

## ON THE SIMPLICITY OF THE EIGENVALUES OF NON-SELF-ADJOINT MATHIEU-HILL OPERATORS

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ABSTRACT. Firstly, we analyze some spectral properties of the non-self-adjoint Hill operator with piecewise continuous even potential. Then using this we find conditions on the potential of the non-self-adjoint Mathieu operator, such that all eigenvalues of the periodic, antiperiodic, Dirichlet, and Neumann boundary value problems are simple.

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### 1. INTRODUCTION AND PRELIMINARY FACTS

Let  $P(q)$ ,  $A(q)$ ,  $D(q)$ ,  $N(q)$  be the operators in  $L_2[0, \pi]$  associated with the equation

$$-y''(x) + q(x)y(x) = \lambda y(x) \tag{1}$$

and the periodic

$$y(\pi) = y(0), \quad y'(\pi) = y'(0), \tag{2}$$

antiperiodic

$$y(\pi) = -y(0), \quad y'(\pi) = -y'(0), \tag{3}$$

Dirichlet

$$y(\pi) = y(0) = 0, \tag{4}$$

Neumann

$$y'(\pi) = y'(0) = 0 \tag{5}$$

boundary conditions respectively, where  $q$  is a piecewise continuous function. The domains of definitions of these operators are the set of all functions  $f \in L_2[0, \pi]$ , satisfying the corresponding boundary conditions and the conditions  $(-f'' + qf) \in L_2[0, \pi]$ ,  $f' \in AC[0, \pi]$ , where  $AC[0, \pi]$  is the set of all absolutely continuous functions on  $[0, \pi]$ .

It is well known that [1], the spectra of the operators  $P(q)$  and  $A(q)$ , consist of the eigenvalues  $\lambda_{2n}$  and  $\lambda_{2n+1}$ , called periodic and antiperiodic eigenvalues, which are respectively the roots of

$$F(\lambda) = 2 \quad \& \quad F(\lambda) = -2, \tag{6}$$

where  $n = 0, 1, \dots$ ,  $F(\lambda) =: \varphi'(\pi, \lambda) + \theta(\pi, \lambda)$  is the Hill discriminant, and  $\varphi(x, \lambda)$ ,  $\theta(x, \lambda)$  are the solutions of the equation (1), satisfying the initial conditions

$$\theta(0, \lambda) = \varphi'(0, \lambda) = 1, \quad \theta'(0, \lambda) = \varphi(0, \lambda) = 0. \tag{7}$$

The eigenvalues of the operators  $D(q)$  and  $N(q)$ , called Dirichlet and Neumann eigenvalues, are the roots of

$$\varphi(\pi, \lambda) = 0 \quad \& \quad \theta'(\pi, \lambda) = 0 \tag{8}$$

respectively. The spectrum of the operator  $L(q)$  associated with (1), and the conditions

$$y(2\pi) = y(0), \quad y'(2\pi) = y'(0) \tag{9}$$

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are the union of the periodic and antiperiodic eigenvalues. In other words, the spectrum of  $L(q)$  consists of the eigenvalues  $\lambda_n$  for  $n = 0, 1, \dots$  which are the roots of the equation

$$(F(\lambda) - 2)(F(\lambda) + 2) = 0. \quad (10)$$

The operators  $P(q)$ ,  $A(q)$ ,  $D(q)$ , and  $N(q)$  are denoted respectively by  $P(a, b)$ ,  $A(a, b)$ ,  $D(a, b)$ , and  $N(a, b)$ , if

$$q(x) = ae^{-i2x} + be^{i2x}, \quad (11)$$

where  $a$  and  $b$  are complex numbers. If  $b = a$ , then, for simplicity of the notations, these operators are redenoted by  $P(a)$ ,  $A(a)$ ,  $D(a)$ , and  $N(a)$ . The eigenvalues of  $P(a)$  and  $A(a)$  are denoted by  $\lambda_{2n}(a)$  and  $\lambda_{2n+1}(a)$  for  $n = 0, 1, \dots$ .

We use the following two classical theorems (see p.8-9 of [4] and p.34-35 of [1]).

**Theorem 1.** If  $q$  is an even piecewise continuous function, then  $\varphi(x, \lambda)$  is an odd function, and  $\theta(x, \lambda)$  is an even function. Periodic solutions are either  $\varphi(x, \lambda)$  or  $\theta(x, \lambda)$ , unless all solutions are periodic (with period  $\pi$  or  $2\pi$ ). Moreover, the following equality holds

$$\varphi'(\pi, \lambda) = \theta(\pi, \lambda). \quad (12)$$

**Theorem 2.** For all  $n$  and for any nonzero  $a$ , the geometric multiplicity of the eigenvalue  $\lambda_n(a)$  of the operators  $P(a)$  and  $A(a)$  is 1 (that is, there exists one eigenfunction corresponding to  $\lambda_n(a)$ ) and the corresponding eigenfunction is either  $\varphi(x, \lambda_n(a))$  or  $\theta(x, \lambda_n(a))$ , where, for simplicity of the notations, the solutions of the equation

$$-y''(x) + (2a \cos 2x)y(x) = \lambda y(x) \quad (13)$$

satisfying (7), are denoted also by  $\varphi(x, \lambda)$  and  $\theta(x, \lambda)$ .

In [1, 4] these theorems were proved for the real-valued potentials. However, the proofs pass through for the complex-valued potentials without any change. The spectrum of  $P(a)$ ,  $A(a)$ ,  $D(a)$ ,  $N(a)$  for  $a = 0$  are

$$\{(2k)^2 : k = 0, 1, \dots\}, \{(2k+1)^2 : k = 0, 1, \dots\}, \{k^2 : k = 1, 2, \dots\}, \{k^2 : k = 0, 1, \dots\}$$

respectively. All eigenvalues of  $P(0)$ , except 0 and  $A(0)$ , are double, while the eigenvalues of  $D(0)$  and  $N(0)$  are simple. We also use the following result of [7].

**Theorem 3.** If  $ab = cd$ , then the Hill discriminants  $F(\lambda, a, b)$  and  $F(\lambda, c, d)$  (see (6)), for the operators  $P(a, b)$  and  $P(c, d)$ , are the same.

By Theorem 2, the geometric multiplicity of the eigenvalues of  $P(a)$  and  $A(a)$ , for any nonzero complex number  $a$ , is 1. However, in the non-self-adjoint case  $a \in \mathbb{C} \setminus \mathbb{R}$ , the multiplicity (algebraic multiplicity) of these eigenvalues, in general, is not equal to their geometric multiplicity, since the operators  $P(a)$  and  $A(a)$  may have associated functions (generalized eigenfunctions). Thus, in the non-self-adjoint case, the multiplicity (algebraic multiplicity) of the eigenvalues may be any finite number when the geometric multiplicity is 1 (see Chapter 1 of [5]). Therefore the investigation of the multiplicity of the eigenvalues for complex-valued potential is more complicated.

In this paper, in Section 2 we first study some spectral properties of the non-self-adjoint Hill operator with piecewise continuous even potential. In Section 3, we use it to find the conditions on  $a$ , such that all eigenvalues of the operators  $P(a)$ ,  $A(a)$ ,  $D(a)$ , and  $N(a)$  are simple. Namely we prove the following

**Theorem 4.** All eigenvalues of the operators  $A(a)$ ,  $D(a)$  and  $P(a)$ ,  $N(a)$  are simple, if  $0 < |a| \leq \frac{8}{\sqrt{6}}$  and  $0 < |a| \leq \frac{4}{3}$  respectively.

This theorem with Theorem 3 implies

**Theorem 5.** All eigenvalues of the operators  $A(a, b)$  and  $P(a, b)$  are simple, if  $0 < |ab| \leq \frac{32}{3}$  and  $0 < |ab| \leq \frac{16}{9}$  respectively.

Note that the estimations of  $ab$  can be improved by using the numerical methods of [2] and [3].

## 2. ON THE EVEN POTENTIALS

In this section we analyze, in general, the Hill operator with even piecewise continuous potentials. In the paper [6] the following statements about the connections of the spectra of the operators  $P(q)$ ,  $A(q)$ ,  $D(q)$ , and  $N(q)$ , where  $q$  is an even potential, were proven.

**Lemma 1 of [6].** If  $\lambda$  is an eigenvalue of both operators  $D(q)$  and  $N(q)$ , then

$$F(\lambda) = \pm 2, \quad \frac{dF}{d\lambda} = 0, \quad (14)$$

that is,  $\lambda$  is a multiple eigenvalue of  $L(q)$ .

**Proposition 1 of [6].** A number  $\lambda$  is an eigenvalue of  $L(q)$ , if and only if  $\lambda$  is an eigenvalue of  $D(q)$  or  $N(q)$ .

Firstly, using (12) and the Wronskian equalities

$$\theta(\pi, \lambda)\varphi'(\pi, \lambda) - \varphi(\pi, \lambda)\theta'(\pi, \lambda) = 1 \quad (15)$$

we prove the following improvements of these statements.

**Theorem 6.** Let  $q$  be an even complex-valued function. A complex number  $\lambda$  is both a Neumann and Dirichlet eigenvalue, if and only if it is an eigenvalue of the operator  $L(q)$  with geometric multiplicity 2.

**Proof.** Suppose  $\lambda$  is both a Neumann and Dirichlet eigenvalue, that is, both equality in (8) hold. On the other hand, it follows from (12), (8), and (15) that

$$\theta(\pi, \lambda) = \varphi'(\pi, \lambda) = \pm 1. \quad (16)$$

Now using (8), (16), and (7), one can easily verify that both  $\theta(x, \lambda)$  and  $\varphi(x, \lambda)$  satisfy either periodic or anti-periodic boundary conditions, that is,  $\lambda$  is an eigenvalue of the operator  $L(q)$  with geometric multiplicity 2.

Conversely, if  $\lambda$  is an eigenvalue of  $L(q)$  with geometric multiplicity 2, then both  $\theta(x, \lambda)$  and  $\varphi(x, \lambda)$  satisfy either periodic or anti-periodic boundary conditions. Therefore by (7), the equalities in (8) hold, that is,  $\lambda$  is both Neumann and Dirichlet eigenvalue.

**Theorem 7.** Let  $q$  be an even complex-valued function. A complex number  $\lambda$  is an eigenvalue of multiplicity  $s$  of the operator  $L(q)$ , if and only if it is an eigenvalue of multiplicities  $u$  and  $v$  of the operators  $D(q)$  and  $N(q)$  respectively, where  $u + v = s$  and  $u = 0$  ( $v = 0$ ) means that  $\lambda$  is not an eigenvalue of  $D(q)$  ( $N(q)$ ).

**Proof.** It is well-known and clear that  $\lambda_0$  is an eigenvalue of multiplicities  $u$ ,  $v$ , and  $s$  of the operators  $D(q)$ ,  $N(q)$ , and  $L(q)$  respectively, if and only if

$$\varphi(\pi, \lambda) = (\lambda_0 - \lambda)^u f(\lambda), \quad \theta'(\pi, \lambda) = (\lambda_0 - \lambda)^v g(\lambda) \quad (17)$$

and

$$(F(\lambda) - 2)(F(\lambda) + 2) = (\lambda_0 - \lambda)^s h(\lambda), \quad (18)$$

where  $f(\lambda_0) \neq 0$ ,  $g(\lambda_0) \neq 0$ , and  $h(\lambda_0) \neq 0$ . On the other hand, by (12) and (15), we have

$$(F(\lambda) - 2)(F(\lambda) + 2) = 4\theta^2(\pi, \lambda) - 4 = 4(\theta(\pi, \lambda)\varphi'(\pi, \lambda) - 1) = 4\varphi(\pi, \lambda)\theta'(\pi, \lambda). \quad (19)$$

Thus the proof of the theorem follows from (17)-(19).

To analyze the periodic and antiperiodic eigenvalues in detail, let us introduce the following notations and definitions.

**Definition 1.** Let  $\sigma(T)$  denote the spectrum of the operator  $T$ . A number  $\lambda$  is called  $PDN(q)$  (periodic, Dirichlet and Neumann) eigenvalue if  $\lambda \in \sigma(P(q)) \cap \sigma(D(q)) \cap \sigma(N(q))$ . A number  $\lambda \in \sigma(P(q)) \cap \sigma(D(q))$  is called  $PD(q)$  (periodic and Dirichlet) eigenvalue if it is not  $PDN(q)$  eigenvalue. A number  $\lambda \in \sigma(P(q)) \cap \sigma(N(q))$  is called  $PN(q)$  (periodic and Neumann) eigenvalue if it is not  $PDN(q)$  eigenvalue. Everywhere replacing  $P(q)$  by  $A(q)$ , we get the definition of  $ADN(q)$ ,  $AD(q)$  and  $AN(q)$  eigenvalues.

Using Theorems 6, 7, Definition 1, and the equality  $\sigma(P(q)) \cap \sigma(A(q)) = \emptyset$ , we obtain

**Theorem 8.** Let  $q$  be an even complex-valued function. Then

(a) The spectrum of  $P(q)$  is the union of the following three pairwise disjoint sets:  $\{PDN(q)$  eigenvalues $\}$ ,  $\{PD(q)$  eigenvalues $\}$ , and  $\{PN(q)$  eigenvalues $\}$ .

(b) A complex number  $\lambda$  is an eigenvalue of geometric multiplicity 2 of the operator  $P(q)$ , if and only if it is  $PDN(q)$  eigenvalue.

(c) A complex number  $\lambda$  is an eigenvalue of geometric multiplicity 1 of the operator  $P(q)$  if and only if it is either  $PD(q)$  or  $PN(q)$  eigenvalue.

The theorem continues to hold if  $P(q)$ ,  $PDN(q)$ ,  $PD(q)$ , and  $PN(q)$  are replaced by  $A(q)$ ,  $ADN(q)$ ,  $AD(q)$ , and  $AN(q)$  respectively.

Now we prove the main theorem of this section.

**Theorem 9.** Let  $q$  be an even complex-valued function, and  $\lambda$  be an eigenvalue of geometric multiplicity 1 of the operator  $P(q)$ . Then the number  $\lambda$  is an eigenvalue of multiplicity  $s$  of  $P(q)$ , if and only if it is an eigenvalue of multiplicity  $s$  either of the operator  $D(q)$  (first case), or of the operator  $N(q)$  (second case). In the first case, the system of the root functions of the operators  $P(q)$  and  $D(q)$  consists of the same eigenfunction  $\varphi(x, \lambda)$  and associated functions

$$\frac{\partial\varphi(x, \lambda)}{\partial\lambda}, \frac{1}{2!} \frac{\partial^2\varphi(x, \lambda)}{\partial\lambda^2}, \dots, \frac{1}{(s-1)!} \frac{\partial^{s-1}\varphi(x, \lambda)}{\partial\lambda^{s-1}}. \tag{20}$$

In the second case, the system of the root function of the operators  $P(q)$  and  $N(q)$ , consists of the same eigenfunction  $\theta(x, \lambda)$  and associated functions

$$\frac{\partial\theta(x, \lambda)}{\partial\lambda}, \frac{1}{2!} \frac{\partial^2\theta(x, \lambda)}{\partial\lambda^2}, \dots, \frac{1}{(s-1)!} \frac{\partial^{s-1}\theta(x, \lambda)}{\partial\lambda^{s-1}}. \tag{21}$$

The theorem continues to hold if  $P(q)$  is replaced by  $A(q)$ .

**Proof.** Let  $\lambda$  be an eigenvalue of geometric multiplicity 1, and multiplicity  $s$  of the operator  $P(q)$ . By Theorem 1 there are two cases.

Case 1. The corresponding eigenfunction is  $\varphi(x, \lambda)$ .

Case 2. The corresponding eigenfunction is  $\theta(x, \lambda)$ .

We consider Case 1 in the same way one can consider Case 2. In Case 1,  $\theta(x, \lambda)$  is not a periodic solution, that is, it does not satisfy the periodic boundary condition (2). On the other hand, the first equality of (6) with (12) and (7) implies that

$$\theta(\pi, \lambda) = 1 = \theta(0, \lambda), \tag{22}$$

that is,  $\theta(x, \lambda)$  satisfies the first equality in (2). Therefore,  $\theta(x, \lambda)$  does not satisfies the second equality of (2), that is,

$$\theta'(\pi, \lambda) \neq 0. \tag{23}$$

This inequality means that  $v = 0$ , where  $v$  is defined in Theorem 7. Therefore, by Theorem 7 we have  $u = s$ , that is,  $\lambda$  is an eigenvalue of multiplicity  $s$  of the operator  $D(q)$ .

Now suppose that  $\lambda$  is an eigenvalue of multiplicity  $s$  of  $D(q)$ . Then by (8) and (7)

$$\varphi(\pi, \lambda) = 0 = \varphi(0, \lambda). \tag{24}$$

On the other hand, using the first equality of (6), (12), and (7) we get

$$\varphi'(\pi, \lambda) = 1 = \varphi'(0, \lambda). \tag{25}$$

Therefore,  $\varphi(x, \lambda)$  is an eigenfunction of  $P(q)$  corresponding to the eigenvalue  $\lambda$ . Then, by Theorem 1,  $\theta(x, \lambda)$  is not a periodic solution. This, as we noted above, implies (23) and the equality  $u = s$ . Thus, by Theorem 7,  $\lambda$  is an eigenvalue of multiplicity  $s$  of  $P(q)$ .

If  $\lambda$  is an eigenvalue of multiplicity  $s$  of the operators  $P(q)$  and  $D(q)$ , then

$$F(\lambda) = 2, \frac{dF}{d\lambda} = 0, \frac{d^2F}{d\lambda^2} = 0, \dots, \frac{d^{s-1}F}{d\lambda^{s-1}} = 0 \tag{26}$$

and

$$\varphi(\pi, \lambda) = 0, \frac{d\varphi(\pi, \lambda)}{d\lambda} = 0, \frac{d^2\varphi(\pi, \lambda)}{d\lambda^2} = 0, \dots, \frac{d^{s-1}\varphi(\pi, \lambda)}{d\lambda^{s-1}} = 0. \tag{27}$$

Since  $\varphi(0, \lambda) = 0$  and  $\varphi'(0, \lambda) = 1$  for all  $\lambda$ , we have

$$\varphi(0, \lambda) = 0, \frac{d\varphi(0, \lambda)}{d\lambda} = 0, \frac{d^2\varphi(0, \lambda)}{d\lambda^2} = 0, \dots, \frac{d^{s-1}\varphi(0, \lambda)}{d\lambda^{s-1}} = 0 \quad (28)$$

and

$$\varphi'(0, \lambda) = 1, \frac{d\varphi'(0, \lambda)}{d\lambda} = 0, \frac{d^2\varphi'(0, \lambda)}{d\lambda^2} = 0, \dots, \frac{d^{s-1}\varphi'(0, \lambda)}{d\lambda^{s-1}} = 0. \quad (29)$$

Moreover, using (26) and (12), we obtain

$$\varphi'(\pi, \lambda) = 1, \frac{d\varphi'(\pi, \lambda)}{d\lambda} = 0, \frac{d^2\varphi'(\pi, \lambda)}{d\lambda^2} = 0, \dots, \frac{d^{s-1}\varphi'(\pi, \lambda)}{d\lambda^{s-1}} = 0. \quad (30)$$

Thus, by (27)-(30),  $\varphi(x, \lambda)$  and the functions in (20), satisfy both the periodic and Dirichlet boundary conditions. On the other hand, differentiating  $s - 1$  times in  $\lambda$  the equation

$$-\varphi''(x, \lambda) + q(x)\varphi(x, \lambda) = \lambda\varphi(x, \lambda) \quad (31)$$

we obtain

$$-\left(\frac{1}{k!} \frac{\partial^k \varphi(x, \lambda)}{\partial \lambda^k}\right)'' + (q(x) - \lambda) \frac{1}{k!} \frac{\partial^k \varphi(x, \lambda)}{\partial \lambda^k} = \frac{1}{(k-1)!} \frac{\partial^{k-1} \varphi(x, \lambda)}{\partial \lambda^{k-1}}$$

for  $k = 1, 2, \dots, (s-1)$ . Therefore  $\varphi(x, \lambda)$ , and the functions in (20) are the root functions of the operators  $P(q)$  and  $D(q)$ . Thus the first case is proved in the same way we proved the second case. The proofs of these results for  $A(q)$  are similar.

### 3. MAIN RESULTS

In this section, we consider the operators  $P(a)$ ,  $A(a)$ ,  $D(a)$ , and  $N(a)$  with potential

$$q(x) = 2a \cos 2x, \quad (32)$$

where  $a$  is a nonzero complex number. By Theorem 2, the geometric multiplicity of the eigenvalues of  $P(a)$  and  $A(a)$  is 1. Therefore, it follows from Theorem 8 that

$$\sigma(P(a)) = \{PD(a) \text{ eigenvalues}\} \cup \{PN(a) \text{ eigenvalues}\}, \quad (33)$$

$$\sigma(A(a)) = \{AD(a) \text{ eigenvalues}\} \cup \{AN(a) \text{ eigenvalues}\}, \quad (34)$$

where  $PD(q)$ ,  $PN(q)$ ,  $AD(q)$ , and  $AN(q)$  (see Definition 1) are denoted by  $PD(a)$ ,  $PN(a)$ ,  $AD(a)$ , and  $AN(a)$  when the potential  $q$  is defined by (32). Moreover, Theorem 7, Theorem 2, and Theorem 9 yield the equalities

$$\sigma(D(a)) = \{PD(a) \text{ eigenvalues}\} \cup \{AD(a) \text{ eigenvalues}\}, \quad (35)$$

$$\sigma(N(a)) = \{PN(a) \text{ eigenvalues}\} \cup \{AN(a) \text{ eigenvalues}\} \quad (36)$$

and the following theorem.

**Theorem 10.** For any  $a \neq 0$  the eigenvalue  $\lambda$  of the operator  $P(a)$  or  $A(a)$  is a multiple, if and only if it is a multiple eigenvalue either of  $D(a)$  or  $N(a)$ . Moreover, the operators  $P(a)$ ,  $A(a)$ ,  $D(a)$ , and  $N(a)$  have associated functions corresponding to any multiple eigenvalues.

Clearly, the eigenfunctions corresponding to  $PN(a)$  eigenvalues,  $PD(a)$  eigenvalues,  $AD(a)$  eigenvalues, and  $AN(a)$  eigenvalues have the forms

$$\Psi_{PN}(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos 2kx, \quad (37)$$

$$\Psi_{PD}(x) = \sum_{k=1}^{\infty} b_k \sin 2kx, \quad (38)$$

$$\Psi_{AD}(x) = \sum_{k=1}^{\infty} c_k \sin(2k-1)x, \quad (39)$$

and

$$\Psi_{AN}(x) = \sum_{k=1}^{\infty} d_k \cos(2k - 1)x \tag{40}$$

respectively. For simplicity of the calculating, we normalize these eigenfunctions as follows

$$\sum_{k=0}^{\infty} |a_k|^2 = 1, \quad \sum_{k=1}^{\infty} |b_k|^2 = 1, \quad \sum_{k=1}^{\infty} |c_k|^2 = 1, \quad \sum_{k=1}^{\infty} |d_k|^2 = 1. \tag{41}$$

Substituting the functions (37)-(40) into (13), we obtain the following equalities

$$\lambda a_0 = \sqrt{2}aa_1, \quad (\lambda - 4)a_1 = a\sqrt{2}a_0 + aa_2, \quad (\lambda - (2k)^2)a_k = aa_{k-1} + aa_{k+1}, \tag{42}$$

$$(\lambda - 4)b_1 = ab_2, \quad (\lambda - (2k)^2)b_k = ab_{k-1} + ab_{k+1}, \tag{43}$$

$$(\lambda - 1)c_1 = ac_1 + ac_2, \quad (\lambda - (2k - 1)^2)c_k = ac_{k-1} + ac_{k+1}, \tag{44}$$

$$(\lambda - 1)d_1 = -ad_1 + ad_2, \quad (\lambda - (2k - 1)^2)d_k = ad_{k-1} + ad_{k+1} \tag{45}$$

for  $k = 2, 3, \dots$ . Here  $a_k, b_k, c_k, d_k$  depend on  $\lambda$ , and  $a_0, b_1, c_1, d_1$  are nonzero constants (see [1] p. 34-35).

By Theorem 10, if the eigenvalue  $\lambda$  corresponding to one of the eigenfunctions (37)-(40), denoted by  $\Psi(x)$ , is a multiple, then there exists associated function  $\Phi$  satisfying

$$-(\Phi(x, \lambda))'' + (q(x) - \lambda)\Phi(x, \lambda) = \Psi(x). \tag{46}$$

Since the boundary conditions (2)-(5) are self-adjoint, the operators adjoint to  $P(a), A(a), D(a)$ , and  $N(a)$  are  $P(\bar{a}), A(\bar{a}), D(\bar{a})$ , and  $N(\bar{a})$  respectively (see Chapter 1 of [5]). Therefore  $\bar{\lambda}$  and  $\bar{\Psi}(x)$  are eigenvalue, and eigenfunctions of the adjoint operator. Using this, and multiplying both sides of (46) by  $\bar{\Psi}$ , we get  $(\Psi, \bar{\Psi}) = 0$ , where  $(\cdot, \cdot)$  is the inner product in  $L_2[0, \pi]$ . Thus, if the eigenvalues corresponding to the eigenfunctions (37)-(40), are multiple, then we have

$$\sum_{k=0}^{\infty} a_k^2 = 0, \quad \sum_{k=1}^{\infty} b_k^2 = 0, \quad \sum_{k=1}^{\infty} c_k^2 = 0, \quad \sum_{k=1}^{\infty} d_k^2 = 0. \tag{47}$$

To prove the simplicity of the eigenvalue  $\lambda$  corresponding, for example to (40), we show that there is no sequence  $\{d_k\}$  satisfying the above 3 equalities: (45), (41) and (47), since these equalities hold if  $\lambda$  is a multiple eigenvalue. For this we use the following proposition, which readily follows from (41) and (47).

**Proposition 1.** If there exists  $n \in \mathbb{N} = \{1, 2, \dots\}$  such that

$$|d_n(\lambda)|^2 > \frac{1}{2}, \tag{48}$$

then  $\lambda$  is a simple  $AN(a)$  eigenvalue, where  $a \neq 0$ . The statement continues to hold for  $AD(a), PD(a)$ , and  $PN(a)$  eigenvalues if  $d_n$  is replaced by  $c_n, b_n$ , and  $a_n$  respectively.

To apply the Proposition 1, we use following lemmas.

**Lemma 1.** Suppose that  $\lambda$  is a multiple  $AN(a)$  eigenvalue corresponding to the eigenfunction (40), where  $a \neq 0$ . Then

(a) For all  $k \in \mathbb{N}, m \in \mathbb{N}, k \neq m$ , the following inequalities hold

$$|d_k|^2 \leq \frac{1}{2}, \tag{49}$$

$$|d_k \pm d_m|^2 \leq 1, \tag{50}$$

$$|d_k|^2 \leq \frac{|a|^2}{|\lambda - (2k - 1)^2|^2}. \tag{51}$$

(b) If  $\text{Re } \lambda < (2p - 1)^2 - 2|a|$  for some  $p \in \mathbb{N}$ , then  $|d_{k-1}| > |d_k| > 0$ , and

$$|d_{k+s}| < \frac{|2a|^{s+1} |d_{k-1}|}{|\lambda - (2k - 1)^2| |\lambda - (2(k + 1) - 1)^2| \dots |\lambda - (2(k + s) - 1)^2|} \tag{52}$$

for all  $k > p$  and  $s = 0, 1, \dots$

(c) Let  $I \subset \mathbb{N}$  and  $d(\lambda, I) =: \min_{k \in I} |\lambda - (2k - 1)^2| \neq 0$ . Then

$$\sum_{k \in I} |d_k|^2 \leq \frac{4|a|^2}{(d(\lambda, I))^2}. \quad (53)$$

(d) If  $\lambda$  is a multiple of the  $AD(a)$  eigenvalue corresponding to the eigenfunction (39), then the inequalities (49)-(53) continue to hold if  $d_j$  is replaced by  $c_j$ .

**Proof.** (a) If (49) does not hold for some  $k$ , then by Proposition 1,  $\lambda$  is a simple eigenvalue that contradicts the assumption of the lemma. Using the last equalities of (47) and (41), we obtain

$$|(d_k \pm d_m)^2| = \left| - \sum_{n \neq k, m} d_n^2 \pm 2d_k d_m \right| \leq \sum_{n \neq k, m} |d_n|^2 + |d_k|^2 + |d_m|^2 = 1,$$

that is, (50) holds. Now (51) follows from (45) and (50).

(b) Suppose that  $|d_k| \geq |d_{k-1}|$  for some  $k > p > 0$ . By (45)

$$|\lambda - (2k - 1)^2| |d_k| \leq |a| |d_{k-1}| + |a| |d_{k+1}|.$$

On the other hand, using the condition on  $\lambda$ , we get  $|\lambda - (2k - 1)^2| > 2|a|$ . Therefore  $|d_{k+1}| \geq 2|d_k| - |d_{k-1}| \geq |d_k|$ . Repeating this process  $s$  times, we obtain  $|d_{k+s}| \geq |d_{k+s-1}|$  for all  $s \in \mathbb{N}$ . But this means that  $\{|d_{k+s}| : s \in \mathbb{N}\}$  is a nondecreasing sequence. On the other hand,  $|d_k| + |d_{k+1}| \neq 0$ , since if both  $d_k$  and  $d_{k+1}$  are zero, then using (45) we obtain that  $d_j = 0$  for all  $j \in \mathbb{N}$ , that is, the solutions (40) are identically zero. Therefore,  $d_k$  does not converge to zero being the Fourier coefficient of the square integrable function  $\Psi_{AN}(x)$ . This contradiction shows that  $\{|d_{k+s}| : s \in \mathbb{N}\}$  is a decreasing sequence. Thus  $|d_k| > 0$  for all  $k > p$ .

Now let us prove (52). Using (45) and the inequality  $|d_{k-1}| > |d_k| > 0$ , we get

$$|d_{k+s}| < \frac{|2a| |d_{k+s-1}|}{|\lambda - (2(k+s) - 1)^2|} \quad (54)$$

for all  $s = 0, 1, \dots$ . Iterating (54)  $s$  times, we obtain (52).

(c) By (45) we have

$$\sum_{k \in I} |d_k|^2 \leq \sum_{k \in I} \frac{|a|^2 (|d_{k-1}| + |d_{k+1}|)^2}{(d(\lambda, I))^2} \leq \sum_{k \in I} \frac{2|a|^2 (|d_{k-1}|^2 + |d_{k+1}|^2)}{(d(\lambda, I))^2}.$$

Note that in case  $k = 1$ , instead of  $d_{k-1}$ , we take  $d_1$  (see the first equality of (45)). Now (53) follows from (41).

(d) Everywhere replacing  $d_k$  by  $c_k$ , we get the proof of the last statement.

In a similar way, we prove the following lemma for  $P(a)$ .

**Lemma 2.** Suppose that  $\lambda$  is a multiple  $PD(a)$  eigenvalue corresponding to the eigenfunction (38), where  $a \neq 0$ . Then

(a) For all  $k \in \mathbb{N}$ ,  $m \in \mathbb{N}$ ,  $n \in \mathbb{N}$ ,  $n \neq m$ , the following inequalities hold

$$|b_m|^2 \leq \frac{1}{2}, \quad |b_n \pm b_m|^2 \leq 1, \quad |b_k|^2 \leq \frac{|a|^2}{|\lambda - (2k)^2|^2}. \quad (55)$$

(b) If  $\operatorname{Re} \lambda < (2p)^2 - 2|a|$  for some  $p \in \mathbb{N}$ , then  $|b_{k-1}| > |b_k| > 0$  and

$$|b_{k+s}| < \frac{|2a|^{s+1} |b_{k-1}|}{|\lambda - (2k)^2| |\lambda - (2(k+1))^2| \dots |\lambda - (2(k+s))^2|} \quad (56)$$

for all  $k > p$  and  $s = 0, 1, \dots$

(c) Let  $I \subset \mathbb{N}$  and  $b(\lambda, I) = \min_{k \in I} |\lambda - (2k)^2| \neq 0$ . Then

$$\sum_{k \in I} |b_k|^2 \leq \frac{4|a|^2}{(b(\lambda, I))^2}. \tag{57}$$

(d) If  $\lambda$  is a multiple  $PN(a)$  eigenvalue corresponding to (37) then the statements (a) and (b) continue to hold for  $k > 1$ ,  $m \geq 0$ , and the statement (c) continues to hold for  $I \subset \{2, 3, \dots\}$  if  $b_j$  is replaced by  $a_j$ .

Introduce the notation  $D_n = \{\lambda \in \mathbb{C} : |\lambda - (2n - 1)^2| \leq 2|a|\}$ .

**Theorem 11.** (a) All eigenvalues of the operator  $A(a)$  lie on the unions of  $D_n$  for  $n \in \mathbb{N}$ .

(b) If  $4n - 4 > (1 + \sqrt{2})|a|$ , where  $a \neq 0$ , then the eigenvalues of  $A(a)$  lying in  $D_n$  are simple.

**Proof.** By (34), if  $\lambda$  is an eigenvalue of the operator  $A(a)$ , then the corresponding eigenfunction is either  $\Psi_{AN}(x)$  or  $\Psi_{AD}(x)$  (see (39) or (40)). Without loss of generality, we assume that the corresponding eigenfunction is  $\Psi_{AN}(x)$ .

(a) Since  $d_k \rightarrow 0$  as  $k \rightarrow \infty$ , there exists  $n \in \mathbb{N}$ , such that  $|d_n| = \max_{k \in \mathbb{N}} |d_k|$ . Therefore (a) follows from (45) for  $k = n$ .

(b) Suppose that  $\lambda \in D_n$  is a multiple eigenvalue corresponding to the eigenfunction  $\Psi_{AN}(x)$ . By definition of  $D_n$  for  $k \neq n$  we have

$$|\lambda - (2k - 1)^2| \geq |(2n - 1)^2 - (2k - 1)^2| - |2a| \geq |(2n - 3)^2 - (2n - 1)^2| - |2a|.$$

This, together with the condition on  $n$  and the definition of  $d(\lambda, I)$  (see Lemma 1(c)), gives  $d(\lambda, \mathbb{N} \setminus \{n\}) > 2\sqrt{2}|a|$ . Thus, using (53) and (41), we get

$$\sum_{k \neq n} |d_k|^2 < \frac{1}{2} \quad \& \quad |d_n|^2 > \frac{1}{2}$$

which contradicts Proposition 1.

Instead of Lemma 1, using Lemma 2 in the same way, we prove the following:

**Theorem 12.** (a) All  $PD(a)$  eigenvalues lie in the unions of  $B = \{\lambda : |\lambda - 4| \leq |a|\}$  and

$B_n = \{\lambda : |\lambda - (2n)^2| \leq 2|a|\}$  for  $n = 2, 3, \dots$ . All  $PN(a)$  eigenvalues lie in the unions of  $A_0 = \{\lambda : |\lambda| \leq \sqrt{2}|a|\}$ ,  $A_1 = \{\lambda : |\lambda - 4| \leq (1 + \sqrt{2})|a|\}$  and  $B_n$  for  $n = 2, 3, \dots$

(b) If  $4n - 2 > (1 + \sqrt{2})|a|$  and  $n > 1$ , where  $a \neq 0$ , then the eigenvalues of  $P(a)$  lying in  $B_n$  are simple.

Now we prove the main result for  $A(a)$ .

**Theorem 13.** If  $0 < |a| \leq \frac{8}{\sqrt{6}}$ , then all eigenvalues of the operator  $A(a)$  are simple.

**Proof.** Since  $8 > \frac{8}{\sqrt{6}}(1 + \sqrt{2})$ , by Theorem 11(b), the ball  $D_n$  for  $n > 2$  does not contain the multiple eigenvalues of the operator  $A(a)$ . Therefore, we need to prove that the ball  $D_n$  for  $n = 1, 2$  also does not contain the multiple eigenvalues. Since the balls  $D_1$  and  $D_2$  are contained in the half plane  $\{\lambda \in \mathbb{C} : \text{Re } \lambda < 16\}$ , we consider the following two strips  $\{\lambda \in \mathbb{C} : 9 < \text{Re } \lambda < 16\}$ ,  $\{\lambda \in \mathbb{C} : 6 < \text{Re } \lambda \leq 9\}$  and half plane  $\{\lambda \in \mathbb{C} : \text{Re } \lambda \leq 6\}$  separately. We consider the  $AN(a)$  eigenvalues, that is, the eigenvalues corresponding to the eigenfunctions (40). Consideration of the  $AD(a)$  eigenvalues are the same.

To prove the simplicity of the eigenvalues lying in the above strips, we assume that  $\lambda$  is a multiple eigenvalue. Using Lemma 1 by direct calculate (see Estimation 1 and Estimation 2 in Appendix), we show that (48) for  $n = 2$  holds, which contradicts Proposition 1.

Investigating the half plane  $\text{Re } \lambda \leq 6$  is more complicated. Here we use the first two equalities of (45)

$$(\lambda - 1)d_1 = -ad_1 + ad_2, \quad (\lambda - 9)d_2 = ad_1 + ad_3. \tag{58}$$

By direct calculating, we get (see Estimation 3 and Estimation 4 in the Appendix)

$$\sum_{k=3}^{\infty} |d_k|^2 < 0.03415, \quad \frac{|d_3|}{|d_2|} < 0.17432. \tag{59}$$



Then by (41) we have

$$|d_1|^2 + |d_2|^2 > 1 - \varepsilon, \quad (60)$$

where  $\varepsilon = 0.03415$ . On the other hand, by (49),  $|d_1|^2 \leq \frac{1}{2}$ ,  $|d_2|^2 \leq \frac{1}{2}$ . These inequalities and (47) imply that

$$|d_1|^2 = \frac{1}{2} - \varepsilon_1, \quad |d_2|^2 = \frac{1}{2} - \varepsilon_2, \quad d_2^2 = -d_1^2 + \varepsilon_3,$$

where  $\varepsilon_1 \geq 0$ ,  $\varepsilon_2 \geq 0$ ,  $\varepsilon_1 + \varepsilon_2 = \varepsilon$ ,  $|\varepsilon_3| < 0.03415$ . Now, one can easily see that

$$\left(\frac{d_2}{d_1}\right)^2 = -1 + \alpha, \quad \frac{d_2}{d_1} = \pm(i + \delta),$$

where  $|\alpha| < \frac{0.03415}{0.5-0.03415} < 0.074$ ,  $|\delta| < \frac{1}{2}|0.074| + \frac{1}{7}|0.074|^2 < 0.04$ . Therefore we have

$$\frac{d_2}{d_1} - \frac{d_1}{d_2} = \pm \frac{(i + \delta)^2 - 1}{i + \delta} = \pm \frac{2i(i + \delta) + \delta^2}{i + \delta} = \pm 2i + \gamma, \quad (61)$$

where  $|\gamma| < \frac{(0.04)^2}{1-0.04} < 0.002$ . On the other hand, dividing the first equality of (58) by  $d_1$ , and the second by  $d_2$ , and then subtracting second from the first and using (61), we get

$$\frac{8}{a} = \pm 2i - 1 + \gamma - \frac{d_3}{d_2}, \quad (62)$$

where by assumption  $\left|\frac{8}{a}\right| \geq \sqrt{6}$ . Therefore, using the second estimation of (59) in (62), we get the contradiction  $2.4495 < \sqrt{6} \leq \left|\frac{8}{a}\right| < \sqrt{5} + 0.17432 + 0.002 < 2.4125$ .

In the same way we consider the simplicity of the eigenvalues of the operators  $P(a)$ ,  $D(a)$ , and  $N(a)$ . First let us investigate the eigenvalues of  $D(a)$ . Since the eigenvalues of  $D(a)$  are the union of  $PD(a)$  and  $AD(a)$  eigenvalues, and the  $AD(a)$  eigenvalues are investigated in Theorem 13, we investigate the  $PD(a)$  eigenvalues.

**Theorem 14.** If  $0 < |a| \leq 5$ , then all  $PD(a)$  eigenvalues are simple. Moreover, if  $0 < |a| \leq \frac{8}{\sqrt{6}}$ , then all eigenvalues of the operator  $D(a)$  are simple.

**Proof.** The second statement follows from the first statement and Theorem 13. Therefore we need to prove the first statement by using (43). Since  $14 > 5(1 + \sqrt{2})$ , by Theorem 12, the  $PD(a)$  eigenvalues lying in the ball  $B_n$  for  $n > 3$  are simple.

If  $\lambda \in B_3$ , then  $26 \leq \operatorname{Re} \lambda \leq 46$ . Using Lemma 2 and (41), we obtain the estimations (see Estimation 5 in Appendix)

$$\sum_{k \neq 3} |b_k|^2 < \frac{1}{2}, \quad |b_3|^2 > \frac{1}{2},$$

which, by Proposition 1, proves the simplicity of the  $PD(a)$  eigenvalues lying in  $B_3$ .

Now we need to prove that the balls  $B$  and  $B_2$  do not contain the multiple  $PD(a)$  eigenvalues. Since these balls are contained in the strip  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq 26\}$ , we consider the following cases:  $16 < \operatorname{Re} \lambda \leq 26$ ,  $12 < \operatorname{Re} \lambda \leq 16$  and  $\operatorname{Re} \lambda \leq 12$ .

In the first two cases, using Lemma 2, we get the inequality (see Estimation 6 and Estimation 7) obtained from (48) for  $n = 2$ , by replacing  $d_n$  with  $b_n$ , which proves, by Proposition 1, the simplicity of the eigenvalues. Now consider the third case  $\operatorname{Re} \lambda \leq 12$ . Using Lemma 2, we obtain (see Estimation 8 and Estimation 9 in Appendix)

$$\sum_{k=3}^{\infty} |b_k|^2 < \frac{1}{15}, \quad \frac{|b_3|}{|b_2|} < 0.2131. \quad (63)$$

The first inequality of (63) with (41) implies that

$$|b_1|^2 + |b_2|^2 > 1 - \beta, \quad (64)$$

where  $\beta < \frac{1}{15}$ . Instead of (60), using (64), and repeating the proof of (61), we obtain

$$\frac{b_2}{b_1} - \frac{b_1}{b_2} = \frac{(i + \delta)^2 - 1}{i + \delta} = \frac{2i(i + \delta) + \delta^2}{i + \delta} = \pm 2i + \gamma_1, \quad (65)$$

where  $|\gamma_1| < 0.01$ . Now dividing the first equality of (43) by  $b_1$ , and the second equality of (43) for  $k = 2$  by  $b_2$ , and then subtracting the second from the first, and using (65), we get

$$\frac{12}{a} = \pm 2i + \gamma_1 - \frac{b_3}{b_2}, \quad (66)$$

where by assumption  $|\frac{12}{a}| \geq 2.4$ . Thus, using (63) in (66), we get the contradiction  $2.4 \leq |\frac{12}{a}| < 2 + 0.2131 + 0.01 = 2.2231$ .

**Theorem 15.** If  $0 < |a| \leq \frac{4}{3}$ , then all eigenvalues of  $P(a)$  and  $N(a)$  are simple.

**Proof.** By Theorem 13 and Theorem 14, we need to prove that if  $|a| \leq \frac{4}{3}$ , then all  $PN(a)$  eigenvalues are simple. Since  $6 > (1 + \sqrt{2})\frac{4}{3}$ , by Theorem 12, the  $PN(a)$  eigenvalues lying in the ball  $B_n$  for  $n > 1$  are simple.

Now we prove that the balls  $A_0$  and  $A_1$  do not contain the multiple  $PN(a)$  eigenvalues. Since these balls are contained in  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 8\}$ , we consider the following cases:

**Case 1:**  $3 \leq \operatorname{Re} \lambda < 8$ . Using (42) and Lemma 2, (see Estimation 10 in Appendix), we obtain  $|a_1|^2 > \frac{1}{2}$ , which by Proposition 1, proves the simplicity of the eigenvalues.

**Case 2:**  $\operatorname{Re} \lambda < 3$ . Using Lemma 2, we obtain ( see Estimations 11 and 12 in Appendix)

$$\sum_{k=2}^{\infty} |a_k|^2 < \frac{1}{58}, \quad \frac{|a_2|}{|a_1|} < 0.10301. \quad (67)$$

The first inequality of (67) with (41) implies that

$$|a_0|^2 + |a_1|^2 > 1 - \rho, \quad (68)$$

where  $\rho < \frac{1}{58}$ . Instead of (60), using (68) and repeating the proof of (61), we obtain

$$\frac{a_1}{a_0} - \frac{a_0}{a_1} = \pm 2i + \gamma, \quad (69)$$

where  $|\gamma| < 0.0006$ . Now dividing the first equality of (42) by  $a_0$ , and the second by  $a_1$ , and then subtracting the second from the first, and taking into account (69), we get

$$\frac{4}{a} = \pm 2\sqrt{2}i + \sqrt{2}\gamma - \frac{a_2}{a_1}, \quad (70)$$

where by assumption  $|\frac{4}{a}| \geq 3$ . Therefore, using (67) we get the contradiction

$$3 \leq |\frac{4}{a}| < \sqrt{2}(2 + 0.0006) + 0.10301 = 2.9323.$$

#### 4. APPENDIX

**Estimation 1:** Let  $9 < \operatorname{Re} \lambda < 16$  and  $|a| \leq \frac{8}{\sqrt{6}}$ . Using (51), (53), and taking into account

that  $d(\lambda, \{4, 5, \dots\}) < 33$ , we get

$$|d_1|^2 \leq \frac{|a|^2}{|\lambda - 1|^2} \leq \frac{1}{6}, \quad |d_3|^2 \leq \frac{|a|^2}{|\lambda - 25|^2} \leq \frac{32}{243}, \quad \sum_{k=4}^{\infty} |d_k|^2 < \frac{128}{3267}, \quad \sum_{k \neq 2} |d_k|^2 < \frac{1}{2}.$$

**Estimation 2.** Let  $6 < \operatorname{Re} \lambda \leq 9$  and  $|a| \leq \frac{8}{\sqrt{6}}$ . By (51), (53), and by  $d(\lambda, \{4, 5, \dots\}) \leq 40$  we have

$$|d_1|^2 \leq \frac{32}{75}, \quad |d_3|^2 \leq \frac{1}{24}, \quad \sum_{k=4}^{\infty} |d_k|^2 \leq \frac{2}{75}, \quad \sum_{k \neq 2} |d_k|^2 < \frac{1}{2}.$$

**Estimation 3.** Let  $\operatorname{Re} \lambda \leq 6$  and  $|a| \leq \frac{8}{\sqrt{6}}$ . By (52) and (49), we have

$$|d_4| \leq \frac{\left|2 \times \frac{8}{\sqrt{6}}\right|^2 |d_2|}{|43| |19|} \leq \frac{\left|2 \times \frac{8}{\sqrt{6}}\right|^2 \frac{\sqrt{2}}{2}}{|43| |19|}, \quad |d_5| \leq \frac{\left|2 \times \frac{8}{\sqrt{6}}\right|^3 |d_2|}{|75| |43| |19|} \leq \frac{\left|2 \times \frac{8}{\sqrt{6}}\right|^3 \frac{\sqrt{2}}{2}}{|75| |43| |19|}. \quad (71)$$

Now using (51) and (53), and taking into account  $d(\lambda, \{6, 7, \dots\}) \leq 115$ , we obtain

$$|d_3|^2 \leq \frac{\left|\frac{8}{\sqrt{6}}\right|^2}{|19|^2} = \frac{32}{1083} \quad \& \quad \sum_{k=6}^{\infty} |d_k|^2 \leq \frac{4 \left|\frac{8}{\sqrt{6}}\right|^2}{|115|^2}, \quad \sum_{k=3}^{\infty} |d_k|^2 < 0.03415.$$

**Estimation 4.** Now we estimate  $\frac{|d_3|}{|d_2|}$  for  $\operatorname{Re} \lambda \leq 6$  and  $|a| \leq \frac{8}{\sqrt{6}}$ . Iterating (45) for  $k = 3$ , we get

$$\begin{aligned} d_3 &= \frac{ad_2 + ad_4}{\lambda - 25} = \frac{ad_2}{\lambda - 25} + \frac{a}{\lambda - 25} \left( \frac{ad_3 + ad_5}{\lambda - 49} \right) \\ &= \frac{ad_2}{\lambda - 25} + \frac{a^3 d_2}{(\lambda - 25)^2 (\lambda - 49)} + \frac{a^3 d_4}{(\lambda - 25)^2 (\lambda - 49)} + \frac{a^2 d_5}{(\lambda - 25) (\lambda - 49)}. \end{aligned} \quad (72)$$

Therefore, dividing both sides of (72) by  $d_2$ , and using (52), we obtain

$$\frac{|d_3|}{|d_2|} \leq \frac{\frac{8}{\sqrt{6}}}{19} + \frac{\left|\frac{8}{\sqrt{6}}\right|^3}{|43| |19|^2} + \frac{4 \left|\frac{8}{\sqrt{6}}\right|^5}{|43|^2 |19|^3} + \frac{8 \left|\frac{8}{\sqrt{6}}\right|^5}{|75| |43|^2 |19|^2} \leq 0.17432.$$

**Estimation 5.** Let  $26 \leq \operatorname{Re} \lambda \leq 46$  and  $|a| \leq 5$ . Using (56) and (58), we get

$$\begin{aligned} |b_1|^2 &\leq \frac{|a|^2}{|\lambda - 4|^2} \leq \frac{|5|^2}{|22|^2} = \frac{25}{484}, \quad |b_2|^2 \leq \frac{|a|^2}{|\lambda - 16|^2} \leq \frac{|5|^2}{|10|^2} = \frac{1}{4}, \\ |b_4|^2 &\leq \frac{|a|^2}{|\lambda - 64|^2} \leq \frac{|5|^2}{|18|^2} = \frac{25}{324}, \quad \sum_{k=5}^{\infty} |b_k|^2 \leq \frac{4|5|^2}{|54|^2} = \frac{25}{729}, \quad \sum_{k \neq 3} |b_k|^2 < \frac{1}{2}. \end{aligned}$$

**Estimation 6.** Let  $16 < \operatorname{Re} \lambda \leq 26$  and  $|a| \leq 5$ . By (55) and (57), we have

$$\begin{aligned} |b_1|^2 &\leq \frac{|a|^2}{|\lambda - 4|^2} \leq \frac{|5|^2}{|12|^2} = \frac{25}{144}, \quad |b_3|^2 \leq \frac{|a|^2}{|\lambda - 36|^2} \leq \frac{|5|^2}{|10|^2} = \frac{1}{4}, \\ \sum_{k=4}^{\infty} |b_k|^2 &\leq \frac{4|5|^2}{|38|^2} = \frac{25}{361}, \quad \sum_{k \neq 2} |b_k|^2 \leq \frac{25}{144} + \frac{1}{4} + \frac{25}{361} = \frac{25621}{51984} < \frac{1}{2}. \end{aligned}$$

**Estimation 7.** Let  $12 < \operatorname{Re} \lambda \leq 16$  and  $|a| \leq 5$ . By (55) and (57), we have

$$\begin{aligned} |b_1|^2 &\leq \frac{|5|^2}{|8|^2} = \frac{25}{64}, \quad |b_3|^2 \leq \frac{|5|^2}{|20|^2} = \frac{1}{16}, \quad |b_4|^2 \leq \frac{|5|^2}{|48|^2} = \frac{25}{2304}, \\ \sum_{k=5}^{\infty} |b_k|^2 &\leq \frac{4|5|^2}{|84|^2} = \frac{25}{1764}, \quad \sum_{k \neq 2} |b_k|^2 \leq \frac{25}{64} + \frac{1}{16} + \frac{25}{2304} + \frac{25}{1764} = \frac{53981}{112896} < \frac{1}{2}. \end{aligned}$$

**Estimation 8.** Let  $\operatorname{Re} \lambda \leq 12$  and  $|a| \leq 5$ . Using (55) and (57), we obtain

$$\begin{aligned} |b_4|^2 &\leq \frac{|5|^2}{|52|^2} = \frac{25}{2704}, \quad |b_3|^2 \leq \frac{|5|^2}{|24|^2} = \frac{25}{576}, \\ \sum_{k=5}^{\infty} |b_k|^2 &\leq \frac{4|5|^2}{|88|^2} = \frac{25}{1936}, \quad \sum_{k=3}^{\infty} |b_k|^2 \leq \frac{25}{2704} + \frac{25}{576} + \frac{25}{1936} = \frac{30495}{465088} < \frac{1}{15}. \end{aligned}$$

**Estimation 9.** Here we estimate  $\frac{|b_3|}{|b_2|}$  for  $\operatorname{Re} \lambda \leq 12$  and  $|a| \leq 5$ . Iterating (43) for  $k = 3$ , we get

$$\begin{aligned} b_3 &= \frac{ab_2 + ab_4}{\lambda - 36} = \frac{ab_2}{\lambda - 36} + \frac{a}{\lambda - 36} \left( \frac{ab_3 + ab_5}{\lambda - 64} \right) = \\ &= \frac{ab_2}{\lambda - 36} + \frac{a^3 b_2}{(\lambda - 36)^2 (\lambda - 64)} + \frac{a^3 b_4}{(\lambda - 36)^2 (\lambda - 64)} + \frac{a^2 b_5}{(\lambda - 36)(\lambda - 64)}. \end{aligned} \quad (73)$$

Now dividing both sides of (73) by  $b_2$ , and using (56) we obtain

$$\frac{|b_3|}{|b_2|} \leq \frac{5}{24} + \frac{|5|^3}{|52| |24|^2} + \frac{4 |5|^5}{|52|^2 |24|^3} + \frac{8 |5|^5}{|88| |52|^2 |24|^2} < 0.2131.$$

**Estimation 10.** Let  $3 \leq \operatorname{Re} \lambda < 8$  and  $|a| \leq \frac{4}{3}$ . By (42), Lemma 2(d), and (55), we have

$$\begin{aligned} |a_0|^2 &\leq \frac{|\sqrt{2}aa_1|^2}{|\lambda|^2} \leq \frac{|\frac{4}{3}|^2}{|3|^2} = \frac{16}{81}, \quad |a_2|^2 \leq \frac{|a|^2}{|\lambda - 16|^2} \leq \frac{|\frac{4}{3}|^2}{|8|^2} = \frac{1}{36}, \\ \sum_{k=3}^{\infty} |a_k|^2 &\leq \frac{4 |\frac{4}{3}|^2}{|28|^2} = \frac{4}{441}, \quad \sum_{k \neq 1} |a_k|^2 \leq \frac{16}{81} + \frac{1}{36} + \frac{4}{441} < \frac{1}{2}. \end{aligned}$$

**Estimation 11.** Let  $\operatorname{Re} \lambda < 3$  and  $|a| \leq \frac{4}{3}$ . By Lemma 2(d), (55), and (57), we have

$$|a_2|^2 \leq \frac{|a|^2}{|\lambda - 16|^2} \leq \frac{16}{1521}, \quad \sum_{k=3}^{\infty} |a_k|^2 \leq \frac{4 |\frac{4}{3}|^2}{|33|^2} = \frac{64}{9801}, \quad \sum_{k=2}^{\infty} |a_k|^2 < \frac{1}{58}.$$

**Estimation 12.** Here we estimate  $\frac{a_2}{a_1}$  for  $\operatorname{Re} \lambda < 3$  and  $|a| \leq \frac{4}{3}$ . Iterating (42) for  $k = 2$ , we get

$$\begin{aligned} a_2 &= \frac{aa_1 + aa_3}{\lambda - 16} = \frac{aa_1}{\lambda - 16} + \frac{a}{\lambda - 16} \left( \frac{aa_2 + aa_4}{\lambda - 36} \right) = \\ &= \frac{aa_1}{\lambda - 16} + \frac{a^3 a_1}{(\lambda - 16)^2 (\lambda - 36)} + \frac{a^3 a_3}{(\lambda - 16)^2 (\lambda - 36)} + \frac{a^2 a_4}{(\lambda - 16)(\lambda - 36)}. \end{aligned} \quad (74)$$

Now dividing both sides of (74) by  $a_1$ , and using Lemma 2(d) and (56), we obtain

$$\frac{|a_2|}{|a_1|} \leq \frac{\frac{4}{3}}{13} + \frac{|\frac{4}{3}|^3}{|33| |13|^2} + \frac{4 |\frac{4}{3}|^5}{|33|^2 |13|^3} + \frac{8 |\frac{4}{3}|^5}{|61| |33|^2 |13|^2} < 0.10301.$$

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