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CHARACTERIZATIONS OF THE IDENTITY COMPONENT IN NEAR-RINGS OF LIPSCHITZ TRANSFORMATIONS

A THESIS

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CHARACTERIZATIONS OF THE IDENTITY
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To my Grandmother,

Mrs. Seaton Taylor Purdom.

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Because of the complexity of the mathematical expressions appearing in the thesis, special permission was given by the Graduate Division to use extra spacing throughout the main body of the text.

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CHAPTER I

INTRODUCTION

Suppose that S is a Banach space and that X is a normed near-ring of Lipschitz transformations from S to S containing the identity mapping I on S. We shall be concerned with characterizing the identity component I(X) of X, i.e., the largest connected set of invertible elements of X containing I.

A partial characterization of I(X) has already been given by J. W. Neuberger [4]. If Q is in X, Neuberger defines $(\text{Exp } Q)\, x = \lim_{n \to +\infty} (I + (1/n)Q)^n x$ for each x in S. It is then shown that each element of I(X) can be approximated, uniformly on bounded subsets of S, by a sequence whose elements are finite products of transformations of the form Exp Q with Q in X.

Using results of J. S. MacNerney [3] in the study of integral equations, we shall see that this density in I(X) of finite products of exponentials of elements of X is a special case of a more general phenomenon. We shall, in fact, obtain a complete characterization of I(X).

CHAPTER II

SOME PRELIMINARY NOTIONS

A precise statement of the setting in which we shall work is now given. Let $\{S, \| \|\}$ be a Banach space. We say that X is a near-ring of Lipschitz transformations from S to S if, and only if, X is a vector space (real or complex) of functions from S to S such that each of the following is true:

(i) $Q(0) = 0$ for each Q in X, where 0 is the additive identity of S.

(ii) The identity mapping, I, on S is in X.

(iii) If each of Q and H is in X, then the composition QH is in X.

(iv) Finally, if Q is in X, then there is a number B such that if each of x and y is in S, it then follows that $\|Qx - Qy\| \leq B \|x - y\|$, i.e., Q is Lipschitz.

The least such number B we call $|Q|$. It is then easy to show that X is normed by $| \ |$ in the following sense:

(i) If Q is in X, $|Q| \geq 0$. Further, $|Q| = 0$ if, and only if, $Q = 0$.

(ii) If each of Q and H is in X, then $|Q + H| \leq |Q| + |H|$, and $|QH| \leq |Q||H|$.

(iii) If β is a number and Q is in X, then $|\beta Q| = |\beta||Q|$.

(iv) Finally, $|I| = 1$.

For our purposes, it will not be enough to consider only sequences which converge in the sense of $| \; |$. In order to make use of MacNerney's results in the case that X is not merely a collection of linear functions, we consider a slight restriction of uniform convergence on bounded subsets of S.

To this end, we say that a sequence $\{A_n\}_{n=1}^{+\infty}$ of elements of X is L-uniformly convergent if, and only if, each of the following is true:

(i) For each $\varepsilon > 0$, there is a positive integer N such that if integers r and m are larger than N, then $\|A_r x - A_m x\| \leq \varepsilon \|x\|$ for each x in S, and

(ii) The number sequence $\{|A_n|\}_{n=1}^{+\infty}$ is bounded. It should be noted that condition (i) guarantees the existence of a pointwise limit and that condition (ii) guarantees that this pointwise limit is Lipschitz. Now, further, we agree that T is the L-uniform limit of $\{A_n\}_{n=1}^{+\infty}$ if, and only if, $\{A_n\}_{n=1}^{+\infty}$ is L-uniformly convergent, and for each $\varepsilon > 0$, there is a positive integer N such that if an integer m is larger than N, then $\|Tx - A_m x\| \leq \varepsilon \|x\|$ for each x in S. Finally, we say that X is L-uniformly complete if, and only if, whenever $\{A_n\}_{n=1}^{+\infty}$ is L-uniformly convergent in X, there is an element T of X which is the L-uniform limit of $\{A_n\}_{n=1}^{+\infty}$.

That L-uniformly complete spaces do indeed exist is the subject of Theorem 1.

<u>THEOREM 1</u>

Let M be the set to which Q belongs only in case

(i) Q is a function from S to S with $Q(0) = 0$,

and

(ii) there is a number B such that if each of x and y is in S, then $\|Qx - Qy\| \leq B\|x - y\|$.

Denote the least such B by $|Q|$. Then $\{M, |\ |\}$ is an

L-uniformly complete near-ring.

PROOF: It is evident that $\{M, |\ |\}$ is a normed near-ring. Suppose that $\{A_n\}_{n=1}^{+\infty}$ is an L-uniformly convergent sequence in M. Since S is complete, the pointwise limit T exists. If $\varepsilon > 0$, there is a positive integer N such that if integers m and n are larger than N, then $\| A_n x - A_m x \| \leq \varepsilon \| x \|$ for each x in S. Let y be in S. If $\delta > 0$, there exists a positive integer P such that if an integer n is larger than P, then $\| A_n y - Ty \| < \delta$. Hence, if m and n are integers with $m > N$ and $n > \max\{N,P\}$, we then have that $\| A_m y - Ty \| \leq \| A_n y - Ty \| + \| A_m y - A_n y \| \leq \varepsilon \| y \| + \delta$. This holds for each $\delta > 0$, so $\| A_m y - Ty \| \leq \varepsilon \| y \|$. Thus $\{A_n\}_{n=1}^{+\infty}$ converges L-uniformly to T.

Now let B be such that $| A_n | \leq B$ for each integer n. Let x and y be in S. If $\varepsilon > 0$, then there is a positive integer N such that if an integer k is larger than N, then $\| Tx - A_k x \| < \varepsilon$ and $\| Ty - A_k y \| < \varepsilon$. It then follows that $\| Tx - Ty \| \leq \| Tx - A_k x \| + \| A_k x - A_k y \| + \| A_k y - Ty \| \leq B \| x - y \| + 2\varepsilon$. Hence $| T | \leq B$ and T is in M.

The proof is complete.

REMARK 1.1 It is worth noting that if $\{A_n\}_{n=1}^{+\infty}$ is a sequence in X which converges in norm, then $\{A_n\}_{n=1}^{+\infty}$ is also L-uniformly convergent. Hence, if X is L-uniformly complete, X is complete in the sense of the norm. The converse, however, is not true, as the following example shows.

Example 1.1 Let Y be the collection to which f belongs only in case f is a Lipschitz function from the real numbers to the real numbers with (i) $f(0) = 0$ and (ii) there is a number C and a number β such that $f(x) = f(C) + \beta(x - C)$ for each $x \geq C$. If A is in Y, let $|A|$ denote the usual Lipschitz norm. Now, if M is the collection of Lipschitz functions from the real numbers to the real numbers to which f belongs only in case $f(0) = 0$, then Theorem 1 shows that M is L-uniformly complete and hence is complete in the normed sense. We may thus let NY denote the normed closure of Y in M and LY denote the L-uniform closure of Y in M.

If, for each positive integer n, we put $f_n(x) = \sin(\pi x)$ for $x < n$ and $f_n(x) = 0$ for $x \geq n$, then the sequence so constructed converges L-uniformly to $Tx = \sin(\pi x)$. Hence, T is in LY. Now, if each of G and H is in M, and each of G' and H' exists on a subset V of the real numbers, then it is easy to verify that $\| G - H \| \geq \sup \{| G'(x) - H'(x) | : x$ is in $V\}$. Hence, if H is in NY, then $\| H - T \| \geq \pi$; so T is not in NY. We see at this point that the requirement that a space be L-uniformly complete is considerably stronger than that of completeness in the sense of the norm.

The present example also furnishes an example in which there is a difference between the L-uniform closure and the closure in the sense that if $\{A_n\}_{n=1}^{+\infty}$ is a sequence in Y which converges, uniformly on bounded subsets of the real numbers, to a Lipschitz function F, then F is in the closure. The latter closure is, in fact, M itself. We may note, however, that if $f(x) = x \sin(\ln | x |)$ for $x \neq 0$ and $f(0) = 0$, then f is in M; and, moreover, if $g(x) = g(C) + \beta(x - C)$ for $x \geq C$, then

$$\lim_{n\to+\infty} | f(e^{((\pi + 4\pi n)/2)}) - g(e^{((\pi + 4\pi n)/2)}) | \cdot e^{-((\pi + 4\pi n)/2)} = | 1 - \beta |, \text{ and}$$

$$\lim_{n\to+\infty} | f(e^{((3\pi + 4\pi n)/2)}) - g(e^{((3\pi + 4\pi n)/2)}) | \cdot e^{-((3\pi + 4\pi n)/2)} = | -1 - \beta | .$$

Hence, no sequence in Y can converge L-uniformly to f.

Henceforth, X will always denote an L-uniformly complete near-ring of functions from S to S. We now state a few fundamental facts concerning L-uniform convergence and normed convergence. The proofs are not difficult and hence are omitted.

<u>REMARK 1.2</u> If $\{A_n\}_{n=1}^{+\infty}$ and $\{B_n\}_{n=1}^{+\infty}$ are sequences in X which converge L-uniformly to V and T, respectively, then $\{A_nB_n\}_{n=1}^{+\infty}$ converges L-uniformly to VT.

<u>REMARK 1.3</u> If $\{A_n\}_{n=1}^{+\infty}$ is a sequence in X which converges to T in norm, and if C is in X, then $\{A_nC\}_{n=1}^{+\infty}$ converges to TC in norm.

<u>REMARK 1.4</u> If $\{A_n\}_{n=1}^{+\infty}$ converges L-uniformly to T, then $|T| \leq \liminf_{n\to+\infty} |A_n|$.

The ideas expressed in Remarks 1.2, 1.3, and 1.4 will be used many times in what follows. To refer to each after every use would only clutter the remaining text. Hence, we shall not make practice of referencing each use of the above remarks.

CHAPTER III

AN INITIAL CHARACTERIZATION

We now move to an initial characterization of I(X) in terms of finite products of elements of a sufficiently small neighborhood of I.

If T is in X, we use T^{-1} to denote the element of X having the property that $TT^{-1} = T^{-1}T = I$, provided that such exists; and we define $T^n = TT^{n-1}$ where $T^o = I$ by definition. In the case that X is a collection of linear functions, we have the well-known result that if $|Q| < 1$, $(I - Q)^{-1}$ exists and is given by $(I - Q)^{-1} = \sum_{k=0}^{+\infty} Q^k$, where the convergence involved is in the sense of the norm, $|\ |$, on X. In the non-linear case, we use L-uniform convergence to obtain an analogous result.

<u>THEOREM 2</u>

If $|Q| < 1$, then $(I - Q)^{-1}$ exists and $|(I - Q)^{-1}| \leq 1/(1 - |Q|)$.

<u>PROOF</u>: Put $A_o = I$ and $A_{n+1} = I + QA_n$ for each non-negative

integer n. Now if each of x and y is in S and if p and q are positive integers, then $\| A_p x - A_p y \| =$

$$\| x + QA_{p-1}x - (y + QA_{p-1}y)\|$$

$$\leq \| x - y \| + | Q | \, \| A_{p-1}x - A_{p-1}y \|$$

$$\leq \| x - y \| \sum_{k=0}^{+\infty} | Q |^k = \| x - y \| \, (1 - | Q |)^{-1}.$$

Hence, the sequence $\{| A_n |\}_{n=0}^{+\infty}$ is bounded above by $(1 - | Q |)^{-1}$. Also, $\| A_p x - A_{p+q} x \|$

$$= \| x + QA_{p-1}x - (x + QA_{p+q-1}x) \|$$

$$\leq | Q | \, \| A_{p-1}x - A_{p+q-1}x \| \leq | Q |^p \, \| x - A_q x \|$$

$$\leq | Q |^p \, (1 + (1 - | Q |)^{-1}) \| x \|.$$

Hence $\{A_n\}_{n=0}^{+\infty}$ is L-uniformly convergent; and if F is the the L-uniform limit, $| F | \leq (1 - | Q |)^{-1}$.

Finally, $\| x - (I - Q)A_p x \| = \| x - A_p x + QA_p x \| = \| A_{p+1}x - A_p x \| \leq | Q |^p \| A_1 x - x \| \leq | Q |^{p+1} \| x \|$; and $\| x - A_p(I - Q)x \| = \| x - (I - Q)x - QA_{p-1}(I - Q)x \| \leq | Q | \, \| x - A_{p-1}(I - Q)x \| \leq | Q |^p \| x - (I - Q)x \| \leq | Q |^{p+1} \| x \|$. Hence, $F(I - Q) = (I - Q)F = I$.

<u>COROLLARY 2.1</u>

If $| Q | < 1$, $| I - (I - Q)^{-1} | (1 - | Q |) \leq | Q |$.

<u>PROOF</u>: Let $\{A_n\}_{n=0}^{+\infty}$ be as in the proof of Theorem 2. Then

$$|I - A_p| = |I - (I + QA_{p-1})| = |QA_{p-1}|$$
$$\leq |Q|(1 - |Q|)^{-1}.$$

COROLLARY 2.2

If T in X is invertible and if $|T - Q||T^{-1}| < 1$, then Q^{-1} exists and $|Q^{-1}|(1 - |I - QT^{-1}|) \leq |T^{-1}|$.

PROOF: $|I - QT^{-1}| < 1$; so by Theorem 2, $(QT^{-1})^{-1}$ exists. Now $I = (QT^{-1})(QT^{-1})^{-1} = Q(T^{-1}(QT^{-1})^{-1})$, and $I = T^{-1}IT$ $= T^{-1}(QT^{-1})^{-1}(QT^{-1})T = (T^{-1}(QT^{-1})^{-1})Q$. Hence $Q^{-1} = T^{-1}(QT^{-1})^{-1}$. Finally, $|Q^{-1}| = |T^{-1}(QT^{-1})^{-1}| \leq |T^{-1}||(QT^{-1})^{-1}|$, so we have $|Q^{-1}| \leq |T^{-1}|(1 - |I - QT^{-1}|)^{-1}$.

COROLLARY 2.3

The set of invertible elements of X is open.

COROLLARY 2.4

I(X) is open.

PROOF: Note that if Q is in X and $\varepsilon > 0$, then the set of all H in X such that $|Q - H| < \varepsilon$ is arcwise connected.

We now move to our initial characterization of I(X). Since finite products will occur frequently in what follows, we make the following definition:

DEFINITION 3.1 If M is a subset of X, then ΠM denotes the set to which Q belongs only in case there is a sequence $\{H_i\}_{i=1}^{n}$ in M such that $Q = \prod_{i=1}^{n} H_i$.

THEOREM 3

Let $0 < \varepsilon \leq 1$. Let $M(\varepsilon)$ be the collection of elements of X to which Q belongs only in case $|I - Q| < \varepsilon$. Then $I(X) = \Pi M(\varepsilon)$.

PROOF: By Theorem 2, $\Pi M(\varepsilon)$ is a set of invertible elements. Suppose that $\{Q_p\}_{p=1}^{n}$ is a sequence in $M(\varepsilon)$. For $0 \leq a \leq b \leq 1$, $(I - a(I - Q_1)) \prod_{p=2}^{n} Q_p$ is in $\Pi M(\varepsilon)$; and the fact that $|(I - b(I - Q_1)) \prod_{p=2}^{n} Q_p - (I - a(I - Q_1)) \prod_{p=2}^{n} Q_p|$ $\leq (b - a)|I - Q_1| \prod_{p=2}^{n} |Q_p|$ gives, inductively that $\Pi M(\varepsilon)$ is arcwise connected. Hence $\Pi M(\varepsilon)$ is a subset of $I(X)$. Also, if H is in $\Pi M(\varepsilon)$, F is in X, and $|H - F| < \varepsilon |H^{-1}|^{-1}$, then $|I - FH^{-1}| < \varepsilon$. Since $F = (FH^{-1})H$, F is in $\Pi M(\varepsilon)$; hence, $\Pi M(\varepsilon)$ is open.

Finally, suppose that there exists an element A of $I(X)$ not in $\Pi M(\varepsilon)$. If $B(A)$ denotes the set of F in X having the property that $|F - A||A^{-1}| < \varepsilon/4$ and U denotes the union of all such $B(A)$ with A in $I(X) - \Pi M(\varepsilon)$, it then follows that U is an open set of invertible elements of X. The fact that each B(A) is arcwise connected implies that U is a subset of I(X). Further, if F is in B(A), then $|I - FA^{-1}| < \varepsilon/4$, so AF^{-1} exists and $|I - AF^{-1}| < \varepsilon/3$, by Corollary 2.1. Since $A = (AF^{-1})F$, F is not in $\Pi M(\varepsilon)$; hence, B(A) does not intersect $\Pi M(\varepsilon)$. Since I is in $\Pi M(\varepsilon)$ and the union of $\Pi M(\varepsilon)$ and U is I(X), we have a contradiction to the fact that I(X) is connected. Hence, there is no element A in I(X) but not in $\Pi M(\varepsilon)$.

The proof is now complete.

<u>COROLLARY 3.1</u> I(X) is a group.

<u>PROOF</u>: Theorem 3 gives that I(X) is closed under multiplication. Also, if Q is in I(X), then there is a sequence $\{H_p\}_{p=1}^{n}$ in X so that $|I - H_p| < 1/4$ for each integer p in [1,n] and so that $Q = \prod_{p=1}^{n} H_p$. By Corollary 2.1, $|I - H_p^{-1}| < 1/3$ for each integer p in [1,n]. Since $Q^{-1} = \prod_{p=0}^{n-1} H_{n-p}^{-1}$, Q^{-1} is in I(X). Hence I(X) is a group.

<u>COROLLARY 3.2</u> If P is a connected set of invertible elements of X and P contains a neighborhood of I, and if P is closed under multiplication, then P = I(X).

CHAPTER IV

A FEW RESULTS ON INTEGRAL EQUATIONS

We now introduce several results developed in the study of integral equations. These, together with Theorem 3, will in Chapter V furnish a complete characterization of I(X). Theorems four, five, six, and seven are due to J. S. MacNerney [3]. Theorem eight is similar to a result due to J. V. Herod [2]. The proofs are the author's.

A body of needed definitions is now given. Suppose that R is a linearly ordered set. If each of x and y is in R, we say that $\{t_p\}_{p=0}^{n}$ is a subdivision of $\{x,y\}$ if, and only if, $t_o = x$, $t_n = y$, and t_k lies between t_{k-1} and t_{k+1} for k = 1,2,...n-1. We say that $\{s_p\}_{p=0}^{m}$ is a refinement of $\{t_p\}_{p=0}^{n}$ if, and only if, $\{s_p\}_{p=0}^{m}$ is a subdivision of $\{x,y\}$ which contains $\{t_p\}_{p=0}^{n}$ as a subsequence.

Continued sums and products will arise frequently in the discussion which follows. To simplify discussion, we make a few notational agreements. Let α be a function from from $R \times R$ to X. If each of x and y is in R, and if $\{t_p\}_{p=0}^{n}$ is a subdivision of $\{x,y\}$, then by $\sum_t \alpha$ we mean $\sum_{p=1}^{n} \alpha(t_{p-1},t_p)$; by $\Pi_t \alpha$ we mean $\Pi_{p=1}^{n} \alpha(t_{p-1},t_p)$. For convenience, we agree that $\sum_{p=j+1}^{j} \alpha(t_{p-1},t_p) = 0$ and that $\Pi_{p=j+1}^{j} \alpha(t_{p-1},t_p) = I$ for each integer j. By $\sum_x^y \alpha$ we mean that element of X having the property that if $\varepsilon > 0$, there is a subdivision $\{t_p\}_{p=0}^{n}$ of $\{x,y\}$ such that whenever $\{s_p\}_{p=0}^{m}$ refines $\{t_p\}_{p=0}^{n}$, $\| (\sum_s \alpha)P - (\sum_x^y \alpha)P \| \leq \varepsilon \| P \|$ for each P in S — provided, of course, that such a limit exists. By $\Pi_x^y \alpha$ we mean that element of X having the property that if $\varepsilon > 0$, there is a subdivision $\{t_p\}_{p=0}^{n}$ of $\{x,y\}$ such that whenever $\{s_p\}_{p=0}^{m}$ refines $\{t_p\}_{p=0}^{n}$, then $\| (\Pi_s \alpha)P - (\Pi_x^y \alpha)P \| \leq \varepsilon \| P \|$ for each P in S — again provided that such a limit exists.

We shall, of course, not wish to consider all functions from R×R to X. We rather restrict our attention to gain assurance that $\sum_x^y [\alpha - I]$ exists or that $\Pi_x^y [I + \alpha]$ exists. We accomplish this as follows:

Suppose that each of α, μ, V, and M is a function with domain R×R.

<u>DEFINITION 4.1</u> We say that α is in OA^+ only in case each of the following is true:

(i) α has values which are non-negative numbers, and

(ii) $\alpha(x,y) + \alpha(y,z) = \alpha(x,z)$ whenever y lies between x and z.

<u>DEFINITION 4.2</u> We say that μ is in OM^+ only in case each of the following is true:

(i) μ has numerical values not less than 1, and

(ii) $\mu(x,y)\mu(y,z) = \mu(x,z)$ whenever y lies between x and z.

<u>DEFINITION 4.3</u> We say that V is in OA only in case each of the following is true:

(i) V has values in X,

(ii) $V(x,y) + V(y,z) = V(x,z)$ whenever y lies between x and z, and

(iii) There is an element β of OA^+ such that $|V(x,y)| \leq \beta(x,y)$ for each $\{x,y\}$ in $R\times R$. (In this event, we say that $|V| \ll \beta$.)

DEFINITION 4.4 We say that M is in OM only in case each of the following is true:

(i) M has values in X,

(ii) $M(x,y)M(y,z) = M(x,z)$ whenever y lies between x and z, and

(iii) There is an element ψ of OM^+ such that $|M(x,y) - I| \leq \psi(x,y) - 1$ for each $\{x,y\}$ in $R\times R$. (In this event we say that $|M - I| \ll \psi - 1$.)

Such restriction of attention produces worthwhile results; for, using continued products and continued sums, we are now able to establish strong connections, first, between OA^+ and OM^+, and, second, between OA and OM. Theorems four, five, and six have as their subjects these relationships.

LEMMA 4.1

Suppose that α is in OA^+. Then $\mu(x,y) = \prod_x^y [1 + \alpha]$ exists and is an element of OM^+. Moreover, the convergence involved is monotonic in the sense that if $\{s_p\}_{p=0}^m$ refines $\{t_p\}_{p=0}^n$, then $\prod_s [1 + \alpha] \geq \prod_t [1 + \alpha]$.

PROOF: Suppose that $\{t_p\}_{p=0}^n$ is a subdivision of $\{x,y\}$. Then $\prod_t [1 + \alpha] \leq \prod_t e^{\alpha} = e^{\alpha(x,y)}$. Hence, we may put

$\mu(x,y) = \sup \{\prod_t [1 + \alpha] : t \text{ is a subdivision of } \{x,y\}\}$.

Now the fact that $(1 + \alpha(t_{p-1},t_p))(1 + \alpha(t_p,t_{p+1}))$

$= 1 + \alpha(t_{p-1},t_{p+1}) + \alpha(t_{p-1},t_p)\,\alpha(t_p,t_{p+1})$

$\geq 1 + \alpha(t_{p-1},t_{p+1})$ gives that $\prod_t [1 + \alpha]$ increases under one-point refinements, and, hence, under refinement. Immediately we have that $\prod_x^y [1 + \alpha]$ exists and gives rise to an element of OM^+.

LEMMA 4.2

Suppose that μ is in OM^+. $\alpha(x,y) = \sum_x^y [\mu - 1]$ exists and is an element of OA^+. Moreover, the convergence involved is monotonic in the sense that if $\{t_p\}_{p=0}^{n}$ is refined by $\{s_p\}_{p=0}^{m}$, then $\sum_s [\mu - 1] \le \sum_t [\mu - 1]$.

<u>PROOF</u>: If $\{t_p\}_{p=0}^{n}$ is a subdivision of $\{x,y\}$, then $\sum_t [\mu - 1] \ge 0$, so put $\alpha(x,y) = \inf \{ \sum_t [\mu - 1] : t$ is a subdivision of $\{x,y\}\}$. Now, the fact that $\mu(t_{p-1},t_{p+1}) - 1 = \mu(t_{p-1},t_p)\,\mu(t_p,t_{p+1}) - 1 = \mu(t_{p-1},t_p)(\mu(t_p,t_{p+1}) - 1) + \mu(t_{p-1},t_p) - 1 \ge \mu(t_{p-1},t_p) - 1 + \mu(t_p,t_{p+1}) - 1$ gives gives that $\sum_t [\mu - 1]$ decreases under one-point refinements and, hence, under refinements. Hence, $\sum_x^y [\mu - 1]$ exists and gives rise to an element of OA^+.

<u>THEOREM 4</u>

Define $E^+(\alpha)(x,y) = \prod_x^y [1 + \alpha]$ for each α in OA^+, each (x,y) in $R\times R$. Define $L^+(\mu)(x,y) = \sum_x^y [\mu - 1]$ for each μ in OM^+, each (x,y) in $R\times R$. Then E^+ and L^+ are inverses in the sense that if α is in OA^+ and μ is in OM^+, then $E^+L^+(\mu) = \mu$ and $L^+E^+(\alpha) = \alpha$.

PROOF: Let $\varepsilon > 0$ be given. Suppose that $L^{+}(\mu) = \alpha$. If (x,y) is in $R \times R$, there is a subdivision $\{t_p\}_{p=0}^{n}$ of $\{x,y\}$ such that if $\{s_p\}_{p=0}^{m}$ refines $\{t_p\}_{p=0}^{n}$, then

$$\mu(x,y)((\sum_s [\mu - 1]) - \alpha(x,y)) < \varepsilon.$$

Since $\mu(s_{p-1},s_p) - 1 - \alpha(s_{p-1},s_p) \geq 0$ for each p in $[1,m]$,

$$0 \leq \prod_{p=1}^{m} \mu(s_{p-1},s_p) - \prod_{p=1}^{m} [1 + \alpha(s_{p-1},s_p)]$$

$$= \sum_{k=1}^{m} \Big[(\prod_{p=k}^{m} \mu(s_{p-1},s_p)) \prod_{p=1}^{k-1} [1 + \alpha(s_{p-1},s_p)]$$

$$- (\prod_{p=k+1}^{m} \mu(s_{p-1},s_p)) \prod_{p=1}^{k} [1 + \alpha(s_{p-1},s_p)] \Big]$$

$$= \sum_{k=1}^{m} \Big[(\prod_{p=k+1}^{m} \mu(s_{p-1},s_p)) \prod_{p=1}^{k-1} [1 + \alpha(s_{p-1},s_p)] \cdot$$

$$(\mu(s_{k-1},s_k) - 1 - \alpha(s_{k-1},s_k)) \Big]$$

$$\leq \sum_{k=1}^{m} \mu(s_k,s_m)\, \mu(s_o,s_{k-1})(\mu(s_{k-1},s_k) - 1 - \alpha(s_{k-1},s_k))$$

$$\leq \mu(x,y)(\sum_s [\mu - 1] - \alpha(x,y)) < \varepsilon.$$

Hence, $E^{+}L^{+}(\mu) = \mu$.

Now suppose that $\mu = E^{+}(\alpha)$. Then if (x,y) is in $R \times R$, there is a subdivision $\{t_p\}_{p=0}^{n}$ such that if $\{s_p\}_{p=0}^{m}$ refines $\{t_p\}_{p=0}^{n}$, then $\mu(x,y) - \prod_s [1 + \alpha] < \varepsilon$. Since

$\mu(s_{p-1},s_p) - (1 + \alpha(s_{p-1},s_p)) \geq 0$ for each p in [1,m], we have $0 \leq \sum_{p=1}^{m} [\mu(s_{p-1},s_p) - 1 - \alpha(s_{p-1},s_p)]$

$$\leq \sum_{p=1}^{m} \left[(\mu(s_{p-1},s_p) - 1 - \alpha(s_{p-1},s_p)) \prod_{k=p+1}^{m} \mu(s_{k-1},s_k) \cdot \prod_{k=1}^{p-1} [1 + \alpha(s_{k-1},s_k)] \right]$$

$$= \left(\prod_{p=1}^{m} \mu(s_{p-1},s_p) \right) - \prod_{p=1}^{m} [1 + \alpha(s_{p-1},s_p)] < \varepsilon.$$

Hence, $L^+E^+(\alpha) = \alpha$.

Thus we have established that each element of OA^+ arises from an element of OM^+ via continued sums and that each element of OM^+ arises from an element of OA^+ via continued products. We will show that an analogous correlation holds between OA and OM, but first we have need of a few lemmas and an existence theorem.

Fundamental to establishing the existence of $\prod_x^y [I + V]$ for V in OA is the following lemma:

LEMMA 5.1

Suppose that V is in OA and α is in OA^+ with $|V| \ll \alpha$, and that x and y are in R. If $\{t_p\}_{p=0}^{n}$ is a subdivision of $\{x,y\}$, and B is in S, and $\mu = E^+(\alpha)$, then

$$\left\| \prod_{p=1}^{n} [I + V(t_{p-1},t_p)]\, B - (I + V(x,y))B \right\|$$

$$\leq (\mu(x,y) - 1 - \alpha(x,y)) \|B\| .$$

<u>PROOF</u>: $\left\| \prod_{p=1}^{n} [I + V(t_{p-1},t_p)]\, B - B - V(x,y)B \right\|$

$$= \left\| \sum_{k=1}^{n} \left[\prod_{p=k}^{n} [I + V(t_{p-1},t_p)] - \prod_{p=k+1}^{n} [I + V(t_{p-1},t_p)] \right] B \right.$$

$$\left. - \sum_{k=1}^{n} V(t_{k-1},t_k)\, B \right\|$$

$$\leq \sum_{k=1}^{n} \left\| (I + V(t_{k-1},t_k) - I) \prod_{p=k+1}^{n} [I + V(t_{p-1},t_p)]\, B \right.$$

$$\left. - V(t_{k-1},t_k) B \right\|$$

$$\leq \sum_{k=1}^{n} \alpha(t_{k-1},t_k) \left\| \left(\prod_{p=k+1}^{n} [I + V(t_{p-1},t_p)] - I \right) B \right\|$$

$$= \sum_{k=1}^{n} \alpha(t_{k-1},t_k) \left\| \sum_{c=k+1}^{n} \left[\prod_{p=c}^{n} [I + V(t_{p-1},t_p)] \right. \right.$$

$$\left. \left. - \prod_{p=c+1}^{n} [I + V(t_{p-1},t_p)] \right] B \right\|$$

$$\leq \sum_{k=1}^{n} \alpha(t_{k-1},t_k) \Big(\sum_{c=k+1}^{n} \| V(t_{c-1},t_c) \cdot$$

$$\prod_{p=c+1}^{n} [I + V(t_{p-1},t_p)]\, B \| \Big)$$

$$\leq \sum_{k=1}^{n} \alpha(t_{k-1},t_k) \Big(\sum_{c=k+1}^{n} \alpha(t_{c-1},t_c) \cdot$$

$$\prod_{p=c+1}^{n} [1 + \alpha(t_{p-1},t_p)] \Big) \|B\|$$

$$= \sum_{k=1}^{n} \alpha(t_{k-1},t_k)\Big(\sum_{c=k+1}^{n} \Big[\prod_{p=c}^{n}[1 + \alpha(t_{p-1},t_p)] - \prod_{p=c+1}^{n}[1 + \alpha(t_{p-1},t_p)]\Big] \| B \|\Big)$$

$$= \sum_{k=1}^{n} \alpha(t_{k-1},t_k)\Big(\prod_{p=k+1}^{n}[1 + \alpha(t_{p-1},t_p)] - 1\Big)\| B \|$$

$$= \Big[\prod_{p=1}^{n}[1 + \alpha(t_{p-1},t_p)] - \alpha(x,y)\Big] \| B \|$$

$$\leq (\mu(x,y) - 1 - \alpha(x,y))\| B \|.$$

Fundamental to establishing the existence of $\sum_{x}^{y}[M - I]$ for M in OM is

LEMMA 5.2

Suppose that M is in OM and μ is in OM^{+} with $|M - I| \ll \mu - 1$, and that each of x and y is in R. If $\{t_p\}_{p=0}^{n}$ is a subdivision of $\{x,y\}$, B is in S, and $\alpha = L^{+}(\mu)$, then $\| (M(x,y) - I)B - \sum_{t}[M - I]B \|$

$$\leq (\mu(x,y) - 1 - \alpha(x,y))\| B \|.$$

PROOF: $\| (M(x,y) - I)B - \sum_{t}[M - I]B \|$

$$= \| \sum_{p=1}^{n}[M(t_{p-1},t_n) - M(t_p,t_n)]B - \sum_{p=1}^{n}[M(t_{p-1},t_p) - I]B \|$$

$$\leq \sum_{p=1}^{n} \| (M(t_{p-1},t_p) - I)M(t_p,t_n)B - (M(t_{p-1},t_p) - I)B \|$$

$$\leq \sum_{p=1}^{n} (\mu(t_{p-1},t_p) - 1) \| (M(t_p,t_n) - I)B \|$$

$$\leq \sum_{p=1}^{n} (\mu(t_{p-1},t_p) - 1)(\mu(t_p,t_n) - 1) \| B \|$$

$$\leq \left[(\mu(x,y) - 1) - \sum_{p=1}^{n} [\mu(t_{p-1},t_p) - 1] \right] \| B \|$$

$$\leq (\mu(x,y) - 1 - \alpha(x,y)) \| B \|.$$

In order to establish in Theorem 5 the existence of $\sum_x^y [W - I]$ and $\prod_x^y [I + V]$ for W in OM and V in OA, we note one more inequality in Lemma 5.3. Establishment of this inequality does not shorten the proof of Theorem 5; it does, however, serve to make the proof more readable. Hence, we include the lemma.

LEMMA 5.3

Suppose that each of $\{A_p\}_{p=1}^{n}$ and $\{B_p\}_{p=1}^{n}$ is a sequence in X and that y is in S. Then

$$\| \prod_{p=1}^{n} A_p\, y - \prod_{p=1}^{n} B_p\, y \| \leq \sum_{k=1}^{n} [(\prod_{p=1}^{n-k} | A_p |) \cdot \| (A_{n-k+1} - B_{n-k+1}) \prod_{p=n-k+2}^{n} B_p\, y \|].$$

<u>PROOF</u>: $\left\| \prod_{p=1}^{n} A_p y - \prod_{p=1}^{n} B_p y \right\|$

$$= \left\| \sum_{k=1}^{n} \left[\prod_{p=1}^{n-k+1} A_p \prod_{p=n-k+2}^{n} B_p y - \prod_{p=1}^{n-k} A_p \prod_{p=n-k+1}^{n} B_p y \right] \right\|$$

$$\leq \sum_{k=1}^{n} \left[\left(\prod_{p=1}^{n} |A_p| \right) \left\| (A_{n-k+1} - B_{n-k+1}) \prod_{p=n-k+2}^{n} B_p y \right\| \right].$$

We now have in Theorem 5 establishment of functions from OA to OM and from OM to OA, similar to those already established from OA^+ to OM^+ and from OM^+ to OA^+.

<u>THEOREM 5</u>

Suppose that V is in OA and that W is in OM. Then

$F(x,y) = \prod_x^y [I + V]$ exists and is in OM; and,

$G(x,y) = \sum_x^y [W - I]$ exists and is in OA.

<u>PROOF</u>: Let $\{t_p\}_{p=0}^{n}$ be a subdivision of $\{x,y\}$ and let s_p be a subdivision of $\{t_{p-1}, t_p\}$ for each p in [1,n]. Let s denote the subdivision of $\{x,y\}$ whose values are precisely the union of the values of each of the sequences s_p, p = 1, 2,...n. Note that s is then a refinement of $\{t_p\}_{p=0}^{n}$ and, further, that if $\{c_p\}_{p=0}^{m}$ is a refinement of $\{t_p\}_{p=0}^{n}$, then

c can be characterized in this fashion (i.e., in terms of subdivisions of $\{t_{p-1}, t_p\}$ for p = 1,2,...n). Finally, we let α be in OA^+ with $|V| \ll \alpha$, let B be in S, and let $E^+(\alpha) = \mu$. Then, using Lemmas 5.1 and 5.3, we get

$$\left\| \prod_t [I + V] B - \prod_s [I + V] B \right\|$$

$$= \left\| \prod_{p=1}^{n} [I + V(t_{p-1}, t_p)] B - \prod_{p=1}^{n} [\prod_{s_p} [I + V]] B \right\|$$

$$\leq \sum_{k=1}^{n} \left[\prod_{p=1}^{n-k} [| I + V(t_{p-1}, t_p) |] \left\| \left[(I + V(t_{n-k}, t_{n-k+1}) - \prod_{s_{n-k+1}} [I + V]) \prod_{p=n-k+2}^{n} [\prod_{s_p} [I + V]] \right] B \right\| \right]$$

$$\leq \sum_{k=1}^{n} \left[(\prod_{p=1}^{n-k} [1 + \alpha(t_{p-1}, t_p)])(\mu(t_{n-k}, t_{n-k+1}) - 1 - \alpha(t_{n-k}, t_{n-k+1})) \left\| \prod_{p=n-k+2}^{n} [\prod_{s_p} [I + V]] B \right\| \right]$$

$$\leq \sum_{k=1}^{n} [\mu(t_o, t_{n-k})(\mu(t_{n-k}, t_{n-k+1}) - 1 - \alpha(t_{n-k}, t_{n-k+1})) \cdot \mu(t_{n-k+1}, t_n) \| B \|]$$

$$\leq (\sum_t [\mu - 1 - \alpha]) \mu(x,y) \| B \| .$$

Since S is complete, the existence of $F(x,y) = \prod_x^y [I + V]$ is clear. Further, $| \prod_t [I + V] | \leq \prod_t [1 + \alpha] \leq \mu(x,y)$ gives that F(x,y) is in X, by L-uniform completeness of X.

Also, if y lies between x and z, we may take subdivisions of $\{x,y\}$, $\{y,z\}$, and $\{x,z\}$ and form their "union" to deduce that $F(x,y)F(y,z) = F(x,z)$. Finally, we have that

$$\left| \prod_{p=1}^{n} [I + V(t_{p-1},t_p)] - I \right|$$

$$\leq \sum_{k=1}^{n} \left| \prod_{p=k}^{n} [I + V(t_{p-1},t_p)] - \prod_{p=k+1}^{n} [I + V(t_{p-1},t_p)] \right|$$

$$\leq \sum_{k=1}^{n} |V(t_{k-1},t_k)| \left| \prod_{p=k+1}^{n} [I + V(t_{p-1},t_p)] \right|$$

$$\leq \sum_{k=1}^{n} [\alpha(t_{k-1},t_k)(\prod_{p=k+1}^{n} [1 + \alpha(t_{p-1},t_p)])]$$

$$= \sum_{k=1}^{n} \left[\prod_{p=k}^{n} [1 + \alpha(t_{p-1},t_p)] - \prod_{p=k+1}^{n} [1 + \alpha(t_{p-1},t_p)] \right]$$

$\leq \mu(t_o,t_n) - 1$. Hence F is in OM.

Now let ρ be in OM^+, let $|W - I| \ll \rho - 1$, and let $\beta = L^+(\rho)$. Then, using Lemma 5.2, we deduce that

$$\left\| \sum_t [W - I] B - \sum_s [W - I] B \right\|$$

$$= \left\| \sum_{p=1}^{n} [W(t_{p-1},t_p) - I]B - \sum_{p=1}^{n} [\sum_{s_p} [W - I]]B \right\|$$

$$\leq \sum_{p=1}^{n} \left\| (W(t_{p-1},t_p) - I)B - \sum_{s_p} [W - I]B \right\|$$

$$\leq \sum_{p=1}^{n} [\rho(t_{p-1},t_p) - 1 - \beta(t_{p-1},t_p)] \| B \|.$$

Since S is complete, the existence of $G(x,y) = \sum_x^y [W - I]$

follows immediately. Further, $| \sum_t [W - I] | \leq \sum_t | W - I |$ $\leq \sum_t [\rho - 1]$ gives that $| \sum_x^y [W - I] | \ll \beta$. As before, it is clear that $G(x,y) + G(y,z) = G(x,z)$ whenever y lies between x and z. Hence G is in OA.

The proof is now complete.

The following are now immediate consequences of Lemmas 5.1 and 5.2 and Theorem 5.

COROLLARY 5.1

Suppose that α is in OA^+, V is in OA, $| V | \ll \alpha$, $E^+(\alpha) = \mu$, and B is in S. Then

$$\| \, {}_x\Pi^y [I + V]\, B - (I + V(x,y))B \, \| \leq (\mu(x,y) - 1 - \alpha(x,y)) \| B \| .$$

COROLLARY 5.2

Suppose that μ is in OM^+, W is in OM with $| W - I | \ll \mu - 1$, $L^+(\mu) = \alpha$, and B is in S. Then

$$\| (W(x,y) - I)B - \sum_x^y [W - I]B \| \leq (\mu(x,y) - 1 - \alpha(x,y)) \| B \| .$$

COROLLARY 5.3

If α is in OA^+, V is in OA with $| V | \ll \alpha$, and

$= E^{+}(\alpha)$, then $\left| \prod_{x}^{y}[I + V] - I \right| \leq \mu(x,y) - 1$.

We now move to the anticipated result, parallel to that obtained in Theorem 4.

THEOREM 6

Define $E(V)(x,y)$ to be $\prod_{x}^{y}[I + V]$ and $L(W)(x,y)$ to be $\sum_{x}^{y}[W - I]$ for each (x,y) in $R\times R$, each V in OA, and each W in OM. Then E and L are inverses in the sense that $E(L(W)) = W$ and $L(E(V)) = V$ for each W in OM and each V in OA.

PROOF: Suppose that α is in OA^{+}, that V is in OA with $|V| \ll \alpha$, and that $E(V) = W$. Then $|W - I| \ll E^{+}(\alpha) - 1$. If $E^{+}(\alpha) = \mu$, (x,y) is in $R\times R$, and B is in S, then for each subdivision $\{t_p\}_{p=0}^{n}$ of $\{x,y\}$, we have

$$\left\| \sum_{t}[W - I]B - V(x,y)B \right\|$$

$$\leq \sum_{p=1}^{n} \left\| W(t_{p-1},t_p)B - B - V(t_{p-1},t_p)B \right\|$$

$$\leq \sum_{p=1}^{n} (\mu(t_{p-1},t_p) - 1 - \alpha(t_{p-1},t_p)) \| B \|$$

by Corollary 5.1. Since $\mu = E^{+}(\alpha)$, we have that $L(W) = V$.

Now suppose that μ is in OM^+, that W is in OM with $|W - I| \ll \mu - 1$, and that $L(W) = V$. Then $|V| \ll L^+(\mu)$. If $L^+(\mu) = \alpha$, (x,y) is in $R \times R$, $\{t_p\}_{p=0}^{n}$ is a subdivision of $\{x,y\}$, and B is in S, we then use Lemma 5.3 and Corollary 5.2 to deduce that $\| W(x,y)B - \Pi_t [I + V] B \|$

$$= \| \Pi_{p=1}^{n} W(t_{p-1},t_p) B - \Pi_{p=1}^{n} [I + V(t_{p-1},t_p)] B \|$$

$$\leq \sum_{k=1}^{n} (\Pi_{p=1}^{n-k} | W(t_{p-1},t_p) |) \| (W(t_{n-k},t_{n-k+1}) - I - V(t_{n-k},t_{n-k+1})) \Pi_{p=n-k+2}^{n} [I + V(t_{p-1},t_p)] B \|$$

$$\leq \sum_{k=1}^{n} (\Pi_{p=1}^{n} \mu(t_{p-1},t_p))(\mu(t_{n-k},t_{n-k+1}) - 1 - \alpha(t_{n-k},t_{n-k+1})) \Pi_{p=n-k+2}^{n} [1 + \alpha(t_{p-1},t_p)] \| B \|$$

$$\leq \sum_{k=1}^{n} \mu(t_o,t_{n-k})(\mu(t_{n-k},t_{n-k+1}) - 1 - \alpha(t_{n-k},t_{n-k+1})) \mu(t_{n-k+1},t_n) \| B \|$$

$$\leq \mu(x,y) \| B \| \sum_t [\mu - 1 - \alpha].$$

Since $\alpha = L^+(\mu)$, we have $E(V) = W$, and the proof is complete.

MacNerney uses the functions E and L in the analysis of Stieljes-type integral equations as follows:

If V is in OA and F is a quasi-continuous function from the real numbers to X, and if x and y are real numbers, we define $(R)\int_x^y VF$ using approximating sums of the form $\sum_{p=1}^{n} V(t_{p-1},t_p)F(t_p)$ corresponding to subdivisions $\{t_p\}_{p=0}^{n}$ of $\{x,y\}$. We then wish to solve the system

$$(*) \qquad F(t) = I + (R)\int_t^0 VF \quad .$$

MacNerney has shown

THEOREM 7

The system (*) has exactly one quasi-continuous solution F. Moreover, if E(V) = W, then F(t) = W(t,0) for each t.

Since the characterizations of I(X) which follow do not depend directly of Theorem 7, we omit the proof. The contents of Theorem 7, however, are of great importance in interpreting these characterizations and so should be kept in mind.

We now move to a result establishing sufficient conditions on V in OA to guarantee that $\prod_x^y [I + V]$ be in I(X) for each pair of numbers (x,y). To this end, we define

$\alpha(x,x^+) = \lim_{y\to x} \{\alpha(x,y) : y > x\}$, $\alpha(x,x^-) = \lim_{y\to x}\{\alpha(x,y) : y < x\}$, $\alpha(x^+,x) = \lim_{y\to x} \{\alpha(y,x) : y > x\}$,and $\alpha(x^-,x) = \lim_{y\to x} \{\alpha(y,x) : y < x\}$ for each number x, and for either α in OA^+ or α in OM^+. We now establish two lemmas which together give a sufficient condition on V in OA to insure that $\prod_x^y [I + V]$ is in I(X).

LEMMA 8.1

Suppose that α is in OA^+ and $\mu = E^+(\alpha)$. Then $\alpha(x,x^+) = \mu(x,x^+) - 1$, $\alpha(x,x^-) = \mu(x,x^-) - 1$, $\alpha(x^+,x) = \mu(x^+,x) - 1$, and $\alpha(x^-,x) = \mu(x^-,x) - 1$ for each number x.

PROOF: We shall prove only that $\alpha(x,x^+) = \mu(x,x^+) - 1$. The other three identities have essentially the same proof.

Since $1 + \alpha(x,y) \leq \mu(x,y)$ for each y, we have that $\alpha(x,x^+) \leq \mu(x,x^+) - 1$. To deduce that $\mu(x,x^+) - 1 \leq \alpha(x,x^+)$, we suppose that $c \gg x$ and that $\varepsilon > 0$. Now, since $\alpha = L^+(\mu)$, there is a subdivision $\{t_p\}_{p=0}^n$ of $\{x,c\}$ such that $\sum_t [\mu - 1 - \alpha] < \varepsilon$. Hence, $\mu(x,t_1) - 1 - \alpha(x,t_1) < \varepsilon$. We also have that if $x < y < t_1$, then

$(\mu(x,y) - 1 - \alpha(x,y)) + (\mu(y,t_1) - 1 - \alpha(y,t_1))$ $\leq (\mu(x,t_1) - 1 - \alpha(x,t_1))$. Thus $\mu(x,x^+) - 1 \leq \alpha(x,x^+) + \varepsilon$ for each $\varepsilon > 0$. So we have $\mu(x,x^+) - 1 \leq \alpha(x,x^+)$ and the lemma is proved.

LEMMA 8.2

If α is in OA^+ with each of $\alpha(x,x^+) < 1$ and $\alpha(x^-,x) < 1$ for each x (resp., $\alpha(x^+,x) < 1$ and $\alpha(x,x^-) < 1$ for each x), and if $a \leq b$ (resp., $a \geq b$), then there is a subdivision $\{t_p\}_{p=0}^{n}$ of $\{a,b\}$ such that if $\mu = E^+(\alpha)$, it then follows that $\mu(t_{p-1},t_p) - 1 < 1$ for each p in $[1,n]$.

PROOF: Suppose that x is in $\{a,b\}$. By Lemma 8.1, there is a $y > x$ and a $c < x$ so that $\mu(x,y) - 1 < 1$ and $\mu(c,x) - 1 < 1$. Let $A(x) = ((x + c)/2, (x + y)/2)$. Then the collection of $A(x)$ for all x in $\{a,b\}$ forms an open covering of $\{a,b\}$. We may take a finite subcovering, say $A(x_o)$, $A(x_1), \ldots A(x_m)$ with $a = x_o < x_1 < \ldots < x_m = b$. Now let $z_o = x_o$. Let $z_k = \sup \{x_j$ so that $A(x_j) \cap A(z_{k-1})$ is not empty$\}$. So doing, we construct a sequence $\{z_p\}_{p=0}^{r}$ so that $a = z_o$,

$b = z_r$, $z_{k-1} < z_k$, and $A(z_{k-1}) \cap A(z_k)$ is not empty for each k in [1,r]. Now if $z_{k-1} < w_k < z_k$ and w_k is in $A(z_{k-1}) \cap A(z_k)$, then $\mu(z_{k-1},w_k) - 1 < 1$ and $\mu(w_k,z_k) - 1 < 1$ for each k in [1,r]. So if $\{t_p\}_{p=0}^{n}$ is a subdivision which contains the points z_k and w_k, then $\{t_p\}_{p=0}^{n}$ has the desired properties. (If $a \geq b$, the proof is almost the same.)

We now define, for V in OA, the variation of V from x to y, $\sum_x^y |V|$, to be the least number B such that $\sum_{p=1}^{n} |V(t_{p-1},t_p)| \leq B$ for each subdivision $\{t_p\}_{p=0}^{n}$ of $\{x,y\}$. Since V in OA gives $|V| \underline{<<} \alpha$ for some α in OA^+, $\sum_x^y |V| \leq \alpha(x,y)$ is immediate. Hence, $\beta(x,y) = \sum_x^y |V|$ is the "smallest" element of OA such that $|V| \underline{<<} \beta$.

We now have easily a sufficient condition on V in OA to assure that $\prod_x^y [I + V]$ is in I(X).

THEOREM 8

Suppose that V is in OA and let β denote $\sum_x^y |V|$ for each pair of numbers (x,y). If $\beta(x,x^+) < 1$ and

$\beta(x^-,x) < 1$ for each x, then $\prod_a^b [I + V]$ is in I(X) whenever $a \leq b$. If $\beta(x^+,x) < 1$ and $\beta(x,x^-) < 1$ for each x, then $\prod_a^b [I + V]$ is in I(X) whenever $a \geq b$.

<u>PROOF</u>: Let $\{t_p\}_{p=0}^{n}$ be the subdivision selected in the proof of Lemma 8.2. $\prod_a^b [I + V] = \prod_{p=1}^{n} \left(\prod_{t_{p-1}}^{t_p} [I + V] \right)$.

Theorem 3, the initial characterization of I(X), gives the desired result immediately.

CHAPTER V

CHARACTERIZATIONS OF THE IDENTITY COMPONENT

We now furnish two characterizations of the identity component. The first is a generalization of a result due to J. W. Neuberger and includes Newberger's result as a special case. The second unites information known about the identity component and about values of solutions of integral equations. Neither result has appeared in the literature.

We have first

THEOREM 9

Suppose that V is in OA with V(0,1) = I. Let M(V) denote the set to which F belongs only in case there is an element H in X such that $F = \prod_0^1 [I + VH]$, where VH denotes that element of OA defined to be V(x,y)H for each pair (x,y). If Q is in I(X), then Q is the L-uniform limit of a sequence in $\Pi M(V)$.

PROOF: Let $\beta(x,y)$ denote the variation of V from x to y. It follows that if B is in S, Q is in X, and $\{t_p\}_{p=0}^{n}$ is a subdivision of $\{x,y\}$, then

$$\| (I + V(0,1)Q)B - \prod_{p=1}^{n}[I + V(t_{p-1},t_p)Q]\, B \|$$

$$= \| (\sum_{p=1}^{n} V(t_{p-1},t_p)QB) + IB - \prod_{p=1}^{n}[I + V(t_{p-1},t_p)Q]\, B \|$$

$$= \| \sum_{p=1}^{n} \Big[(V(t_{p-1},t_p)QB) - V(t_{p-1},t_p)Q \cdot \prod_{k=p+1}^{n}[I + V(t_{k-1},t_k)Q]\, B\Big] \|$$

$$\leq \sum_{p=1}^{n} \Big[\beta(t_{p-1},t_p) \mid Q \mid \| (I - \prod_{k=p+1}^{n}[I + V(t_{k-1},t_k)Q])\, B \|\Big]$$

$$\leq \sum_{p=1}^{n} \Big[\beta(t_{p-1},t_p) \mid Q \mid \beta(t_p,1) \mid Q \mid \prod_{t_p}^{1}[1 + \mid Q \mid \beta]\ \| B \|\Big]$$

$$\leq \mid Q \mid^2 \beta(0,1)^2 \prod_{0}^{1}[1 + \mid Q \mid \beta]\ \| B \|.$$ Hence,

$$\| (I + Q)B - \prod_{0}^{1}[I + VQ]\, B \| \leq \mid Q \mid^2 \beta(0,1)^2 \prod_{0}^{1}[1 + \mid Q \mid \beta]\ \| B \|.$$

Now suppose that $\mid Q \mid < 1/2$. If n is a positive integer, $I + Q = \prod_{k=0}^{n-1}[(I + ((n-k)/n)Q)(I + ((n-k-1)/n)Q)^{-1}]$

and $\mid (I + (k/n)Q)(I + ((k-1)/n)Q)^{-1} - I \mid =$

$$| (I + (k/n)Q - I - ((k-1)/n)Q)(I + ((k-1)/n)Q)^{-1} |$$

$$\leq (1/n) | Q | | (I + ((k-1)/n)Q)^{-1} |$$

$$\leq (1/n) | Q | (1 - | Q |)^{-1} < (1/n).$$

Put $C_k = (I + ((n-k+1)/n)Q)(I + ((n-k)/n)Q)^{-1} - I$ for each k in $[1,n]$. Then if B is in S, we have

$$\| (I + Q)B - \prod_{k=1}^{n} [\prod_{0}^{1} [I + VC_k]] B \|$$

$$\leq \sum_{k=1}^{n} \Big[(\prod_{p=1}^{n-k} | I + C_p |) \| (I + C_{n-k+1} - \prod_{0}^{1} [I + VC_{n-k+1}]) \prod_{p=n-k+2}^{n} [\prod_{0}^{1} [I + VC_p]] B \| \Big]$$

$$\leq \| B \| \sum_{k=1}^{n} \Big[(\prod_{p=1}^{n-k} [1 + (1/n)]) | C_{n-k+1} |^2 \beta(0,1)^2 \cdot (\prod_{0}^{1} [1 + | C_{n-k+1} | \beta]) \prod_{p=n-k+2}^{n} [\prod_{0}^{1} [1 + | C_p | \beta]] \Big]$$

$$\leq \sum_{k=1}^{n} \Big[(1 + (1/n))^{n-k} (1/n)^2 \beta(0,1)^2 \exp[(1/n)\beta(0,1)] \cdot \exp[(k-1)\beta(0,1)/n] \| B \| \Big]$$

$$\leq (1/n)\beta(0,1)^2 \exp[1 + \beta(0,1)] \| B \| .$$

Hence, the sequence $\{\prod_{k=1}^{n} [\prod_{0}^{1} [I + VC_k]]\}_{n=1}^{+\infty}$ has L-uniform limit $(I + Q)$. Theorem 9 now follows immediately from the initial characterization of $I(X)$ given in Theorem 3.

Note that if, in Theorem 9, we have $|V(x,x^+)| < 1$ and $|V(x^-,x)| < 1$ for each x, then each element of the approximating sequence is in I(X) by Corollary 5.3. Hence we have a partial characterization of I(X).

J. W. Neuberger's result now comes as a corollary to Theorem 9. If we put $V(x,y) = (y - x)I$ in Theorem 9, we have

COROLLARY 9.1

Suppose that H is in I(X). Then H can be written as the L-uniform limit of a sequence whose elements are finite products of elements of the form Exp (Q), where $(\text{Exp}(Q))B = \lim_{n \to +\infty}(I + (1/n)Q)^n B$ for each B in S.

It is important to note that the characterization given in Theorem 9 is only partial, in the sense that the following conjecture due to Neuberger remains unanswered:

Conjecture Suppose that H is in X, that H^{-1} exists, and that H is the L-uniform limit of a sequence in I(X). Then H is in I(X).

This difficulty is avoided altogether in the next theorem.

THEOREM 10

Suppose that H is in **X**. These are equivalent:

(i) There is a V in OA with V(,0) continuous and with $H = {}_1\prod^0 [I + V]$.

(ii) H is in I(X).

PROOF: Theorem 8 gives that (i) implies (ii). If Q is in X and $\|Q\| < 1$, we may put $W(x,y) = (I + xQ)(I + yQ)^{-1}$ for each of x and y in [0,1]. Then if z lies between x and y, $W(x,z)W(z,y) = W(x,y)$. Moreover, we have that

$\|W(x,y) - I\| = |(I + xQ)(I + yQ)^{-1} - (I + yQ)(I + yQ)^{-1}|$

$= \|(x - y)Q(I + yQ)^{-1}\| \leq |x - y||Q|(1 - |yQ|)^{-1}$

$\leq (1 - |yQ|)^{-1}(1 + |x - y||Q|) - 1$

$\leq (1 - |Q|)^{-1} \exp[\,|x - y|\|Q\|\,] - 1$. Hence W is in OM.

Further, if V = L(W), V(,0) is continuous by Lemma 8.1.

Finally, if H is in I(X), then $H = \prod_{p=1}^{n} [I + Q_p]$ for some sequence $\{Q_p\}_{p=0}^{n}$ in X with $|Q_p\| < 1$ for each p.

In similar fashion to the above, we may construct a W in OM such that $W(1 - (p-1)/n, 1 - p/n) = (I + Q_p)$ for each p.

Again, if V = L(W), then V(,0) is continuous.

In view of Theorem 7, we may interpret Theorem 10 as follows:

COROLLARY 10.1

Consider the system

$$(**) \quad U(x) = I + (R)\int_x^0 VU\,, \quad U(1) = H.$$

If H is in I(X), then there is a V in OA with V(,0) continuous such that (**) has a solution. If H is not in I(X), no such V exists.

CHAPTER VI

LOGARITHMS, EXPONENTIALS, AND THE IDENTITY COMPONENT IN LINEAR SPACES

In case X is a collection of linear functions, we can sharpen our results considerably. We first put $H(Q)(t) = \sum_{n=0}^{+\infty}[(1/n!)t^nQ^n]$ for each Q in X and note that the convergence is absolute in the norm of X. Now, if $V(x,y) = (x - y)Q$, then $U(t) = \text{Exp}(tQ)$ is the solution to

$$(*) \quad U(t) = I + (R)\int_t^0 VU.$$

If we observe that $I + (R)\int_t^0 V[H(Q)]$

$$= I + \int_t^0 (-dx)Q\sum_{n=0}^{+\infty}[(1/n!)x^nQ^n] = I + \sum_{n=0}^{+\infty}[\int_t^0 (-dx)x^nQ^{n+1}/n!]$$

$= I + \sum_{n=1}^{+\infty}(1/n!)t^nQ^n = H(Q)(t)$, Theorem 7 gives then that $\text{Exp}Q = \lim_{n\to+\infty}(I + (1/n)Q)^n = \sum_{n=0}^{+\infty}[(1/n!)Q^n]$. It is, further, easy to show that Exp( ) is a continuous function from X to X such that whenever AB = BA, Exp(A + B) = Exp(A)Exp(B). Moreover, we may put

$$\mathrm{Ln}(I - Q) = \lim_{n\to+\infty}\sum_{k=0}^{n}[(I - ((n-k)/n)Q)(I - ((n-k-1)/n)Q)^{-1} - I],$$

and the convergence involved is in norm from the proof of Theorem 10 and the linearity of X. Using further the ideas of the proofs of Theorems 9 and 10, we have

THEOREM 11

If Q is in X, X is linear, and $|Q| < 1$, then $\mathrm{Exp}(\mathrm{Ln}(I - Q)) = I - Q$.

PROOF: In a linear space we have $\mathrm{Exp}Q = \sum_{n=0}^{+\infty}[(1/n!)Q^n]$.

It follows that $|\mathrm{Exp}Q - (I + Q)| \leq (1/2)|Q|^2\exp(|Q|)$.

If n is a positive integer, we then have

$$|\mathrm{Exp}[(I - ((n-k)/n)Q)(I - ((n-k-1)/n)Q)^{-1} - I]$$
$$- (I - ((n-k)/n)Q)(I - ((n-k-1)/n)Q)^{-1}|$$
$$= |\mathrm{Exp}[(1/n)Q(I - ((n-k-1)/n)Q)^{-1}]$$
$$- (I + (1/n)Q(I - ((n-k-1)/n)Q)^{-1})|$$
$$\leq (1/2)(1/n)^2|Q|^2|(I - ((n-k-1)/n)Q)^{-1}|^2 \cdot$$
$$\exp[|Q|(1/n)|(I - ((n-k-1)/n)Q)^{-1}|]$$
$$\leq (1/2)(1/n)^2|Q|^2(1 - |Q|)^{-2}\exp[|Q|/n(1 - |Q|)]$$

for each k in [0,n-1]. Now the terms of the sequence $\{(I - ((n-k)/n)Q)(I - ((n-k-1)/n)Q)^{-1} - I\}_{k=0}^{n-1}$ commute with each other under multiplication. Hence we have that if

$C_k = (I - ((n-k)/n)Q)(I - ((n-k-1)/n)Q)^{-1}$, then

$$|\,\mathrm{Exp}(\sum_{k=0}^{n-1}[C_k - I]) - \prod_{k=0}^{n-1} C_k\,| = |\,\prod_{k=0}^{n-1}\mathrm{Exp}(C_k - I) - \prod_{k=0}^{n-1} C_k\,|$$

$$\leq \sum_{p=1}^{n} (\prod_{k=0}^{n-p-1} |\,\mathrm{Exp}(C_k - I)\,|)(\,|\,\mathrm{Exp}(C_{n-p} - I) - C_{n-p}\,|\,) \cdot$$

$$(\prod_{k=n-p+1}^{n-1} |\,C_k\,|)$$

$$\leq \sum_{p=1}^{n} (\prod_{k=0}^{n-p-1}[\exp((1/n)\,|\,Q\,|\,(1 - |\,Q\,|)^{-1}])((1/n)^2(1/2) \cdot$$

$$|\,Q\,|^2(1 - |\,Q\,|)^{-2}\exp[(1/n)\,|\,Q\,|\,(1 - |\,Q\,|)^{-1}]) \cdot$$

$$(\prod_{k=n-p+1}^{n-1} |\,C_k\,|)$$

$$\leq (1/n)\,|\,Q\,|^2(1 - |\,Q\,|)^{-2}\exp[\,|\,Q\,|\,(1 - |\,Q\,|)^{-1}](1/2),$$

using linearity of X and Lemma 5.3.

Continuity of Exp( ) now gives the theorem.

We may now appeal to Theorem 3, the initial characterization of I(X), to obtain a sharper characterization of I(X) for complete spaces of linear functions.

<u>THEOREM 12</u>

If X is a complete space of linear functions from S to S, then H is in I(X) only in case there is a sequence

$\{Q_p\}_{p=1}^{n}$ in X such that $H = \prod_{p=1}^{n} \mathrm{Exp}(Q_p)$.

In the case that the underlying space S is a finite dimensional vector space over the complex field and X is the collection of all linear transformations from S to S, we have an even stronger result from functional analysis. If we put $\mathrm{Ln}(Q) = (1/2\pi i) \int_C (zI - Q)^{-1} \ln(z)\, dz$, where C is any rectifiable simple closed curve, containing in its interior all characteristic roots of Q but not the origin, we then have that $\mathrm{Exp}(\mathrm{Ln}(Q)) = Q$ for each invertible Q. In this setting one then has $I(X) = \mathrm{Exp}(X) = \{$all Q in X such that Q^{-1} exists$\}$ [1].

We might hope that this result would generalize at least to finite dimensional spaces X of linear functions. Such, however, is not the case.

If X is the collection of all real 2 by 2 matrices, we can show that $I(X) = \{$all M in X such that $\det(M) > 0\}$. Now, if A is in X, then $Y(t) = \mathrm{Exp}(tA)$ is the solution to the matrix differential equation $Y' = AY$, $Y(0) = I$. This we shall use to show that $\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ is not an exponential.

We consider $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. If $b = 0$, we have then that $v_1' = av_1$. If v_1 is the upper left entry of of the solution of $Y' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} Y$, $Y(0) = I$, we then have $v_1(t) = \exp(at)$, and $v_1(1) \neq -2$. Similarly, we may assume that $c \neq 0$. We solve the coupled system of differential equations to obtain

$$v_k'' - (\text{Tr } A)v_k' + (\det A)v_k = 0, \quad k = 1, 2;$$

$$v_2 = (1/b)(v_1' - av_1); \text{ and}$$

$$v_1 = (1/c)(v_2' - dv_2) \quad —$$

where A denotes the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let B denote $(1/2)\text{Tr } A$. There are three cases. If $Y' = AY$, $Y(0) = I$, and $B^2 = \det A$, it then follows that

$Y(t) = e^{Bt} \begin{bmatrix} (B-d)t + 1 & bt \\ ct & (B-a)t + 1 \end{bmatrix}$. Since neither b nor c is 0, $Y(1) \neq \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$. If $B^2 < \det A$, put $\varepsilon = \sqrt{\det A - B^2}$. The solution Y is then given by

$$Y(t) = e^{Bt} \begin{bmatrix} ((B-d)/\varepsilon)\sin(\varepsilon t) + \cos(\varepsilon t) & (b/\varepsilon)\sin(\varepsilon t) \\ (c/\varepsilon)\sin(\varepsilon t) & ((B-a)/\varepsilon)\sin(\varepsilon t) + \cos(\varepsilon t) \end{bmatrix}.$$

Now $\sin(\varepsilon) = 0$ forces the diagonal entries of $Y(1)$ to be the same. Finally, if $B^2 > \det A$, put $\varepsilon = \sqrt{B^2 - \det A}$. Then we have that the solution matrix $Y(t)$ is given by

$$e^{(B-\varepsilon)t} \begin{bmatrix} ((\varepsilon+B-a)/2\varepsilon) + ((\varepsilon+a-B)/2\varepsilon)e^{2\varepsilon t} & (b/2\varepsilon)(e^{2\varepsilon t} - 1) \\ (c/2\varepsilon)(e^{2\varepsilon t} - 1) & ((\varepsilon+B-d)/2\varepsilon) + ((\varepsilon+d-B)/2\varepsilon)e^{2\varepsilon t} \end{bmatrix}.$$

Since neither b nor c is 0, and since $(e^{2\varepsilon t} - 1)$ is monotonic, $Y(1)$ cannot be $\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$. Hence $\mathrm{Exp}(X)$ is properly contained in $I(X)$.

CHAPTER VII

CONCLUDING REMARKS

We may summarize as follows:

If X is an L-uniformly complete near-ring of Lipschitz transformations from a Banach space S to S, then

(i) The identity component I(X) is precisely that subset of X to which H belongs only in case H can be written as a finite product of elements of X whose normed distance from I is less than 1 (Theorem 3).

(ii) If V is in OA with V(0,1) = I, and if M(V) is the subset of X to which Q belongs only in case there is an element P of X such that $Q = \prod_0^1 [I + VP]$, then each H in I(X) can be written as the L-uniform limit of a sequence, each of whose members is a finite product of elements of M(V) (Theorem 9). Most notably, if in Theorem 9 we put V(x,y) = (y - x)I, we then have Neuberger's characterizations. Each element of I(X) can be written as the L-uniform limit of a sequence whose members are finite products of exponentials in X. If X is a collection of linear functions, then each element of I(X) can be written precisely as a finite product of exponentials.

(iii) H is in I(X) only in case H lies in the range of the solution U to some integral equation of the form $U(x) = I + (R)\int_x^0 VU$, with V in OA and with V(,0) continuous. A change of scale gives this version of Corollary 10.1.

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