $g \circ f(x) = g(f(x))$ is continuous on B_n, and $g \circ f: B_n \rightarrow S_{n-1} \subsetneq B_n$. Now for $x \in B_n - S_{n-1}$, $g \circ f(x) \neq x$. For $x \in S_{n-1}$, $\|x\| = 1$. So $x \neq 0$, and $g \circ f(x) = g(f(x)) = g(x) = -x \neq x$. Hence, $g \circ f$ has no fixed point. This contradicts I. ▯

II $\Longrightarrow$ I. Assume $f: B_n \rightarrow B_n$ is a continuous mapping and $f(x) \neq x$ for any $x \in B_n$. Consider the line $L = \{L(t): L(t) = (1-t)x + tf(x), -\infty<t<\infty\}$. Clearly L is the line through x and f(x), and there exist exactly two values of t so that $\|L(t)\| = 1$. (See diagram.) In proving this, we use



the notation $\langle x,y \rangle$ to mean the usual inner product of x and y in E^n, and $\|x\|^2 = \langle x,x\rangle$. We want to solve the equation $\|(1-t)x + tf(x)\| = 1$. We have

$$0 = \|(1-t)x + tf(x)\|^2 - 1$$

$$= \langle (1-t)x + tf(x), (1-t)x + tf(x)\rangle - 1$$

$$= (1-t)^2\|x\|^2 + 2t(1-t)\langle x,f(x)\rangle + t^2\|f(x)\|^2 - 1$$

$$= t^2(\|x\|^2 - 2\langle x,f(x)\rangle + \|f(x)\|^2) + 2t(\langle x,f(x)\rangle - \|x\|^2) + (\|x\|^2 - 1)$$

$$= \|x - f(x)\|^2t^2 + 2\langle x,f(x)-x\rangle t + (\|x\|^2 - 1)$$

Therefore,

$$t = \frac{-\langle x,f(x)-x\rangle \pm \sqrt{\langle x,f(x)-x\rangle^2 - \|x-f(x)\|^2(\|x\|^2-1)}}{\|x-f(x)\|^2}.$$

By assumption, $\|x-f(x)\| > 0$. If $\|x\| = 1$, the two solutions are

$$t_1 = 0 \quad \text{and} \quad t_2 = \frac{-\langle x,f(x)-x\rangle}{\|x-f(x)\|^2}.$$

We want to show that $t_2 > 0$. By the Schwarz inequality and since $\|f(x)\| \le 1$,

$$\langle x,f(x)\rangle \le |\langle x,f(x)\rangle| \le \|x\|\|f(x)\| \le \|x\| \cdot 1 = \|x\|^2.$$

Furthermore, equality holds in the Schwarz inequality if, and only if, x and f(x) are linearly dependent, or $f(x) = \pm x$. Since $f(x) \neq x$, if $f(x) = -x$, then $t_2 = 1$. Otherwise, $\langle x,f(x)\rangle < \|x\|^2$, and $-\langle x,f(x)-x\rangle = \|x\|^2 - \langle x,f(x)\rangle > 0$. So $t_2 > 0$.

On the other hand, if $\|x\| < 1$, then $t = 0$ is not a solution. We then have the discriminant

$$\langle x,f(x)-x\rangle^2 - \|x-f(x)\|^2(\|x\|^2-1) > \langle x,f(x)-x\rangle^2$$

and there exist solutions $t_1 < 0$ and $t_2 > 0$ which make $\|L(t)\| = 1$.

We use the solution t_1 in both cases, and define

$$t(x) = \frac{-\langle x,f(x)-x\rangle - \sqrt{\langle x,f(x)-x\rangle^2 - \|x-f(x)\|^2(\|x\|^2-1)}}{\|x-f(x)\|^2}$$

Since the inner product, as a function of two variables, is continuous on $E^n \times E^n$, then $t(x)$ is continuous on B_n.

Now define $g(x) = (1-t(x))x + t(x)f(x)$ on B_n. Clearly g is continuous on B_n, and for $x \in S_{n-1}$, or $\|x\| = 1$, we have $g(x) = x$. Thus, g leaves each point of S_{n-1} fixed. This contradicts II. ▯

II $\Rightarrow$ III. Suppose there exists a function $f(t,x)$: $[0,1] \times S_{n-1} \rightarrow S_{n-1}$ such that $f(1,x) = x$ and $f(0,x) = x_o$, where x_o is fixed in S_{n-1}, and f is continuous in the pair (t,x).

Let $y \in B_n$. Consider the mapping

$$F(y) = \begin{cases} x_o & \text{if } y = 0 \\ f(\|y\|, \frac{y}{\|y\|}) & \text{if } y \neq 0. \end{cases}$$

Clearly F maps B_n into S_{n-1} and leaves each point of S_{n-1} fixed. Using the properties of f, we want to show that F is continuous on B_n. It is evident that F is continuous at all $y \neq 0$. So consider a sequence $\{y_i\}$ in B_n such that $y_i \to 0$ and $y_i \neq 0$ for any i. Then $y_i/\|y_i\| \in S_{n-1}$ and $\|y_i\| \to 0$. By uniform continuity of f,

$$\|f(\|y_i\|, \frac{y_i}{\|y_i\|}) - f(0, \frac{y_i}{\|y_i\|})\| \to 0.$$

But for all i, $f(0, \frac{y_i}{\|y_i\|}) = x_o$. Thus, F is continuous at y = 0, also. However, we now have conditions on F which contradict II. So, there is no such function f. ▯

III ⇒ II. Suppose there exists a continuous mapping $F: B_n \to S_{n-1}$ such that $F(x) = x$ for each $x \in S_{n-1}$. Fix $x_o \in S_{n-1}$. Define the mapping f by

$$f(t,x) = F((1-t)x_o + tx)$$

for $t \in [0,1]$ and $x \in S_{n-1}$. Continuity of f follows immediately from the continuity of F. Furthermore,

$$f(0,x) = F(x_o) = x_o$$

and

$$f(1,x) = F(x) = x$$

for all $x \varepsilon X_{n-1}$. This contradicts III. ▯

This completes the proof of the equivalences.

CHAPTER III

EXTENSIONS OF BROUWER'S FIXED POINT THEOREM

One of the first generalizations to the Brouwer Fixed Point Theorem was due to Schauder. The Schauder Fixed Point Theorem deals with a normed linear space with no restriction to finite dimensionality. However, to compensate for this, Schauder requires that the mapping be more than just continuous. Before stating this theorem, we need to develop a few concepts.

Definition. A subset C in a topological space X is *compact* if, and only if, for every set $\{U_\alpha : \alpha \in A,\ U_\alpha \text{ is open in } X,\ C \subset \bigcup_{\alpha\in A} U_\alpha\}$ there is a finite subcollection $\{U_1, U_2, \cdots, U_n\}$ such that $C \subset \bigcup_{i=1}^{n} U_i$. More briefly, every open covering of C has a finite subcovering.

Definition. A subset K of a topological space X is *relatively compact* if and only if the closure of K in X, denoted by $\bar{K}$, is compact.

Lemma. A compact set C in a normed linear space X is totally bounded, i.e., given $\varepsilon > 0$, there is a finite set of elements $v_1, v_2, \cdots, v_n$ in C such that for each $y \in C$ there is at least one v_i such that $\|y - v_i\| < \varepsilon$. The set $\{v_1, v_2, \cdots, v_n\}$ is called an ε-net of C.

Proof. Let $\varepsilon > 0$ be given. For $v \in C$, define

$$N(v;\varepsilon) = \{y : y \in X,\ \|y - v\| < \varepsilon\}.$$

Then $\{N(v;\varepsilon) : v \in C\}$ is an open covering of C. So, there is a finite subcovering, say $N(v_1;\varepsilon)$, $N(v_2;\varepsilon)$, $\cdots$, $N(v_n;\varepsilon)$. Thus, $C \subset \bigcup_{i=1}^{n} N(v_i;\varepsilon)$, and for $y \in C$, we have $y \in N(v_i;\varepsilon)$ for some $1 \le i \le n$, or $\|y-v_i\| < \varepsilon$. $\square$

We now need to define a special type of continuous mapping in a normed linear space.

Definition. Let E be a subset of a normed linear space X. The transformation $T: E \to X$ is *completely continuous* if

(i) T is continuous, and

(ii) for each bounded subset M of E, T(M) is relatively compact.

Now suppose K is relatively compact in a normed linear space X. Let $v_1, v_2, \cdots, v_n$ be an ε-net for $\bar{K}$. For each $x \in \bar{K}$, define

$$F_\varepsilon(x) = \frac{\sum_{i=1}^{n} m_i(x) v_i}{\sum_{i=1}^{n} m_i(x)}$$

where
$$m_i(x) = \begin{cases} \varepsilon - \|x-v_i\| & \text{if } \|x-v_i\| \le \varepsilon \\ 0 & \text{if } \|x-v_i\| > \varepsilon \end{cases}$$

Lemma. Let T be a completely continuous transformation defined on a bounded set E in a normed linear space X. Let K be a relatively compact set in X, and let $T(E) \subset K$. If $F_\varepsilon(x)$ is defined on $\bar{K}$ as described above, then for each $x \in E$, we have

$$\|T(x) - F_\varepsilon \circ T(x)\| < \varepsilon.$$

Proof. Since $\|T(x)-v_i\| < \varepsilon$ for some v_i, then $m_i(T(x)) > 0$. Thus, $\sum_{i=1}^{n} m_i(T(x)) > 0$. Hence,

$$\|T(x) - F_\varepsilon \circ T(x)\| = \frac{\left\|T(x)\cdot\sum_{i=1}^{n} m_i(T(x)) - \sum_{i=1}^{n} m_i(T(x))v_i\right\|}{\sum_{i=1}^{n} m_i(T(x))}$$

$$\leq \frac{\sum_{i=1}^{n} m_i(T(x))\|T(x)-v_i\|}{\sum_{i=1}^{n} m_i(T(x))}$$

$$< \frac{\sum_{i=1}^{n} m_i(T(x))\varepsilon}{\sum_{i=1}^{n} m_i(T(x))}$$

$$= \varepsilon$$ ▯

We now state the Schauder Fixed Point Theorem. The proof presented here is basically that of Cronin [4].

Theorem 3.1. Let K be a closed bounded convex set in a real normed linear space X, and let T be a completely continuous transformation on K such that $T(K) \subset K$. Then T has a fixed point.

Proof. Since K is bounded, we have that T(K) is relatively compact, or $\overline{T(K)}$ is compact. Furthermore, since K is closed and $T(K) \subset K$, it follows that $\overline{T(K)} \subset K$.

Let $\{\varepsilon_n\}$ be a monotonic decreasing sequence with $\varepsilon_n \to 0$. Since $\overline{T(K)}$ is compact, there exists an ε_n-net, $v_1^n, v_2^n, \cdots, v_{i_n}^n$, of $\overline{T(K)}$ for each $n \geq 1$. We may now define $F_{\varepsilon_n}(y)$, as described above, for each $y \in \overline{T(K)}$ and each $n \geq 1$. It is easy to check that each F_{ε_n} is continuous on $\overline{T(K)}$. Now define

$$T_n(x) = F_{\varepsilon_n} \circ T(x)$$

for each $x \in K$. Clearly each T_n is continuous on K. Letting $M(x) = \sum_{i=1}^{i_n} m_i(T(x))$, since $\frac{m_i(T(x))}{M(x)} \geq 0$ for each i and $\sum_{i=1}^{i_n} \frac{m_i(T(x))}{M(x)} = 1$, by convexity of K we have that

$$T_n(x) = F_{\varepsilon_n}(T(x)) = \sum_{i=1}^{i_n} \frac{m_i(T(x))}{M(x)} v_i^n \qquad \varepsilon\ K$$

for each $x \in K$. That is, $T_n(K) \subset K$.

Consider the finite dimensional subspace X_n of X which is spanned by $v_1^n, v_2^n, \cdots, v_{i_n}^n$, the ε_n-net of $\overline{T(K)}$. Then X_n is linearly homeomorphic to some E^k, and is complete and, hence, closed. Define

$$K_n = K \cap X_n.$$

Since both K and X_n are closed and convex, K_n is also closed and convex. The fact that K is bounded in X implies K_n is bounded in X_n. Thus, K_n is homeomorphic to the closed unit ball $B = \{x: x \in X_n, \|x\| \leq 1\}$ in X_n, which is, in turn, homeomorphic to the closed unit ball B_k in E^k. So, K_n has the fixed point property. Observe that for $y \in X_n$, and $M(y) = \sum_{i=1}^{i_n} m_i(T(y))$

$$T_n(y) = F_\varepsilon(T(y)) = \sum_{i=1}^{i_n} \frac{m_i(T(y))v_i}{M(y)} \qquad \varepsilon\ X_n.$$

Thus, $T_n(X_n) \subset X_n$. It then follows that

$$T_n(K_n) = T_n(K \cap X_n) \subset T_n(K) \cap T_n(X_n) \subset K \cap X_n = K_n$$

We now have T_n continuous on K_n and $T_n(K_n) \subset K_n$. Hence, there exists a point $x_n \in K_n$ such that $T_n(x_n) = x_n$. This is true for each $n \geq 1$. The sequence $\{x_n\}$ is in K, so the sequence $\{T(x_n)\}$ is in $\overline{T(K)}$, which is compact. Hence, there exists a subsequence of $\{T(x_n)\}$ which converges to some point x_o in $\overline{T(K)}$, and hence, in K. For simplicity of notation assume the sequence $\{T(x_n)\}$ itself converges to x_o. Our aim is to show that $T(x_o) = x_o$.

Given $\varepsilon > 0$, there exists an integer N so that if $n \geq N$, then

$$\|T(x_n) - x_o\| < \frac{\varepsilon}{2}, \tag{1}$$

and ε_n (from above) is less than $\frac{\varepsilon}{2}$. Hence, by the previous lemma,

$$\|T(x_n) - T_n(x_n)\| < \varepsilon_n < \frac{\varepsilon}{2}. \tag{2}$$

Adding (1) and (2), we have that for $n \geq N$,

$$\|T_n(x_n) - x_o\| < \varepsilon.$$

Since $T_n(x_n) = x_n$, for such n,

$$\|x_n - x_o\| < \varepsilon.$$

Thus, $x_n \to x_o$. By continuity of T, we have $T(x_n) \to T(x_o)$. Previously, we had $T(x_n) \to x_o$. By uniqueness of limits in a Hausdorff space, we must have $T(x_o) = x_o$. □

It is interesting to note that in dropping the restriction that the normed linear space be of finite dimension, we possibly lose compactness of the closed unit ball. To compensate, Schauder needs the mapping to be completely continuous to prove his generalization.

Another generalization of Brouwer's Fixed Point Theorem, which requires less structure on the space on which the continuous mapping is defined, is due to Tychonoff. The proof used here was furnished by W. J. Kammerer. We first review three concepts which shall arise in the ensuing discussion.

Definition. A non-negative real-valued function p(x) defined on a linear space X over a field F is a *semi-norm* if the following conditions hold:

(i) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in F$, all $x \in X$.

(ii) $p(x+y) \le p(x) + p(y)$ for all $x,y \in X$.

If, in addition, $p(x) = 0$ if and only if $x = 0$, then p(x) is a norm.

Definition. A linear topological space X is *locally convex* if for every open set N containing the origin, there is a convex open set U containing the origin with $U \subset N$.

It can be shown that a linear topological space X is locally convex, if and only if, there is a family of semi-norms defined on X which generates the topology on X. To clarify this, let p_α be a semi-norm on X, and let T_α be the topology generated by p_α. The topology generated by a family of semi-norms $\{p_\alpha: \alpha \varepsilon \Lambda\}$ has as a subbase the union of all sets in each T_α, i.e., $\cup\{T_\alpha: \alpha \varepsilon \Lambda\}$ A thorough discussion of the equivalence above can be found in Yosida [15]. With this background, we are ready to state Tychonoff's theorem.

Theorem 3.2. Let X be a locally convex Hausdorff linear topological space. If K is a non-empty compact convex set in X, then every continuous mapping from K into K has a fixed point.

Before proceeding directly to the proof, we prove a helpful lemma.

Lemma. Let X and K be as stated in the above theorem. If f is a continuous real-valued function defined on K × K such that for every fixed y ε K, f(x,y) is a convex function of x, then there exists a point $\tilde{y}$ ε K such that

$$f(\tilde{y},\tilde{y}) \leq f(x,\tilde{y}) \quad \text{for all } x \varepsilon K.$$

Remark. To say that f(x,y) is a convex function of x for each fixed y means that for x_1, x_2 ε K and $0 \leq t \leq 1$,

$$f(tx_1 + (1-t)x_2, y) \leq tf(x_1,y) + (1-t)f(x_2,y).$$

It is easy to show that under such conditions, for any

$x_1, x_2, \cdots, x_n \in K$, and any real numbers $t_1, t_2, \cdots, t_n$ such that each $t_i \geq 0$ and $\sum_{i=1}^{n} t_i = 1$,

$$f(\sum_{i=1}^{n} t_i x_i, t) \leq \sum_{i=1}^{n} t_i f(x_i, y).$$

Proof of the Lemma. For $x \in K$, define $C_x = \{y: y \in K$ and $f(y,y) - f(x,y) \leq 0\}$. We want to show that $\bigcap_{x \in K} C_x$ is non-empty. Since $x \in C_x$, each C_x is non-empty. To show each C_x is closed, let $\{y_k\}$ be a convergent sequence in C_x with $y_k \longrightarrow y$. Then for each k,

$$f(y_k, y_k) - f(x, y_k) \leq 0$$

By the continuity of f,

$$\lim_{k \to \infty} f(y_k, y_k) - f(x, y_k) \leq 0.$$

Thus,

$$f(y,y) - f(x,y) \leq 0.$$

So, $y \in C_x$, and C_x is closed. The collection $\{C_x: x \in K\}$ is, therefore, a family of non-empty closed sets in K. We now appeal to a condition which is equivalent to that of K being compact, namely the finite intersection property. This states that if $\{A_\alpha: \alpha \in \Lambda\}$ is a collection of non-empty closed sets in K such that for any finite subcollection

$A_1, A_2, \cdots, A_n$, their intersection $\bigcap_{i=1}^{n} A_i$ is non-empty, then $\bigcap_{\alpha\varepsilon\Lambda} A_\alpha$ is non-empty. We proceed to show that $\{C_x : x \varepsilon K\}$ is such a collection.

Let $\{x_1, x_2, \cdots, x_n\} \subset K$. Let H be the convex hull of $\{x_1, x_2, \cdots, x_n\}$. Since K is convex, then $H \subset K$. We will show there is a point $y \varepsilon H$ such that $y \varepsilon \bigcap_{i=1}^{n} C_{x_i}$. For any $y \varepsilon H$, $y = \sum_{i=1}^{n} t_i x_i$ where each $t_i \geq 0$ and $\sum_{i=1}^{n} t_i = 1$. Define for each $y \varepsilon H$

$$g_i(y) = \max\{f(y,y) - f(x_i,y), 0\} \qquad \text{for } i=1,2,\cdots,n.$$

Clearly each g_i is continuous on H. Also, observe that the following three conditions are equivalent:

(a) $\qquad f(y,y) \leq f(x_i,y) \qquad$ for $i=1,2,\cdots,n$

(b) $\qquad g_i(y) = 0 \qquad$ for $i=1,2,\cdots,n$

(c) $\qquad t_i \sum_{k=1}^{n} g_k(y) = g_i(y) \qquad$ for $i=1,2,\cdots,n$

Clearly (a) is equivalent to (b), and (b) implies (c). The only difficulty arises in showing (c) implies (b). Let $y = \sum_{i=1}^{n} t_i x_i$, where each $t_i \geq 0$, and $\sum_{i=1}^{n} t_i = 1$. Then

$$f(y,y) = f\left(\sum_{i=1}^{n} t_i x_i, y\right) \leq \sum_{i=1}^{n} t_i f(x_i, y).$$

Note that $f(y,y) = \sum_{i=1}^{n} t_i f(y,y)$. Therefore,

$$\sum_{i=1}^{n} t_i(f(y,y) - f(x_i,y)) \leq 0. \qquad (*)$$

Consider those $t_i > 0$. If $f(y,y) - f(x_i,y) > 0$ for each such t_i, we then have a contradiction with (*). Thus, for some $t_j > 0$, $f(y,y) - f(x_j,y) \leq 0$. Thus $g_j(y) = 0$. By (c), $t_j \cdot \sum_{k=1}^{n} g_k(y) = g_j(y) = 0$. Since each $g_k(y) \geq 0$, we have that $g_k(y) = 0$ for each k.

Now consider the (n-1)simplex $S = \{(t_1,t_2,\cdots,t_n): t_i \geq 0, \sum_{i=1}^{n} t_i \leq 1\}$. Define the mapping ϕ: $S \to S$ as follows: Letting $y = \sum_{j=1}^{n} t_j x_j$,

$$\phi(t_1,t_2,\cdots,t_n) = (t_1',t_2',\cdots,t_n')$$

where

$$t_i' = \frac{t_i + g_i(y)}{1 + \sum_{k=1}^{n} g_k(y)} \quad \text{for } i=1,2,\cdots,n.$$

It is easily verified that $(t_1',t_2',\cdots,t_n') \in S$. Since each g_i is continuous, ϕ is continuous. So by Brouwer's Fixed Point Theorem there is a point $(\tilde{t}_1,\tilde{t}_2,\cdots,\tilde{t}_n)$ in S such that

$$\tilde{t}_i = \frac{\tilde{t}_i + g_i(\tilde{y})}{1 + \sum_{k=1}^{n} g_k(\tilde{y})} \quad \text{for each } i,$$

where

$$\tilde{y} = \sum_{i=1}^{n} \tilde{t}_i x_i .$$

Therefore,

$$\tilde{t}_i \sum_{k=1}^{n} g_k(\tilde{y}) = g_i(\tilde{y}) \quad \text{for } i=1,2,\cdots,n$$

This is condition (c). Consequently,

$$g_i(\tilde{y}) = 0 \quad \text{for } i=1,2,\cdots,n$$

This says $\tilde{y} \in C_{x_i}$ for each i. So, $\tilde{y} \in \bigcap_{i=1}^{n} C_{x_i}$. By the finite intersection property, we have there is a $y \in \bigcap_{x \in K} C_x$, or

$$f(y,y) \leq f(x,y) \quad \text{for all } x \in K.$$

This proves the lemma.

Proof of Tychonoff's Fixed Point Theorem. Let $\{p_\alpha: \alpha \in \Lambda\}$ be a family of semi-norms which generate the topology on X. Then each p_α is continuous on X. Define $C_\alpha = \{y \in K: p_\alpha(y-f(y)) = 0\}$. Clearly each C_α is closed since p_α and f are continuous. Since X is Hausdorff, if $x \neq y$, there exists a p_α so that $p_\alpha(x-y) > 0$. Thus, if $p_\alpha(y-f(x)) = 0$ for all $\alpha \in \Lambda$, then $y = f(y)$. So, we want to show $\bigcap\{C_\alpha: \alpha \in \Lambda\}$ is non-empty. Again we appeal to the finite intersection property since K is compact. Let $\{\alpha_1,\alpha_2,\cdots,\alpha_n\} \subset \Lambda$. Define

$$g(x,y) = \sum_{i=1}^{n} p_{\alpha_i}(x-f(y)).$$

Then $g(x,y)$ is continuous on $K \times K$ and convex in x. This follows, since for $x_1, x_2 \in K$ and $0 \le t \le 1$, $p_\alpha(tx_1 + (1-t)x_2 - f(y)) \le tp_\alpha(x_1 - f(y)) + (1-t)p_\alpha(x_2 - f(y))$. The conditions of the lemma are satisfied. Hence, there is a $y \in K$ such that $g(y,y) \le g(x,y)$ for all $x \in K$. That is,

$$\sum_{i=1}^{n} p_{\alpha_i}(y-f(y)) \le \sum_{i=1}^{n} p_{\alpha_i}(x-f(y)) \quad \text{for all } x \in K.$$

Note that $f(y) \in K$. So for $x = f(y)$, $p_{\alpha_i}(x-f(y)) = 0$ for $i=1,2,\cdots,n$. Thus, $p_{\alpha_i}(y-f(y)) = 0$ for $i=1,2,\cdots,n$, or $y \in \bigcap_{i=1}^{n} C_{\alpha_i}$. We, therefore, conclude that there exists a point $y \in \bigcap\{C_\alpha : \alpha \in \Lambda\}$. So, $f(y) = y$. $\square$

The generalizations presented up to this point have dealt with changes in the hypotheses concerning the structure of the space and the properties of the function. We now present an extension which considers changes in the hypotheses concerning the set on which the mapping is defined. This theorem is due to Brown [2].

Theorem 3.3. Let S be a compact set in E^n, and let C be an n-cell or a single point in E^n with $C \subset S$. Let f be a continuous map from S into E^n which carries the boundary B of S into C. Then f has a fixed point.

Proof. Let T be a Euclidean n-cell containing S and $f(S)$, and let $T_1 = T$ minus all interior points of S. Now B is a compact subset of T_1, and f maps B into C. Applying Tietze's extension theorem we can extend f to a continuous mapping f^* from T_1 into C. Now define

$$f' = \begin{cases} f & \text{on } S \\ \\ f^* & \text{on } T_1 \end{cases} .$$

It is easy to show that f' is continuous on T, and $f'(T) \subset T$. By Brouwer's theorem, there is a point $x_o \in T$ such that $f'(x_o) = x_o$. Note that if $x \notin S$, $f'(x) \in C \subset S$. Thus, $x_o \in S$, and on S, $f' = f$. So $f(x_o) = x_o$. □

Our next generalization actually generalizes both the Banach Fixed Point Theorem for contraction mappings and the Schauder Fixed Point Theorem. This theorem is due to Krasnosel'skii [11,12].

We first state the Banach Fixed Point Theorem and prove a lemma which will be useful in the proof of the generalization.

Banach Fixed Point Theorem. If g is a contraction mapping with domain D, a closed subset of a Banach space X, such that $g(D) \subset D$, then g has a unique fixed point.

Recall that g is a contraction mapping on D if, and only if, there is a number $\alpha \in [0,1)$ such that for any pair $x,y \in D$, it follows that

$$\|g(x)-g(y)\| \le \alpha\|x-y\| .$$

The following lemma gives an equivalent statement to the definition of a completely continuous operator. Recall that in a normed linear space X if $E \subset X$, an operator $h: E \to X$ is completely continuous if h is continuous and for each bounded subset M of E, h(M) is relatively compact.

Lemma. Let E be a subset of a normed linear space X. An operator h: E→X is completely continuous on E if, and only if,

(i) h is continuous, and

(ii) for every bounded sequence $\{x_k\}$ in E, the sequence $\{h(x_k)\}$ contains a subsequence converging to some point in X.

Proof. This lemma follows easily if we use the fact that in a normed linear space compactness is equivalent to sequential compactness. We say that a set C is sequentially compact if every sequence in C contains a subsequence which converges to a point in C.

Assume h is completely continuous on E. Let $\{x_k\}$ be a bounded sequence in E. Then $\{x_k\}$ is contained in some bounded set M in E. We know that $\overline{h(M)}$ is compact. So the sequence $\{h(x_k)\}$ contains a subsequence which converges to a point in $\overline{h(M)}$, by sequential compactness.

Now assume conditions (i) and (ii) above hold. We need to show that the image of every bounded subset of E is relatively compact. Let M be a bounded subset of E. We will show that $\overline{h(M)}$ is sequentially compact. Let $\{y_k\}$ be a sequence in $\overline{h(M)}$. We consider two cases.

Case 1. If there is a subsequence $\{y_{k_i}\}$ such that each $y_{k_i} \in h(M)$, then $y_{k_i} = h(x_i)$ for some $x_i \in M$, for $i=1,2,\cdots$. By condition (ii), since the sequence $\{x_i\}$ is bounded in E, there is a subsequence of $\{y_{k_i}\}$ which converges to some point y in X. Since $\overline{h(M)}$ is closed, $y \in \overline{h(M)}$.

Case 2. If there is a subsequence $\{y_{k_i}\}$ in $\overline{h(M)}$, but not in $h(M)$, for each i there is a point $x_i \in M$ such that

$$\|y_{k_i} - h(x_i)\| < \frac{1}{i} .$$

Now by condition (ii) there is a subsequence $\{h(x_{i_j})\}$ which converges to a point y in X. Again, $y \in \overline{h(M)}$. Then given $\varepsilon > 0$, there is an integer J such that for all $j \geq J$,

$$\|h(x_{i_j}) - y\| < \frac{\varepsilon}{2} .$$

Choosing an integer i_j such that $i_j > \frac{2}{\varepsilon}$ and $j \geq J$, we have

$$\|y_{k_{i_j}} - y\| \leq \|y_{k_{i_j}} - h(x_{i_j})\| + \|h(x_{i_j}) - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

So, $y_{k_{i_j}} \to y$.

Thus, $\overline{h(M)}$ is sequentially compact, and h is completely continuous. □

We are now ready to prove the generalization due to Krasnosel'-skii.

Theorem 3.4. Let X be a Banach space and B be a closed, bounded, convex set in X. Let $f: B \to B$ be a mapping such that $f = g + h$, where g is contractive on B and h is completely continuous on B, and such that for every $x,y \in B$, $g(x) + h(y) \in B$. Then f has a fixed point.

Proof. Fix $x_o \varepsilon B$. Consider the mapping g^* given by $g^*(x) = g(x) + h(x_o)$ for $x \varepsilon B$. Since g is contractive on B, there exists an α $(0<\alpha<1)$ such that $\|g(x) - g(y)\| \leq \alpha\|x - y\|$ for all $x,y \varepsilon B$. Then it is easy to see that g^* is also contractive on B, and the same α works for it. Thus, by the Banach Fixed Point Theorem, there is a unique fixed point for g^*, call it $\Psi(x_o)$. That is, $\Psi(x_o) = g(\Psi(x_o)) + h(x_o)$, and by the hypotheses, $\Psi(x_o) \varepsilon B$. This procedure defines a mapping Ψ on B such that $\Psi(B) \subset B$. We want to show that Ψ is a completely continuous operator.

To show Ψ is continuous on B, let x_o be an arbitrary point in B. By continuity of h, given $\varepsilon > 0$, there exists a $\delta > 0$ so that if $\|y - x_o\| < \delta$ and $y \varepsilon B$, then $\|h(y) - h(x_o)\| < \varepsilon(1-\alpha)$. Note $1-\alpha>0$. Then

$$\begin{aligned}\|\Psi(y) - \Psi(x_o)\| &= \|g(\Psi(y)) + h(y) - [g(\Psi(x_o)) + h(x_o)]\| \\ &\leq \|g(\Psi(y)) - g(\Psi(x_o))\| + \|h(y) - h(x_o)\| \\ &< \alpha\|\Psi(y) - \Psi(x_o)\| + \varepsilon(1-\alpha).\end{aligned}$$

Therefore,

$$(1-\alpha)\|\Psi(y) - \Psi(x_o)\| < \varepsilon(1-\alpha),$$

or

$$\|\Psi(y) - \Psi(x_o)\| < \varepsilon.$$

Hence, Ψ is continuous on B.

Recalling the condition which is equivalent to the definition of a completely continuous operator, we want to show that for every bounded sequence $\{x_k\}$ in B, the sequence $\{\Psi(x_k)\}$ has a convergent subsequence. Note that by the completeness of X, any Cauchy sequence in X converges to a point in X. Thus, any Cauchy sequence in $\overline{\Psi(B)}$ converges to a point in $\overline{\Psi(B)}$. Also observe that every sequence in B is bounded, because B is bounded.

Let $\{x_k\}$ be a sequence in B. Consider the sequence $\{\Psi(x_k)\}$ in $\Psi(B)$. Then for each k,

$$\Psi(x_k) = g(\Psi(x_k)) + h(x_k).$$

So,

$$h(x_k) = \Psi(x_k) - g(\Psi(x_k)).$$

Since h is completely continuous on B, the sequence $\{h(x_k)\}$ has a convergent subsequence, say $\{h(x_{k_i})\}$. We show this implies that the sequence $\{\Psi(x_{k_i})\}$ is convergent in $\overline{\Psi(B)}$. Let $\varepsilon > 0$ be given. There exists an integer N so that if $k_i, k_j \geq N$, then $\|h(x_{k_i}) - h(x_{k_j})\| < \varepsilon(1-\alpha)$. Thus, for $k_i, k_j \geq N$,

$$(1-\alpha)\varepsilon > \|[\Psi(x_{k_i}) - g(\Psi(x_{k_i}))] - [\Psi(x_j) - g(\Psi(x_{k_j}))]\|$$

$$\geq \|\Psi(x_{k_i}) - \Psi(x_{k_j})\| - \|g(\Psi(x_{k_i})) - g(\Psi(x_{k_j}))\|$$

$$\geq \|\Psi(x_{k_i}) - \Psi(x_{k_j})\| - \alpha\|\Psi(x_{k_i}) - \Psi(x_{k_j})\|$$

$$= (1-\alpha)\|\Psi(x_{k_i}) - \Psi(x_{k_j})\|.$$

Therefore,

$$\|\Psi(x_{k_i}) - \Psi(x_{k_j})\| < \varepsilon.$$

So, $\{\Psi(x_{k_i})\}$ is a convergent sequence in $\overline{\Psi(B)}$. By previous remarks, the sequence converges to a point in $\overline{\psi(B)}$. This proves that Ψ is completely continuous.

Now applying the Schauder Fixed Point Theorem to Ψ, we have that there is a point $x_o \varepsilon B$ such that $\Psi(x_o) = x_o$. Thus,

$$x_o = \Psi(x_o) = g(\Psi(x_o)) + h(x_o) = g(x_o) + h(x_o)$$

or

$$f(x_o) = x_o. \qquad \Box$$

The fixed point theorem of Brouwer has been further generalized to certain point-to-set mappings by S. Kakutani [9], who uses his generalization to prove some theorems due to J. von Neumann which are applicable to the theory of games. We will present the generalization here, and in the following chapter present the theorems due to von Neumann, with proofs essentially those of Kakutani.

In the course of this discussion, we adopt the following notation. If S is a closed bounded convex set in E^n, let K(S) denote the set of all non-empty closed convex subsets of S. We also need the following definition.

Definition. A mapping $\Phi: S \to K(S)$ is called *upper semi-continuous* if given a sequence $\{x_n\}$ in S with $x_n \to x_o$, $y_n \in \Phi(x_n)$, and $y_n \to y_o$ it follows that $y_o \in \Phi(x_o)$.

Kakutani's generalization may then be stated as follows:

Theorem 3.5. If S is an r-dimensional closed simplex and $\Phi: S \to K(S)$ is an upper semi-continuous point-to-set mapping, then there exists an $x_o \in S$ such that $x_o \in \Phi(x_o)$.

Proof. Let $S^{(n)}$ be the nth barycentric simplical subdivision of S. We want to define a continuous mapping from S into S in terms of the vertices of $S^{(n)}$. For each vertex e^n of $S^{(n)}$ define $\Phi_n(e^n)$ to be some arbitrary point $y^n \in \Phi(e^n)$. Now if $x \in S$, then x is in at least one r-subsimplex in $S^{(n)}$. If x is in only one such r-subsimplex T whose vertices are $e_o^n, e_1^n, \cdots, e_r^n$, the x may be written as $x = \sum_{i=0}^{r} \alpha_i(x) e_i^n$, where $\sum_{i=0}^{r} \alpha_i(x) = 1$ and each $\alpha_i(x) \geq 0$. Since ϕ_n is already defined on the vertices $e_o^n, e_1^n, \ldots, e_n^n$, we can extend ϕ_n linearly to all of T. That is, let

$$\phi_n(x) = \sum_{i=0}^{r} \alpha_i(x) \phi_n(e_i^n).$$

This mapping is clearly well-defined and continuous on the interior of each subsimplex of $S^{(n)}$. Suppose, however, that x is contained in two subsimplexes T_1 and T_2. Then x must be on a boundary (or face) shared by both simplexes, whose vertices are, say, $e_0^n, e_1^n, \cdots, e_k^n$, $k<r$. These vertices are common to both T_1 and T_2. So, if $T_1 = |e_0^n, e_1^n, \cdots, e_k^n, a_1, \cdots, a_{r-k}|$ and $T_2 = |e_0^n, e_1^n, \cdots, e_k^n, b_1, \cdots, b_{r-k}|$, then x has the representation $x = \sum_{i=0}^{k} \alpha_i(x) e_i^n + \sum_{i=1}^{r-k} \alpha_{k+i}(x) a_i$ where $\sum_{i=0}^{r} \alpha_i(x) = 1$ and each $\alpha_i(x) \geq 0$. Moreover, for $i \geq k+1$, $\alpha_i(x) = 0$. Similarly, $x = \sum_{i=0}^{k} \beta_i(x) e_i^n + \sum_{i=1}^{r-k} \beta_{k+i}(x) b_i$ and for each $i \geq k+1$, $\beta_i(x) = 0$. Thus, by linear independence of the e_i^n, $\alpha_i(x) = \beta_i(x)$ for $i=0,1,\cdots,r$. Therefore, $\phi_n(x)$ is well-defined and continuous on all of S.

We may now apply Brouwer's fixed point theorem to ϕ_n for each $n \geq 1$ and conclude that there exists an $x_n \in S$ such that $\phi_n(x_n) = x_n$. The sequence $\{x_n\}$ which is in the compact set S must have a convergent subsequence $\{x_{n_\nu}\}$, where $x_{n_\nu} \to x_o$, and $x_o \in S$. We will show that $x_o \in \Phi(x_o)$.

Let T_n be an r-dimensional subsimplex in $S^{(n)}$ which contains the point x_n and has vertices $e_0^n, e_1^n, \cdots, e_r^n$. Recall that for any set A, the diameter of A, d(A), is defined by $d(A) = \sup\{\|x-y\| : x,y \in A\}$. Thus, it is certainly true that $d(T_n) \leq \left(\frac{r-1}{r}\right)^n d(S)$ for $n=1,2,\cdots$. We use this fact to show that $e_i^{n_\nu} \to x_o$ for each $i=0,1,\cdots,r$. Let $\varepsilon > 0$ be given. Choose an integer N so that if $n_\nu \geq N$, then $\left(\frac{r-1}{r}\right)^{n_\nu} d(S) < \frac{\varepsilon}{2}$, and $\|x_{n_\nu} - x_o\| < \frac{\varepsilon}{2}$. Then $\|e_i^{n_\nu} - x_o\| \leq \|e_i^{n_\nu} - x_{n_\nu}\| + \|x_{n_\nu} - x_o\| < d(T_{n_\nu}) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, when $n_\nu \geq N$. Consequently, $e_i^{n_\nu} \to x_o$ for each $i=0,1,\cdots,r$.

For each x_n, there exist unique scalars $\lambda_i^n \geq 0$, $i=0,1,\cdots,r$ with $\sum_{i=0}^{r} \lambda^n = 1$ such that $x_n = \sum_{i=0}^{r} \lambda_i^n e_i^n$. Let $y_i^n = \phi_n(e_i^n)$ for $i=0,1,\cdots,r$; $n = 1,2,\cdots$. Since each e_i^n is a vertex in $S^{(n)}$, then $y_i^n \in \Phi(e_i^n)$. Also, $x_n = \phi_n(x_n) = \phi_n\left(\sum_{i=0}^{r} \lambda_i^n e_i^n\right) = \sum_{i=0}^{r} \lambda_i^n \phi_n(e_i^n) = \sum_{i=0}^{r} \lambda_i^n y_i^n$ for each $n \geq 1$.

Now consider a further subsequence $\{x_{n'_\nu}\}$ of $\{x_{n_\nu}\}$ such that the sequences $\{y_i^{n'_\nu}\}$ and $\{\lambda_i^{n'_\nu}\}$ for $i=0,1,\cdots,r$ converge. Denote their limits by $\lim_{\nu\to\infty} y_i^{n'_\nu} = y_i^o$ and $\lim_{\nu\to\infty} \lambda_i^{n'_\nu} = \lambda_i^o$, $i=0,1,\cdots,r$. Then $\lambda_i^o \geq 0$ and $\sum_{i=0}^{r} \lambda_i^o = \sum_{i=0}^{r} \lim_{\nu\to\infty} \lambda_i^{n'_\nu} = \lim_{\nu\to\infty} \sum_{i=0}^{r} \lambda_i^{n'_\nu} = 1$. Consequently, $x_o = \lim_{\nu\to\infty} x_{n'_\nu} = \lim_{\nu\to\infty} \sum_{i=0}^{r} \lambda_i^{n'_\nu} y_i^{n'_\nu} = \sum_{i=0}^{r} \lim_{\nu\to\infty}(\lambda_i^{n'_\nu} y_i^{n'_\nu}) = \sum_{i=0}^{r} \lambda_i^o y_i^o$. Gathering conclusions, we have $e_i^{n'_\nu} \to x_o$, $y_i^{n'_\nu} \in \Phi(e_i^{n'_\nu})$, and $y_i^{n'_\nu} \to y_i^o$ for $i=0,1,\cdots r$. By upper semi-continuity of Φ, these imply $y_i^o \in \Phi(x_o)$ for $i=0,1,\cdots,r$. Since $\Phi(x_o)$ is a convex set, and $x_o = \sum_{i=0}^{r} \lambda_i^o y_i^o$, it follows that $x_o \in \Phi(x_o)$. ▯

Corollary. Theorem 3.5 is true for any closed bounded convex set S in a Euclidean space.

Proof. Let S be a closed bounded convex set in E^n. Let S' be a closed simplex which contains S. Let Ψ be a mapping of S' onto S defined by the following procedure.

Since S is a compact convex set in E^n, for each $y \varepsilon E^n$ there is a unique closest point $x \varepsilon S$ to y. That is,

$$d(y,x) < d(y,s) \quad \text{for all } s \neq x,\ s \varepsilon S.$$

We define the mapping Ψ on S' to take a point $y \varepsilon S'$ into its unique closest point in S. Note that for $y \varepsilon S$, $\Psi(y) = y$. We want to show that Ψ is continuous on S'. Let $y_o \varepsilon S'$ and let $\Psi(y_o) = x_o$. Let $\{y_n\}$ be a sequence in S' such that $d(y_o,y_n) < \frac{1}{n}$ for $n=1,2,\cdots$. Let $\Psi(y_n) = x_n$. Then $\{x_n\}$ is a sequence in the compact set S. Thus, there is a convergent subsequence. For simplicity of notation, assume $\{x_n\}$ itself converges to some point $z \varepsilon S$. We will show that $z = x_o$. Since $y_n \rightarrow y_o$ and $x_n \rightarrow z$, continuity of the distance function implies that

$$d(y_n,x_n) \rightarrow d(y_o,z). \tag{1}$$

We now show that $d(y_n,x_n) \rightarrow d(y_o,x_o)$. By definition of Ψ and by the triangle inequality, we have that

$$d(y_o,x_o) \leq d(y_o,x_n) \leq d(y_o,y_n) + d(y_n,x_n).$$

Therefore,

$$d(y_o,x_o) - d(y_n,x_n) \leq d(y_o,y_n) < \frac{1}{n}.$$

Similarly,

$$d(y_n,x_n) \le d(y_n,x_o) \le d(y_n,y_o) + d(y_o,x_o),$$

and

$$d(y_n,x_n) - d(y_o,x_o) \le d(y_n,y_o) < \frac{1}{n}.$$

Hence,

$$|d(y_n,x_n) - d(y_o,x_o)| < \frac{1}{n} \quad \text{for each } n \ge 1.$$

So, $d(y_n,x_n) \to d(y_o,x_o)$. Combining this with (1), we have that

$$d(y_o,x_o) = d(y_o,z).$$

Since $x_o \in S$ and $z \in S$, and there is a unique closest point to y_o in S, we must have $z = x_o$. Hence, Ψ is continuous on S' and leaves each point in S fixed.

Now $\Phi \circ \Psi$: $S' \to K(S) \subset K(S')$, where $\Phi \circ \Psi(x) = \Phi(\Psi(x))$. Our aim is to show $\Phi \circ \Psi$ is upper semi-continuous on S'. Let $\{x_n\}$ be a sequence in S' such that $x_n \to x_o$, with $y_n \in \Phi \circ \Psi(x_n)$ and $y_n \to y_o$. We need to show these conditions imply $y_o \in \Phi \circ \Psi(x_o)$. By continuity of Ψ on S', $x_n \to x_o$ implies $\Psi(x_n) \to \Psi(x_o)$. Since Φ is upper semi-continuous on S, this sequence $\{\Psi(x_n)\}$ and the previous conditions on $\{y_n\}$ imply $y_o \in \Phi(\Psi(x_o)) = \Phi \circ \Psi(x_o)$. So $\Phi \circ \Psi$ is an upper semi-continuous mapping from S' into K(S'). Thus, by Theorem 3.5 there exists an $x_o \varepsilon S'$ such that $x_o \varepsilon \Phi \circ \Psi(x_o)$. Looking back at the definition of Ψ, since $\Phi \circ \Psi(x_o) \varepsilon K(S)$ implies $x_o \varepsilon S$, we have that $\Psi(x_o) = x_o$. Hence, $x_o \varepsilon \Phi(x_o)$. $\square$

CHAPTER IV

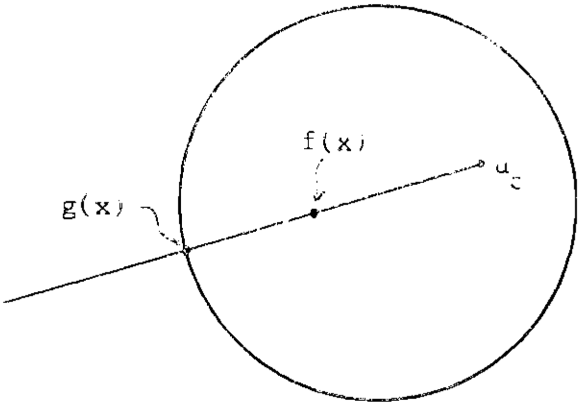
APPLICATIONS

Our first consideration involves a direct application of the Brouwer Fixed Point Theorem. Let C and D be two n-cells in E^n, and let $f: C \to D$ be a continuous mapping. If u_o is an interior of D, we are interested in the conditions we might impose upon f in order to assure a solution $x_o \in C$ to the equation $f(x) = u_o$. Using Brouwer's theorem we have determined some conditions on f to insure such a solution. The next three theorems, which deal with possible conditions for f, were suggested by R. H. Kasriel.

Theorem 4.1. Let $B = \{x: \|x\| \leq 1\}$ be the unit ball in E^n, and let u_o be an interior point of B. Let $i: B \to B$ be the identity mapping. Then there exists a $\delta > 0$ such that wherever a continuous mapping $f: B \to B$ satisfies the condition that $\|f(x) - i(x)\| < \delta$ for all $x \in B$, there exists a point $x_o \in B$ such that $f(x_o) = u_o$.

Proof. Suppose that for every $\delta > 0$ there exists at least one continuous mapping $f: B \to B$ which satisfies $\|f(x) - i(x)\| < \delta$ for all $x \in B$, but for which there is no solution in B to the equation $f(x) = u_o$. In particular for $\delta_o = \frac{1}{3} d(u_o, S_{n-1})$, where $S_{n-1} = \{x: x \in E^n, \|x\| = 1\}$, there must be such a function f. Using this function, we aim to construct a continuous mapping from B into B which has no fixed point.

We first project the point f(x) onto the sphere S_{n-1} in the manner we shall describe below. Consider the "half-line" through u_o and f(x) described by the set $\{y: y \in E^n, y = u_o + \lambda(f(x)-u_o)\}$. (See the diagram.) We shall show for each x, there is a unique $\lambda > 0$ such that $\|u_o + \lambda(f(x)-u_o)\| = 1$



We use the notation $u_o = (u_1,u_2,\cdots,u_n)$ and $f(x) = (y_1,y_2,\cdots,y_n)$. Then

$$1 = \|u_o + \lambda(f(x) - u_o)\|^2$$

$$= \sum_{i=1}^{n} (u_i + \lambda(y_i - u_i))^2$$

$$= \sum_{i=1}^{n} [\lambda^2(y_i-u_i)^2 + \lambda(2u_i(y_i-u_i)) + u_i^2]$$

$$= \lambda^2 \sum_{i=1}^{n} (y_i-u_i)^2 + \lambda(2 \sum_{i=1}^{n} u_i(y_i-u_i)) + \sum_{i=1}^{n} u_i^2$$

$$= \lambda^2\|f(x) - u_o\|^2 + \lambda(2 \sum_{i=1}^{n} u_i(y_i-u_i)) + \|u_o\|^2.$$

Hence,

$$\lambda^2\|f(x) - u_o\|^2 + \lambda(2 \sum_{i=1}^{n} u_i(y_i - u_i)) + \|u_o\|^2 - 1 = 0.$$

Let $a = \|f(x) - u_o\|^2$, $b = 2 \sum_{i=1}^{n} u_i(y_i - u_i)$, and $c = \|u_o\|^2 - 1$. The equation

$$a\lambda^2 + b\lambda + c = 0$$

implies that

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Clearly $a > 0$ and $c < 0$. Thus,

$$b^2 - 4ac > b^2.$$

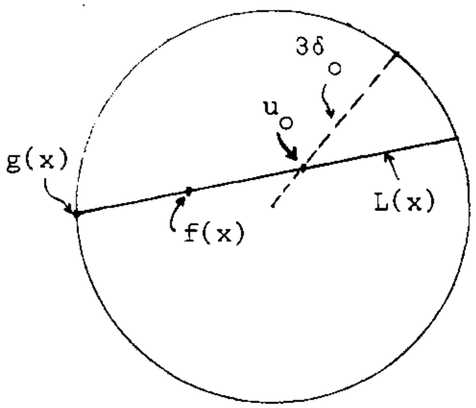
To solve the equation $\|u_o + \lambda(f(x)-u_o)\| = 1$ subject to the condition that $\lambda > 0$, we must choose the particular λ, call it $\lambda(x)$, defined by

$$\lambda(x) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Clearly a, b, and c are continuous functions defined on B. Thus, $\lambda(x)$ is continuous on B.

Now define the mapping $g\colon B \to B$ by $g(x) = u_o + \lambda(x)(f(x)-u_o)$. Then g is continuous on B and $\|g(x)\| = 1$ for all $x \in B$. The effect of g is to project f(x) radially outward from u_o onto the sphere S_{n-1}. The diagram above indicates this.

Next consider the continuous mapping h: $S_{n-1} \to S_{n-1}$ defined by $h(y) = -y$. Then the composite mapping $h \circ g(x) = h(g(x))$ is defined and continuous from B into S_{n-1}. We shall show that $h \circ g$ has no fixed point. Let L(x) be that part of the line through x_o, f(x), and g(x) which is interior to B. (See diagram below.) Let $\|L(x)\|$ denote its length. Note that $\|L(x)\| \leq$ diameter of B = 2.



We have

$$\|g(x) - f(x)\| + \|f(x) - u_o\| + 3\delta_o \leq \|L(x)\| \leq 2.$$

Thus,

$$\|g(x) - f(x)\| \leq (2-3\delta_o) - \|f(x) - u_o\| < 2 - 3\delta_o.$$

Recalling that $\|f(x) - x\| < \delta_o$, it follows that

$$\|g(x) - x\| \le \|g(x) - f(x)\| + \|f(x) - x\| < (2-3\delta_o) + \delta_o = 2 - 2\delta_o.$$

Now

$$\begin{aligned} \|h \circ g(x) - x\| &= \|h \circ g(x) - g(x) + g(x) - x\| \\ &\ge \|h(g(x)) - g(x)\| - \|g(x) - x\| \\ &= \|-2g(x)\| - \|g(x) - x\| \\ &= 2 - \|g(x) - x\| \\ &> 2 - (2 - 2\delta_o) \\ &= 2\delta_o \\ &> 0. \end{aligned}$$

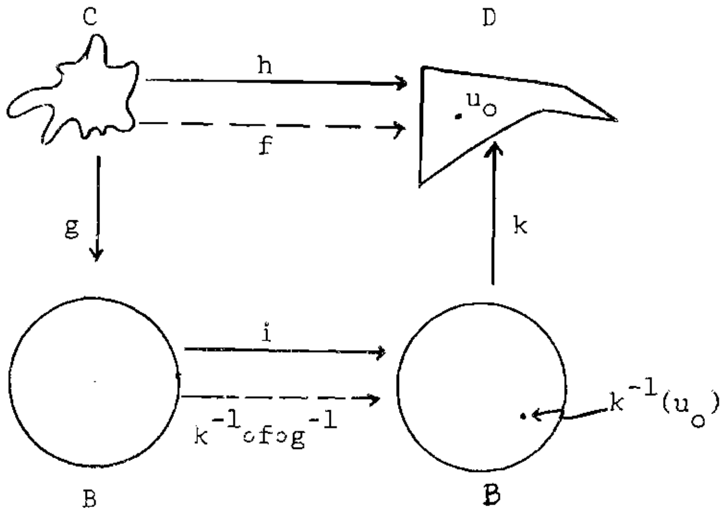
This says that $h \circ g$ has no fixed points, which contradicts Brouwer's Fixed Point Theorem. Hence, there exists a $\delta > 0$ so that if $f\colon B \to B$ is continuous and $\|f(x) - i(x)\| < \delta$ for all $x \in B$, then there is a solution $x_o \in B$ for the equation $f(x) = u_o$. ▯

Using Theorem 4.1 we can get some less restrictive conditions on the mapping to insure the desired solution. In order to prove our next theorem, we need to appeal to a theorem on invariance of domain, which may be found in Bers [1].

Theorem. (Invariance of an interior point.) Let A be a set in E^n. If $f: A \to E^n$ is continuous and one-to-one, then an interior point of A is mapped into an interior point of f(A).

Theorem 4.2. Let C and D be two n-cells in E^n, and let h be a homeomorphism from C onto D. If u_o is an interior point of D, there exists a $\delta > 0$ such that for each continuous mapping $f: C \to D$ which satisfies $\|f(x) - h(x)\| < \delta$ for all $x \in C$, there is a point $x_o \in C$ such that $f(x_o) = u_o$.

Proof. Let $B = \{x: x \in E^n, \|x\| \leq 1\}$. From the definition of an n-cell, there exist homeomorphisms g and k where $g: C \to B$, $k: B \to D$, and i is the identity mapping on B such that $h = k \circ i \circ g$. (See diagram below.)



Now let u_o be an interior point of D, and assume $f: C \to D$ is continuous. Since k is a homeomorphism, k^{-1} is continuous and one-to-one. By the theorem stated above, $k^{-1}(u_o)$ is an interior point of B. Moreover, the mapping $k^{-1} \circ f \circ g^{-1}: B \to B$ is continuous. Thus, by Theorem 4.1 there exists a $\delta > 0$ such that if

$$\|k^{-1} \circ f \circ g^{-1}(y) - i(y)\| < \delta \text{ for all } y \in B$$

then there exists a $y_o \in B$ such that $k^{-1} \circ f \circ g^{-1}(y_o) = k^{-1}(u_o)$. Hence, $f \circ g^{-1}(y_o) = u_o$. Letting $x_o = g^{-1}(y_o)$, which is in C, we have $f(x_o) = u_o$.

We now must find a δ_o such that if $\|f(x) - h(x)\| < \delta_o$ for all $x \in C$, then $\|k^{-1} \circ f \circ g^{-1}(y) - i(y)\| < \delta$ for all $y \in B$. The mapping k^{-1} is continuous on the compact set D and is, therefore, uniformly continuous. So, there exists a $\delta_o > 0$ such that if, for $x \in C$

$$\|f(x) - h(x)\| < \delta_o$$

then

$$\|k^{-1} \circ f(x) - k^{-1} \circ h(x)\| < \delta.$$

Note that if $y \in B$, there is a unique $x \in C$ with $x = g^{-1}(y)$. Assuming

$$\|f(x) - h(x)\| < \delta_o \quad \text{for all } x \in C,$$

we have

$$\|k^{-1} \circ f \circ g^{-1}(y) - i(y)\| = \|k^{-1} \circ f \circ g^{-1}(y) - k^{-1} \circ h \circ g^{-1}(y)\|$$

$$= \|k^{-1} \circ f(x) - k^{-1} \circ h(x)\|$$

$< \delta$ for all $y \in B$.

Thus, if $\|f(x) - h(x)\| < \delta_o$ for all $x \in C$, we are guaranteed a solution $x_o \in C$ to the equation $f(x) = u_o$. ▯

Our final result in this direction reads as follows.

Theorem 4.3. Let C and D be n-cells in E^n, and let h be a homeomorphism from C onto D. Let $\{f_n\}$ be a sequence of continuous mappings from C into D such that $f_n \to h$ uniformly. If u_o is an interior point of D, then there exists an integer N such that if $n \geq N$, $f_n(x) = u_o$ has a solution in C.

Proof. Since u_o is interior to D, by Theorem 4.2 there exists a $\delta > 0$ such that if $\|f_n(x) - h(x)\| < \delta$ for all $x \in C$, then $f_n(x) = u_o$ has a solution in C. By uniform convergence, there exists an integer N such that if $n \geq N$, then $\|f_n(x) - h(x)\| < \delta$ for all $x \in C$. Thus, for all $n \geq N$, $f_n(x) = u_o$ has a solution in C. ▯

Continuing in the vein of trying to find solutions to functional equations, we consider an application of the Schauder Fixed Point Theorem in proving a theorem due to Peano dealing with differential equations. We first discuss some facts which will be used in the proof.

Let X be a topological space and (Y,d) be a metric space. A set $\mathcal{F}$ of continuous mappings of X into Y is called equicontinuous at $x \in X$ if for every $\varepsilon > 0$, there is an open set $U \subset X$ containing x such that the image of U under each $f \in \mathcal{F}$ is a subset of the ball $B = \{y: y \in Y, d(y,f(x)) < \varepsilon\}$. If $\mathcal{F}$ is equicontinuous at each point of X, we say that $\mathcal{F}$ is equicontinuous on X.

We shall be interested in knowing if a set $\mathcal{F}$ of continuous mappings of a compact interval I in E^1 into a Banach space F is a compact subset of the set of all continuous mappings of I into F, which we denote by C[I,F]. The use the topology on C[I,F] generated by the norm $\|f\| = \max\{\|f(x)\|: x \in I\}$. The following theorem gives us one criterion for determining if $\mathcal{F}$ is compact in C[I,F]. A proof of this may be found in Yosida [15].

Theorem (Ascoli-Arzelà). Let I be a closed bounded interval in E^1 and let F be a Banach space. If a set $\mathcal{F} \subset C[I,F]$ is closed, bounded, and equicontinuous, then $\mathcal{F}$ is compact.

With this as background we proceed to our second application. The proof is essentially that found in Edwards [6].

Theorem 4.4 (Peano). Let T be a closed bounded interval in E^1, and let F be a finite dimensional normed linear space. Let $r > 0$ and $y_o \in F$. Let $B = \{y: y \in F, \|y-y_o\| \leq r\}$ and $t_o \in T$. Let $f: T \times B \rightarrow F$ be a continuous mapping. Then there exists a solution to the differential equation

$$\frac{dx}{dt} = f(t,x), \quad x(t_o) = y_o.$$

Proof. Since F is a finite dimensional normed linear space, F is linearly homeomorphic to some Euclidean space E^k and is, therefore, complete. Also, every closed bounded set in F is compact. Thus, B is compact in F, and since T is compact in E^1, we have that $T \times B$ is compact in $E^1 \times F$. Furthermore, the continuity of f on $T \times B$ implies there exists an $M > 0$ such that $\|f(t,x)\| \leq M$ for all $t \in T$, all $x \in B$. In the course of this proof it will become evident that the smaller we can get the bound M, the "wider" the interval is on which a solution exists.

Let $c = \frac{r}{M}$ and $T_1 = T \cap [t_0-c, t_0+c]$. Denote the set of continuous functions defined on T_1 with values in F by $C[T_1,F]$. Note that if $x \in C[T_1,F]$, then $\|x(t)\|$ takes on a maximum value (finite), since T_1 is compact. We first show that $C[T_1,F]$ is a Banach space with norm $\|x\| = \sup\{\|x(t)\|: t \in T_1\}$. It is easily verified that $C[T_1,F]$ is a normed linear space. The only real issue is to show that $C[T_1,F]$ is complete.

Let $\{x_k\}$ be a Cauchy sequence in $C[T_1,F]$. That is, $\|x_n - x_m\| \longrightarrow 0$ as $m,n\to\infty$. Fix $t \in T_1$. Then since $\|x_n(t) - x_m(t)\| \leq \|x_n - x_m\|$, the sequence $\{x_k(t)\}$ is a Cauchy sequence in F, which is complete. This is true for each $t \in T_1$. Now define x by

$$x(t) = \lim_{k\to\infty} x_k(t) \qquad \text{for } t \in T_1.$$

We must show that $x \in C[T_1,F]$ and $\|x_k-x\| \to 0$ as $k\to\infty$. Let $\varepsilon > 0$ be given. Then there exists an integer N such that if $m,n \geq N$, then $\|x_n - x_m\| < \frac{\varepsilon}{3}$. Therefore, $\|x_n(t) - x_m(t)\| < \frac{\varepsilon}{3}$ for all $t \in T_1$. Fix $n \geq N$. Then

$$\lim_{m\to\infty} \|x_n(t) - x_m(t)\| \leq \frac{\varepsilon}{3} \qquad \text{for all } t \in T_1. \tag{*}$$

Let $t_1 \in T_1$. By continuity of x_n (n is still fixed here), there exists a $\delta > 0$ such that whenever $|t-t_1| < \delta$ and $t \in T_1$, then

$$\|x_n(t) - x_n(t_1)\| < \frac{\varepsilon}{3}.$$

To show that x is continuous at t_1, let $t \in T_1$ with $|t-t_1| < \delta$. We have

$$\|x(t)-x(t_1)\| \leq \|x(t)-x_n(t)\| + \|x_n(t)-x_n(t_1)\| + \|x_n(t_1)-x(t_1)\|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Therefore, x is continuous on T_1. So $x \in C[T_1,F]$.

By (*), $\|x_n(t)-x(t)\| \leq \frac{\varepsilon}{3}$ for all $t \in T_1$, which implies that $\|x_n-x\| \leq \frac{\varepsilon}{3}$. Thus, $\|x_n-x\| \to 0$ as $n\to\infty$, and we have shown that $C[T_1,F]$ is complete and, therefore, a Banach space.

Define the set $A = \{x\colon x \in C[T_1,F], \|x(t)-y_o\| \leq r \text{ for all } t \in T_1\}$. Clearly, for $x \in A$, $\|x\| \leq \|y_o\| + r$. So A is bounded. Also, observe that the set A is closed. That is, if $\{x_n\}$ is a sequence in A and $\|x_n-x\| \to 0$ as $n\to\infty$, then given $\varepsilon > 0$, there exists an N such that if $n \geq N$, one has

$$\|x(t)-y_o\| \leq \|x(t)-x_n(t)\| + \|x_n(t)-y_o\| < \varepsilon + r.$$

This is true for every $\varepsilon > 0$. So $\|x(t)-y_o\| \leq r$, and $x \in A$.

That A is convex follows easily. Let $x,y \in A$ and $0\leq\alpha\leq 1$. Then

$$\|\alpha x(t) + (1-\alpha)y(t) - y_o\| \leq \|\alpha(x(t) - y_o)\| + \|(1-\alpha)(y(t) - y_o)\|$$

$$= \alpha\|x(t) - y_o\| + (1-\alpha)\|y(t) - y_o\|$$

$$\leq \alpha r + (1-\alpha)r$$

$$= r,$$

and we have $\alpha x + (1-\alpha)y \in A$.

Now consider the mapping u defined on A as follows:

$$u(x)(t) = y_o + \int_{t_o}^{t} f(s,x(s))ds$$

where $t \in T_1$, $x \in A$. Since $x(s) \in B$ for all $s \in T_1$, then $\|f(s,x(s))\| \leq M$ for all $s \in T_1$. It then follows that for $t \in T_1$,

$$\|u(x)(t) - y_o\| = \|\int_{t_o}^{t} f(s,x(s))ds\| \leq M|t-t_o| \leq Mc = r.$$

Thus, $u(x) \in A$ for each $x \in A$.

We aim to show that $u: A \rightarrow A$ has a fixed point in A. We do so by showing that u is a completely continuous operator on A. We will then have all the conditions necessary to appeal to the Schauder Fixed Point Theorem.

Let $\{x_n\}$ be a sequence in A and let $x \in A$ with $x_n \rightarrow x$ as $n \rightarrow \infty$; i.e., $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$. Since f is continuous on $T \times B$, given $\varepsilon > 0$, there exists a $\delta > 0$ so that whenever $|t_1 - t_2| < \delta$ and $\|x_2 - x_1\| < \delta$, then $\|f(t_1,x_1) - f(t_2,x_2)\| < \frac{\varepsilon}{c}$. Choose an integer N so that if $n \geq N$, $\|x_n - x\| < \delta$. Letting $n \geq N$, it follows that $\|f(s,x_n(s)) - f(s,x(s))\| < \frac{\varepsilon}{c}$ for all $s \in T_1$, and for all $t \in T_1$,

$$
\begin{aligned}
\|u(x_n)(t) - u(x)(t)\| &= \left\|\int_{t_o}^{t} f(s,x_n(s))ds - \int_{t_o}^{t} f(s,x(s))ds\right\| \\
&= \left\|\int_{t_o}^{t} [f(s,x_n(s) - f(s,x(s))]ds\right\| \\
&< |t-t_o| \cdot \frac{\varepsilon}{c} \\
&\leq c \cdot \frac{\varepsilon}{c} \\
&= \varepsilon.
\end{aligned}
$$

Thus, $\|u(x_n) - u(x)\| < \varepsilon$, and u is continuous on A.

Finally we need to show that u takes bounded subsets of A into relatively compact sets in A. Since A is bounded, every subset of A is bounded. Moreover, if D is any subset of A, then $u(D) \subset u(A)$, and $\overline{u(D)} \subset \overline{u(A)}$. If we show $\overline{u(A)}$ is compact, then $\overline{u(D)}$ is compact, since a closed subset of a compact set is compact.

We have that $u(A) \subset A$, and A is closed and bounded. Therefore, $\overline{u(A)} \subset A$ is bounded. To show $\overline{u(A)}$ is compact, by the Ascoli-Arzelà theorem it suffices to show that $\overline{u(A)}$ is equicontinuous. Note that $y \in u(A)$ if, and only if, there is an $x \in A$ such that $y = u(x)$. Thus,

$y \in \overline{u(A)}$ if, and only if, there exists a sequence $\{x_n\}$ in A such that $\|u(x_n) - y\| \to 0$.

Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{3M}$. Let $t_1, t_2 \in T_1$ with $|t_1 - t_2| < \delta$. Let $y \in \overline{u(A)}$ and $\{x_n\}$ be a sequence in A such that $u(x_n) \to y$. Then there exists an integer N such that if $n \geq N$, we have $\|u(x_n) - y\| < \frac{\varepsilon}{3}$. Hence,

$$\begin{aligned}
\|y(t_1)-y(t_2)\| &\leq \|y(t_1)-u(x_n)(t_1)\| + \|u(x_n)(t_1)-u(x_n)(t_2)\| \\
&\quad + \|u(x_n)(t_2)-y(t_2)\| \\
&< \frac{\varepsilon}{3} + \left\|\int_{t_2}^{t_1} f(s,x(s))ds\right\| + \frac{\varepsilon}{3} \\
&< \frac{2\varepsilon}{3} + M \cdot |t_1-t_2| \\
&< \frac{2\varepsilon}{3} + M \cdot \frac{\varepsilon}{3M} \\
&= \varepsilon.
\end{aligned}$$

This shows that $\overline{u(A)}$ is equicontinuous. So u is a completely continuous operator mapping the closed, bounded, convex set A into itself. By the Schauder Fixed Point Theorem, there exists an $x_o \in A$ such that $u(x_o) = x_o$. Therefore, for all $t \in T_1$,

$$x_o(t) = u(x_o)(t) = y_o + \int_{t_o}^{t} f(s,x(s))ds$$

or equivalently,

$$\frac{dx_o}{dt} = f(t,x_o(t)), \quad x_o(t_o) = y_o. \quad \square$$

This completes our discussion of solutions to functional equations. We now proceed to applications of Kakutani's generalization of Brouwer's theorem. The following two theorems are due to J. von Neumann with the proofs, presented here, essentially those of Kakutani [9]. After proving these theorems, we shall interpret the second one in terms of game theory.

Theorem 4.5. Let K and L be two non-empty closed bounded convex sets in E^m and E^n, respectively, and consider their Cartesian product $K \times L$ in E^{m+n}. Let U and V be two closed subsets of $K \times L$ such that for any $x_o \in K$ the set $U_{x_o} = \{y: y \in L \text{ and } (x_o,y) \in U\}$ is non-empty, closed, and convex, and such that for any $y_o \in L$ the set $V_{y_o} = \{x: x \in K \text{ and } (x,y_o) \in V\}$ is non-empty, closed, and convex. Under these assumptions, U and V have a point in common.

Proof. Let $S = K \times L$. We define a point-to-set mapping Φ on S by the following: For $z = (x,y) \in S$ where $x \in K$, $y \in L$

$$\Phi(z) = V_y \times U_x.$$

We want to show that $\Phi(S) \subset K(S)$ and that $\Phi:S \to K(S)$ is upper semi-continuous on S.

To show $\Phi(z) \in K(S)$ where $z = (x,y)$, we must show that $V_y \times U_x$ is non-empty, closed, and convex. Since V_y and U_x are non-empty and closed in K and L, respectively, for each $(x,y) \in S$, then $V_y \times U_x$ is non-empty and closed in S. For convexity, let (x_1,y_1) and (x_2,y_2) be in $V_y \times U_x$, and $0 \le t \le 1$. Consider $(x_t,y_t) = t(x_1,y_1) + (1-t)(x_2,y_2)$. Since $x_1,x_2 \in V_y$, which is convex, then $x_t = tx_1 + (1-t)x_2 \in V_y$. Likewise, $y_1,y_2 \in U_x$ implies $y_t \in U_x$. So, $(x_t,y_t) \in V_y \times U_x$.

To show that Φ is upper semi-continuous, let $\{z_n\}$ be a sequence in S with $z_n \to z_o$, $\omega_n \in \Phi(z_n)$, and $\omega_n \to \omega_o$. We must show $\omega_o \in \Phi(z_o)$. Let $z_n = (x_n,y_n)$, $z_o = (x_o,y_o)$, $\omega_n = (r_n,s_n)$, and $\omega_o = (r_o,s_o)$. Then $\omega_n \in \Phi(z_n) = V_{y_n} \times U_{x_n}$ if and only if $r_n \in V_{y_n}$ and $s_n \in U_{x_n}$. Thus, $(r_n,y_n) \in V$ and $(x_n,s_n) \in U$. Furthermore, $\omega_n \to \omega_o$ if and only if $r_n \to r_o$ and $s_n \to S_o$; $z_n \to z_o$ if and only if $x_n \to x_o$ and $y_n \to y_o$. Since U and V are closed, we have that $(r_o,y_o) \in V$ and $(x_o,s_o) \in U$. So, $(r_o,s_o) \in V_{y_o} \times U_{x_o}$, or, equivalently, $\omega_o \in \Phi(z_o)$.

Having satisfied all the conditions of Theorem 3.5, we know there exists a point $z_o \in S$ such that $z_o \in \Phi(z_o)$. That is, there exist $x_o \in K$, $y_o \in L$ so that $x_o \in V_{y_o}$ and $y_o \in U_{x_o}$. Equivalently, $(x_o,y_o) \in V$ and $(x_o,y_o) \in U$. Thus, $U \cap V$ is non-empty. $\square$

We now use Theorem 4.5 to prove

Theorem 4.6. Let $f(x,y)$ be a continuous real-valued function defined for $x \in K$ and $y \in L$, where K and L are arbitrary closed bounded convex sets in two Euclidean spaces E^m and E^n. If for every $x_o \in K$ and for every real number α, the set $\{y\colon y \in L \text{ and } f(x_o,y) \le \alpha\}$ is convex, and if for every $y_o \in L$ and for every real number β, the set $\{x\colon x \in K \text{ and } f(x,y_o) \ge \beta\}$ is convex, then we have

$$\max_{x \in K} \min_{y \in L} f(x,y) = \min_{y \in L} \max_{x \in K} f(x,y).$$

Proof. Let U and V be subsets of K × L defined as follows:

$$U = \{(x_o,y_o): x_o \in K,\ y_o \in L, \text{ and } f(x_o,y_o) = \min_{y \in L} f(x_o,y)\}$$

$$V = \{(x_o,y_o): x_o \in K,\ y_o \in L, \text{ and } f(x_o,y_o) = \max_{x \in K} f(x,y_o)\}$$

We will show that U and V satisfy the conditions of Theorem 3.2. First we need to show U and V are closed in K × L. Let $\{(x_n,y_n)\}$ be a sequence in U with $x_n \rightarrow x_o$, $y_n \rightarrow y_o$. Let y be any point in L. Then

$$f(x_n,y_n) \leq f(x_n,y) \quad \text{for all } n.$$

Therefore, by continuity of f,

$$f(x_o,y_o) = \lim_{n\to\infty} f(x_n,y_n) \leq \lim_{n\to\infty} f(x_n,y) = f(x_o,y).$$

This is true for all $y \in L$. So, $(x_o,y_o) \in U$, and U is closed. A similar argument holds for V being closed.

Now let $x_o \in K$ and $U_{x_o} = \{y: y \in L \text{ and } (x_o,y) \in U\}$. We need to show U_{x_o} is non-empty, closed, and convex. Since f(x,y) is a continuous function of two variables, $f(x_o,y)$ is continuous in the second variable and is defined on the compact set L. Therefore, $f(x_o,y)$ takes on a minimum value in L. That is, there exists a point $y_o \in L$ such that $f(x_o,y_o) = \min_{y\varepsilon L} f(x_o,y)$. So, $(x_o,y_o) \varepsilon U$, and U_{x_o} is non-empty.

Let $\{y_n\}$ be a sequence in U_{x_o} such that $y_n \to \tilde{y}$. Then $\{(x_o,y_n)\}$ is a sequence in U, and $(x_o,y_n) \to (x_o,\tilde{y})$. Since U is closed, $(x_o,\tilde{y}) \varepsilon U$, which implies $\tilde{y} \varepsilon U_{x_o}$. Thus, U_{x_o} is closed.

For convexity of U_{x_o}, let $y_1,y_2 \varepsilon U_{x_o}$, and $0 \le t \le 1$. Let $y_t = t_{y_1} + (1-t)y_2$, and let $\alpha = \min_{y\varepsilon L} f(x_o,y)$. Then by the assumption that the set $\{y: y \varepsilon L \text{ and } f(x_o,y) \le \alpha\}$ is convex, and since $f(x_o,y_1) = f(x_o,y_2) = \alpha$, we have that $f(x_o,y_t) \le \alpha$. However, $f(x_o,y) \ge \alpha$ for all $y \varepsilon L$. So, $f(x_o,y_t) = \alpha$, and $y_t \varepsilon U_{x_o}$.

Now let $y_o \varepsilon L$ and define $V_{y_o} = \{x: x \varepsilon K \text{ and } (x,y_o) \varepsilon V\}$. By arguments similar to those above, using the condition that for $\beta = \max_{x\varepsilon K} f(x,y_o)$ the set $\{x: x \varepsilon K \text{ and } f(x,y_o) \ge \beta\}$ is convex, we conclude that V_{y_o} is non-empty, closed, and convex.

Hence, by Theorem 4.5, there exists a point $(x_o,y_o) \varepsilon K \times L$ such that $(x_o,y_o) \varepsilon U \cap V$, or equivalently,

$$f(x_o,y_o) = \min_{y\varepsilon L} f(x_o,y) = \max_{x\varepsilon K} f(x,y_o).$$

Consequently, we have

$$\min_{y\varepsilon L} \max_{x\varepsilon K} f(x,y) \le \max_{x\varepsilon K} f(x,y_o) = \min_{y\varepsilon L} f(x_o,y) \le \max_{x\varepsilon K} \min_{y\varepsilon L} f(x,y).$$

That is,

$$\min_{y\varepsilon L} \max_{x\varepsilon K} f(x,y) \le \max_{x\varepsilon K} \min_{y\varepsilon L} f(x,y). \tag{1}$$

To show the inequality in the opposite direction, fix $y \in L$. We then have for each $x \in K$

$$f(x,y) \leq \max_{x \in K} f(x,y).$$

Therefore,

$$\min_{y \in L} f(x,y) \leq \min_{y \in L} \max_{x \in K} f(x,y). \qquad (*)$$

This is true for all $x \in K$. Thus, the right side of the inequality (*) is an upper bound for the quantity $\min_{y \in L} f(x,y)$ for each $x \in K$. Hence,

$$\max_{x \in K} \min_{y \in L} f(x,y) \leq \min_{y \in L} \max_{x \in K} f(x,y). \qquad (2)$$

Combining (1) and (2) we have

$$\max_{x \in K} \min_{y \in L} f(x,y) = \min_{y \in L} \max_{x \in K} f(x,y). \qquad \square$$

In order to illustrate the meaning of Theorem 4.6 in the setting of game theory, we introduce a simple game known as a two-person zero-sum game. This is a game in which there are exactly two participants with one participant gaining what the other loses. For a more complete discussion, we refer the reader to Karlin [10], on which the present discussion is based.

Let A and B be the two players involved in the game. A fundamental concept in game theory is that of strategy. A strategy for A is a complete enumeration of all moves A will make for any possible situation which might arise, whether the situation arises accidentally or is due to a move by B. Moreover, A's strategy is a rule which determines A's next move by taking into account all that has happened previously.

We now give a formal definition of a two-person zero-sum game. This game is defined to be a triplet $\{K,L,f\}$, where K denotes the space of strategies for A, L denotes the space of strategies for B, and f is a real-valued function defined on $K \times L$. Assume A selects a strategy x from K, and B chooses a strategy y from L. For the pair (x,y) the pay-off to A is f(x,y), and the pay-off to B is -f(x,y).

We make the further assumption that K and L are closed, bounded, convex sets in E^n and E^m, respectively. We then have a strategy representable as a point in a finite dimensional space. Justification for such an assumption lies in the fact that many actual games fall in this category. Restrictions on f, such as those in Theorem 4.6, also arise in actual games.

We now face the problem of choosing strategies. Suppose the rules require B to tell A the strategy he is going to use; call it y_o. Then A will try to maximize his own pay-off by choosing a strategy $x_o \in K$ so that $f(x_o,y_o) = \max_{x \in K} f(x,y_o)$. Realizing that A will do this, B should have chosen $y_o \in L$ so that

$$\max_{x \in K} f(x,y_o) = \min_{y \in L} \max_{x \in K} f(x,y) = \bar{v}.$$

Then $\bar{v}$ is the most that A can benefit, if B chooses strategy y_o.

Suppose, on the other hand, that A must announce his strategy x_o to B. Then B is certain to choose a strategy y_o to maximize his returns and minimize his pay-off to A. That is, B wants

$$f(x_o,y_o) = \min_{y \in L} f(x_o,y).$$

So A can best protect his possible profit by announcing x_o so that

$$\min_{y \in L} f(x_o,y) = \max_{x \in K} \min_{y \in L} f(x,y) = \underline{v}.$$

Then $\underline{v}$ is the most that A can guarantee himself independent of B's choice of strategy.

Assuming f satisfies the conditions imposed in Theorem 4.6, we have that $\bar{v} = \underline{v} = v$. This common value v is called the value of the game to A, and -v is the value of the game to B. That is, by an appropriate choice of strategy, A can guarantee winning at least the amount $\underline{v} = v$, and by judicious play B can prevent A from gaining more than $\bar{v} = v$.

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