View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Dogus University Institutional Repository

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 415479, 8 pages
http://dx.doi.org/10.1155/2013/415479

Research Article

Asymptotic and Numerical Methods in Estimating Eigenvalues

Guldem Yıldız,[1] Bulent Yılmaz,[2] and O. A. Veliev[3]

[1] *Department of Mathematics, Nigde University, 51200 Nigde, Turkey*
[2] *Department of Math, Faculty of Science and Letters, Marmara University, Göztepe Kampüsü, Kadiköy, 81040 Istanbul, Turkey*
[3] *Department of Mathematics, Dogus University, Acıbadem, Kadiköy, 81010 Istanbul, Turkey*

Correspondence should be addressed to Guldem Yıldız; guldem.yildiz@nigde.edu.tr

Received 17 January 2013; Accepted 8 March 2013

Academic Editor: Safa Bozkurt Coskun

Copyright © 2013 Guldem Yıldız et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Asymptotic formulas and numerical estimations for eigenvalues of SturmLiouville problems having singular potential functions, with Dirichlet boundary conditions, are obtained. This study gives a comparison between the eigenvalues obtained by the asymptotic and the numerical methods.

1. Introduction

Let $L(q)$ be an operator generated in $L_2[0,1]$ by the expression

$$L(q) = -y''(x) + q(x)\, y(x), \quad 0 \le x \le 1, \tag{1}$$

and by Dirichlet boundary conditions

$$y(0) = y(1) = 0, \tag{2}$$

where $q(x)$ is a complex-valued summable function.

In this paper, we consider the small and large eigenvalues of the operator $L(q)$ when $q(x)$ has a finite number of singularities. The large eigenvalues are investigated by the asymptotic method given in [1, 2]. Note that in classical investigations in order to obtain the asymptotic formulas of order $O(n^{-l})$ it is required that $q(x)$ be $(l-1)$ times differentiable (see [3–10]). The method of [1] gives the possibility of obtaining the asymptotic formulas of order $O(n^{-l})$ of eigenvalues and eigenfunctions of $L(q)$ when $q(x)$ is an arbitrary summable complex-valued function. The small eigenvalues are investigated by numerical and asymptotic methods. Then, we compare the results with the ones obtained by the other methods.

Expression of differential equations in matrix form and the advances in the field of the computers have led to major developments in numerical methods. Regarding the numerical solution of the Sturm-Liouville problems, finite difference method is amongst the popular methods (see [11, 12]). Finite difference method can give effective results for the eigenvalues when it is used in connection with asymptotic correction technique. In [13] and [14] the Sturm-Liouville problems with Dirichlet and the general boundary conditions were studied, respectively. Andrew and Paine [15] found the approximate eigenvalues of regular Sturm-Liouville problem by using the finite element method. Chen and Ho [16] used the differential transform method to solve the eigenvalue problems. Ghelardoni [17] named some linear multistep methods as boundary value methods and found the approximate eigenvalues of Sturm-Liouville problem. Ghelardoni and Gheri [18] used the shooting technique for the calculation of the eigenvalues of Sturm-Liouville problem by considering the Prüfer transformation given in [19]. Kumar [20], Kumar and Aziz [21] gave numerical examples to linear or nonlinear boundary value problems by using finite differences method for singular boundary value problems. Kumar and Singh [22] made a study which collected and classified various calculation techniques for the solution of singular boundary value problems.

2. Asymptotic Formulas for Eigenvalues

It is well known that (see formulas (47a), (47b) in page 65 of [7]) the eigenvalues of the operator $L(q)$, where $q(x)$ is a complex-valued summable function, consist of the sequence $\{\lambda_n\}$ satisfying

$$\lambda_n = (n\pi)^2 + O(1). \tag{3}$$

In [1] (see Theorem 1 of [1]), it is proved that the eigenvalues λ_n satisfy the following formula

$$\lambda_n = (n\pi)^2 + C_0 + F_m + O\left(\left(\frac{\ln|n|}{n}\right)^{m+1}\right), \quad \forall m = 0, 1, 2, \ldots, \tag{4}$$

where $F_0 = -C_{2n}$, $C_n = \int_0^1 q(x)\cos n\pi x\, dx$,

$$F_1 = A_1\left((n\pi)^2\right) = -C_{2n} + \sum_{\substack{n_1=-\infty \\ n_1 \neq -2n}}^{\infty} \frac{C_{n_1}\left(C_{n_1} - C_{n_1+2n}\right)}{\left[(n\pi)^2 - \left(\pi\left(n+n_1\right)\right)^2\right]}, \tag{5}$$

$$F_k = A_k\left((n\pi)^2 + F_{k-1}\right), \quad \forall k = 2, 3, \ldots. \tag{6}$$

Note that in [1], without loss of generality, it was assumed that $C_0 = 0$. Then using (4), the cases $q(x) = p(x) + c/x^{\alpha}$ and $q(x) = a_0/\sqrt{x} + a_1/\sqrt{1-x} + b_0\sqrt{x} + b_1\sqrt{1-x}$ (where c, a_0, a_1, b_0, b_1 are complex numbers) are investigated in detail.

In this paper, we consider the case

$$q(x) = \sum_{k=0}^{\nu} \frac{c_k}{\left|x - t_k\right|^{\alpha_k}}, \quad 0 < \alpha_k < 1, \tag{7}$$

where ν is a positive integer and c_k is a complex number. First using (4) we prove the following.

Theorem 1. *The eigenvalue λ_n of the operator $L(q)$ with potential* (7) *satisfies the asymptotic formula:*

$$\begin{aligned}\lambda_n &= (n\pi)^2 + C_0 \\ &\quad - \sum_{k=0}^{\nu} \frac{c_k}{(2n)^{1-\alpha_k}}\left(\cos 2n\pi t_k\left(d_{4k} + d_{4k+2}\right)\right. \\ &\qquad \left. - \sin 2n\pi t_k\left(d_{4k+1} + d_{4k+3}\right)\right) \\ &\quad + O\left(\frac{\ln|n|}{n}\right),\end{aligned} \tag{8}$$

where

$$\begin{aligned} d_{4k} &= \int_0^{\infty} \frac{\cos \pi s}{s^{\alpha_k}} ds, \qquad d_{4k+2} = \int_{-\infty}^{0} \frac{\cos \pi s}{s^{\alpha_k}} ds, \\ d_{4k+1} &= \int_{-\infty}^{0} \frac{\sin \pi s}{s^{\alpha_k}} ds, \qquad d_{4k+3} = \int_0^{\infty} \frac{\sin \pi s}{s^{\alpha_k}} ds. \end{aligned} \tag{9}$$

Proof. At (4) for $m = 0$, let us use the formula

$$\lambda_n = (\pi n)^2 + C_0 - C_{2n} + O\left(\frac{\ln|n|}{n}\right), \tag{10}$$

where

$$C_n = \int_0^1 q(x)\cos n\pi x\, dx = \sum_{k=0}^{\nu} \int_0^1 \frac{c_k \cos n\pi x}{\left(x - t_k\right)^{\alpha_k}} dx. \tag{11}$$

In the last equality, using the transformations $t = x - t_k$ and $nt = s$ we obtain

$$\begin{aligned} C_n &= \sum_{k=0}^{\nu} \int_{-t_k}^{1-t_k} \frac{c_k \cos n\pi\left(t + t_k\right)}{t_k^{\alpha_k}} dt \\ &= \sum_{k=0}^{\nu} \int_{-t_k}^{1-t_k} \frac{c_k \cos n\pi t \cos n\pi t_k}{t_k^{\alpha_k}} dt \\ &\quad - \sum_{k=0}^{\nu} \int_{-t_k}^{1-t_k} \frac{c_k \sin n\pi t \sin n\pi t_k}{t_k^{\alpha_k}} dt, \\ C_n &= \sum_{k=0}^{\nu} \frac{c_k}{n^{1-\alpha_k}}\left(\cos n\pi t_k \int_{-nt_k}^{n(1-t_k)} \frac{\cos \pi s}{s^{\alpha_k}} ds\right. \\ &\qquad \left. - \sin n\pi t_k \int_{-nt_k}^{n(1-t_k)} \frac{\sin \pi s}{s^{\alpha_k}} ds\right). \end{aligned} \tag{12}$$

Let $n(1 - t_k) = n_k$. By (9) we have

$$\begin{aligned} C_n = \sum_{k=0}^{\nu} \frac{c_k}{n^{1-\alpha_k}} &\left(\cos n\pi t_k\left(d_{4k} - \int_{n_k}^{\infty} \frac{\cos \pi s}{s^{\alpha_k}} ds\right.\right. \\ &\qquad \left. + d_{4k+2} - \int_{-\infty}^{-nt_k} \frac{\cos \pi s}{s^{\alpha_k}} ds\right) \\ &\quad - \sin n\pi t_k\left(d_{4k+1} - \int_{-\infty}^{-nt_k} \frac{\sin \pi s}{s^{\alpha_k}} ds\right. \\ &\qquad \left.\left. + d_{4k+3} - \int_{n_k}^{\infty} \frac{\sin \pi s}{s^{\alpha_k}} ds\right)\right). \end{aligned} \tag{13}$$

Arguing as in the proof of (41) of [1] one can readily see that

$$\begin{aligned} &\int_{-\infty}^{-nt_k} \frac{\cos \pi s}{s^{\alpha_k}} ds = O\left(\frac{1}{n_k^{\alpha_k}}\right), \qquad \int_{n_k}^{\infty} \frac{\cos \pi s}{s^{\alpha_k}} ds = O\left(\frac{1}{n_k^{\alpha_k}}\right), \\ &\int_{-\infty}^{-nt_k} \frac{\sin \pi s}{s^{\alpha_k}} ds = O\left(\frac{1}{n_k^{\alpha_k}}\right), \qquad \int_{n_k}^{\infty} \frac{\sin \pi s}{s^{\alpha_k}} ds = O\left(\frac{1}{n_k^{\alpha_k}}\right). \end{aligned} \tag{14}$$

Therefore

$$C_n = \sum_{k=0}^{\nu} \frac{c_k}{n^{1-\alpha_k}} \left(\cos n\pi t_k \left(d_{4k} + d_{4k+2}\right) - \sin n\pi t_k \left(d_{4k+1} + d_{4k+3}\right)\right) + O\left(\frac{1}{n}\right). \tag{15}$$

Thus (8) follows from (4) for $m = 0$. The theorem is proved. □

Now assuming that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_\nu = \frac{1}{2} \tag{16}$$

we obtain more precise asymptotic formula by using more subtle estimations.

Theorem 2. *If* (16) *holds, then the eigenvalue* λ_n *of the operator* $L(q)$ *with potential* (7) *satisfies the asymptotic formula:*

$$\lambda_n = (n\pi)^2 + C_0 - C_{2n} + O\left(\left(\frac{\ln|n|}{n}\right)^2\right). \tag{17}$$

Proof. To prove the theorem we use (4) for $m = 1$, (5) and prove that

$$\sum_{\substack{n_1=-\infty \\ n_1 \neq 0,-2n}}^{\infty} \frac{C_{n_1}^2}{n_1 (2n + n_1)} = O\left(\left(\frac{\ln|n|}{n}\right)^2\right),$$
$$\sum_{\substack{n_1=-\infty \\ n_1 \neq 0,-2n}}^{\infty} \frac{C_{n_1} C_{n_1+2n}}{n_1 (2n + n_1)} = O\left(\left(\frac{\ln|n|}{n}\right)^2\right). \tag{18}$$

In (15), instead of n and α_k taking n_1 and $1/2$ we get

$$C_{n_1} = \sum_{k=0}^{\nu} \frac{c_k}{(n_1)^{1/2}} \left(\cos n_1\pi t_k \left(d_{4k} + d_{4k+2}\right) - \sin n_1\pi t_k \left(d_{4k+1} + d_{4k+3}\right)\right) + O\left(\frac{1}{n_1}\right). \tag{19}$$

From (19) one can readily see that there exists a constant r such that

$$\left|C_{n_1}\right|^2 < r\frac{1}{n_1} \tag{20}$$

for $n_1 = 1, 2, \ldots$. Therefore, instead of equation (56) of [1], using (20) and repeating the proof of equation (55) of [1] we get the proof of (18). Thus the proof of the theorem follows from (4), (5), and (18). The theorem is proved. □

3. Numerical Approximation

Now, we consider the small eigenvalues of the $L(q)$ operator by a numerical method.

For the finite difference method [11, 19] take an equally spaced mesh ($m \geqslant 2$)

$$0 = x_0 < x_1 < \cdots < x_{m+1} = 1, \tag{21}$$

where

$$x_j = jh, \qquad h = \frac{1}{m+1}. \tag{22}$$

Writing $y(x_j)$ as y_j, $q(x_j)$ as q_j , and $y''(x_j)$ as y_j'' , we use the centered difference approximation

$$-y_j'' \approx \frac{-y_{j-1} + 2y_j - y_{j+1}}{h^2}. \tag{23}$$

Substituting in (1) we obtain the approximating scheme

$$\frac{-Y_{j-1} + 2Y_j - Y_{j+1}}{h^2} + q_j Y_j = \Lambda Y_j, \quad j = 1, 2, \ldots, m. \tag{24}$$

Incorporating the boundary conditions, we get

$$Y_0 = 0, \qquad Y_{m+1} = 0. \tag{25}$$

This can be written in matrix form as

$$TY = \Lambda Y, \tag{26}$$

where

$$T = \frac{1}{h^2} K + Q^*, \tag{27}$$

T is a tridiagonal matrix and

$$Y = \begin{pmatrix} Y_1 \\ \cdot \\ \cdot \\ \cdot \\ Y_m \end{pmatrix},$$

$$K = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}, \tag{28}$$

$$Q^* = \begin{pmatrix} q_1 & & & & & \\ & q_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & q_m \end{pmatrix}.$$

The eigenvalues of (1), (2) are approximated by the eigenvalues of matrix T.

In the previous section, the asymptotic formulas for eigenvalues of the operator $L(q)$ (1), (2) with the potential (7) are investigated. In this section, we will find the eigenvalues of the operator $L(q)$ by using the finite difference method when $c_k = 1$, $\alpha_k = 1/2$ for $k = 0, 1, \ldots, \nu$, and $t_k = k/\nu$. Let us introduce the notation

$$q_k(x) = \frac{1}{|x - t_k|^{1/2}} \quad (29)$$

and denote the nth eigenvalue of the operator $L(q_k)$ by λ_n^k. The nth eigenvalue of the operator $L(Q_\nu)$, where

$$Q_\nu(x) = \sum_{k=0}^{\nu} q_k(x), \quad (30)$$

is denoted by Λ_n^ν.

In order to be able to apply the Finite Difference method, the nodes should not coincide with the singular points. Let $m_\nu = \nu(m+1)$ and x_j nodal points be

$$x_j = \frac{j}{m_\nu}, \qquad j = 1, 2, \ldots, m_\nu - 1; \;\; j \neq k(m+1). \quad (31)$$

Then $x_j \neq t_k$.

The approximate eigenvalues of the operators $L(q_k)$ and $L(Q_\nu)$ obtained by the numerical method are denoted s_n^k and S_n^ν, respectively.

Example 3. In this example we find the eigenvalues of the following boundary value problem

$$-y'' + \sum_{k=0}^{5} \frac{1}{|(x - t_k)|^{1/2}} y = \lambda y, \qquad y(0) = y(1) = 0 \quad (32)$$

for $m_\nu = 150$, $t_k = k/\nu$, and $\nu = 5$ by using Finite Difference method. In Table 1 an example of the computation of the eigenvalues of the operators $L(q_k)$ and $L(Q_\nu)$ is given.

One can see from Table 1 that for $n \geq 30$ the eigenvalues of the operators $L(q_k)$ and $L(Q_\nu)$ are close to each other. This shows that the effect of the potential to the large eigenvalues is small. Moreover the eigenvalues in first, second, and third columns coincide with the eigenvalues in the sixth, fifth, and fourth columns, respectively, since the potential $q_k(x)$ can be reduced to $q_{\nu-k}(x)$ by using the transformation $x = 1 - z$.

4. Comparison of the Asymptotic and Numerical Methods

In this section we compare the estimations obtained by numerical and asymptotic methods of the eigenvalues Λ_n^2 of the operator $L(Q_2)$, where Q_2 is defined by (30) and (29). The nth eigenvalue of the operator $L(0)$ is $(n\pi)^2$. The effect of the potential q_k on the nth eigenvalue λ_n^k of the operator $L(q_k)$, that is, the perturbation of the nth eigenvalue when $L(0)$ is perturbed by q_k is

$$p_n^k = \lambda_n^k - (n\pi)^2. \quad (33)$$

Similarly, the effect of Q_2 on the nth eigenvalue Λ_n^2, that is, the perturbation of the nth eigenvalue when $L(0)$ is perturbed by Q_2 is

$$P_n^2 = \Lambda_n^2 - (n\pi)^2. \quad (34)$$

The perturbations P_n^2, p_n^k evaluated by the numerical and asymptotic methods are denoted by $P_n^2(s)$, $p_n^k(s)$, $P_n^2(a)$, and $p_n^k(a)$, respectively.

According to Theorem 2 we define the approximate eigenvalues, denoted by a_n^k and A_n^2, of the operators $L(q_k)$ and $L(Q_2)$ obtained by the asymptotic method as follows

$$a_n^k = (n\pi)^2 + \int_0^1 q_k(x)\,dx - \int_0^1 q_k(x)\cos 2\pi nx\,dx, \quad (35)$$

$$A_n^2 = (n\pi)^2 + \int_0^1 Q_2(x)\,dx - \int_0^1 Q_2(x)\cos 2\pi nx\,dx. \quad (36)$$

Therefore it is natural to define $P_n^2(a)$ and $p_n^k(a)$ by

$$P_n^2(a) = A_n^2 - (n\pi)^2 = \int_0^1 Q_2(x)\,dx - \int_0^1 Q_2(x)\cos 2\pi nx\,dx,$$
$$p_n^k(a) = a_n^k - (n\pi)^2 = \int_0^1 q_k(x)\,dx - \int_0^1 q_k(x)\cos 2\pi nx\,dx. \quad (37)$$

It readily follows from formulas (37) and (30) that

$$P_n^2(a) = \sum_{k=0}^{2} p_n^k(a). \quad (38)$$

It means that for the large eigenvalues the effect of Q_2 is asymptotically equal to the sum of the effects of the potentials q_k.

The perturbations $P_n^2(s)$ and $p_n^k(s)$ evaluated via the finite difference method are given in Table 2. In order to see the effect of the singular points, the number of subintervals is taken as $m_\nu = 20000$.

Table 2 shows that the effect of Q_2 is approximately within the value range of 5.10^{-5} and 2.10^{-2}, equal to the sum of the effects of the potentials q_k. Thus the perturbation estimations by the numerical methods validate the naturality of (38).

TABLE 1

n	s_n^0	s_n^1	s_n^2	s_n^3	s_n^4	s_n^5	S_n^5
1	11,3346	12,0617	13,6112	13,6112	12,0617	11,3346	24,8946
2	41,0082	42,3680	42,5655	42,5655	42,3680	41,0082	54,4909
3	90,3543	92,1439	91,4556	91,4556	92,1439	90,3543	103,912
4	159,389	161,282	161,330	161,330	161,282	159,389	174,822
5	248,089	249,774	249,278	249,278	249,774	248,089	261,654
6	356,418	357,740	357,883	357,883	357,740	356,418	369,832
7	484,332	485,347	486,023	486,023	485,347	484,332	497,594
8	631,775	632,690	632,941	632,941	632,690	631,775	643,821
9	789,683	799,727	800,334	800,334	799,727	789,683	811,972
10	984,984	986,288	986,440	986,440	986,288	984,984	998,400
20	3892,79	3864,330	3894,39	3894,39	3864,330	3892,79	3906,97
30	8599,57	8600,85	8601,04	8601,04	8600,85	8599,57	8612,95
40	14902,3	14903,5	14903,7	14903,7	14903,5	14902,3	14915,3
50	22529,2	22530,6	22530,7	22530,7	22530,6	22529,2	22542,7
60	31151,4	31152,8	31152,8	31152,8	31152,8	31151,4	31165,1
70	40396,8	40398	40398,3	40398,3	40398	40396,8	40410,1
80	49866,8	49867,9	49868,2	49868,2	49867,9	49866,8	49879,9
90	59152,9	59154,3	59154,3	59154,3	59154,3	59152,9	59166,4
100	67854,6	67856,1	67856,1	67856,1	67856,1	67854,6	67868,3

TABLE 2

n	$P_n^2(s)$	$p_n^0(s)$	$p_n^1(s)$	$p_n^2(s)$	$\sum_{k=0}^{2} p_n^k(s)$	$\lvert p_n^2(s) - \sum_{k=0}^{2} p_n^k(s)\rvert$
1	6,853010948	1,507680075	3,819522987	1,507680075	6,834883138	0,018127810
2	5,436436039	1,649044169	2,135885787	1,649044169	5,433974124	0,002461915
3	6,833127542	1,712792234	3,412323071	1,712792234	6,837907540	0,004779998
4	5,833596917	1,750983257	2,332560213	1,750983257	5,834526726	0,000929810
5	6,821354115	1,777106646	3,269640370	1,777106646	6,823853662	0,002499546
6	6,014222685	1,796412736	2,422163800	1,796412736	6,014989272	0,000766587
7	6,817074514	1,811422205	3,195628783	1,811422205	6,818473194	0,001398680
8	6,122590551	1,823514738	2,476005854	1,823514738	6,123035330	0,000444779
9	6,814993899	1,833516325	3,148677685	1,833516325	6,815710336	0,000716437
10	6,196675434	1,841954226	2,512821873	1,841954226	6,196730326	0,000054891
20	6,378247333	1,885045278	2,601825385	1,885045278	6,371915940	0,006331392

It is well known that if we consider the Sturm-Liouville operator

$$L(\varepsilon Q_2) = -\frac{d}{dx^2} + \varepsilon Q_2(x), \qquad y(0) = y(1) = 0, \tag{39}$$

where ε is a small positive parameter, then the asymptotic methods can be applied more successfully. The nth eigenvalue of the operators $L(\varepsilon Q_\nu)$ is denoted by $\Lambda_n^\nu(\varepsilon)$. The approximate eigenvalues obtained by the asymptotic and numerical methods are denoted by $A_n^\nu(\varepsilon)$ and $S_n^\nu(\varepsilon)$, respectively.

It follows from Theorem 2 and formulas (36), (30) that

$$\Lambda_n^2(\varepsilon) = A_n^2(\varepsilon) + O\left(\left(\frac{\varepsilon \ln|n|}{n}\right)^2\right), \tag{40}$$

where

$$\begin{aligned} A_n^2(\varepsilon) &= (\pi n)^2 + \varepsilon \int_0^1 Q_2(x)\,dx - \varepsilon C_{2n} \\ &= (\pi n)^2 + \varepsilon \int_0^1 \left(\frac{1}{\sqrt{|x-1|}} + \frac{1}{\sqrt{|x-1/2|}} + \frac{1}{\sqrt{|x|}}\right) dx \\ &\quad - \varepsilon \int_0^1 \left(\frac{1}{\sqrt{|x-1|}} + \frac{1}{\sqrt{|x-1/2|}} + \frac{1}{\sqrt{|x|}}\right) \cos 2n\pi x\,dx. \end{aligned} \tag{41}$$

In Tables 3, 4, and 5 the approximate eigenvalues $A_n^2(\varepsilon)$ obtained by the asymptotic method and their comparison

Table 3

n	$\lambda_n(0)$	$A_n^2(1)$	$S_n^2(1)$	$\lvert A_n^2(1)-S_n^2(1)\rvert$	$\lvert A_n^2(1)-\lambda_n(0)\rvert$	$\lvert S_n^2(1)-\lambda_n(0)\rvert$
1	9,8696	16,779308	16,7201	0,059208	6,909708	6,8505
2	39,4784	44,915438	44,9149	0,000538	5,437038	5,4365
3	88,8264	95,665322	95,6569	0,008422	6,838922	6,8305
4	157,9137	163,748052	163,7473	0,000752	5,834352	5,8336
5	246,7401	253,572371	253,5588	0,013571	6,832271	6,8187
6	355,3058	361,320361	361,3199	0,000461	6,014561	6,0141
7	483,6106	490,44101	490,4249	0,01611	6,83041	6,8143
8	631,6547	637,77751	637,7770	0,00051	6,12281	6,1223
9	799,4379	806,267576	806,2502	0,017376	6,829676	6,8123
10	986,9604	993,157379	993,1570	0,000379	6,196979	6,1966
20	3947,8418	3954,223216	3954,2175	0,005216	6,381416	6,3757
30	8882,6440	8889,107347	8889,0782	0,029347	6,463347	6,4342
40	15791,3670	15797,8793	15797,7870	0,0893	6,51236	6,42
50	24674,0110	24680,55663	24680,3312	0,22663	6,54563	6,3202
60	35530,5758	35537,1461	35536,6786	0,4661	6,5703	6,1028
70	48361,0616	48367,65097	48366,7849	0,87097	6,58937	5,7233
80	63165,4682	63172,073	63170,5955	1,473	6,6048	5,1273
90	79943,7956	79950,41327	79948,0466	2,36327	6,61767	4,251
100	98696,0440	98702,67245	98699,0652	3,60245	6,62845	3,0212

Table 4

n	$\lambda_n(0)$	$A_n^2(0,1)$	$S_n^2(0,1)$	$\lvert A_n^2(0,1)-S_n^2(0,1)\rvert$	$\lvert A_n^2(0,1)-\lambda_n(0)\rvert$	$\lvert S_n^2(0,1)-\lambda_n(0)\rvert$
1	9,8696	10,5605748	10,5582	0,0023748	0,6909748	0,6886
2	39,4784	40,0221196	40,0221	$1{,}96\times 10^{-5}$	0,5437196	0,5437
3	88,8264	89,5103278	89,5085	0,0018278	0,6839278	0,6821
4	157,9137	158,4971086	158,4971	$8{,}6\times 10^{-6}$	0,5834086	0,5834
5	246,7401	247,4233362	247,4214	0,0019362	0,6832362	0,6813
6	355,3058	355,9072187	355,9072	$1{,}87\times 10^{-5}$	0,6014187	0,6014
7	483,6106	484,2936551	484,2916	0,0020551	0,6830551	0,681
8	631,6547	632,2669645	632,2668	0,0001645	0,6122645	0,6121
9	799,4379	800,1209185	800,1187	0,0022185	0,6830185	0,6808
10	986,9604	987,580134	987,5798	0,000334	0,619734	0,6194
20	3947,8418	3948,479906	3948,4741	0,005806	0,638106	0,6323
30	8882,6440	8883,2903	8883,2611	0,0291996	0,6463	0,6171
40	15791,3670	15792,01827	15791,9259	0,0923677	0,65127	0,5589
50	24674,0110	24674,66557	24674,4401	0,225465	0,65457	0,4291
60	35530,5758	35531,23287	35530,7654	0,4674694	0,65707	0,1896
70	48361,0616	48361,72051	48360,8544	0,8661055	0,65891	0,2072
80	63165,4682	63166,12865	63164,6511	1,4775505	0,66045	0,8171
90	79943,7956	79944,45741	79942,0907	2,3667109	0,66181	1,7049
100	98696,0440	98696,70685	98693,0996	3,6072546	0,66285	2,9444

with $S_n^2(\varepsilon)$ and nonperturbated eigenvalues $\lambda_n(0) = (n\pi)^2$ for $\varepsilon = 1$, $\varepsilon = 0,1$, and $\varepsilon = 0,01$, respectively, are given.

Table 3 shows the eigenvalues of $L(\varepsilon Q_2)$ operator obtained by asymptotic method and finite difference method, respectively, for $\varepsilon = 1$. Here the number of subintervals is taken as $m_\gamma = 15000$.

Table 4 shows the eigenvalues of $L(\varepsilon Q_2)$ operator obtained by asymptotic method and finite difference method, respectively, for $\varepsilon = 0,1$. Here the number of subintervals is taken as $m_\gamma = 15000$.

Table 5 shows the eigenvalues of $L(\varepsilon Q_2)$ operator obtained by asymptotic method and finite difference method, respectively, for $\varepsilon = 0,01$. Here the number of subintervals is taken as $m_\gamma = 15000$.

5. Conclusion

It is natural and well known that for small values of the parameter ε and for large eigenvalues the asymptotic method

Table 5

n	$\lambda_n(0)$	$A_n^2(0,01)$	$S_n^2(0,01)$	$\lvert A_n^2(0,01) - S_n^2(0,01)\rvert$	$\lvert A_n^2(0,01) - \lambda_n(0)\rvert$	$\lvert S_n^2(0,01) - \lambda_n(0)\rvert$
1	9,8696	9,93870144	9,9385	0,00020144	0,06910144	0,0689
2	39,4784	39,53278781	39,5328	$1{,}219 \times 10^{-5}$	0,05438781	0,0544
3	88,8264	88,89482843	88,8946	0,00022843	0,06842843	0,0682
4	157,9137	157,9720142	157,9720	$1{,}423 \times 10^{-5}$	0,0583142	0,0583
5	246,7401	246,8084326	246,8082	0,00023264	0,0683326	0,0681
6	355,3058	355,3659045	355,3659	$4{,}46 \times 10^{-6}$	0,0601045	0,0601
7	483,6106	483,6789196	483,6786	0,0003196	0,0683196	0,068
8	631,6547	631,71591	631,7158	0,00010995	0,06121	0,0611
9	799,4379	799,5062527	799,5058	0,00045269	0,0683527	0,0679
10	986,9604	987,0224095	987,0220	0,00040949	0,0620095	0,0616
20	3947,8418	3947,905575	3947,8998	0,00577499	0,063775	0,058
30	8882,6440	8882,708595	8882,6794	0,02919485	0,064595	0,0354
40	15791,3670	15791,43216	15791,3398	0,09236434	0,06516	0,0272
50	24674,0110	24674,07646	24673,8510	0,22545896	0,06546	0,16
60	35530,5758	35530,64155	35530,1740	0,46754647	0,06575	0,4018
70	48361,0616	48361,12746	48360,2614	0,86605935	0,06586	0,8002
80	63165,4682	63165,53422	63164,0567	1,47751533	0,06602	1,4115
90	79943,7956	79943,86183	79941,4951	2,36672503	0,06623	2,3005
100	98696,0440	98696,1103	98692,5031	3,60719526	0,0663	3,5409

gives us approximations with smaller errors. The numerical method, in general, gives better results for smaller eigenvalues. The tables show that the results of the asymptotic method also give quiet acceptable results for small eigenvalues, since $A_n^2(\varepsilon) - S_n^2(\varepsilon)$ is small.

Therefore we can easily observe that both of two methods give high-precision results for the calculation of the small eigenvalues. Additionally while the perturbation parameter tends to zero both of the methods are enhanced for smaller eigenvalues, but while this fact is limited to $n = 10$ for the numerical approximation, the enhancement continues for the asymptotic method applied to higher eigenvalues. Thus we can conclude that the asymptotic method coupled with a perturbation parameter near to zero provides us a better approximation quality in calculating eigenvalues.

In Tables 3–5 there are two observations to be considered: the first observation is that for small eigenvalues the perturbated results by numerical and asymptotic methods are close to each other for all ε. The second observation is that for the large eigenvalues the perturbated results obained by asymptotic methods decrease linearly with respect to small ε, while the perturbated results obtained by numerical methods are almost the same for all values of ε. This shows that for small values of the perturbation parameter ε the asymptotic method is preferable.

Conflict of Interests

The authors of the paper do not have any direct or indirect financial relation with the commercial identities mentioned in the paper.

References

[1] B. Yilmaz and O. A. Veliev, “Asymptotic formulas for Dirichlet boundary value problems,” *Studia Scientiarum Mathematicarum Hungarica*, vol. 42, no. 2, pp. 153–171, 2005.

[2] O. A. Veliev and M. Toppamuk Duman, “The spectral expansion for a nonself-adjoint Hill operator with a locally integrable potential,” *Journal of Mathematical Analysis and Applications*, vol. 265, no. 1, pp. 76–90, 2002.

[3] G. D. Birkhoff, “Boundary value and expansion problems of ordinary linear differential equations,” *Transactions of the American Mathematical Society*, vol. 9, no. 4, pp. 373–395, 1908.

[4] N. Dunford and J. T. Schwartz, *Linear Operators. Part III*, Spectral Operators, Wiley-Interscience, New York, NY, USA, 1988.

[5] W. N. Everitt, J. Gunson, and A. Zettl, “Some comments on Sturm-Liouville eigenvalue problems with interior singularities,” *Journal of Applied Mathematics and Physics*, vol. 38, no. 6, pp. 813–838, 1987.

[6] V. A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhäuser, Basel, Switzerland, 1986.

[7] M. A. Naimark, *Linear Differential Operators*, George G. Harrap and Company, 4th edition, 1967.

[8] B. N. Parlett, *The Symmetric Eigenvalue Problem*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1980.

[9] J. D. Tamarkin, “Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions,” *Mathematische Zeitschrift*, vol. 27, no. 1, pp. 1–54, 1928.

[10] E. C. Titchmarsh, *Eigenfunction Expansions*, vol. I, Oxford University Press, 1962.

[11] R. L. Burden, *Numerical Analysis*, Brooks Cole, Pacific Grove, Calif, USA, 7th edition, 2001.

[12] A. L. Andrew, "Correction of finite difference eigenvalues of periodic Sturm-Liouville problems," *Australian Mathematical Society Journal Series B*, vol. 30, no. 4, pp. 460–469, 1989.

[13] J. W. Paine, F. R. de Hoog, and R. S. Anderssen, "On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems," *Computing*, vol. 26, no. 2, pp. 123–139, 1981.

[14] R. S. Anderssen and F. R. de Hoog, "On the correction of finite difference eigenvalue approximations for Sturm-Liouville problems with general boundary conditions," *BIT*, vol. 24, no. 4, pp. 401–412, 1984.

[15] A. L. Andrew and J. W. Paine, "Correction of finite element estimates for Sturm-Liouville eigenvalues," *Numerische Mathematik*, vol. 50, no. 2, pp. 205–215, 1986.

[16] C.-K. Chen and S.-H. Ho, "Application of differential transformation to eigenvalue problems," *Applied Mathematics and Computation*, vol. 79, no. 2-3, pp. 173–188, 1996.

[17] P. Ghelardoni, "Approximations of Sturm-Liouville eigenvalues using boundary value methods," *Applied Numerical Mathematics*, vol. 23, no. 3, pp. 311–325, 1997.

[18] P. Ghelardoni and G. Gheri, "Improved shooting technique for numerical computations of eigenvalues in Sturm-Liouville problems," *Nonlinear Analysis*, vol. 47, pp. 885–896, 2001.

[19] J. D. Pryce, *Numerical Solution of Sturm-Liouville Problems*, Clarendon Press, Oxford, UK, 1993.

[20] M. Kumar, "A new finite difference method for a class of singular two-point boundary value problems," *Applied Mathematics and Computation*, vol. 143, no. 2-3, pp. 551–557, 2003.

[21] M. Kumar and T. Aziz, "A uniform mesh finite difference method for a class of singular two-point boundary value problems," *Applied Mathematics and Computation*, vol. 180, no. 1, pp. 173–177, 2006.

[22] M. Kumar and N. Singh, "A collection of computational techniques for solving singular boundary-value problems," *Advances in Engineering Software*, vol. 40, no. 4, pp. 288–297, 2009.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization