N-GAIR: Non-Greedy Asynchronous Interference Reduction Algorithm in Wireless Networks

ZEKERIYA UYKAN

Control and Automation Eng. Dept. Dogus University Kadiköy, 34722 Istanbul TURKEY
E-mail: zuykan@dogus.edu.tr

In this paper, we consider optimum channel/frequency allocation problem in wireless networks by reducing total network interference signal powers, which is an NP-complete problem. Its optimum solution for general wireless networks for even 2-channel case is not known. Turning the channel/frequency allocation problem into a maxCut graph partitioning problem, we i) propose a spectral clustering based channel allocation algorithm, called SpecPure, and ii) propose and analyze a novel Non-Greedy Asynchronous Interference Reduction Algorithm for Wireless Networks, called N-GAIR, and iii) extend the results in [1] to the case where the number of channels is arbitrary. By simulating various CDMA based ad-hoc networks, we examine various scenarios to compare the performances of the proposed algorithms with the reference algorithm. We draw various conclusions for different network scenarios. For example, the results show that the SpecPure algorithm performs well for "symmetric" base locations scenarios, while the N-GAIR performs best for random base locations scenarios. The results confirm the effectiveness of the proposed algorithms, which can be adopted by any cellular, cognitive, ad-hoc or mesh type radio networks.

Keywords: Mobile radio systems; optimum channel/frequency allocation; max cut, weighted graph partitioning; spectral clustering; minimum-interference-channel-allocation algorithm.

I. INTRODUCTION

Channel/frequency allocation is an important and essential mechanism in order to mitigate the interference in a wireless network, and has been a focus of intensive research in both academia and industry in the last two decades.

As various new types of wireless networks with/without infrastructure, like cognitive, ad-hoc or mesh type radio systems, among others, are about to emerge in the coming decade, the optimum channel allocation problem remains to be a hot research topic in both academia and industry. However, the channel/frequency allocation problem in wireless cellular system is known to be NP-complete (see e.g. [2], [3], among others). This means that no polynomial-time algorithms are available for that. The optimum general solution for any mobile network for even 2-channel case is not known.

There is a vast literature in the area of channel/frequency allocation/ assignment in various wireless radio systems. For a survey, and further references see e.g. [4], [5], among others. Various algorithms used in practical systems are based on a simple heuristics that the mobile/base is assigned to the channel in a distributed fashion where it experiences minimum interference (e.g. [6-8], etc). Although these algorithms perform well in practice in general, their solutions do not give any guarantee on the global performance because their performance may typically depend on the initial states and may suffer from local minima problem for many various location distributions. Two asynchronous minimum-interference algorithms, called basic GADIA and soft GADIA, are examined in details in [9] for symmetric link matrix case by Babadi and Tarokh. They analytically prove in [9] that the performance of the GADIA is close to that of an optimum centralized frequency allocation algorithm for symmetric case. Re-investigating the GADIA from a game-theoretic perspective, Wu *et.al.* show in [10] that the soft GADIA is shown to converge to a global minimum with arbitrarily high probability for a sufficiently large learning parameter (and for symmetric case).

Channel allocation problem is also examined as a graph multi-coloring problem, which is NP-complete as well. For various algorithms used in graph multi-coloring and further references see e.g. [7] and [3] among others. The multi-coloring problem for channel allocation can be summarized as follows [7]: Let $G = (V, E)$ denote an interference graph, where the node set V denotes cells or base stations, and the edge set E represents geographical proximity of cells and therefore the possibility of co-channel interference. A static snapshot of the network at a fixed instant of time is given by a weighted graph. The goal of an algorithm for the channel allocation problem, at that instant in time, is to be able to allocate distinct channels to each node such that no two adjacent nodes have channels in common. In graph-theory terminology, what is required is a proper multi-coloring of G with distinct colors representing distinct channels. In this paper, we follow this path which formulates the average network/channel interference minimization problem as a graph partitioning problem, in general. However, in this paper, our main contribution is that turning the channel allocation problem into a maxCut problem, we propose and examine two channel allocation algorithms. Furhermore, while typically in literature, set *E* represents geographical proximity of the base stations which yields symmetric adjacency matrix, in our approach in this

paper set E represents link gains (received-signal-strength) yielding asymmetric system matrix. One of the proposed solutions is based on the maxCut approximation, while the other one is a hybrid algorithm which first finds the maximum eigenvector of network link gain matrix followed by a novel interference-reduction algorithm. We examine the advantages and drawbacks of the proposed solutions with respect to the minimum-interference algorithm as the main reference algorithm.

The rest of the paper is organized as follows: Section II gives the formulation of the problem. The spectral based solutions for 2-channel case and *arbitrary number of channel case* are presented in section III and IV, respectively. The proposed method called N-GAIR is presented and analyzed in section V. The computer simulations are shown in section VI, followed by the conclusions in section VII.

II. PROBLEM FORMULATION

In this section, we formulate the total and average network interference to be minimized by channel/ frequency allocation. We would like to formulate it on a general level so that the results can be adopted by different cellular, cognitive, ad-hoc, sensor, etc type radio networks: Let us consider a general mobile radio system consisting of N transmitters which are possibly mobile. Let's call the transmitter as Mobile Base (MB), and the receiver as Mobile Station (MS). Downlink is considered without loss of generality. We assume the flat fading and slow fading channel case, which means the channel coherence time is much higher than the channel allocation algorithm runtime. In practice, this includes, for example a scenario where MBs are either resting or are moving with relatively small speed, or pedestrian speed.

Let the number of co-channel MB's be N. Then the received Signal-to-Noise-Ratio (SINR) at receiver i is given by (see. e.g. [8], [9], [10])

$$\theta_i(k) = \frac{g_{ii} p_i(k)}{\phi_i + \sum_{j=1}^{N} g_{ij} p_j(k)}, \qquad i = 1, 2, \cdots, Na \qquad (1)$$

where $\theta_i(k)$ denotes the received SINR at receiver i at time k; $p_i(k)$ is the transmit power of transmitter i at time k; g_{ij} is link gain from transmitter j to receiver i (involving path loss, shadowing, etc), and ϕ_i is the thermal noise at receiver i. The matrix whose entries are the link gains, i.e. $[\mathbf{G}]_{ij} = g_{ij}$ is called as system link gain matrix. The link gain g_{ij} can be modeled as follows

$$g_{ij} = \frac{s_{ij} \, c_{ij}}{d^{\beta}_{ij}}, \qquad i, j = 1, 2, \cdots, N \qquad (2)$$

where s_{ij} is the shadow fading term, d_{ij}^{β} is propagation loss with pathloss exponent β, and c_{ij} is multipath fading factor [8]. For information about modeling of radio wave propagation, see e.g. [8], [11]. Since, in this paper, we focus on the optimum channel allocation problem, we assume that the powers of the transmitters are fixed. Let the number of MBs is N and that of channels/frequencies be L (where $N > L$). So, we need to allocate the N MB's to L channels/frequencies. Without loss of generality and for the sake of brevity, we assume that N is an integer multiple of L. Once allocation of N MB's to L channels is performed, then total co-channel interference in the network, denoted as I_{tot}^{ntw}, is given by

$$
\begin{aligned}
I_{tot}^{ntw} &= \sum_{s \in 1}^{L} I_s \qquad\qquad\qquad s = 1, \cdots, L \\
&= p_{tx} \sum_{s=1}^{L} \sum_{\substack{j \in C_s}}^{N_s} \sum_{\substack{i \in C_s \\ (i \neq j)}}^{N_s} g_{ij}
\end{aligned}
\tag{3}
$$

Where p_{tx} is MB's fixed transmit power, C_s represents the set of MBs assigned to channel/frequency s; N_s is the length of the set C_s (i.e., the number of MBs in channel s), and g_{ij} represents the corresponding link gains, I_s is the sum of the interference signal powers experienced by those MBs using the same channel s, and $\sum_{s=1}^{L} N_s = N$. Similarly, we define average co-channel interference, I_{ave}^{ntw}, as

$$
\begin{aligned}
I_{ave}^{ntw} &= \sum_{s \in 1}^{L} \frac{1}{N_s} I_s \qquad\qquad\qquad s = 1, \cdots, L \\
&= p_{tx} \sum_{s=1}^{L} \frac{1}{N_s} \sum_{\substack{j \in C_s}}^{N_s} \sum_{\substack{i \in C_s \\ (i \neq j)}}^{N_s} g_{ij}
\end{aligned}
\tag{4}
$$

where N_s is the number of MBs in channel s. The lower the interference, the better performance (e.g. data rate) of the wireless network for the same radio resources. It's also expected that the lower the interference in (3)-(4), the higher the SINR in (1). Therefore it is aimed to minimize the total/average interference in (3)-(4) by improving the channel allocation mechanism: Thus, from (1)-(3), we formulate the channel allocation problem as determining the sets C_s of mobiles ($s = 1, ..., L$) which minimizes the total network interference I_{tot}^{ntw} in (3), i.e.:

$$
\min_{\text{determine } C_S, \ (s=1, \cdots, L)} I_{tot}^{ntw}
\tag{5}
$$

Similarly, for the average co-channel interference, I_{ave}^{ntw} in (4), channel allocation problem is

$$\min_{\text{determine } C_S,\ (s=1,\cdots,L)} I_{ave}^{ntw} \tag{6}$$

Let's assume that there should be at least one mobile in each channel due to the assumption $N>L$, and assume that neighboring channel interference can be omitted. Once the channel allocation determines the sets C_S, then for each channel, we have an $N_S \times N_S$ dimensional L different co-channel link gain matrices for calculating the (total/average) network interference. For example, let $N=6$, and $L=2$: In this case, if the channel allocation algorithm allocates one MB into one channel, and the remaining 5 MBs into the other channel, then it means we have 1×1 and 5×5 dimensional two different link gain matrices to calculate the network interference. If the allocation were 3 MBs into one channel and other 3 MBs into the other channel, then we would have 3×3 and 3×3 dimensional two different link gain matrices. The total network interference in (5) would have 20 entries (because in one channel zero interference, and in the other $(5\times 5)-5 = 20$ nondiagonal elements) in the first case, and would have only 12 entries in the latter case (because $\big((3\times 3)-3\big) + \big((3\times 3)-3\big) = 12$).

Channel allocation procedure determines which g_{ij}'s to be chosen in minimizing the total network interferences I_{tot}^{ntw} or I_{ave}^{ntw} in (3) and (4), respectively, and this is an NP-complete problem for even $L=2$ case. The amount of time for exhaustive search goes roughly exponentially by the network size N (for proof, see e.g. p.359 of [12]). In what follows in section III, we examine the $L=2$ case.

III. SPECTRAL CLUSTERING BASED SOLUTIONS FOR $L=2$ CASE

In this section, we examine the optimization problem defined in section 2 for the case $L=2$, because *i)* it's general solution is not known (NP-complete), and *ii)* for $L=2^q$, $q\geq 2$, the same algorithm can be iteratively applied $2^q - 1$ times to find to solution.

A. maxCut Solution: Approximation by Maximum-Eigenvector (SpecPure Algorithm)

Let's first define the following matrices:

Definition: Denoting the received signal strength (RSS) at MS *i* of the pilot signal from MB *j* as r_{ij}, we define so-called *received-signal-strength (RSS) based adjacency system matrix* as follows

$$\mathbf{R} = \left[r_{ij}\right]_{N \times N} = \left[g_{ij} p_{tx}\right] \tag{7}$$

where g_{ij} is given by (2), and p_{tx} is the fixed transmit power of MB j. Because we assume that all the (downlink) transmit powers are fixed and the same, the r_{ij} and g_{ij} gives the same information. So, the link matrix $\mathbf{G}$ obtained from (2) and the matrix R in (7) gives the same information. The RSS info is readily available in current cellular radio systems (GSM, 3G, etc). Every mobile measures the pilot signals of all the neighboring base stations, and reports the sorted measurement list to its serving base. This info is used for e.g. handover process. We assume that the bases can exchange the RSS info so that the RSS based system matrix can be available in a centralized base.

In graph partitioning literature, the similarity (or dissimilarity) matrix is symmetric because a metric applied when constructing the matrix. As in [7], in most of the works treating the channel allocation problem as a graph multi-coloring problem, the edges represent geographical adjacency of the cells. In this paper, our approach is based on the link gain matrix (i.e., RSS info because the transmit powers are fixed), which is naturally asymmetric. Indeed, in the wireless network case, the defined *RSS based non-adjacency matrix* in (7), and thus the matrix $\mathbf{G}$, is naturally and may be highly asymmetric because of the random MB and MS locations as well as the randomness in the channel.

Laplacian matrix, sometimes called admittance matrix or Kirchhoff matrix, is a matrix representation of a graph, and has various interesting mathematical properties (see e.g. [13]). Unnormalized Laplacian matrix (e.g. [13], [14]) for the link matrix is defined as

$$\mathbf{L} = \mathbf{D} - \mathbf{G} \tag{8}$$

where diagonal matrix $\mathbf{D} = \left[d_{mn}\right] = \begin{cases} \displaystyle\sum_{j=1,\ (j \neq i)}^{N} g_{ij}, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}$.

Lemma 1. Let's define a discrete-value vector $\mathbf{x} = \left[x_1 \ \cdots \ x_N\right]$ such that $x_i, x_j \in \{-1, +1\}$. For the defined (asymmetric) link-gain matrix $\mathbf{G}$ by (2) and the Laplacian matrix in (8), we obtain

$$\mathbf{x}^T \mathbf{L} \mathbf{x} \ = \ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \left(x_i - x_j\right)^2 \tag{9}$$

Proof: Examining $\mathbf{x}^T \mathbf{L} \mathbf{x}$ using the matrix $\mathbf{G}$ and the diagonal matrix $\mathbf{D}$ gives

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \mathbf{x}^T (\mathbf{D} - \mathbf{G}) \mathbf{x}$$
$$= \sum_{i=1}^{N} d_{ii} x_i^2 - \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} x_i x_j \tag{10}$$

Writing $d_{ii} = \sum_{j=1}^{N} g_{ij}$ in (10) gives

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} g_{ij} \right) x_i^2 - \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} x_i x_j \tag{11}$$

Introducing the constraint that $x_i^2 = x_j^2$ (because $x_i, x_j \in \{-1, +1\}$) for i, $j = 1,2,...,N$, in eq.(11) yields

$$\mathbf{x}^T \mathbf{L} \mathbf{x} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \left(x_i - x_j \right)^2 \tag{12}$$

The equation (12) is obtained using the fact that if $x_i, x_j \in \{-1, +1\}$, then $\left(x_i - x_j \right)^2 = 2 \left(x_i^2 - x_i x_j \right)$. This completes the proof.

∎

In graph theory, a cut means a partition of the vertices of graph into two sets; the size of the cut is the sum of the edges with one vertex on either side of the partition. A maximum cut is a cut whose size is not smaller than the size of any other cut. The problem of finding a maximum cut in a graph is known as the max-cut problem, which is NP-complete.

Proposition 1. The optimum channel/frequency allocation defined above:

i)   is equal to optimum maxCut solution of the proposed RSS-based *adjacency* matrix with zero diagonal (i.e., $g_{ii} = 0$) in (7); and
ii) can be approximated by the maximum eigenvector computation of the matrix $\mathbf{L}$ in (8) after relaxation.

Proof. Excluding the own channel gains from the system link gain matrix $\mathbf{G}$ (i.e., making the main diagonal zero, $g_{ii} = 0$), we write the 1-norm of $\mathbf{G}$ with zero diagonal as

$$\| \mathbf{G} \|_1 = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \tag{13}$$

Then, considering the grouping of MBs into two sets C_1 and C_2, we write

$$\|\mathbf{G}\|_1 = \sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij} = \sum_{i \in C_1}^{N_1}\sum_{j \in C_1}^{N_1} g_{ij} + \sum_{i \in C_1}^{N_1}\sum_{j \in C_2}^{N_2} g_{ij} + \sum_{i \in C_2}^{N_2}\sum_{j \in C_2}^{N_2} g_{ij} + \sum_{i \in C_2}^{N_2}\sum_{j \in C_1}^{N_1} g_{ij}$$

(14)

Eq.(14) can be written as

$$\|\mathbf{G}\|_1 = \text{constant} = I_1 + J_1 + I_2 + J_2$$

(15)

where $I_S = \sum_{i \in C_S}^{N_S}\sum_{j \in C_S}^{N_S} g_{ij}$ is the total co-channel interference for channel $s = 1$

and 2, and $J_1 = \sum_{i \in C_1}^{N_1}\sum_{j \in C_2}^{N_2} g_{ij}$, and $J_2 = \sum_{i \in C_2}^{N_2}\sum_{j \in C_1}^{N_1} g_{ij}$, represents the total interfer-

ence which is eliminated once C_1 and C_2 are determined by the channel allo-
cation process. As a brief example for the notation C_1 and C_2, let $N = 6$, and
let MB 1, 2 and 3 be in C_1, and let MB 4, 5 and 6 be in C_2. In this
case $C_1 = \{1, 2, 3\}$, $C_2 = \{4, 5, 6\}$; $I_1 = g_{12} + g_{13} + g_{21} + g_{23} + g_{31} + g_{32}$;
$I_2 = g_{45} + g_{46} + g_{54} + g_{56} + g_{64} + g_{65}$; $J_1 = g_{14} + g_{15} + g_{16} + g_{24} + g_{25} + g_{26}$
$+ g_{34} + g_{35} + g_{36}$; and $J_2 = g_{41} + g_{42} + g_{43} + g_{51} + g_{52} + g_{53} + g_{61} + g_{62} + g_{63}$.
Note that we treat own link gains $g_{ii} = 0$ in this formulation because we only
consider minimizing the total/average network interference.

In graph theory, a *cut* means a partition of the vertices of graph into two
sets (C_1 and C_2); and the size of the cut is the sum of the edges with one vertex
on either side of the partition. This means

$$cut(C_1, C_2) = \sum_{i \in C_1}^{N_1}\sum_{j \in C_2}^{N_2} g_{ij} \text{ ''' and ''' } cut(C_2, C_1) = \sum_{i \in C_2}^{N_2}\sum_{j \in C_1}^{N_1} g_{ij}$$

(16)

Volume (*vol*) of a set is equal to the sum of all the edges whose vertices are in
the same set (e.g. [13], [16]). This means

$$vol(C_1) = \sum_{i \in C_1}^{N_1}\sum_{j \in C_1}^{N_1} g_{ij} \text{ ' and '' } vol(C_2) = \sum_{i \in C_2}^{N_2}\sum_{j \in C_2}^{N_2} g_{ij}$$

(17)

So, from (13)-(17) and $vol(G) = \sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}$, we can write

$$\|\mathbf{G}\|_1 = vol(G) = vol(C_1) + cut(C_1, C_2) + vol(C_2) + cut(C_2, C_1)$$

(18)

where $vol(C_1) = I_1$, $vol(C_2) = I_2$, and $J_1 = cut(C_1, C_2)$ and $J_2 = cut(C_2, C_1)$.

Let's define a discrete-value vector $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix}$ such that, $x_i =$
$\begin{cases} -1, & \text{if vertex (MB) } i \text{ in set (channel) } C_1 \\ +1, & \text{if vertex (MB) } i \text{ in set (channel) } C_2 \end{cases}$ $i = 1,2,...N.$ Then, we can write

$$J_1 + J_2 = \sum_{i \in C_1}^{N_1} \sum_{j \in C_2}^{N_2} g_{ij} + \sum_{i \in C_2}^{N_2} \sum_{j \in C_1}^{N_1} g_{ij} = \frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \left(x_i - x_j \right)^2 \qquad (19)$$

From (14)-(15) ,

$$\min_{C_1,C_2} \left(I_1 + I_2 \right) \equiv \max_{C_1,C_2} \left(J_1 + J_2 \right) = \max_{C_1,C_2} \left(cut(C_1,C_2) + cut(C_2,C_1) \right) \qquad (20)$$

This is equal to well-known weighted maxCut problem in graph theory (e.g. [13]). From eq.(12) (Lemma 1), eq.(19) and (20), we obtain

$$\min_{C_1,C_2} \left\{ I_1 + I_2 \right\} = \max_{C_1,C_2} \left(cut(C_1,C_2) + cut(C_2,C_1) \right) = \max \left\{ \mathbf{x}^T \mathbf{L} \mathbf{x} \right\}$$

$$= \quad \max \left\{ \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \left(x_i - x_j \right)^2 \right\}$$

where $x_i, x_j \in \left\{ -1, +1 \right\}$ \qquad (21)

Let's denote the eigenvector corresponding the maximum eigenvalue of matrix $\mathbf{L}$ in (8) as $\mathbf{e}_{max}$. Relaxing the optimization in (21) such that $\mathbf{x} \in \Re^{N \times 1}$ (instead of being integers), we know from linear algebra theory that the optimum solution which maximizes $\mathbf{x}^T \mathbf{L} \mathbf{x}$ in (21) with unit norm constraint is equal to

$$\mathbf{x}^{opt} = sign\left(\mathbf{e}_{max} \right) \qquad (22)$$

which completes the proof.

∎

It should be noted that the unnormalized Laplacian matrix in (8) is not the only choice in implementation. Different normalized Laplacian matrices can be used in practice after defining symmetric link gain matrix $\bar{\mathbf{G}} = 0.5 \left(\mathbf{G} + \mathbf{G}^T \right)$ i.e., $\bar{g}_{ij} = 0.5 \left(g_{ij} \right) + g_{ji}$. In Lemma 1 and Proposition 1 above, we proceed with asymmetric link gain matrix $\mathbf{G}$ for the sake of clarity and brevity. Equivalently, we may alternatively define the matrix in (8) as $\bar{\mathbf{L}} = \mathbf{D} - \bar{\mathbf{G}}$, where

$$\mathbf{D} = \left[d_{mn} \right] = \begin{cases} \displaystyle\sum_{j=1,\ (j\neq i)}^{N} \overline{g}_{ij}\ , & \text{if } m = n \\ 0, & \text{otherwise} \end{cases} \quad,\ \text{and obtain the same results. This is}$$

because $\left\| \mathbf{G} \right\|_1 = \left\| \overline{\mathbf{G}} \right\|_1$ and $\displaystyle\sum_{i\in C_1}^{N_1} \sum_{j\in C_1}^{N_1} \overline{g}_{ij} + \sum_{i\in C_2}^{N_2} \sum_{j\in C_2}^{N_2} \overline{g}_{ij} = \sum_{i\in C_1}^{N_1} \sum_{j\in C_1}^{N_1} g_{ij} + \sum_{i\in C_2}^{N_2} \sum_{j\in C_2}^{N_2} g_{ij}$,

and $\displaystyle\sum_{i\in C_1}^{N_1} \sum_{j\in C_2}^{N_2} \overline{g}_{ij} + \sum_{i\in C_2}^{N_2} \sum_{j\in C_1}^{N_1} \overline{g}_{ij} = \sum_{i\in C_1}^{N_1} \sum_{j\in C_2}^{N_2} g_{ij} + \sum_{i\in C_2}^{N_2} \sum_{j\in C_1}^{N_1} g_{ij}$. In this paper, we only

use the Laplacian matrix in (8) for the sake of its simplicity. Alternatively, some of the Laplacian matrices which can be used in practice are "symmetric" Laplacian $\mathbf{L}_{sym} = \mathbf{D}^{-1/2} \overline{\mathbf{L}} \mathbf{D}^{-1/2}$, or "random-walk" Laplacian $\mathbf{L}_{rw} = \mathbf{D}^{-1} \overline{\mathbf{L}}$ (e.g. [13], [14]). It's reported that the results may depend on the choice of the Laplacian matrix.

Corollary 1. The solution in Proposition 1 which minimizes the total co-channel interference in the network I_{tot}^{ntw} in (3) also minimizes the average co-channel interference I_{ave}^{ntw} in (4) if it's required that the total number of MBs should be evenly distributed over the channels.

Proof: If $N_1 = N_2 = (N/2)$, from eq.(3), (4) and Proposition 1, it's straightforward to obtain

$$\min\left\{ I_{ave}^{ntw} \right\} = \min\left\{ \frac{2}{N} \left(\left\| \mathbf{G} \right\|_1 - \left(cut(A,B) + cut(B,A) \right) \right) \right\} \qquad (23)$$

which gives Corollary 1.

∎

In Proposition 1, we turn the the channel allocation problem into a maxCut graph partitioning problem. Therefore, any commercial applications developed for general maxCut problems, e.g., semiconductor design [16], quantum computing [17], semi-definite programming (e.g. [18], [19]) to name a few, could also be used for channel allocation problem. In what follows, we propose spectral clustering as an approximation to the maxCut solution due to its simplicity and good performance for various scenarios as shown by simulation results in section VI.

IV. SPECTRAL CLUSTERING BASED SOLUTIONS FOR ARBITRARY L CASE

The case for $L = 2$ is considered in [1]. The analysis of the case $L = 2$ is presented above. In this section, we examine the same problem for arbitrary L. As in (14)-(15), let's define I_l and J_l , $l = 1,..., L$ as

$$I_l = \sum_{i \in C_l}^{N_l} \sum_{j \in C_l}^{N_l} g_{ij} \cdot \text{and} \cdot J_l = \sum_{i \in C_l}^{N_l} \sum_{j \in \bar{C}_l}^{\bar{N}_l} g_{ij} \quad (24)$$

where set C_l consists of the indices of those MBs which are in set l, and set $\bar{C}_l$ represents the indices of the rest of the MBs, N_l is the length of C_l (i.e., the number of MBs in set C_l), and $\bar{N}_l$ is the length of $\bar{C}_l$ (and thus $\bar{N}_l = N - N_l$). Allocating N MBs into L channels (i.e. determining the sets $\{C_1, C_2, \cdots, C_L\}$), the volume (i.e., entrywise 1-norm) of the link gain matrix $\mathbf{G}$ with zero diagonal can be written as

$$vol(\mathbf{G}) = \sum_{l=1}^{L} (I_l + J_l) \quad (25)$$

where I_l and J_l are defined in (24). From (24) and (25)

$$vol(\mathbf{G}) = \sum_{l=1}^{L} \left\{ \sum_{i \in C_l}^{N_l} \sum_{j \in C_l}^{N_l} g_{ij} + \sum_{i \in C_l}^{N_l} \sum_{j \in \bar{C}_l}^{\bar{N}_l} g_{ij} \right\} \quad (26)$$

Using the definition of *vol* and *cut* as defined above, eq.(26) can be written as

$$vol(\mathbf{G}) = \text{constant} = \sum_{l=1}^{L} vol(C_l) + \sum_{l=1}^{L} cut(C_l, \bar{C}_l) \quad (27)$$

where $vol(C_l) = \sum_{i \in C_l}^{N_l} \sum_{j \in C_l}^{N_l} g_{ij}$ is the sum of *interference in channel l*, and

$cut(C_l, \bar{C}_l) = \sum_{i \in C_l}^{N_l} \sum_{j \in \bar{C}_l}^{\bar{N}_l} g_{ij}$ represents the sum of the *interference that is elimi-*

nated in channel l due to the channel allocation result . Comparing (25) and (27)

$$I_l = vol(C_l) = \sum_{i \in C_l}^{N_l} \sum_{j \in C_l}^{N_l} g_{ij} \quad \text{and} \quad J_l = cut(C_l, \bar{C}_l) = \sum_{i \in C_l}^{N_l} \sum_{j \in \bar{C}_l}^{\bar{N}_l} g_{ij} \quad (28)$$

where $l = 1,2,...,N$. From (27)

$$\min_{\{C_l\}_{l=1}^{L}} \left\{ \sum_{l=1}^{L} vol(C_l) \right\} \equiv \max_{\{C_l\}_{l=1}^{L}} \left\{ \sum_{l=1}^{L} cut(C_l, \bar{C}_l) \right\} \quad (29)$$

From eq.(29), minimizing the total co-channel interference is equal to well-known weighted *maxCut* problem in graph theory as explained in Section III. For the unnormalized Laplacian matrix in (8) it's known from the

linear algebra theory that the *maxCut* problem in (29) can be formulated as matrix trace maximization problem for general L (see e.g. [14]) as follows

$$\text{maxCut}_{C_1,\dots,C_L} \{\mathbf{G}\} = \max \left\{ Tr\left(\mathbf{H}^T \mathbf{L} \mathbf{H}\right)\right\} \qquad (30)$$

where $Tr(\cdot)$ represents trace operation for a matrix, and where

$$\mathbf{H} = [h_{ij}] = \begin{cases} 1/\sqrt{L}, & \text{if } i\text{'th MB is in } C_j \\ 0, & \text{otherwise} \end{cases} \qquad (31)$$

$$i = 1,\cdots,N, \qquad j = 1,\cdots,L$$

Equation (30) is standard trace maximization problem, and relaxing the discrete-solution constraint of (31), it is well-known from linear algebra that the optimum solution is given by choosing $\mathbf{H}$ as the matrix which contains the greatest k eigenvectors of matrix $\mathbf{L}$ as columns (see e.g.[14]). Thus, the first proposed algorithm, the SpecPure algorithm, for arbitrary L is given in TABLE 1.

V. N-GAIR: SPECTRAL-BASED NON-GREEDY ASYNCHRONOUS INTERFERENCE REDUCTION ALGORITHM

The proposed algorithm in this paper is called N-GAIR (spectral-based Non-Greedy Asynchronous Interference Reduction algorithm) and is summarized in TABLE 2.

Proposition 2: In the second phase of the N-GAIR summarized in TABLE 2, it converges to a (local) minimum of the total co-channel interference in the

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1. Every mobile base (MB) transmits a pilot signal.
2. Every MS measures the received signal strength (RSS), , from all other MBs. The average link gains g_{ij} are calculated from r_{ij} because pilot signal power is known.
3. Sends the RSS information to the center MB.
4. The center MB establishes the RSS based adjacency matrix in (7); and finds the greatest L eigenvectors of the Laplacian matrix $\mathbf{L}_{N\times N}$ in (8).
5. Establish a matrix $\mathbf{Y}_{N\times L}$ whose columns are the greatest L eigenvectors.
6. Run k-means algorithm to the row vectors $\left\{\mathbf{y}_i\right\}_{i=1}^{N} \in \Re^L \cdots$ of $\cdots \mathbf{Y}_{N\times L}$ of $\mathbf{Y}_{N\times L}$ and determine the clusters $C_1,\cdots,C_L \cdot .$

TABLE 1
SpecPure algorıtm for Arbıtrary L.

Phase 1: Central rough estimate

Run a standard spectral clustering for L clusters as in Step 1 in TABLE 1 and determine the clusters $C_1, \cdots, C_L$.

Phase 2: Asynchronously tuned solution

Defining $I_i = vol\left(C_i\right) = \sum_{i \in C_i}^{N_{s(i)}} \sum_{j \in C_i}^{N_{s(i)}} g_{ij}$, $i = 1,...L$, update the channel allocation by the following procedure:

for $i = 1:N$

    $s(i) =$ the channel index for MB i.

    $I_{s(i)} = vol\left(C_{s(i)}\right) = \sum_{i \in C_{s(i)}}^{N_{s(i)}} \sum_{j \in C_{s(i)}}^{N_{s(i)}} g_{ij}$; %Determine the total interference in channel $s(i)$

    $d = 1;$

    while $d<L$ (such that $\left(s(i) + d\right) \in \left\{1, 2, \cdots, L\right\}$)

        $I_{s(i)+d} = \sum_{i \in C_{s(i)+d}}^{N_{s(i)+d}} \sum_{j \in C_{s(i)+d}}^{N_{s(i)+d}} g_{ij}$% the total interference in channel $s(i) + d$ if the MB

        i were assigned to $(s(i) + d)$.

        if $(I_{s(i)+d} < I_{s(i)})$

            $s(i) \leftarrow s(i) + d$; % update the channel because total network interference in

            (3) is reduced!

            break; % the channel is found, and break the while loop

        end

        $d \leftarrow d + 1$

    end

end

TABLE 2

N-GAIR: Spectral-Based Non-Greedy Asynchronous Interference Reduction Algorithm.

network I_{tot}^{ntw} in (3) for any initial condition within a finite number of iterations.

Proof: Let's denote $I_{tot}^{ntw}(n)$ as the total network (co-channel) interference at iteration n. Thus, $I_{tot}^{ntw}(0)$ indicates the initial total interference corresponding to the initial channel allocation at iteration 0. From (3), (25) and (28) $I_{tot}^{ntw} = \sum_{l=1}^{L}(I_l + J_l)$. Representing the channel index for MB i as $s(i)$ such that $s(i) \in \left\{1, 2, \cdots, L\right\}$, we can write from (28) that $I_{s(i)} = vol\left(C_{s(i)}\right) = \sum_{i \in C_{s(i)}}^{N_{s(i)}} \sum_{j \in C_{s(i)}}^{N_{s(i)}} g_{ij}$.

The N-GAIR allocates the MBs to L channels according to the following simple rule: If, for MB i, there exists a channel $s(i) + d \in \left\{1, 2, \cdots, L\right\}$ such that $I_{s(i)+d} < I_{s(i)}$, then MB i is allocated from channel $s(i)$ to channel $s(i) + d$,

and stops searching for further channels, and goes to the next step. Examining the fact that $I_{s(i)+d} < I_{s(i)}$ in eq.(3) yields in

$$I_{tot}^{ntw}(n) < I_{tot}^{ntw}(n-1) \tag{32}$$

This means that for every iteration n, at which an MB is allocated to a new channel, the total network interference of eq.(3) is further minimized. And because the number of all possible channel combinations is limited, the N-GAIR will converge to a local minimum of the cost function (total network interference in eq.(3)) within a finite number of iterations. When none of the MBs finds a better channel with less interference any more, a (local) minimum is reached. This completes the proof.

∎

As compared to the GADIA algorithm in [6], the proposed algorithm N-GAIR updates the channel for the first better channel encountered, while the GADIA checks all possible L channels to determine the best one in a greedy manner. This implies that the N-GAIR would require less number of measurements in an epoch than the GADIA does in symmetric link gain network scenarios. However, the disadvantage of N-GAIR is that it needs more interference information: For every channel, while the basic GADIA needs only the co-channel interference measurements, the N-GAIR needs not only co-channel interference from other cells but also the portion of the interference it causes to other cells.

VI. SIMULATION RESULTS

We compare the performances of the proposed SpecPure and N-GAIR algorithms with the basic GADIA in [9] as a representative of the minimum interference algorithms like those in [6]-[8], among others. Remind that the GADIA is originally designed and analyzed for symmetric link gain case. In the ad-hoc network case examined in this paper, the link matrix is (and may be highly) asymmetric. Without loss of generality, direct-sequence (W) CDMA wireless system and downlink transmission is considered in all the examples. For link gains modeling, attenuation factor $\beta = 3$, the log-normally distributed s_{ij} in (2) is generated according to the model in [20], and the lognormal variance is 1 dB. (W)CDMA chiprate is 3.84 MHz. All the transmit powers of MBs are fixed to 2 mW for all simulations. In order to give an insight into some of the cases where the SpecPure outperforms the GADIA which gets stuck into a local minimum, some illustrative sample snapshots with various numbers of MBs are examined in examples 1 to 3.

In the simulations, we examine to the average interference power obtained from (4) which is calculated from (7), as well as the average Shannon channel

capacity which is calculated using the SINR in (1). The channel capacity is the bit rate at which data can be sent along a channel with a negligible error rate. For the MB i the Shannon channel capacity is

$$C_i = W \log_2 \left(1 + \theta_i\right) \tag{33}$$

where $W = 3.84$ MHz is the bandwidth and θ_i is the Signal-to-Noise-Ratio in (1). Then the average Shannon capacity over the N MBs is

$$C = \frac{1}{N} \sum_{i=1}^{N} C_i \tag{34}$$

A. *Example 1 (relatively small N, L = 2):* In this simple example, there are 10 clusters of MBs (i.e., $N = 10$). The center MBs are located on a straight line as shown in Figure 1. a and 2.b. The channel allocation results of the GADIA [6] and the SpecPure in Table 1 are also indicated in Figure 1. a and Figure 1. b, respectively. The red and the green colors show the channel allocation. Figure 1. shows the greatest eigenvector with respect to the smallest one of the Laplacian matrix in (8), according to which the SpecPure allocates the MBs to the channels. As seen from the Figure 1. a, the MBs 3 and 4, which are next to each other, are allocated into the same channel, which deteriorates the performance. On the other hand, the SpecPure gives the globally optimum solution for this scenario as seen in Figure 1. b and Figure 1. c. The greatest eigenvector with respect to the least significant eigenvector of the Laplacian matrix in (8) is shown in Fig.1.(c). Fig.1.c. shows that the signs of the dominant eigenvector indicates the globally optimum channel allocation solution for this particular example.

The average interference power and the average Shannon channel capacity results are presented in Fig. 2 over 10000 snapshots for the wireless network in Fig.1. As seen from Fig.2, the SpecPure and the N-GAIR outperforms the GADIA: The average interference powers of the SpecPure and N-GAIR is about 3.34 dB lower than the GADIA case; and this translates into 377.2 kbps channel capacity increase on the average as compared to the GADIA case for this network configuration.

B. *Example 2 (relatively small N, L = 2):* In example 1 above, the center MBs were located on a straight line (in one dimension). In this illustrative example, an ad-hoc network with 6 MBs in 2-dimensional center MB locations is shown in Figure 3. The channel allocation results are also given in Figure 3.a and 3.b for the GADIA and the SpecPure, respectively. As seen from the figure, the GADIA gets stuck at a local minima, while the SpecPure finds the globally optimum solution using the greatest eigenvector shown in Figure 3.c. Figure 3.c. shows that the signs of the dominant eigenvector indicates the globally optimum channel allocation solution for this particular example.
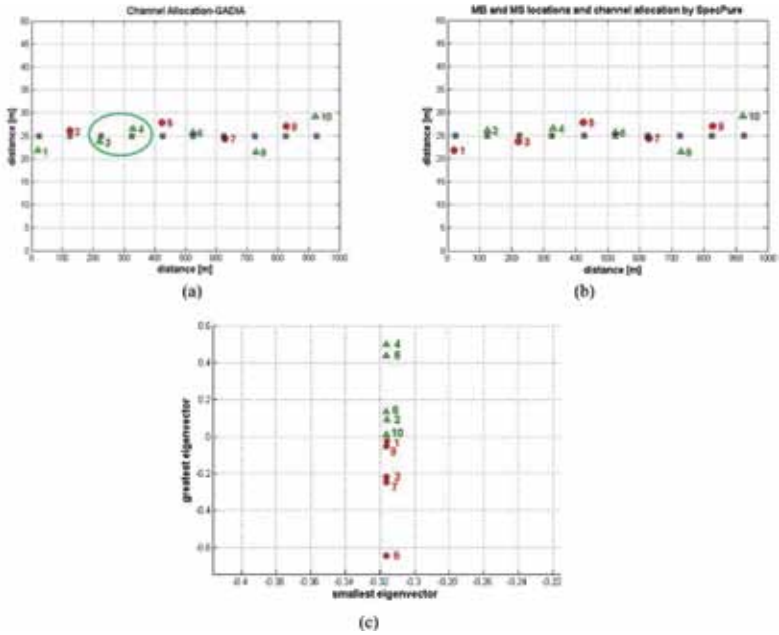
FIGURE 1
A snapshot of the network in Example 1 with $N = 10$. The center base locations are shown as stars in squares; and the circles and triangles in different colors indicate the MB locations with their channel allocations by (a) GADIA and (b) SpecPure. The greatest eigenvector of the corresponding Laplacian matrix in the SpecPure is show in (c).
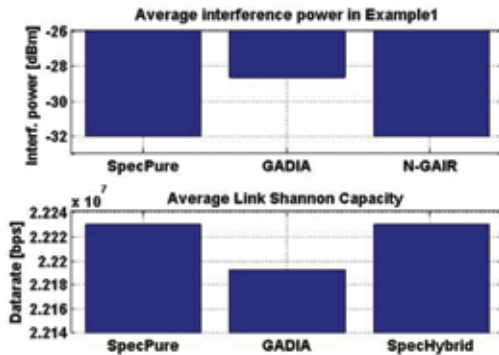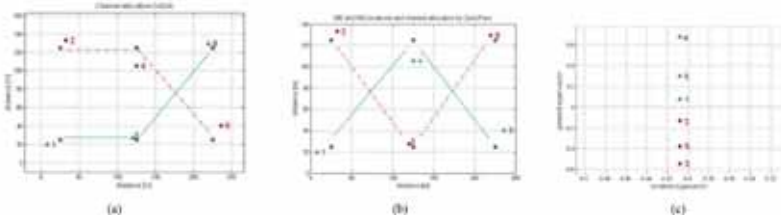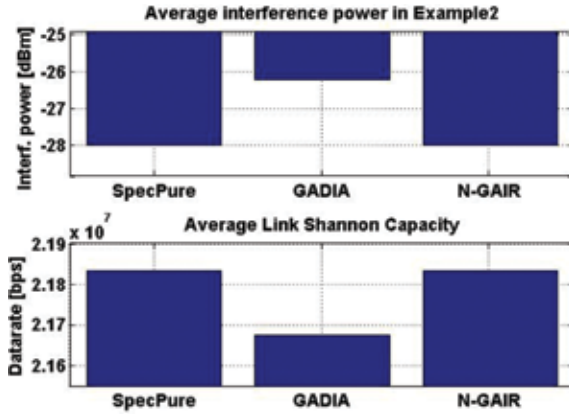

FIGURE 2
Average interference power and average Shannon channel capacity over the 10000 snapshots in Example 1.

The average interference power and the average Shannon channel capacity results are presented in Fig. 4 over 10000 snapshots for the wireless network in Fig.3.a. As seen from Fig.4, the SpecPure and the N-GAIR outperforms the

FIGURE 3

The channel allocation results in Example 2: The circles and triangles in different colors indicate the MB locations with their channel allocations by (a) GADIA and (b) SpecPure. The greatest eigenvector of the corresponding Laplacian matrix in the SpecPure is show in (c).

FIGURE 4

Average interference power and average Shannon capacity over the 10000 snapshots in Example 2.

GADIA for this scenario: The average interference powers of the SpecPure and N-GAIR is about 1.75 dB lower than the GADIA case; and this translates into a 155 kbps channel capacity increase as compared to the GADIA case for this network configuration.

C. Example 3 (relatively large N, L=2): In this example, we present two illustrative examples by increasing the number of MBs to 36 and 48. In the 36-MB case, the MBs are located on a square shape as shown in Figure 5. a. The channel allocations are shown by different colors in the figure. Comparing Figure 5. a and Figure 5. b, where the MBs 2, 3, 8, 9, 14, 15, 28, 29, 34 and 35 are differently allocated by the GADIA and the SpecPure, we see that the GADIA gets stuck at a local minima while the SpecPure finds the globally optimum solution for this example. The largest eigenvector (w.r.t. to the smallest one) of the corresponding Laplacian matrix in (8), which is used by the SpecPure is shown in Figure 5. c.

The average interference power and the average Shannon channel capacity results over 10000 snapshots for the wireless network in Fig.5.a are presented
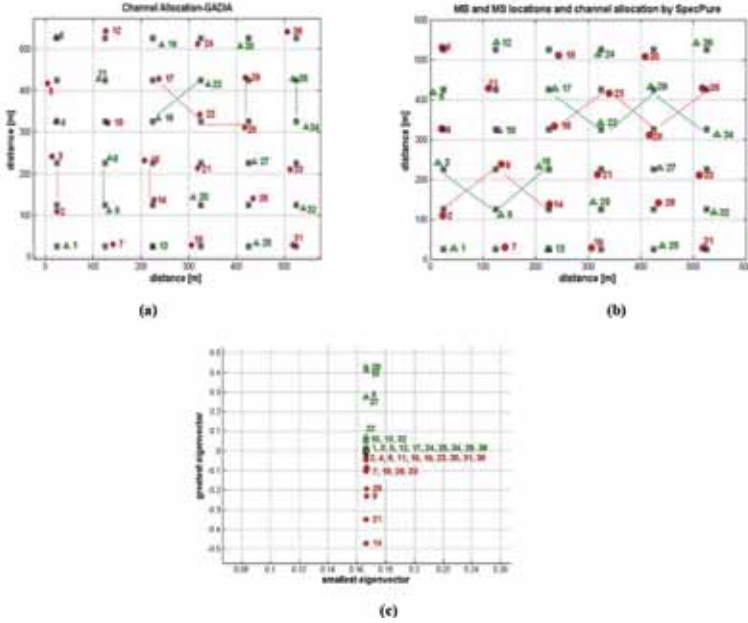
FIGURE 5
A snapshot of the ad-hoc network with $N = 36$. The circles and diamonds indicates the MB locations with their channel allocations by (a) GADIA and (b) SpecPure. The greatest eigenvector (wrt to the smallest one) of the corresponding Laplacian matrix in the SpecPure is show in (c).

in Fig. 6. As seen from Fig.6, the SpecPure and the N-GAIR outperforms the GADIA for this scenario: The average interference power of the SpecPure and N-GAIR is about 0.635 dB lower than the that of the GADIA case; and this translates into 87.95 kbps channel capacity increase as compared to the GADIA case for this network configuration.

A snapshot of the 48-MB network is shown in Figure 7. Comparing Figure 7. a and Figure 7. b, where the MBs 1-2, 5-13, 7-8, 9-17, 16-24 are differently allocated by the GADIA and the SpecPure, we see that the GADIA gets stuck at a local minima while the SpecPure finds the globally optimum solution for this example.

The average interference power and the average Shannon channel capacity results are presented in Fig. 8 over 10000 snapshots for the wireless network in Fig.7.a. As seen from Fig.8, the average interference power of the SpecPure is about 0.59 dB lower, and that of N-GAIR is about 0.83 dB lower than that of the GADIA; and this translates into 95.29 kbps and 98.17 kbps channel capacity increase for the SpecPure and N-GAIR respectively as compared to the GADIA case.

D. Example 4 (N = 4 to 14, L = 2, random locations): In all the examples above, the center MBs are located "symmetrically". In this example, the loca-
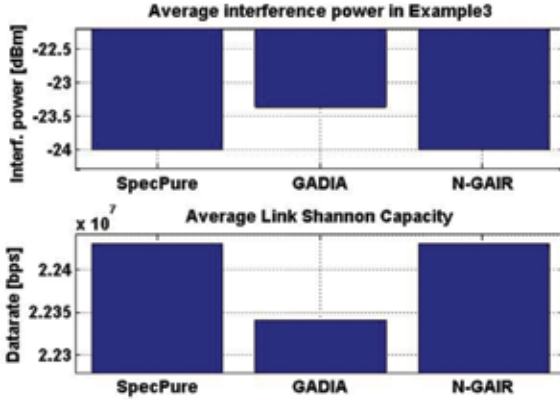
FIGURE 6
Average interference power and average Shannon capacity over the 10000 snapshots in Fig.5.a. in Example 3.
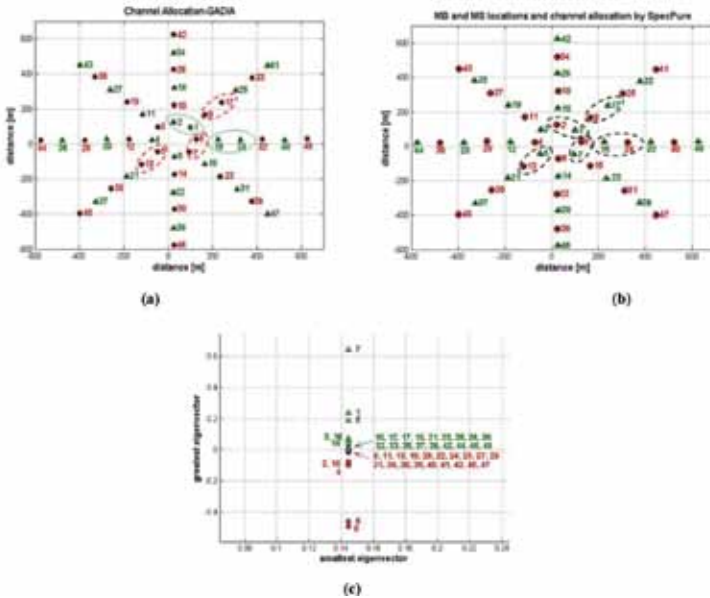

FIGURE 7
A snapshot of the network with $N = 48$. The circles and diamonds indicate the MB locations with their channel allocations by (a) GADIA and (b) SpecPure. The greatest eigenvector (wrt to the smallest one) of the corresponding Laplacian matrix in the SpecPure is show in (c).

tions are fully random. First we examine moderate size ad-hoc networks ($N = 4$ to 14). Average interference powers are shown in Fig.9 for $N = 4$ to 14. We observe from the figure that $i)$ the N-GAIR clearly outperforms the
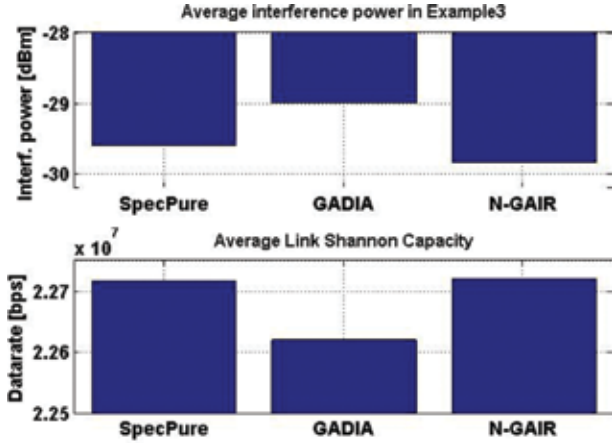
FIGURE 8
Average interference power and average Shannon capacity over the 10000 snapshots in the network in Fig.7.a.
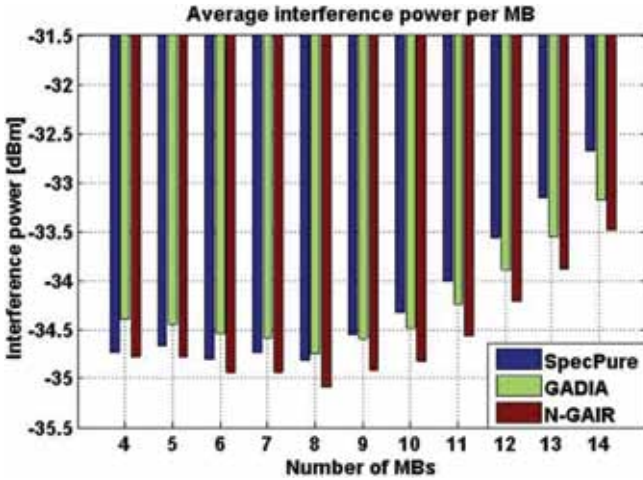

FIGURE 9
Average interference power and average Shannon capacity over the 10000 snapshots in Exampe 4.

GADIA, i.e., the average interference power of the N-GAIR is lower than that of the GADIA in all cases. *ii)* The performance of N-GAIR is better than that of the GADIA only for $N = 4$ to 8; but the NGAIR deteriorates as N increases as compared to the GADIA.

E. Example 5: (Large N (60 to 150), arbitrary locations, arbitrary L (4, 5, 6)): This example examines the randomly located large-size networks (i.e., $N = 50, 100, 150$) for arbitrary number of channel case (L>3). Average inter-

ference powers and average Shannon capacity results are shown in Fig.10 and Fig.11, respectively. In both figures, case A refers to $N = 60$, $L = 4$; case B refers to $N = 60$, $L = 6$; case C: $N = 100$, $L = 4$; case D: $N = 100$, $L = 6$, case E: $N = 150$, $L = 4$, and case F: $N = 150$, $L = 6$. Both figures suggest that *i)* the SpecPure algorithm performs poorly as compared to the GADIA and the proposed algorithm N-GAIR. The reason is because the performance of the *k*-means algorithm in the eigenspace is generally quite poor for large N and L, and therefore does not provide any good initial condition for the second phase for large N and L. *ii)* The proposed method N-GAIR and the
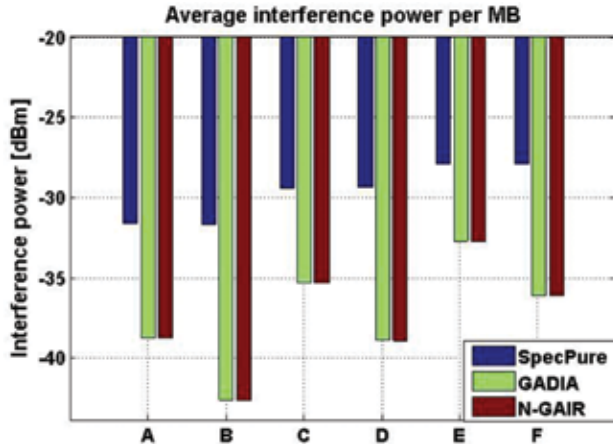

FIGURE 10
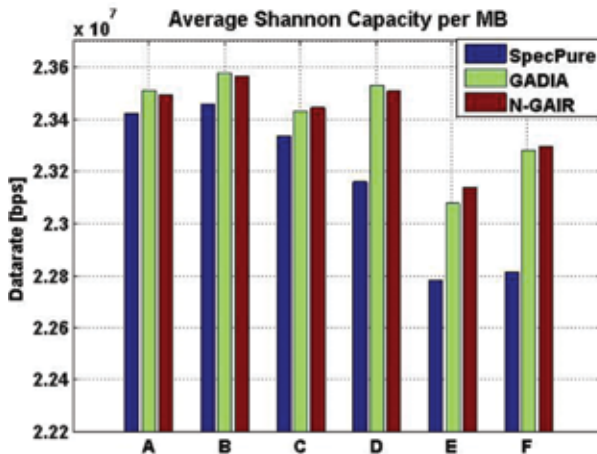Average interference power over 1000 snapshots in Exampe 5.

FIGURE 11
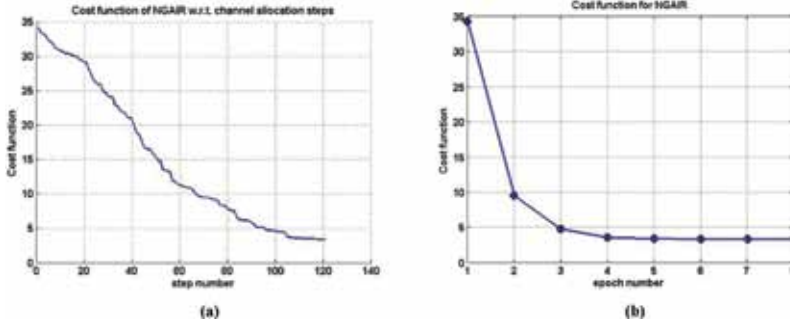Average Shannon capacity over 1000 snapshots in Exampe 5.

FIGURE 12

A typical plot of the evolution of the total network interference in eq.(3) for the normalized link gains with respect to **(a)** step numbers and **(b)** epoch numbers for $N = 120$ and $L = 6$.

GADIA give comparable results in large-size networks for arbitrary number of channel greater than 3.

In order to give an insight into the evolution of the total network interference in eq.(3) with respect to step numbers (at which a channel update is performed) as well as with respect to epoch numbers, Figure 12 shows a typical example for $N = 120$ and $L = 6$. Figure 12 implies that, as explained in section V, the N-GAIR converges to a (local) minimum.

CONCLUSIONS

In this paper, we examine the channel allocation problem in wireless networks, which is known to be NP-complete, and thus its optimum solution of general wireless networks for even 2-channel case is not known. In this paper, turning the channel/frequency allocation problem into a maxCut graph partitioning problem, we

i)   propose a spectral clustering based channel allocation algorithm, named as SpecPure,

ii) propose and analyze a novel Non-Greedy Asynchronous Interference Reduction Algorithm for Wireless Networks, named as N-GAIR,

iii) extend the results in [1] to the case where L is arbitrary, (in [1], $L = 2$ case is examined),

iv) show that the proposed N-GAIR minimizes the total network interference, and

v)   show that the N-GAIR can be applied to any maxCut graph partitioning problem.

The proposed solutions can be adopted by any cellular, cognitive, ad-hoc or mesh type radio networks. By simulating a (W)CDMA based ad-hoc net-

work, we examine various scenarios to compare the performances of the proposed algorithms with those of reference algorithm. Simulation results suggest that

i)   the proposed SpecPure gives the globally optimum solution either for special cases like symmetrical center MB locations e.g. square-type, circle-type, etc scenarios, or for *L = 2;*

ii) However, the SpecPure may give relatively poor solutions for random MB locations and relatively large *N and L,*

iii) The proposed algorithm N-GAIR clearly outperforms the standard minimum-interference algorithm (like basic GADIA) for arbitrarily large *N* and *L = 2* case for symmetric MB locations,

iv) For random MB locations and for arbitrarily large *N* and *L,* the N-GAIR and GADIA give comparable results, which outperform the SpecPure,

v)   The N-GAIR results confirm its effectiveness for ad-hoc wireless networks with large number of nodes.

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REFERENCES

[1] Z. Uykan, "Spectral Based Solutions for (Near) Optimum Channel/Frequency Allocation", Proc. of IWSSIP 2011 (18th International Conference on Systems, Signals and Image Processing), June 2011, Sarajevo, Bosnia and Herzegovina.

[2] D.C. Cox and D.O. Reudink, "Dynamic Channel Assignment in High-Capacity Mobile Communications Systems," Bell System Technical Journal, vol. 50, no. 6, July-August 1971.

[3] I. Stojmenovic (Editor), "Handbook of Wireless Networks and Mobile Computing", John Wiley & Sons, 2002.

[4] I. Katzela, and M. Naghshineh, "Channel assignment schemes for cellular mobile telecommunication systems: A comprehensive survey", IEEE Communicaitons, Surveys & Tutorials, vol.3, no.2, pp. 10–31, 2000.

[5] S. Chieochan, E. Hossain, and J. Diamond, "Channel assignment schemes for infrastructure-based 802.11 WLANs: A survey", IEEE Communicaitons, Surveys & Tutorials, vol.12, no.1, pp. 124–136, 2010.

[6] J.C.I. Chuang, "Autonomous adaptive frequency assignment for TDMA portable radio systems", IEEE Trans. on Vehicular Technology, vol.40, nr.3, pp. 627–635, 1991.

[7] K. Sivarajah, H.S. Al-Raweshidy, "Dynamic channel allocation for ongoing calls in UTRA TDD system", Electronics Letters, vol. 40, nr. 19, pp.1197–1198, 2004.

[8] H.L. Nguyen; U.T. Nguyen, "Minimum Interference Channel Assignment for Multicast in Multi-Radio Wireless Mesh Networks", in Proc. IWCMC '08, pp. 626 – 631, 2008.

[9] B. Babadi and V. Tarokh, "GADIA: A Greedy Asynchronous Distributed Interference Avoidance Algorithm", *IEEE Transactions on Information Theory,* vol. 56, no. 12, Dec. 2010.

[10] Q. Wu, Y. Xu, Yuhua, L. Shen, and J. Wang, "Investigation on GADIA Algorithms for Interference Avoidance: A Game-Theoretic Perspective", IEEE Communications Letters, vol. 16, No.7, pp. 1041–1043, 2012.

[11] L. Narayanan, "Channel Assignment and Graph Multicoloring", *Handbook of Wireless Networks and Mobile Computing, (Editor I. Stojmenovic),* ch.4, pp.71-94. (ISBN: 0-471-41902-8).

[12] T.S. Rappaport, Wireless Communications: Principles & Practice, Second Edition, Prentice Hall. 2002.

[13] L.D. Livnat, and U. Zwick, "Improved rounding techniques for the MAX-2SAT and MAX-DICUT problems", In Proc. 9th Ann. IPCO, pp. 67–82, 2002.

[14] J. Zander, "Performance of optimum transmitter power control in cellular radio systems," *IEEE Trans. Veh. Technol.*, vol. 41, no. 1, pp. 57–62, Feb. 1992.

[15] L.C.Y. Lee, Mobile Communications Engineering, McGraw Hill, New York, 1982.

[16] M.E.J. Newman, "Networks, An Introduction", Oxford University Press, NY, 2011.

[17] U.v. Luxburg, M. Belkin, and O. Bousquet, "Consistency of spectral clustering", *Annals of Statistics*, vol.36, no. 2, pp.555–586, 2008.

[18] U.v. Luxburg, "A Tutorial on Spectral Clustering", Technical Report TR-149, Max-Planck Institute for Biological Cybernetics, August 2006.

[19] L. Gao , Y. Zengb and A. Donga, "An ant colony algorithm for solving Max-cut problem", *Progress in Natural Science,* vol. 18, no. 9, pp. 1173-1178, Sep 2008.

[20] B.M.Fialho dos Anjos. New Convex Relaxations for the Maximum Cut and VLSI Layout Problems. PhD thesis, University of Waterloo, Ontario Canada, Department of Combinatorics and Optimization, 2001.

[21] E. H. Miller, N. Yamamoto and M. Fazel. Computational approach to quantum encoder design for purity optimization. Physical Review A, 76(1), July 2007. (available also at http://arxiv.org/abs/quant-ph/0606106) .

[22] M. Goemans and D. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming". *Journal of the ACM*, vol.42, pp.1115–1145, 1995.

[23] J. Kim and S. Choi, "Semidefinite spectral clustering," *Pattern Recognition*, vol. 39, no. 11, pp. 2025-2035, Nov. 2006.

[24] A.J. Viterbi, "*CDMA: Principles of Spread Spectrum Communication"*, Reading, MA: Addison-Wesley, 1995.