SOME REPRESENTATION THEOREMS IN ANALYSIS

A THESIS

Presented to

The Faculty of the Graduate Division

by

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

June, 1964

SOME REPRESENTATION THEOREMS IN ANALYSIS

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Date approved by Chairman: May 20, 1964

ACKNOWLEDGMENTS

I wish to express my sincere gratitude to Dr. Eric R. Immel, my thesis advisor, for his watchful guidance and expert help in the preparation of this thesis. I wish to thank the other two members of my reading committee, Dr. James A. Walker and Dr. Joseph Ford, for their many helpful suggestions. I also wish to thank Mrs. Peggy Weldon for her excellent and prompt typing of this thesis. In conclusion, I wish to express my heartfelt appreciation to Dr. Marvin B. Sledd for his advice, encouragement and administrative assistance, without which this undertaking would not have been possible.

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CHAPTER I

INTRODUCTION

This study is devoted largely to a presentation of three theorems of modern analysis: The Stone-Weierstrass approximation theorem, the Stone representation theorem for Boolean algebras, and a representation for L- spaces of abstract measure spaces based on a theorem by Kakutani. The proofs of these theorems illustrate some applications of topology and algebra to analysis.

Chapter II contains an expository account of the Stone-Weierstrass theorem. In the theorem necessary and sufficient conditions are given that a continuous real-valued function defined on a compact topological space be the uniform limit of a sequence of linear combinations of products of functions in some given set of continuous functions on the space. The analog of the theorem for the complex case is presented, as well as the extension of the theorem to locally compact spaces. The proof, a modified version of that given by Stone, makes use of minimality arguments applied repeatedly to lattices, vector lattices, and algebras of functions.

Chapter III contains a proof of the Stone representation theorem for Boolean algebras. The Stone representation theorem states that given any Boolean algebra there exists a totally disconnected, compact Hausdorff space such that the original Boolean algebra is a lattice isomorphic image of the Boolean algebra of all open-closed subsets of the space. The proof is a modification of that found in WALLMAN [1]. The usefulness of this theorem first becomes evident in Chapter IV, where it plays a major role in the proof of the representation theorem of that chapter.

If $(X, \mathcal{A}, \mu)$ is a complete measure space, the L space of the measure space is the space of all equivalence classes of functions whose integrals, with respect to the measure μ, exist and are finite. Two functions are said to be equivalent if they agree almost everywhere on X with respect to μ. In Chapter IV it is proved that the L space of any complete measure space is isometric and isomorphic, as a normed linear space, to the L space of a complete measure space whose σ-ring of measurable sets is a completion of the Baire σ-ring in some locally compact Hausdorff space. This measure space has the property that all continuous functions which vanish at infinity are measurable.

The statement and proof of this theorem are based upon a much more general theorem of Kakutani. In his original paper, Kakutani proves that certain types of partially ordered Banach spaces are isomorphic and isometric, as normed linear spaces, to L spaces of complete measure spaces in locally compact, totally disconnected Hausdorff spaces.

The representation theorem of Chapter IV is less general than the original theorem of Kakutani. For this reason a more elementary proof is possible. The theorem of Chapter IV is formulated in measure-theoretic terms and is proved largely by appeals to measure-theoretic properties of the spaces concerned. Of necessity, the proof of the original theorem employs an elaborate superstructure manufactured from algebraic and order-theoretic properties of the space to be represented. The present proof does not seem to appear anywhere else in the literature.

All lattice-theoretic terms are defined either in the text or in the glossary. Non-standard definitions from topology and real variable theory are also included. Symbols such as (x.y), where x and y are integers, are used to refer to theorems, lemmas and remarks. The first integer denotes the chapter in which the numbered item appears. The second integer indicates the position of the item within the chapter. The end of a proof is indicated by the symbol ∎. The symbol $\Longrightarrow$ is to be read "implies," and the symbol $\Longleftrightarrow$ is to be read "if and only if."

CHAPTER II

THE STONE-WEIERSTRASS APPROXIMATION THEOREM

This chapter contains an expository account of the Stone-Weierstrass approximation theorem. The theorem, a generalization of the classical approximation theorem of Weierstrass, was first proved by Stone in 1937. The theorems and proofs presented in this chapter are modified versions of those in STONE [1].

The Weierstrass approximation theorem states that any continuous function defined on a closed and bounded interval of the real line is the uniform limit of a sequence of polynomials. The setting for the Stone-Weierstrass theorem is a compact topological space X. In the theorem necessary and sufficient conditions are given that a continuous function on X be the uniform limit of a sequence of linear combinations of products of functions in some given set of continuous functions on X.

Both the real and the complex cases of the theorem are presented. In the latter part of the chapter the extension of the Stone-Weierstrass theorem to locally compact spaces is given.

The terms lattice, vector lattice, algebra, topological space, compact, and locally compact are used throughout this chapter. Definitions of these terms appear in the glossary.

(2.1) Definitions and Conventions

Let X be a compact topological space and let $C_r(X)$ denote the space of all real-valued functions defined and continuous on X.

For $f, g \in C_r(X)$ the symbol $f \leq g$ will mean $f(x) \leqq g(x)$ for all $x \in X$. Under the relation $\leqq$ defined above, the space $C_r(X)$ is a partially ordered set. If $f, g \in C_r(X)$ then the function $(f \wedge g)$ defined on X by $(f \wedge g)(x) = \min.\ (f(x), g(x))$ is clearly the greatest lower bound in $C_r(X)$ for f and g. The function $f \vee g$ defined on X by $(f \vee g)(x) = \max.\ (f(x), g(x))$ is the least upper bound for f and g in $C_r(X)$. Thus $C_r(X)$ is a lattice and, with algebraic operations defined pointwise, $C_r(X)$ is a vector lattice and a real algebra.

For each $f \in C_r(X)$, let $\|f\| = \sup \{|f(x)| : x \in X\}$. A short computation shows that the function $\| \ \|$ on $C_r(X)$ is a norm on $C_r(X)$. In the topology induced on $C_r(X)$ by the norm $\| \ \|$, $C_r(X)$ is a closed lattice, a closed vector lattice, and a closed algebra. Whenever closed lattices, vector lattices or algebras are mentioned in this chapter it will be tacitly assumed that the word "closed" means closed with respect to the topology induced on $C_r(X)$ by the norm $\| \ \|$.

The proofs of all the assertions made above are straightforward computations based on definitions listed in the glossary.

If A is any non-empty subset of $C_r(X)$, let

$L(A)$ be the intersection of all sublattices of $C_r(X)$ containing A,

$V(A)$ be the intersection of all vector sublattices of $C_r(X)$ containing A,

$\mathfrak{a}(A)$ be the intersection of all subalgebras of $C_r(X)$ containing A,

$\bar{L}(A)$ be the intersection of all closed sublattices of $C_r(X)$ containing A,

$\bar{V}(A)$ be the intersection of all closed vector sublattices of $C_r(X)$ containing A, and

$\bar{\mathfrak{a}}(A)$ be the intersection of all closed subalgebras of $C_r(X)$ containing A.

It is clear that $L(A)$ is a sublattice of $C_r(X)$; $V(A)$ is a vector sublattice of $C_r(X)$; $\mathcal{A}(A)$ is a subalgebra of $C_r(X)$; $\overline{L}(A)$ is a closed sublattice of $C_r(X)$; $\overline{V}(A)$ is a closed vector sublattice of $C_r(X)$; and $\overline{\mathcal{A}}(A)$ is a closed subalgebra of $C_r(X)$. Each of these sets contains A, and each is minimal in the following sense: If L' is a sublattice of $C_r(X)$ and $L' \supseteq A$, then $L' \supseteq L(A)$. Analogous assertions are valid for $V(A)$, $\mathcal{A}(A)$, $\overline{L}(A)$, $\overline{V}(A)$, and $\overline{\mathcal{A}}(A)$.

(2.2) <u>Definition</u>: A non-empty subset A of $C_r(X)$ has <u>separation property</u> I if and only if: Given $x, y \in X$ with $x \neq y$, and any real numbers a, b, there exists a function $f \in A$ such that $f(x) = a$ and $f(y) = b$.

A non-empty subset B of $C_r(X)$ has <u>separation property</u> II if and only if: Given $x, y \in X$ with $x \neq y$, there exists a function $f \in A$ such that $f(x) \neq f(y)$.

(2.3) <u>Definition</u>: If A is a non-empty subset of $C_r(X)$, the set of all <u>non-negative linear relations</u> satisfied by A is denoted by $\Lambda^+(A)$, and defined to be the following subset of $(X \times X) \times (R \times R)$:[1]

$$\Lambda^+(A) = \left\{ \begin{array}{l} (x, y; r, r') : x, y \in X;\ r, r' \in R;\ r \cdot r' \geq 0 \text{ and} \\ \qquad\qquad r\, f(x) = r' f(y) \quad \text{for all } f \in A\ . \end{array} \right\}$$

(2.4) <u>Definition</u>: Let A be a non-empty subset of $C_r(X)$, and let $x, y \in X$. Let

$$A(x, y) = \{(f(x), f(y)) : f \in A\}\ ,$$

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[1]In this chapter the letter R will denote the real line. The symbol $R \times R$ then denotes the Cartesian product of the real line with itself.

and

$[Q(A)](x, y) = \{(f(x), f(y)) : f \in Q\}$, where Q is any one of the symbols $L, V, \mathcal{A}, \bar{L}, \bar{V}, \bar{\mathcal{A}}$. Note that $A(x, y) \subseteq R \times R$ and $[Q(A)](x, y) \subseteq R \times R$.

(2.5) Lemma: Let the order relation "$\leqq$" be defined on $R \times R$ by

$$(a, b) \leqq (c, d) \Longleftrightarrow \quad a \leqq c \text{ and } b \leqq d ,$$

where (a, b) and (c, d) are points in $R \times R$. If (a, b), $(c, d) \in R \times R$ then

$$\text{g.l.b.}\{(a, b), (c, d)\} = (\text{min.}\{a, b\}, \text{min.}\{c, d\})$$
$$\text{l.u.b.}\{(a, b), (c, d)\} = (\text{max.}\{a, b\}, \text{max.}\{c, d\}) .$$

The vector space $R \times R$ with its usual topology becomes a closed vector lattice in which

(2.5.1) $$(a, b) \wedge (c, d) = (\text{min.} \{a, b\}, \text{min.} \{c, d\}),$$

(2.5.2) $$(a, b) \vee (c, d) = (\text{max.} \{a, b\}, \text{max.} \{c, d\})$$

are valid for all $(a, b), (c, d) \in R \times R$.

The proof is a straightforward computation and is omitted.

(2.6) Lemma: Every closed subalgebra of $C_r(X)$ is also a closed vector sublattice of $C_r(X)$.

Proof: Let $\bar{\mathcal{A}}$ be a closed subalgebra of $C_r(X)$. Since $\bar{\mathcal{A}}$ is, by definition, a closed vector subspace of $C_r(X)$, it suffices to show that $\bar{\mathcal{A}}$ is a sublattice of $C_r(X)$. Since for $f, g \in C_r(X)$, and for each $x \in X$,

(2.6.1) $$(f \vee g)(x) = \max(f(x), g(x)) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2},$$

(2.6.2) $(f \wedge g)(x) = \min.\ (f(x), g(x)) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$,

it suffices to show that if $f \varepsilon \bar{\mathcal{A}}$, then the function $|f|$, defined on X by $|f|(x) = |f(x)|$, is also in $\bar{\mathcal{A}}$.

Let $\varepsilon > 0$ be given. Since $|t|$ is a continuous function of the real variable t, by the Weierstrass approximation theorem there exists a polynomial P' with the property that $|\,|t| - P'(t)\,| < \frac{\varepsilon}{2}$ for every t in the closed interval $[-\|f\|, \|f\|]$. If P is the polynomial which results from replacing the constant term $P'(0)$ of P' by 0, then P is a polynomial with 0 as its constant term which has the property that $|\,|f| - P(t)| < \varepsilon$ for every t in $[-\|t\|, \|f\|]$. Since $\bar{\mathcal{A}}$ is an algebra, the function $P(f)$ in $C_r(X)$ is in $\bar{\mathcal{A}}$.

By the way P was selected $|\,|f(x)| - P(f(x))| < \varepsilon$ for all $x \varepsilon X$, and from this it follows that $\|\,|f| - P(f)\| < \varepsilon$. This implies that if $f \varepsilon \bar{\mathcal{A}}$ then $|f|$ is a limit point of $\bar{\mathcal{A}}$ and since $\bar{\mathcal{A}}$ is closed, $|f| \varepsilon \bar{\mathcal{A}}$. ∎

(2.7) <u>Lemma</u>. If A is a non-empty subset of $C_r(X)$, then for any given $x, y \varepsilon X$,

(2.7.1) $[\bar{V}(A)](x,y) = \left\{(\alpha,\beta) : r_{x,y}\alpha = r'_{x,y}\beta \text{ for some fixed } r_{x,y}, r'_{x,y} \text{ such that } r_{x,y} \cdot r'_{x,y} \geq 0\right\}$

<u>Proof</u>: It will first be shown that $[\bar{V}(A)](x,y)$ is a vector sublattice of $R \times R$.

Let $(\alpha,\beta), (\gamma,\delta) \varepsilon [\bar{V}(A)](x,y)$. Then $\alpha = f(x)$, $\beta = f(y)$, $\gamma = g(x)$ and $\delta = g(y)$, for some $f, g \varepsilon \bar{V}(A)$. Thus $(\alpha + \gamma, \beta + \delta) = ((f + g)x, (f + g)y) \varepsilon [\bar{V}(A)](x,y)$. If λ is real, then $(\lambda\alpha, \lambda\beta) =$

$= ((\lambda f)x, (\lambda f)y) \varepsilon [\bar{V}(A)](x,y)$, so that $[\bar{V}(A)](x,y)$ is a vector subspace of $R \times R$. Also

$$(\alpha, \beta) \vee (\gamma, \delta) = (\max.(f(x), g(x)), \max.(f(y), g(y)) =$$

$$= ((f \vee g)\, x, (f \vee g)y) \varepsilon [\bar{V}(A)](x, y),$$

and similarly $(\alpha, \beta) \wedge (\gamma, \delta) \varepsilon [\bar{V}(A)](x, y)$. It follows that $[\bar{V}(A)](x,y)$ is a vector sublattice of $R \times R$.

Since $[\bar{V}(A)](x,y)$ is a vector subspace of $R \times R$ it must have the form

$$(2.7.2) \quad [\bar{V}(A)](x, y) = \{(\alpha, \beta) : r_{x,y}\alpha = r'_{x,y}\beta, \text{ for some fixed real } r_{x,y}, r'_{x,y}\}.$$

Since $\bar{V}(A)$ is a vector sublattice of $C_r(X)$, it contains the zero function. Hence, if $f \varepsilon \bar{V}(A)$, then $|f| = 2(f \vee 0) - f$ so that $|f| \varepsilon \bar{V}(A)$. This implies that if $(\alpha, \beta) \varepsilon [\bar{V}(A)](x,y)$ then $(|\alpha|, |\beta|) \varepsilon [\bar{V}(A)](x, y)$. With $r_{x,y}, r'_{x,y}$ as in (2.7.2), if $r_{x,y} \cdot r'_{x,y} < 0$ then $(r'_{x,y}, r_{x,y}) \varepsilon [\bar{V}(A)](x,y)$ and hence $(|r'_{x,y}|, |r_{x,y}|) \varepsilon [\bar{V}(A)](x,y)$. This would imply that $r_{x,y}|r'_{x,y}| = r'_{x,y}|r_{x,y}|$, thus contradicting the assumption that $r_{x,y} \cdot r'_{x,y} < 0$. It follows that $r_{x,y} \cdot r'_{x,y} \geq 0$ must be the case, and hence that $[\bar{V}(A)](x, y)$ must have the form (2.7.1). ∎

(2.8) <u>Lemma</u>: If A is a non-empty subset of $C_r(X)$, then $\bar{L}(A) = \overline{L(A)}$ and $\bar{\mathcal{A}}(A) = \overline{\mathcal{A}(A)}$. [The symbols $\overline{L(A)}$ and $\overline{\mathcal{A}(A)}$ denote the closures of the sets $L(A)$ and $\mathcal{A}(A)$ in the norm topology on $C_r(x)$.]

<u>Proof</u>: If $f, g \varepsilon \overline{L(A)}$, then there exist sequences $\{f_n\} \subseteq L(A)$ and $\{g_n\} \subseteq L(A)$ such that $f_n \xrightarrow{U} f$ and $g_n \xrightarrow{U} g$. Since

$$(f_n \vee g_n)(x) = f_n(x) \vee g_n(x) = \max(f_n(x),\ g_n(x)) = \frac{f_n(x) + g_n(x) + |f_n(x)-g_n(x)|}{2},$$

it follows that $f_n \vee g_n \xrightarrow{U} \frac{f+g}{2} + \frac{|f-g|}{2} = f \vee g$. Similarly $(f_n \wedge g_n) \xrightarrow{U} f \wedge g$. Thus $f \vee g$ and $f \wedge g$ are limits of sequences of members of $L(A)$, and hence $f \vee g$ and $f \wedge g$ are in $\overline{L(A)}$. By the preceding argument, it follows that $\overline{L(A)}$ is a lattice. Since $\overline{L(A)}$ is closed and contains A, $\overline{L(A)}$ contains $\bar{L}(A)$. However $\bar{L}(A)$ is a lattice containing A, so that $\bar{L}(A)$ contains $L(A)$. Since $\bar{L}(A)$ is closed this implies that $\bar{L}(A) = \overline{\bar{L}(A)} \supseteq \overline{L(A)}$. Thus $\bar{L}(A) \supseteq \overline{L(A)} \supseteq \bar{L}(A)$, and hence $\bar{L}(A) = \overline{L(A)}$.

By an analogous argument $\bar{\mathfrak{a}}(A) = \overline{\mathfrak{a}(A)}$. ∎

(2.9) <u>Lemma</u>: Let A be a non-empty subset of $C_r(X)$. Then $\bar{L}(A) = \{f : f \in C_r(X)$ and, given $\varepsilon > 0$ and $x,y \in X$, there exists a function $f_{x,y} \in L(A)$ such that $|f(x) - f_{x,y}(x)| < \varepsilon$ and $|f(y) - f_{x,y}(y)| < \varepsilon\}$

<u>Proof</u>: Let the set on the right be denoted by S. Since by Lemma (2.8) $\bar{L}(A) = \overline{L(A)}$, it follows that $\bar{L}(A) \subseteq S$. Now let $f \in S$; let $\varepsilon > 0$ be given; let $x \in X$ be fixed; and for each $y \in X$ let G_y denote the open set

$$G_y = \{z : f(z) - f_{x,y}(z) < \varepsilon\}.$$

Since $f \in S$, it follows that $x, y \in G_y$ and hence that $X = \bigcup_{y \in X} G_y$. By compactness of X there exist points $y_1, \ldots, y_n$ such that $\bigcup_{i=1}^{n} G_{y_i} = X$. Let $g_x = \bigvee_{i=1}^{n} f_{xy_i}$. If $z \in X$ then $z \in G_{y_k}$ for some $k = 1,2,\ldots,n$, and hence $g_x(z) \geq f_{xy_k}(z) > f(z) - \varepsilon$. Since $f \in S$, $|f(x) - f_{xy}(x)| < \varepsilon$ for each $y \in X$, and hence $f_{xy_i}(x) < f(x) + \varepsilon$ for each $i = 1,\ldots,n$. This implies $g_x(x) < f(x) + \varepsilon$.

For each $x \in X$ let H_x denote the open set

$$H_x = \left\{ z : g_x(z) < f(z) + \varepsilon \right\}.$$

By the argument just given $x \in H_x$ for each $x \in X$ and hence $\bigcup_{x \in X} H_x = X$. By compactness of X there exist points $x_1,\ldots,x_m$ such that $\bigcup_{i=1}^{m} H_{x_i} = X$.

Let $h = \bigwedge_{i=1}^{m} g_{x_i}$. If $z \in X$ then $z \in H_{x_k}$ for some $k = 1,2,\ldots,m$, and hence $h(z) \leq g_{x_k}(z) < f(z) + \varepsilon$. Since by the preceding $g_x(z) > f(z) - \varepsilon$ for all $z \in X$ and for any given $x \in X$, it follows that $h(z) > f(z) - \varepsilon$ for all $z \in X$. This implies $f(z) - \varepsilon < h(z) < f(z) + \varepsilon$ for all $z \in X$, and hence $|f(z) - h(z)| < \varepsilon$ for all $z \in X$.

Since $h = \bigwedge_{i=1}^{m} g_{x_i} = \bigwedge_{i=1}^{m} (\bigvee_{j=1}^{n} f_{x_i y_j})$, with $f_{x_i,y_j} \in L(A)$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$, it follows that $h \in L(A)$. Since ε is an arbitrary positive number this implies that $f \in \overline{L(A)}$ and by Lemma (2.8) $f \in \bar{L}(A)$. Thus $S \subseteq \bar{L}(A)$ and, in view of the preceding, $S = \bar{L}(A)$. ∎

(2.9.1) <u>Corollary</u>: If $L(A)$ has separation property (I), then $\bar{L}(A) = C_r(X)$.

<u>Proof</u>: Let $f \in C_r(X)$ and let $x, y \in X$. By definition (2.2) there exists a function $f_{x,y} \in L(A)$ such that $f_{x,y}(x) = f(x)$ and $f_{x,y}(y) = f(y)$.

(2.9.2) <u>Corollary</u>: $\bar{L}(A) = \left\{ f : (f(x), f(y)) \in \overline{[L(A)](x,y)} \text{ for all } x, y \in X \text{ such that } x \neq y \right\}$.

Proof: If $f \in \bar{L}(A)$, then $(f(x), f(y)) \in [L(A)](x,y) \subseteq \overline{[L(A)](x,y)}$ for all $x, y \in X$. Conversely, if $(f(x), f(y)) \in \overline{[L(A)](x,y)}$ for all $x, y \in X$ such that $x \neq y$, then given $\varepsilon > 0$ and $x, y \in X$, there exists a function $f_{x,y} \in L(A)$ such that $|f(x) - f_{x,y}(x)| < \varepsilon$ and $|f(y) - f_{x,y}(y)| < \varepsilon$. By Lemma (2.9), $f \in \bar{L}(A)$. ∎

(2.9.3) Corollary: If $\bar{L}(A) = A$, then

$$A = \left\{f : f \in C_r(X),\ (f(x), f(y)) \in A(x,y) \text{ for all } x, y \in X \text{ such that } x \neq y\right\}.$$

Proof: Let the set on the right in the above equation be denoted by B. By definition (2.4), $A \subseteq B$. Let $f \in B$. Then

$$(f(x), f(y)) \in A(x,y) \subseteq [L(A)](x,y) \subseteq \overline{[L(A)](x,y)}$$

for all $x, y \in X$, such that $x \neq y$. By Corollary (2.9.2), $f \in \bar{L}(A) = A$. ∎

(2.10) Theorem $\bar{V}(A) = \left\{f : f \in C_r(X) \text{ and } \Lambda^+(\{f\}) \supseteq \Lambda^+(A)\right\}$.

Proof: It will first be shown that $\Lambda^+(\bar{V}(A)) = \Lambda^+(A)$. Since $A \subseteq \bar{V}(A)$, it follows that $\Lambda^+(\bar{V}(A)) \subseteq \Lambda^+(A)$. If $\Lambda^+(A)$ is empty this part of the proof is complete. Otherwise let

$$V = \left\{f : f \in C_r(X) \text{ and } \Lambda^+(\{f\}) \supseteq \Lambda^+(A)\right\}.$$

Now let $(x, y; r, r') \in \Lambda^+(A)$, and let $f, g \in V$. Then $(x, y; r,r') \in \Lambda^+(A) \subseteq \Lambda^+(\{f\}) \cap \Lambda^+(\{g\})$, so that $rf(x) = r'f(y)$ and $rg(x) = r'g(y)$. Hence for any real scalars α and β

$$r(\alpha f + \beta g)(x) = r(\alpha f(x) + \beta g(x)) = r'(\alpha f(y) + \beta g(y)) = r'(\alpha f + \beta g)(y).$$

It follows that $\alpha f + \beta g \varepsilon V$. If $\{f_n\}_{n=1}^{\infty} \subseteq V$ and $f_n \xrightarrow{U} f$, then $rf_n(x) = r'f_n(y)$ for every n, and hence $rf(x) = r'f(y)$. Thus $f \varepsilon V$. This implies that V is a closed vector subspace of $C_r(X)$.

Now recall that, by Definition (2.3), $r \cdot r' \geq 0$. If $f, g \varepsilon V$ and $r \geq 0$, then $r(f \vee g)(x) = r \cdot \max(f(x), g(x)) = \max(r \cdot f(x), r \cdot g(x) = \max(r'f(y), r'g(y)) = r' \cdot \max(f(y), g(y)) = r'(f \vee g)(y)$. If $f, g \varepsilon V$ and $r \leq 0$, then $r \cdot (f \vee g)x = r \cdot \max(f(x), g(x)) = \min(r f(x), r g(x) = \min(r'f(y), r'g(y)) = r'\max(f(y), g(y)) = r'(f \vee g)(y)$. Thus if $f, g \varepsilon V$, then $f \vee g \varepsilon V$. A similar argument implies that if $f, g \varepsilon V$, then $f \wedge g \varepsilon V$. It follows that V is a closed vector sublattice of $C_r(X)$ containing A. By minimality of $\bar{V}(A)$, $V \supseteq \bar{V}(A)$. This implies that $\Lambda^+(\bar{V}(A)) \supseteq \Lambda^+(V)$. Since for each $f \varepsilon V$, $\Lambda^+(\{f\}) \supseteq \Lambda^+(A)$, it follows that $\Lambda^+(V) \supseteq \Lambda^+(A)$, and hence that $\Lambda^+(\bar{V}(A)) \supseteq \Lambda^+(A)$. Since the reverse inclusion has already been established, $\Lambda^+(A) = \Lambda^+(\bar{V}(A))$.

In the course of the argument just given the inclusion $V \supseteq \bar{V}(A)$ was established. To complete the proof it suffices to show that $V \subseteq \bar{V}(A)$.

By Lemma (2.7), if $x, y \varepsilon X$ then

$$[\bar{V}(A)](x, y) = \left\{(\alpha, \beta) : r_{x,y}\alpha = r'_{x,y}\beta, \text{ for some fixed } r_{x,y}, r'_{x,y} \text{ such that } r_{x,y} \cdot r'_{x,y} \geq 0\right\}.$$

Let $f \varepsilon V$. Then $\Lambda^+(\{f\}) \supseteq \Lambda^+(A) = \Lambda^+(\bar{V}(A))$. If $g \varepsilon \bar{V}(A)$, then $(g(x), g(y)) \varepsilon [\bar{V}(A)](x, y)$ for each x and y in X, and hence $r_{x,y}\, g(x) = r'_{x,y}\, g(y)$. It follows that $(x, y; r, r') \varepsilon \Lambda^+(\bar{V}(A)) \subseteq \Lambda^+(\{f\})$. Thus $r_{x,y} \cdot f(x) = r'_{x,y} \cdot f(y)$, and hence $(f(x), f(y)) \varepsilon [\bar{V}(A)](x, y) \subseteq \subseteq \overline{[\bar{V}(A)](x, y)}$.

Hence if $f \in V$, then $(f(x), f(y)) \in \overline{[\bar{V}(A)](x, y)}$ for each x and y in X. Since $\bar{L}(\bar{V}(A)) = \bar{V}(A) = L(\bar{V}(A))$, Corollary (2.9.2) implies that $f \in \bar{V}(A)$. Thus $V \subseteq \bar{V}(A)$, and the theorem is proved. ∎

<u>(2.10.1) Corollary</u>: If A has separation property (I) or (II), and contains a non-zero constant function, then $\bar{V}(A) = C_r(X)$.

<u>Proof</u>: Let $f \in C_r(X)$, and let $(x, y; r, r') \in \Lambda^+(A)$. Then $rh(x) = r'h(y)$ for all $h \in A$. If either $r = 0$ or $r' = 0$, then both $r = 0$ and $r' = 0$, since A contains a non-zero constant function. Thus if either $r = 0$ or $r' = 0$ then $r = r' = 0$ and trivially $(x,y; r, r') \in \Lambda^+(\{f\})$. Now suppose that $r \neq 0$ and $r' \neq 0$. Then $r = r'$, because A contains a non-zero constant function. Thus $h(x) = h(y)$ for all $h \in A$. This is impossible because A has separation property (I) or (II). It follows that $\Lambda^+(A) = (x, y; 0, 0) \subseteq \Lambda^+(\{f\})$ for all $f \in C_r(X)$, and by the theorem just proved $C_r(X) = \bar{V}(A)$. ∎

(2.11) (The Stone-Weierstrass Approximation Theorem) Let A be a non-empty subset of $C_r(X)$. A necessary and sufficient condition that a function f in $C_r(X)$ be in $\bar{\mathcal{A}}(A)$ is: If $g(x_o) = 0$ for all $g \in A$ and some $x_o \in X$, then $f(x_0) = 0$; and if $g(y_o) = g(z_o)$ for all $g \in A$ and some y_o, $z_o \in X$ such that $y_o \neq z_o$, then $f(y_o) = f(z_o)$.

<u>Proof</u>: The necessity of the two stated conditions will be demonstrated first. Suppose that x_o, y_o, z_o are points in X such that $y_o \neq z_o$, $g(x_o) = 0$, and $g(y_o) = g(z_o)$ for all $g \in A$. Let

$$P = \left\{ f : f \in C_r(X),\ \ f(x_o) = 0 \right\}$$

and

$$Q = \left\{ f : f \in C_r(X),\ \ f(y_o) = f(z_o) \right\}.$$

A short computation shows that P and Q are both closed subalgebras of $C_r(X)$, and hence $P \cap Q$ is a closed subalgebra of $C_r(X)$ containing A. Hence $P \cap Q$ contains $\bar{\mathcal{a}}(A)$. This implies $f(x_o) = 0$ and $f(y_o) = f(z_o)$ for all $f \varepsilon \bar{\mathcal{a}}(A)$. It follows that the two conditions are necessary.

The sufficiency of the two conditions will now be established. Let f be a function in $C_r(X)$ for which:

(2.11.1) $\quad g(x_o) = 0$ for all $g \varepsilon A$ implies $f(x_o) = 0$

and

(2.11.2) $\quad g(y_o) = g(z_o)$ for all $g \varepsilon A$ implies $f(y_o) = f(z_o)$

for every x_o, y_o, $z_o \varepsilon X$ such that $y_o \neq z_o$.

Let $(x, y; r,r') \varepsilon \Lambda^+(\bar{\mathcal{a}}(A))$. Then $rh(x) = r'h(y)$ for all $h \varepsilon \bar{\mathcal{a}}(A)$. If $r = r' = 0$, then clearly $(x,y;r,r') \varepsilon \Lambda^+(\{f\})$. If $r \neq 0$, $r' = 0$, then $h(x) = 0$ for all $h \varepsilon \bar{\mathcal{a}}(A)$. Hence $h(x) = 0$ for all $h \varepsilon A$, so that $f(x) = 0$ and $rf(x) = 0 = r'f(y)$. This implies that $(x,y;r,r') \varepsilon \Lambda^+(\{f\})$. In like manner, $r = 0$, $r' \neq 0$ implies that $(x,y; r,r') \varepsilon \Lambda^+(\{f\})$. The only remaining possibility is that $r \neq 0$ and $r' \neq 0$. In this case if $h(x) = 0$ for all $h \varepsilon \bar{\mathcal{a}}(A)$ then $h(y) = 0$ for all $h \varepsilon \bar{\mathcal{a}}(A)$. Hence $f(x) = 0 = f(y)$, so that $rf(x) = r'f(y)$ and $(x,y; r,r') \varepsilon \Lambda^+(\{f\})$. If $h_o(x) \neq 0$ for some $h_o \varepsilon \bar{\mathcal{a}}(A)$, then $h_o(y) \neq 0$. Since $\bar{\mathcal{a}}(A)$ is an algebra $h_o^2 \varepsilon \bar{\mathcal{a}}(A)$ and thus $rh_o^2(x) = r'h_o^2(y)$. But $r^2h_o^2(x) = r'^2h_o^2(y)$ and hence $rr'h_o^2(y) = r^2h_o^2(x) = r'^2h_o^2(y)$, which implies that $r = r'$. It follows that $rh(x) = r'h(y) = rh(y)$ for all $h \varepsilon \bar{\mathcal{a}}(A)$ and, since $r \neq 0$, $h(x) = h(y)$ for all $h \varepsilon \bar{\mathcal{a}}(A)$. Thus $f(x) = f(y)$, $rf(x) = r'f(y)$, and hence $(x,y; r,r') \varepsilon \Lambda^+(\{f\})$.

By the argument just given if f satisfies (2.11.1) and (2.11.2) then $\Lambda^+(\bar{\mathcal{A}}(A)) \subseteq \Lambda^+(\{f\})$. By Lemma (2.6), $\bar{\mathcal{A}}(A)$ is a vector sub-lattice of $C_r(X)$. Thus $\bar{V}(\bar{\mathcal{A}}(A)) = \bar{\mathcal{A}}(A)$ and by Theorem (2.10),

$$\bar{\mathcal{A}}(A) = \bar{V}(\bar{\mathcal{A}}(A)) = \left\{ f : f \in C_r(X) \text{ and } \Lambda^+(\{f\}) \supseteq \Lambda^+(\bar{\mathcal{A}}(A)) \right\} .$$

Thus if f satisfies (2.11.1) and (2.11.2) then $f \in \bar{\mathcal{A}}(A)$. This implies the sufficiency of the two stated conditions and concludes the proof. ∎

(2.11.3) <u>Corollary</u>: A necessary and sufficient condition that $\bar{\mathcal{A}}(A)$ contain a non-zero constant function is that for each $x \in X$ there exist some $f \in A$ such that $f(x) \neq 0$.

<u>Proof</u>: Let $g \in \bar{\mathcal{A}}(A)$, where $g(x) = c \neq 0$ for all $x \in X$. By the theorem just proved there exists no $x \in X$ such that $f(x) = 0$ for all $f \in A$.

Now suppose that for each $x \in X$ there exists some $f_x \in A$ such that $f_x(x) \neq 0$. Let g be any real-valued constant function defined on X. Since $g(x) = g(y)$ for all $x, y \in X$, the theorem just proved implies $g \in \bar{\mathcal{A}}(A)$. It follows that $\bar{\mathcal{A}}(A)$ contains all real-valued constant functions defined on X. ∎

(2.11.4) <u>Corollary</u>: If A has separation property (I), or if A has separation property (II) and contains a non-zero constant function, then $\bar{\mathcal{A}}(A) = C_r(X)$.

<u>Proof</u>: If A has separation property (I), then there exists no $x \in X$ such that $g(x) = 0$ for all $g \in A$, and there exist no two distinct points $y, z \in X$ such that $g(y) = g(z)$ for all $g \in A$. Theorem (2.11) implies that $C_r(X) = \bar{\mathcal{A}}(A)$.

If A has separation property (II) and contains a non-zero constant function then the assertion follows immediately from Theorem (2.11) and Corollary (2.11.3). ∎

(2.11.5) Corollary: If $\mathcal{A}(A)$ has separation property (II) and contains a non-zero constant function, then $\mathcal{A}(A)$ has separation property (I).

Proof: Let $\mathcal{A}(A)$ have separation property (II) and contain a non-zero constant function. Let $x_o, y_o \in X$ be such that $x_o \neq y_o$. Then there exists a function $f \in \mathcal{A}(A)$ such that $f(x_o) \neq f(y_o)$. Let a and b be any two real numbers and let

$$g(x) = \left[\frac{a - b}{f(x_o) - f(y_o)} \right] \cdot f(x) + \left[\frac{bf(x_o) - af(y_o)}{f(x_o) - f(y_o)} \right] .$$

for all $x \in X$. Then $g \in \mathcal{A}(A)$ since $\mathcal{A}(A)$ is an algebra. Since $g(x_o) = a$ and $g(y_o) = b$, it follows that $\mathcal{A}(A)$ has separation property (I). ∎

(2.12) Definition. Let X be a compact topological space. The symbol $C_c(X)$ will denote the space of all complex-valued functions defined and continuous on X. With algebraic operations defined pointwise, $C_c(X)$ is a complex algebra. The function $\| \ \|$ defined on $C_c(X)$ by $\|f\| = \sup \{ |f(x)| : x \in X \}$ is a norm on $C_c(X)$ and $C_c(X)$ equipped with this norm is a complete normed vector space. In the topology induced by this norm $C_c(X)$ is a closed complex algebra.

(2.13) Definition: If $f \in C_c(X)$, let $\bar{f}$ be the function defined on X by $\bar{f}(x) = \overline{f(x)}$ for all $x \in X$. The function $\bar{f}$ will be called the conjugate of the function f.

For A, a non-empty subset of $C_c(X)$, define $\bar{\mathcal{a}}_c(A)$ to be the intersection of all closed subalgebras of $C_c(X)$ which contain A and contain the conjugate of each of their members. Clearly $C_c(X)$ has all these properties so that the intersection is not vacuous. Also $\bar{\mathcal{a}}_c(A)$ has all these properties and is contained in any closed subalgebra of $C_c(X)$ which contains A and contains the conjugate of each of its members.

For $f \varepsilon C_c(X)$ the symbol $\mathcal{R}(f)$ will denote the <u>real part</u> of f and the symbol $\mathcal{I}(f)$ will denote the <u>imaginary part</u> of f.

(2.14) <u>Definition</u>: If A is a non-empty subset of $C_c(X)$ let

$$A_r = \left\{ f : f = \mathcal{R}(g) \quad \text{or} \quad f = \mathcal{I}(g) \quad \text{for some } g \varepsilon A \right\},$$

and let

$$B(A) = \left\{ f : f = g + ih \quad \text{where} \quad g, h \varepsilon \bar{\mathcal{a}}(A_r) \right\}.$$

Since $f \varepsilon C_c(X)$ if and only if $\mathcal{R}(f) \varepsilon C_r(X)$ and $\mathcal{I}(f) \varepsilon C_r(X)$, it follows that $A_r \subseteq C_r(X)$.

(2.15) <u>Lemma</u>: If A is a non-empty subset of $C_c(X)$ then $\bar{\mathcal{a}}_c(A) = B(A)$.

<u>Proof</u>: It will be shown first that $\bar{\mathcal{a}}_c(A) \supseteq B(A)$. Let D be any closed complex subalgebra of $C_c(X)$ containing A, and closed under the formation of conjugates. If $f \varepsilon D$, then $\mathcal{R}(f) = \frac{1}{2}(f + \bar{f})$ and $\mathcal{I}(f) = \frac{1}{2i}(f - \bar{f})$ are both in D. Let D_r be as in Definition (2.14). A tedious but perfectly straightforward computation shows that D_r is a real closed subalgebra of $C_r(X)$. Since $D_r \supseteq A_r$, it follows that $D_r \supseteq \bar{\mathcal{a}}(A_r)$. By arguments given above every member of D_r is in D.

This implies that if $f \varepsilon \bar{\mathcal{a}}(A_r)$, then $f \varepsilon D$. Now let $f \varepsilon B(A)$. Then $f = g + ih$ for some g, $h \varepsilon \bar{\mathcal{a}}(A_r)$. By the preceding, g and h are in D so that $f \varepsilon D$. This implies that $D \supseteq B(A)$. In particular this implies that $\bar{\mathcal{a}}_c(A) \supseteq B(A)$.

A short computation shows that $B(A)$ is a closed complex subalgebra of $C_c(X)$ containing A and closed under the formation of conjugates. This implies that $B(A) \supseteq \bar{\mathcal{a}}_c(A)$ and concludes the proof. ∎

(2.16) The Stone-Weierstrass Approximation Theorem. (Complex Case)

Let X be a compact topological space and let A be a non-empty subset of $C_c(X)$. A necessary and sufficient condition that a function $f \varepsilon C_c(X)$ be in $\bar{\mathcal{a}}_c(A)$ is: If x_o is any point in X for which $g(x_o) = 0$ for all $g \varepsilon A$, then $f(x_o) = 0$; and if y_o and z_o are any two distinct points in X for which $g(y_o) = g(z_o)$ for all $g \varepsilon A$, then $f(y_o) = f(z_o)$.

Proof: The necessity of the above conditions will be established first. Let $f \varepsilon \bar{\mathcal{a}}_c(A) = B(A)$. Then $f = h_1 + ih_2$, for $h_1, h_2 \varepsilon \bar{\mathcal{a}}(A_r)$. Now let x_o be a point in X for which $g(x_o) = 0$ for all $g \varepsilon A$. Then $g(x_o) = 0$ for all $g \varepsilon A_r$ and hence, by Theorem (2.11), $h(x_o) = 0$ for all $h \varepsilon \bar{\mathcal{a}}(A_r)$. It follows that $f(x_o) = 0$. Let y_o and z_o be any two distinct points in X for which $g(y_o) = g(z_o)$ for all $g \varepsilon A$. Then the set

$$F = \{h : h \varepsilon C_c(X) \text{ and } h(y_o) = h(z_o)\}$$

is a closed complex subalgebra of $C_c(X)$ which contains A and is closed under the formation of conjugates. This implies that $F \supseteq \bar{\mathcal{a}}_c(A)$ and hence $f(y_o) = f(z_o)$.

Conversely, let f be a function in $C_c(X)$ such that

$$x_o \varepsilon X \text{ and } g(x_o) = 0 \text{ for all } g \varepsilon A \Longrightarrow f(x_o) = 0$$

and

$$y_o, z_o \varepsilon X, \; y_o \neq z_o, \text{ and } g(y_o) = g(z_o) \text{ for all } g \varepsilon A \Longrightarrow f(y_o) = f(z_o) .$$

Then

$$x_o \varepsilon X \text{ and } h(x_o) = 0 \text{ for all } h \varepsilon A_r \Longrightarrow \mathcal{R}(f)(x_o) = \mathcal{I}(f)(x_o) = 0,$$

and

$$y_o, z_o \varepsilon X, \; y_o \neq z_o, \text{ and } h(y_o) = h(z_o) \text{ for all } h \varepsilon A_r \Longrightarrow$$
$$g(y_o) = g(z_o) \text{ for all } g \varepsilon A \Longrightarrow f(y_o) = f(z_o) \Longrightarrow$$
$$\mathcal{R}(f)(y_o) = \mathcal{R}(f)(z_o) \text{ and } \mathcal{I}(f)(y_o) = \mathcal{I}(f)(z_o) .$$

By Theorem (2.11), $\mathcal{R}(f)$ and $\mathcal{I}(f)$ are in $\bar{\mathfrak{a}}(A_r)$, and thus $f \varepsilon B(A) = \bar{\mathfrak{a}}_c(A)$. ∎

(2.17) <u>Remark</u>: In what follows, the symbol X will denote a fixed but otherwise arbitrary <u>locally compact topological space</u>. The symbol X_∞ will denote the one-point compactification of X obtained by adjoining to X the point x_∞ not in X.[1] The set $C_r^o(X)$ is defined to be the set of all continuous, real-valued functions defined on X and satisfying the following condition: For every $\varepsilon > 0$ there exists a closed compact subset $F_{\varepsilon,f}$ of X such that $|f(x)| < \varepsilon$ for all $x \varepsilon X - F_{\varepsilon,f}$.

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[1]See Glossary, page 81.

Functions satisfying this condition will be said to vanish at infinity. The set $C_c^o(X)$ is defined to be the set of all continuous, complex-valued functions defined on X and vanishing at infinity.

Let $C_r(X)$ denote the space of all bounded, continuous, real-valued functions defined on X. Let $C_c(X)$ denote the space of all bounded, continuous, complex-valued functions defined on X. With algebraic operations defined pointwise $C_r(X)$ is a real algebra and $C_c(X)$ is a complex algebra. The function $\| \ \|$ defined on $C_r(X)$ and on $C_c(X)$ by $\|f\| = \sup \{|f(x)| : x \in X\}$ is a norm on $C_r(X)$ and on $C_c(X)$.

A long but nonetheless straightforward computation shows that $C_r^o(X)$ is a closed subalgebra of $C_r(X)$ and that $C_c^o(X)$ is a closed subalgebra of $C_c(X)$.

(2.18) Definition

$$C_r^o(X_\infty) = \{f : f \text{ is a continuous, real-valued function defined on } X_\infty \text{ and } f(x_\infty) = 0\}$$

$$C_c^o(X_\infty) = \{f : f \text{ is a continuous, complex-valued function defined on } X_\infty \text{ and } f(x_\infty) = 0\}$$

(2.19) Lemma: Every function in $C_r^o(X)$ may be uniquely extended to a function in $C_r^o(X_\infty)$, and every function in $C_c^o(X)$ may be uniquely extended to a function in $C_c^o(X_\infty)$. Conversely, the restriction to X of any function in $C_r^o(X_\infty)$ is in $C_r^o(X)$, and the restriction to X of any function in $C_c^o(X_\infty)$ is in $C_c^o(X)$.

Proof: Let $f \in C_c^o(X)$ and define f_∞ on X_∞ by: $f_\infty(x) = f(x)$ if $x \in X$, and $f_\infty(x_\infty) = 0$. The function f_∞ is an extension of f to a function defined on X_∞. Since $f \in C_c^o(X)$ it follows that if $\varepsilon > 0$ is given, then there exists a closed, compact subset $F_\varepsilon \subseteq X$ such that

$|f(x)| < \varepsilon$ for $x \in X - F_\varepsilon$. This implies that $X_\infty - F_\varepsilon$ is an open set containing x_∞ such that $f_\infty(X_\infty - F_\varepsilon) \subseteq N(0; \varepsilon)$. It follows that f_∞ is continuous at x_∞. Since f_∞ is continuous on X, $f_\infty \in C_c^o(X_\infty)$. Any extension of a function f in $C_c^o(X)$ to a function in $C_c^o(X_\infty)$ must agree with f on X and must assume the value 0 at X_∞. Thus uniqueness is a consequence of existence.

Now let $f_\infty \in C_c^o(X_\infty)$ and let f denote the restriction of f_∞ to X. Let $\varepsilon > 0$ be given. Since $f_\infty(x_\infty) = 0$ and since f_∞ is continuous on X_∞, there exists an open set G containing x_∞ such that $f_\infty(G) \subseteq N(0;\varepsilon)$. By definition of the topology on X_∞, this open set G must have the form $G = X_\infty - F$, where F is a closed, compact subset of X. Thus for all $x \in X - F$, $|f(x)| = |f_\infty(x)| < \varepsilon$, and hence $f \in C_c^o(X)$.

The proof of the lemma for $C_r^o(X)$ and $C_r^o(X_\infty)$ is an exact duplication of the proof just given. ∎

(2.20) <u>Definition</u>: For each $f \in C_c^o(X)$ let f_∞ denote the unique extension of f to a function in $C_c^o(X_\infty)$. For each $f \in C_r^o(X)$ let f_∞ denote the unique extension of f to a function in $C_r^o(X_\infty)$.

For each non-empty subset A of $C_c^o(X)$ or of $C_r(X)$, let the set A_∞ be defined as follows:

$$A_\infty = \{f_\infty : f \in A\}.$$

For each non-empty subset B of $C_c^o(X_\infty)$ or of $C_r^o(X_\infty)$, let B_X be the set of all restrictions to X of functions in B.

(2.21) <u>Definition</u>: If A is a non-empty subset of $C_c^o(X)$, define $\bar{\mathcal{a}}_c(A)$ to be the intersection of all closed, complex subalgebras of $C_c^o(X)$ which contain A and are closed under the formation of conjugates. The space $C_c^o(X)$ has all these properties, so that the intersection is not vacuous. The set $\bar{\mathcal{a}}_c(A)$ has all these properties and is contained in any closed complex subalgebra of $C_c^o(X)$ which contains A and is closed under the formation of conjugates.

(2.22) <u>Theorem</u>: If A is a non-empty subset of $C_c^o(X)$, then a necessary and sufficient condition that a function f in $C_c^o(X)$ be in $\bar{\mathcal{a}}_c(A)$ is: If x_o is any point of X for which $g(x_o) = 0$ for all $g \in A$, then $f(x_o) = 0$, and if y_o and z_o are any two distinct points of X for which $g(y_o) = g(z_o)$ for all $g \in A$, then $f(y_o) = f(z_o)$.

<u>Proof</u>: Denote the class of all functions which satisfy the above condition by Q. It must be shown that $Q = \bar{\mathcal{a}}_c(A)$.

It will be established first that $\bar{\mathcal{a}}_c(A) = [\bar{\mathcal{a}}_c(A_\infty)]_X$. A short computation shows that $[\bar{\mathcal{a}}_c(A_\infty)]_X$ is a closed subalgebra of $C_c^o(X)$ which contains A and is closed under the formation of conjugates. This implies that $[\bar{\mathcal{a}}_c(A_\infty)]_X \supseteq \bar{\mathcal{a}}_c(A)$. By the same argument $[\bar{\mathcal{a}}_c(A)]_\infty$ is a closed subalgebra of $C_c^o(X_\infty)$ which contains A_∞ and is closed under the formation of conjugates. This implies that $[\bar{\mathcal{a}}_c(A)]_\infty \supseteq \bar{\mathcal{a}}_c(A_\infty)$, and hence that every function in $\bar{\mathcal{a}}_c(A_\infty)$ is the extension of some function in $\bar{\mathcal{a}}_c(A)$ to a function defined on X_∞. It follows that $[\bar{\mathcal{a}}_c(A_\infty)]_X \subseteq \bar{\mathcal{a}}_c(A)$. Thus $[\bar{\mathcal{a}}_c(A_\infty)]_X = \bar{\mathcal{a}}_c(A)$.

By Theorem (2.16) and Lemma (2.19) $Q_\infty = \bar{\mathcal{a}}_c(A_\infty)$, and hence $Q = [\bar{\mathcal{a}}_c(A_\infty)]_X = \bar{\mathcal{a}}_c(A)$. ∎

(2.22.1) <u>Corollary</u>: If A is any non-empty subset of $C_c^o(X)$ which has separation property (I) or has separation property (II) and for each $x_o \varepsilon X$ contains a function g_{x_o} such that $g_{x_o}(x_o) \neq 0$, then $\bar{\mathcal{a}}_c(A) = C_c^o(X)$.

(2.23) <u>Theorem</u>: If A is a non-empty subset of $C_r^o(X)$, then a necessary and sufficient condition that a function f in $C_r^o(X)$ be in $\bar{\mathcal{a}}(A)$ is: If x_o is any point in X for which $g(x_o) = 0$, for all $g \varepsilon A$, then $f(x_o) = 0$; and if y_o and z_o are any two distinct points in X for which $g(y_o) = g(z_o)$ for all $g \varepsilon A$, then $f(y_o) = f(z_o)$.

The proof is an exact duplicate of the proof of Theorem (2.22) and is omitted.

(2.23.1) <u>Corollary</u>: If A is any non-empty subset of $C_r^o(X)$ which has separation property (I) or has separation property (II) and for each $x_o \varepsilon X$ contains a function g_{x_o} such that $g_{x_o}(x_o) \neq 0$, then $\bar{\mathcal{a}}(A) = C_r^o(X)$.

CHAPTER III

THE STONE REPRESENTATION THEOREM FOR BOOLEAN ALGEBRAS

This chapter contains a proof of the Stone representation theorem for Boolean algebras. The proof presented here is a modification of that given in WALLMAN [1]. The Stone representation theorem states that given any Boolean algebra E there exists a totally disconnected compact Hausdorff space S such that E is a lattice-isomorphic image of the Boolean algebra of all open-closed subsets of S.

The Stone representation theorem of this chapter plays a major role in the proof of the representation theorem of Chapter IV.

(3.1) Definition: A lattice L is said to be distributive whenever

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

and

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

are valid for all $a, b, c \in L$. An element $0 \in L$ is said to be a zero element of L if $a \wedge 0 = 0$ and $a \vee 0 = a$ for each $a \in L$. An element $1 \in L$ is said to be a unit element of L if $a \wedge 1 = a$ and $a \vee 1 = 1$ for each $a \in L$. It is easily seen that there can be at most one zero element and at most one unit element in any given lattice.

Unless otherwise noted the symbol L will denote a fixed but otherwise arbitrary distributive lattice with distinct zero and unit elements.

(3.2) Definition: A non-empty subset H of L is said to have property F if and only if

$$\{a_i\}_{i=1}^{n} \subseteq H \Longrightarrow \bigwedge_{i=1}^{n} a_i \neq 0 .$$

A non-empty subset M of L will be called an ideal in L whenever

$$a, b \in M \Longrightarrow a \wedge b \in M$$

and

$$c \in M \Longrightarrow \text{if } d \in L \text{ and } d > c, \text{ then } d \in M.$$

An ideal M will be called maximal if there exists no ideal N properly containing M and properly contained in L. Note that if M is any maximal ideal in L then $0 \notin M$ and $1 \in M$.

(3.3) Lemma: Let $\mathcal{F}$ be the family of subsets of L with property F, partially ordered by inclusion. If $H \in \mathcal{F}$ then H is contained in some maximal element of .

Proof: By assumption the zero and unit elements of L are distinct. This implies $\{1\} \in \mathcal{F}$ and hence $\mathcal{F}$ is not empty. If $\mathcal{C}$ is any chain in $\mathcal{F}$ it is easily seen that $\cup \mathcal{C}$ is an upper bound for $\mathcal{C}$ in $\mathcal{F}$. The conclusion follows from Zorn's lemma.

(3.3.1) Remark: The lemma just proved is equivalent to the Axiom of Choice. This point is discussed in SIKORSKI [1].

(3.4) Lemma: A non-empty subset G of L is a maximal element of $\mathcal{F}$ if and only if G satisfies the following two conditions:

(3.4.1) $$a, b \in G \Longrightarrow a \wedge b \in G$$

(3.4.2) $c \in G \Longleftrightarrow a \wedge c \neq 0$, for all $a \in G$.

<u>Proof</u>: Let G be a maximal element of $\mathfrak{F}$. Then $G \neq \emptyset$. If $a, b \in G$, then $a \wedge b \neq 0$. This implies that

$$G \cup \{c \wedge d : c, d \in G\}$$

has property F and hence does not properly contain G. Thus G satisfies (3.4.1).

If $c \in L$ and $a \wedge c \neq 0$ for every $a \in G$, then $G \cup \{c\}$ has property F, and hence $c \in G$. Conversely, if $c \in G$ then $a \wedge c \neq 0$ for every $a \in G$. Thus G satisfies (3.4.2).

Now let G be any non-empty subset of L satisfying (3.4.1) and (3.4.2). By finite induction, G has property F. If H is any subset of L containing G and possessing property F then, by (3.4.2), $H \subseteq G$. Thus G is a maximal element of $\mathfrak{F}$. ∎

(3.5) <u>Lemma</u>: Every maximal element of $\mathfrak{F}$ is a maximal ideal in L and conversely.

<u>Proof</u>: Let G be a maximal element of $\mathfrak{F}$. By Lemma (3.4), if $a, b \in G$, then $a \wedge b \in G$. If $d \in L$ and $d > c$ for some $c \in G$, then $d \wedge a \geq c \wedge a > 0$ for all $a \in G$. This implies $d \wedge a \neq 0$ for each $a \in G$. Lemma (3.4) implies $d \in G$. It follows from Definition (3.2) that G is an ideal in L.

Now suppose that there exists an ideal N in L such that N properly contains G. Then there exists an element $n \in L$ such that $n \notin G$. Lemma (3.4) implies that $n \wedge a = 0$ for some $a \in G \subseteq N$. Hence $0 = n \wedge a \in N$. Definition (3.2) together with the fact that $x \geq 0$ for all $x \varepsilon L$ implies $N = L$. It follows that G is a maximal ideal in L.

Conversely let M be a maximal ideal in L. It follows from the maximality of M that $0 \notin M$. Hence by Definition (3.2) and finite induction M has property F. Thus $M \varepsilon \mathcal{F}$ and by Lemma (3.3) M is contained in some maximal element H of $\mathcal{F}$. The preceding paragraph implies that H is a maximal ideal in L. Since M and H are maximal ideals and $H \supseteq M$, it follows that $M = H$ and consequently that M is a maximal element of $\mathcal{F}$. ∎

(3.6) <u>Definition</u>: Let S denote the collection of all maximal elements of $\mathcal{F}$. By Lemma (3.3) S is non-empty. Each element $p \varepsilon S$ will be called a <u>point</u> in S. For each $a \varepsilon L$ the set B_a defined by

$$B_a = \{p : p \varepsilon S, a \varepsilon p\}$$

will be called the <u>basic</u> <u>a-set</u>. By Lemmas (3.3) and (3.5), $B_a = \emptyset$ if and only if $a = 0$. If $p \varepsilon S$ and $a \varepsilon p$ then $1 \wedge a = a \neq 0$. It follows from Lemma (3.4) that $1 \varepsilon p$. This implies that $1 \varepsilon p$ for all $p \varepsilon S$. and hence $B_1 = S$.

(3.7) <u>Lemma</u>: If $p \varepsilon S$ then $\{p\} = \bigcap_{a \varepsilon p} B_a$.

Proof: Let $p \in S$. If $a \in p$, then $p \in B_a$, and hence $\{p\} \subseteq \bigcap_{a \in p} B_a$. Conversely, if $q \in B_a$ for all $a \in p$, then $a \in q$ for each $a \in p$, and hence $q \supseteq p$. Since both q and p are maximal elements of $\mathcal{F}$ it follows that $q = p$. Thus $\{p\} = \bigcap_{a \in p} B_a$. ∎

(3.8) Lemma: If $a, b \in L$, then

(3.8.1) $$B_{a \wedge b} = B_a \cap B_b$$

(3.8.2) $$B_{a \vee b} = B_a \cup B_b$$

Proof: The first assertion follows from the implications

$$p \in B_a \cap B_b \Longleftrightarrow a, b \in p \Longleftrightarrow a \wedge b \in P \Longleftrightarrow p \in B_{a \wedge b}.$$

The second assertion is proved in two steps. That $B_a \cup B_b \subseteq B_{a \vee b}$ follows from the implications

$$p \in B_a \cup B_b \Longrightarrow a \in p \text{ or } b \in p \Longrightarrow a \vee b \in p \Longrightarrow p \in B_{a \vee b}.$$

Now suppose that $a \notin p$ and $b \notin p$. There exist $c, d \in p$ such that $c \wedge a = 0$ and $c \wedge b = 0$. This implies

$$(c \wedge d) \wedge (a \vee b) = [(c \wedge d) \wedge a] \vee [(c \wedge d) \wedge b] = [(a \wedge c) \wedge d] \vee [c \wedge (d \wedge b)]$$

$$= (0 \wedge d) \vee (c \wedge 0) = 0 .$$

By Definition (3.2) $c \wedge d \in p$ and thus, by Lemma (3.4), $a \vee b \notin p$. If $p \in B_{a \vee b}$. then $a \vee b \in p$. The argument just given implies either $a \in p$ or $b \in p$, and hence $p \in B_a \cup B_b$. In view of the preceding, $B_{a \vee b} = B_a \cup B_b$. ∎

(3.9) <u>Lemma</u>: Let

$$\mathcal{U} = \left\{ \bigcap_{a \varepsilon Q} B_a : Q \subseteq L \right\}.$$

Then $\mathcal{U}$ contains $\emptyset$ and S, and is closed under the formation of finite unions and arbitrary intersections.

<u>Proof</u>: If $p \varepsilon \bigcap_{a \varepsilon L} B_a$ then $p \varepsilon B_a$ for all $a \varepsilon L$ and hence $p = L$. This implies that $0 \varepsilon p$ thereby contradicting Lemma (3.5). It follows that $\emptyset = \bigcap_{a \varepsilon L} B_a$. By convention $\bigcap_{a \varepsilon \emptyset} B_a = S$. Thus $\emptyset \varepsilon \mathcal{U}$ and $S \varepsilon \mathcal{U}$.

Let $A = \bigcap_{a' \varepsilon Q_1} B_a$ and $C = \bigcap_{a'' \varepsilon Q_2} B_{a'}$ be sets in $\mathcal{U}$.

Then by Lemma (3.8)

$$A \cup C = \bigcap_{a' \varepsilon Q_1} \bigcap_{a'' \varepsilon Q_2} B_{a'} \cup B_{a''} = \bigcap_{a' \varepsilon Q_1} \bigcap_{a'' \varepsilon Q_2} B_{a' \vee a''} .$$

If

$$Q_3 = \left\{ a' \vee a'' : a' \varepsilon Q_1 \text{ and } a'' \varepsilon Q_2 \right\},$$

then

$$A \cup C = \bigcap_{a \varepsilon Q_3} B_a ,$$

and hence $A \cup C \varepsilon \mathcal{U}$. By induction it follows that $\mathcal{U}$ is closed under the formation of finite unions.

Let T be an arbitrary index set and let $\{A_t : t \varepsilon T\} \subseteq \mathcal{U}$. If $A_t = \bigcap_{a \varepsilon Q_t} B_a$ for each $t \varepsilon T$, then

$$\bigcap_{t \varepsilon T} A_t = \bigcap_{t \varepsilon T} \bigcap_{a \varepsilon Q_t} B_a = \bigcap_{a \varepsilon \bigcup_{t \varepsilon T} Q_t} B_a$$

and hence $\bigcap_{t \varepsilon T} A_t \ \varepsilon\ \mathcal{U}$. ∎

(3.10) <u>Lemma</u>: Let $\mathcal{J} = \{A : S - A\ \varepsilon\ \mathcal{U}\}$. Then $\mathcal{J}$ is a topology on the space S.

<u>Proof</u>: The assertion is an immediate consequence of the definition of a topology on the space S. This definition may be found in the glossary.

(3.10.1) <u>Remark</u>: The sets in $\mathcal{J}$ will henceforth be called open sets in S. Sets in $\mathcal{U}$ will henceforth be called closed sets in S. If $A \subseteq S$ the symbol A^c will denote the set $S - A$.

(3.11) <u>Theorem</u>: The space S is a compact T_1-space in the topology $\mathcal{J}$.

<u>Proof</u>: Lemmas (3.7) and (3.9) imply that each singleton in S is a closed set. It follows that S is a T_1-space.

Let $\mathcal{U}' = \{A_t : t\ \varepsilon\ T\}$ be a subcollection of $\mathcal{U}$ with the finite intersection property. By definition of $\mathcal{U}$ each set A_t is of the form $A_t = \bigcap_{a \varepsilon Q_t} B_a$, where $Q_t \subseteq L$. Suppose that the collection

$$\mathcal{P} = \left\{B_a : a\ \varepsilon \bigcup_{t \varepsilon T} Q_t\right\}$$

does not have the finite intersection property. Then there exists a finite subcollection $B_{a_1},\dots,B_{a_n}$ of $\mathcal{P}$ such that $\bigcap_{i=1}^{n} B_{a_i} = \emptyset$. Each B_{a_i} contains some A_i, so that $\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{n} B_{a_i} = \emptyset$. This contradicts the original hypothesis and implies that the collection $\mathcal{P}$ has the finite intersection property.

Now let $\mathcal{a} = \bigcup_{t \varepsilon T} Q_t$. If $a_1,\dots,a_n$ is any finite subcollection of $\mathcal{A}$, then by Lemma (3.8)

$$B_{\bigwedge_{i=1}^{n} a_i} = \bigcap_{i=1}^{n} B_{a_i} \neq \emptyset .$$

It follows from the discussion accompanying Definition (3.6) that $\bigwedge_{i=1}^{n} a_i \neq 0$. This implies that the set $\mathcal{A} \subseteq L$ has property F. By Lemma (3.3) and Definition (3.6) there exists a point $p \in S$ such that $p \supseteq \mathcal{A}$. Thus $a \in p$ for each $a \in \mathcal{A}$, and hence $p \in B_a$ for each $a \in \bigcup_{t \in T} Q_t$. This implies that

$$p \in \bigcap_{a \in \bigcup_{t \in T} Q_t} B_a = \bigcap_{t \in T} A_t ,$$

and hence that the collection $\mathcal{U}'$ has non-empty intersection.

The preceding argument has established that any collection of closed sets in S with the finite intersection property has non-empty intersection. It follows that S is compact. ∎

(3.12) <u>Definition</u>: A distributive lattice L is said to have the Wallman disjunction property if, whenever a and b are any two distinct elements of L, there exists an element c of L such that one of $a \wedge c$, $b \wedge c$ is zero and the other is not zero.

(3.13) <u>Lemma</u>: A necessary and sufficient condition that the correspondence $a \leftrightarrow B_a$ from L onto $B = \{B_a : a \in L\}$ be one-to-one is that L have the Wallman disjunction property.

<u>Proof</u>: Suppose that L has the Wallman disjunction property. Let $a, b \in L$ and $a \neq b$. For convenience let $a \wedge c = 0$ and $b \wedge c \neq 0$.

Since the set $\{b, c\} \subseteq L$ has property F, Lemma (3.3) implies the existence of a point $p \varepsilon S$ such that $p \supseteq \{b, c\}$. Thus $p \varepsilon B_b \cap B_c$. Since $a \wedge c = 0$, it follows from Lemma (3.8) and Definition (3.6) that $\emptyset = B_{a \wedge c} = B_a \cap B_c$. This implies that $p \varepsilon B_b$ but $p \notin B_a$. Thus if $a,b \varepsilon L$ and $a \neq b$, then $B_a \neq B_b$.

Conversely, suppose that the correspondence $a \longleftrightarrow B_a$ is one-to-one. Let $a,b \varepsilon L$ and $a \neq b$. By assumption $B_a \neq B_b$. For convenience suppose that $p \varepsilon B_b$ but $p \notin B_a$. Then $b \varepsilon p$ but $a \notin p$. By Lemma (3.4) the relations $a \notin p$ and $b \varepsilon p$ imply that there exists some $c \varepsilon p$ such that $a \wedge c = 0$ but $b \wedge c \neq 0$. It follows that L has the Wallman disjunction property. ∎

(3.14) Definition: Let L be a lattice with zero and unit elements. If a and b are elements of L such that $a \wedge b = 0$ and $a \vee b = 1$, then b is called a complement of a, and a is called a complement of b. A lattice L with distinct zero and unit elements is called a complemented lattice provided each of its members has at least one complement in L.

A Boolean algebra is a distributive, complemented lattice.

(3.15) Lemma: If E is a Boolean algebra, then the following statements about E are true.

(3.15.1) For each $a \varepsilon E$ there exists a unique element a' in E such that $a \wedge a' = 0$ and $a \vee a' = 1$. [If E is a Boolean algebra and $a \varepsilon E$, the unique complement of a in E will henceforth be denoted by a'.]

(3.15.2) If $a, b \in E$ and $a \wedge b = 0$, then $a' = b \vee a'$.

(3.15.3) If $a \in E$, then $(a')' = a$.

(3.15.4) If $a,b \in E$, then $(a \vee b)' = a' \wedge b'$ and $(a \vee b)' = a' \wedge b'$.

Proof: Let $a,b,c \in E$ be such that $a \wedge b = 0$, $a \wedge c = 0$, $a \vee b = 1$, and $a \vee c = 1$. Then

(1) $c = 0 \vee c = (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c) = 1 \wedge (b \vee c) = b \vee c$

and

(2) $b = 0 \vee b = (a \wedge c) \vee b = (a \vee b) \wedge (c \vee b) = 1 \wedge (c \vee b) = b \vee c$.

Thus $b = b \vee c = c$. This proves (3.15.1). Since identity (1) makes use only of the relation $a \wedge b = 0$, the assertion (3.15.2) follows. The assertion (3.15.3) is a direct consequence of (3.15.1), and (3.15.4) follows from (3.15.1), (3.15.3), and the identities $(a \wedge b) \wedge (b' \vee a') = 0$ and $(a \wedge b) \vee (b' \vee a') = 1$. ∎

(3.16) Lemma: If E is a Boolean algebra, then E has the Wallman disjunction property.

Proof: Let a and b be distinct elements of E, and suppose that both $a \wedge b' = 0$ and $b \wedge a' = 0$. It follows from (3.15.2) that $a' = b' \vee a' = b'$. By (3.15.3) $a = b$, thus contradicting the assumption that a and b are distinct. This implies that either $a \wedge b' \neq 0$ or $b \wedge a' \neq 0$. Since $b \wedge b' = 0$ and $a \wedge a' = 0$, the proof is complete. ∎

(3.17) Remark: Since a Boolean algebra is, in particular, a distributive lattice with the Wallman disjunction property, all of the results established thus far are applicable. More precisely, let E be a Boolean algebra and let S be the collection of all maximal ideals in E. By Lemmas (3.3) and (3.5) S is non-empty. If the space S is endowed with the topology $\mathcal{J}$, defined in Lemma (3.10), it follows from Lemma (3.11) that S is a compact T_1-space. The topological space $(S, \mathcal{J})$ will be called the Stone space of the Boolean algebra E.

(3.18) Definition: Let L and L' be lattices. A one-to-one mapping f from L onto L' is called an isomorphism from L onto L' provided

$$f(a \wedge b) = f(a) \wedge f(b)$$

and

$$f(a \vee b) = f(a) \vee f(b)$$

for every $a,b \in L$. Two lattices are said to be isomorphic if there exists an isomorphism from one onto the other.

(3.19) Lemma: Let L and L' be lattices and let f be an isomorphism from L onto L'. Then for $a,b \in L$

$$a < b \Leftrightarrow f(a) < f(b).$$

Proof: Let $a,b \in L$ and $a < b$. Then $a \wedge b = a$, and hence $f(a) \wedge f(b) = f(a \wedge b) = f(a)$. It follows that $f(a) \leqq f(b)$. Since f is one-to-one $f(a) \neq f(b)$, and hence $f(a) < f(b)$. Conversely, if $a,b \in L$ and $f(a) < f(b)$, then $f(a \wedge b) = f(a) \wedge f(b) = f(a)$. Since f is one-to-one it follows that $a \wedge b = a$. This implies $a \leq b$. Since $f(a) \neq f(b)$ and since f is one-to-one, $a \neq b$. It follows that $a < b$. ∎

(3.20) Lemma: Let E be a Boolean algebra and let $(S, \mathcal{J})$ be the Stone space of E. For each $a \in E$ let B_a be as defined in Definition (3.6). The collection

$$\mathcal{B} = \{B_a : a \in E\}$$

partially ordered by set inclusion is a Boolean algebra. The mapping f defined on E by $f(a) = B_a$ for each $a \in E$ is an isomorphism from E onto $\mathcal{B}$.

Proof: Two immediate consequences of Lemma (3.8) are that $\mathcal{B}$ is a lattice under set inclusion and that $f(a) \wedge f(b) = f(a \wedge b)$ and $f(a \vee b) = f(a) \vee f(b)$ for each $a, b \in E$. By definition of $\mathcal{B}$, f is an onto mapping. Lemmas (3.13) and (3.16) imply that f is one-to-one. By Definition (3.18) f is an isomorphism from E onto $\mathcal{B}$. Since $\mathcal{B}$ and E are isomorphic and E is a Boolean algebra, it follows that $\mathcal{B}$ is a Boolean algebra. ∎

(3.2.1) Definition: Let $(X, \mathcal{J})$ be a topological space. A collection $\mathcal{B}$ of open sets in X is called an open base for the topology $\mathcal{J}$ if every set in $\mathcal{J}$ is the union of some subcollection of sets in $\mathcal{B}$. A collection $\mathcal{V}$ of closed sets in X is called a closed base for the topology $\mathcal{J}$ if every closed set in X is the intersection of some subcollection of sets in $\mathcal{V}$. A collection $\mathcal{B}$ which is both an open base and a closed base for $\mathcal{J}$ is said to be an open-closed base for $\mathcal{J}$. A set $A \subseteq X$ which is both open and closed is said to be an open-closed set in X.

(3.22) Definition: A topological space X is said to be totally

<u>disconnected</u> if for any two distinct points $x, y \in X$ there exists an open-closed set $F \subseteq X$ such that $x \in F$ and $y \notin F$.

(3.22.1) <u>Remark</u>: An immediate consequence of the above definition is that a totally disconnected T_1-space is necessarily a Hausdorff space.

(3.23) <u>Lemma</u>: If X is a T_1-space such that any closed subset of X is the intersection of open-closed subsets of X, then X is totally disconnected.

<u>Proof</u>: If X contains only one point the theorem is trivially true. Otherwise, let x and y be two distinct points in X. Then the set $\{x\}$ is closed since X is a T_1-space. By hypothesis there exists a collection $\{F_i : i \in I\}$ of open-closed sets in X such that $\bigcap_{i \in I} F_i = \{x\}$. Since $y \neq x$ there exists some open-closed set F_i in the above collection such that $y \notin F_i$ and $x \in F_i$. This implies that X is totally disconnected. ∎

(3.24) <u>Theorem</u>: Let E, S, $\mathcal{J}$, and $\mathcal{B}$ be as defined in Lemma (3.20). Then $\mathcal{B}$ is exactly the collection of all open-closed subsets of S.

<u>Proof</u>: Let $B \in \mathcal{B}$. Then $B = B_a$ for some $a \in E$. By Definition (3.6) and Lemmas (3.8), (3.15), and (3.20), $\emptyset = B_0 = B_{a \wedge a'} = B_a \cap B_{a'}$ and $S = B_1 = B_{a \vee a'} = B_a \cup B_{a'}$. This implies that $B_{a'} = B_a^c$. Since $(a')' = a$, the relation $B_a = B_{a'}^c$ follows. Hence $B_a \in \mathcal{B}$ if and only if $B_a^c \in \mathcal{B}$. By the definition of $\mathcal{J}$ given by Lemmas (3.9) and (3.10) each set $B_a \in \mathcal{B}$ is closed. The preceding arguments imply that each set in $\mathcal{B}$ is the complement of some other set in $\mathcal{B}$. It follows that each set in $\mathcal{B}$ is an open-closed subset of S.

By the definition of the topology $\mathcal{J}$ on S each set $G \varepsilon \mathcal{J}$ is of the form $G = \bigcup_{a \varepsilon A} B_a^c$ where $A \subseteq E$. Let F be any open-closed subset of S. Since F is open, $F = \bigcup_{a \varepsilon A} B_a^c$ for some $A \subseteq E$. By Theorem (3.11) S is compact. Since F is a closed subset of a compact space, F is compact. The collection $\{B_a^c : a \varepsilon A\}$ is an open cover of F. By compactness of F there exists a finite subcollection $B_{a_1}^c, \ldots, B_{a_n}^c$ such that

$$F = \bigcup_{i=1}^{n} B_{a_i}^c = \bigcup_{i=1}^{n} B_{a_i'} = B_{\bigvee_{i=1}^{n} a_i'} = B_{\left(\bigwedge_{i=1}^{n} a_i\right)'} \varepsilon \mathcal{B} . \blacksquare$$

(3.2.4.1) <u>Corollary</u>: The collection $\mathcal{B}$ is an open-closed base for the topology $\mathcal{J}$ on S.

<u>Proof</u>: As noted in the proof of the preceding theorem, Lemmas (3.9) and (3.10) together imply that each closed set in S is the intersection of sets in $\mathcal{B}$. Thus $\mathcal{B}$ is a closed base for the topology $\mathcal{J}$ on S. Since $\mathcal{B}$ is closed under complementation, it follows that $\mathcal{B}$ is also an open base for $\mathcal{J}$. The results of this chapter are summarized in the following theorem.

(3.2.5) <u>The Stone Representation Theorem</u>. Let E be a Boolean algebra and let S be the space whose points are the maximal ideals in E. For each $a \varepsilon E$ define the set B_a in S to be the set of all maximal ideals containing a. Let $\mathcal{B} = \{B_a : a \varepsilon E\}$. Then $\mathcal{B}$ is an open-closed base for the uniquely determined topology $\mathcal{J}$ on S defined by

$$\mathcal{J} = \left\{ \bigcup_{a \varepsilon A} B_a : A \subseteq E \right\} .$$

In the topology $\mathcal{J}$ the space S is a compact, totally disconnected Hausdorff space. The collection $\mathcal{B}$ partially ordered by set inclusion is a Boolean algebra. The mapping f defined on E by $f(a) = B_a$ for each $a \in E$ is an isomorphism from E onto $\mathcal{B}$.

Proof: By Corollary (3.24.1) the above definition of the topology on S agrees with the definition of $\mathcal{J}$ given by Lemmas (3.9) and (3.10). Since a non-empty collection of sets in a non-empty space can be an open base for at most one topology on the space, it follows that the topology $\mathcal{J}$ is uniquely determined by the base $\mathcal{B}$. By Lemma (3.11), S is a compact T_1-space. By Lemmas (3.9), (3.10), (3.23) and (3.24), S is totally disconnected. It follows from (3.22.1) that S is Hausdorff. The concluding assertion of the theorem is a re-statement of Lemma (3.20). ∎

For future reference the following properties of the isomorphism are listed below.

(3.25.1) Corollary: If $a, b \in E$ then

(3.25.1.1) $$B_{a \vee b} = B_a \cup B_b$$

(3.25.1.2) $$B_{a \wedge b} = B_a \cap B_b$$

(3.25.1.3) $$B_{a'} = B_a^c$$

(3.25.1.4) $$B_a \subseteq B_b \Longleftrightarrow a \leqq b$$

Proof: The first two identities follow from Lemma (3.8). The third identity follows from the proof of Theorem (3.24). The fourth identity follows from Lemmas (3.19) and (3.20).

CHAPTER IV

A REPRESENTATION THEOREM BASED ON A THEOREM BY KAKUTANI

If $(X, \mathcal{A}, \mu)$ is a complete measure space, the L space of the measure space is the space of all equivalence classes of functions whose integrals, with respect to the measure μ, exist and are finite. Two functions are said to be equivalent if they agree almost everywhere on X with respect to μ. In Chapter IV it is proved that the L space of any complete measure space is isometric and isomorphic, as a normed linear space, to the L space of a complete measure space whose σ-ring of measurable sets is a completion of the Baire σ-ring in some locally compact Hausdorff space. This measure space has the property that all continuous functions which vanish at infinity are measurable.

The statement and proof of this theorem are based upon a much more general theorem of Kakutani. In his original paper, Kakutani proves that certain types of partially ordered Banach spaces are isomorphic and isometric, as normed linear spaces, to L spaces of complete measure spaces in locally compact, totally disconnected Hausdorff spaces.

The representation theorem of this Chapter is less general than the original theorem of Kakutani. For this reason a more elementary proof is possible. The theorem is formulated in measure-theoretic terms and is proved largely by appeals to measure-theoretic properties of the spaces concerned. The proof of the original theorem employs an elaborate superstructure manufactured from algebraic and order-theoretic properties of the space to be represented. The present proof does not seem to appear anywhere else in the literature.

(4.1) Conventions to be Used in this Chapter

The symbol X will denote a fixed, non-empty space, $\mathcal{R}$, a σ-ring of subsets of X, and μ, a measure defined on all sets in $\mathcal{R}$. The ordered triplet $(X, \mathcal{R}, \mu)$ will be assumed to be a complete measure space. Sets in $\mathcal{R}$ will be called measurable sets, and X will be called the ground set of the measure space $(X, \mathcal{R}, \mu)$.

The symbol $L(X)$ will denote the L space of X as defined in the introduction. Elements of $L(X)$ will be denoted by symbols such as $[f]$, and will be called functions in $L(X)$. It should be noted that a function $[f]$ in $L(X)$ is actually an equivalence class of μ-summable functions g defined on X such that $f = g$ almost everywhere (a.e.) on X with respect to the measure μ.

A function $[f]$ will be called a non-negative measurable simple function in $L(X)$ whenever a member $\hat{f}$ of the equivalence class $[f]$ is a non-negative measurable simple function on X such that $\int \hat{f}\, d\mu$ is finite. By this convention if $B \in R$ and $\mu(B) < +\infty$, then the symbol $[K_B]$ will be called the characteristic function of B in $L(X)$ and will denote the equivalence class of all measurable functions equal almost everywhere to the characteristic function K_B of B.

The symbols $[f] < [g]$ and $[f] \leq [g]$ will mean respectively that $f(x) < g(x)$ a.e. on X and $f(x) \leq g(x)$ a.e. on X for all functions $f \in [f]$ and $g \in [g]$. It is easily verified that the relations "$<$" and "$\leq$" are transitive and that if $[f] \leq [g]$ and $[g] \leq [f]$, then $[g] = [f]$.

To avoid trivialities it will be assumed throughout this chapter that there exists some set $A \varepsilon \mathcal{Q}$ such that $0 < \mu(A) < +\infty$. The L spaces of measure spaces for which such a statement is not valid consist of the zero function alone and are trivially isomorphic and isometric to each other.

If $(Y, \mathcal{Q}, \nu)$ is any measure space and $A, B \varepsilon \mathcal{Q}$, then the symbol $A \underset{\nu}{=} B$ will mean $\nu(A \Delta B) = 0$, where $A \Delta B = (A - B) \cup (B - A)$ is the <u>symmetric</u> <u>difference</u> of A and B. Whenever $A \underset{\nu}{=} B$ it will be said that A and B are <u>equal mod ν</u>. It is easily verified that equality mod ν is symmetric, reflexive, and transitive on $\mathcal{Q}$ so that equality mod ν is an equivalence relation on $\mathcal{Q}$.

(4.2) <u>Lemma</u>: Let $\mathcal{F}$ denote the family of all collections $\mathcal{N}$ of sets in $\mathcal{Q}$ such that

(4.2.1) $\mathcal{N}$ is non-empty,

(4.2.2) if $A \varepsilon \mathcal{N}$, then $0 < \mu(A) < +\infty$,

(4.2.3) if $A, B \varepsilon \mathcal{N}$, then $\mu(A \cap B) = 0$.

Then $\mathcal{F}$ has a maximal element relative to the partial order induced on $\mathcal{F}$ by the set inclusion relation.

<u>Proof</u>: The collection $\mathcal{F}$ is non-empty by the assumption adopted in (4.1) to exclude the trivial case in which L consists only of the zero function. Let $\mathcal{C} = \{\mathcal{N}_i : i \varepsilon I\}$ be a chain in $\mathcal{F}$. If A and B are sets in $\bigcup_{i \varepsilon I} \mathcal{N}_i$, then $A \varepsilon \mathcal{N}_i$ and $B \varepsilon \mathcal{N}_j$ for some $i, j \varepsilon I$. Since $\mathcal{C}$ is a chain this implies that either $\mathcal{N}_i \subseteq \mathcal{N}_j$ or $\mathcal{N}_j \subseteq \mathcal{N}_i$ and

consequently that A and B satisfy (4.2.2) and (4.2.3). Since $\bigcup_{i\in I} \mathcal{N}_i$ is clearly non-empty, it is a member of $\mathcal{F}$ and an upper bound in $\mathcal{F}$ for $\mathcal{C}$. Zorn's lemma implies that $\mathcal{F}$ has a maximal element. ∎

(4.3) Throughout this chapter $\mathcal{M} = \{A_t : t \in T\}$ will denote a fixed maximal element of $\mathcal{F}$.

(4.4) <u>Lemma</u>: Let $\{B_n\}_{n=1}^{\infty}$ be a collection of sets in the σ-ring $\mathcal{R}$ such that $\mu(B_n) < +\infty$ for each $n = 1,2,\ldots,$ and $\mu(B_n \cap B_m) = 0$ if $m \neq n$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) .$$

<u>Proof</u>: A short calculation shows that if $A, B \in \mathcal{R}$ and $\mu(A \cap B) < +\infty$, then

(4.4.1) $$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

Set $A_1 = B_1$ and $A_n = B_n - \bigcup_{k=1}^{n-1} B_k$, for $n > 1$. Then $A_n \cap A_m = \emptyset$ if $m \neq n$, $\{A_n\}_{n=1}^{\infty}$ is a collection of sets in $\mathcal{R}$,

$$\sum_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k, \text{ for every } n \geq 1, \text{ and } \sum_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k .$$

Note that $\mu(A_1) = \mu(B_1)$. Suppose now that

$$\sum_{k=1}^{n} \mu(B_k) = \sum_{k=1}^{n} \mu(A_k) .$$

Then

$$\sum_{k=1}^{n+1} \mu(B_k) = \mu(B_{n+1}) + \sum_{k=1}^{n} \mu(A_k) = \mu(B_{n+1}) + \mu\left(\sum_{k=1}^{n} A_k\right) =$$

$$= \mu(B_{n+1}) + \mu\left(\bigcup_{k=1}^{n} B_k\right).$$

But, using subadditivity of μ,

$$0 \leq \mu\left(B_{n+1} \cap \left(\bigcup_{k=1}^{n} B_k\right)\right) \leq \sum_{k=1}^{n} \mu(B_k \cap B_{n+1}) = 0,$$

so that, by (4.4.1),

$$\mu(B_{n+1}) + \mu\left(\bigcup_{k=1}^{n} B_k\right) = \mu\left(\bigcup_{k=1}^{n+1} B_k\right) = \mu\left(\sum_{k=1}^{n+1} A_k\right) = \sum_{k=1}^{n+1} \mu(A_k),$$

and hence $\sum_{k=1}^{n+1} \mu(B_k) = \sum_{k=1}^{n+1} \mu(A_k)$.

By induction it follows that for every $n = 1,2,\ldots,$

$$\sum_{k=1}^{n} \mu(B_k) = \sum_{k=1}^{n} \mu(A_k),$$

and hence, taking limits on both sides,

$$\sum_{k=1}^{\infty} \mu(B_k) = \sum_{k=1}^{\infty} \mu(A_k) = \mu\left(\sum_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} B_k\right). \blacksquare$$

(4.5) Lemma: If $A \in \mathcal{A}$, $0 < \mu(A) < +\infty$, and if $\mathcal{M}$ is as defined in (4.3), then there exists a uniquely determined countable collection of indices $\{t_n\}_{n=1}^{\infty} \subseteq T$, depending on A, such that $A \supseteq \bigcup_{n=1}^{\infty} (A \cap A_{t_n})$, and such that

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap A_{t_n}) .$$

Proof: Let $A \in \mathcal{Q}$, $0 < \mu(A) < +\infty$ and $\mu(A \cap A_t) = 0$ for all $t \in T$. If $A \notin \mathcal{M}$, then $\mathcal{M} \cup \{A\}$ is a member of the family $\mathcal{F}$ which properly contains $\mathcal{M}$. This is impossible since $\mathcal{M}$ is a maximal member of $\mathcal{F}$. Hence $A \in \mathcal{M}$ and thus $\mu(A \cap A_t) > 0$ for at least one index $t \in T$. This implies that if $A \in \mathcal{Q}$ and $0 < \mu(A) < +\infty$ then $\mu(A \cap A_t) > 0$ for at least one index $t \in T$.

For each $n \geq 1$, let $C_n = \left\{t : \mu(A \cap A_t) \geq \frac{1}{n}\right\}$. If for some n, C_n is infinite, there would exist a sequence $\{t'_m\}_{m=1}^{\infty}$ of distinct indices in C_n such that $\bigcup_{m=1}^{\infty} (A \cap A_{t'_m}) \subseteq A$. Since $\mu(A) < +\infty$, Lemma (4.4) implies that

$$\mu(A) \geq \sum_{m=1}^{\infty} \mu(A \cap A_{t'_m}) ,$$

which is impossible, since $\mu(A \cap A_{t'_m}) \geq \frac{1}{n}$, for $m = 1,2,\ldots$. Hence C_n is finite for each n and thus $\bigcup_{n=1}^{\infty} C_n$ is countable.

Since $\bigcup_{n=1}^{\infty} C_n = \{t : \mu(A \cap A_t) > 0\}$, it follows that $\mu(A \cap A_t) = 0$ for all $t \notin \bigcup_{n=1}^{\infty} C_n$.

Let the non-empty, countable collection $\{t_n\}_{n=1}^{\infty} = \bigcup_{n=1}^{\infty} C_n$ be assigned to A, and consider the set $A - \bigcup_{n=1}^{\infty} (A \cap A_{t_n})$. If $t \in \{t_n\}_{n=1}^{\infty}$, then $A_t \cap (A - \bigcup_{n=1}^{\infty} (A \cap A_{t_n})) \subseteq A_t \cap (A - (A \cap A_t)) = \emptyset$. If $t \in T - \{t_n\}_{n=1}^{\infty}$, then $A_t \cap (A - \bigcup_{n=1}^{\infty} (A \cap A_{t_n})) \subseteq A_t \cap A$, so that $\mu(A_t \cap (A - \bigcup_{n=1}^{\infty} (A \cap A_{t_n}))) \leq \mu(A \cap A_t) = 0$.

Thus for each $t \in T$, $\mu(A_t \cap (A - \bigcup_{n=1}^{\infty} (A \cap A_{t_n}))) = 0$. Consider the set $B = A - \bigcup_{n=1}^{\infty} (A \cap A_{t_n})$. Now $\mu(B) < +\infty$ since $B \subset A$. If $\mu(B) > 0$, then $B \notin \mathcal{M}$, since $\mu(A_t \cap B) = 0$ for each $t \in T$, and if B were in $\mathcal{M}$ then $\mu(A_t \cap B)$ would be strictly positive for at least one index $t \in T$. Since $\mu(B) < +\infty$, the preceding implies that if $\mu(B) > 0$ then $B \notin \mathcal{M}$ and hence $\mathcal{M} \cup \{B\}$ would be a member of $\mathcal{F}$ which properly contains $\mathcal{M}$. This is impossible because $\mathcal{M}$ is a maximal member of $\mathcal{F}$. Thus $\mu(B) = \mu(A - \bigcup_{n=1}^{\infty} (A \cap A_{t_n})) = 0$. This implies that $A \doteq \bigcup_{n=1}^{\infty} (A \cap A_{t_n})$. Since $\mu(A) < +\infty$, it follows that $\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n})\right)$, and hence by Lemma (4.4) that $\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap A_{t_n})$. ∎

(4.6) <u>Lemma</u>: Under the order relation "$<$" defined in (4.1) the linear space $L(X)$ becomes a lattice. For each $t \in T$ the collection L_t of all characteristic functions $[K_{C_t}]$ in $L(X)$ of measurable subsets of A_t is a sub-lattice of $L(X)$. With the definitions $1_t = [K_{A_t}]$ and $0_t = [K_\emptyset]$, L_t becomes a Boolean σ-algebra for which the following relations are valid (where all sets concerned are measurable subsets of A_t):

(4.6.1) $$\bigvee_{n=1}^{\infty} [K_{C_n}] = \left[K_{\bigcup_{n=1}^{\infty} C_n} \right]$$

(4.6.2) $$\bigwedge_{n=1}^{\infty} [K_{C_n}] = \left[K_{\bigcup_{n=1}^{\infty} C_n} \right]$$

(4.6.3) $$[K_C]' = [K_{A_t - C}]$$

Proof: If f and g are μ-summable functions defined on X, let $(f \wedge g)(x) = \text{g.l.b.}\ (f(x), g(x))$ and $(f \vee g)(x) = \text{l.u.b.}\ (f(x), g(x))$ for all $x \varepsilon X$. The functions $f \wedge g$ and $f \vee g$ are also summable and are, respectively, the greatest lower bound and least upper bound for the functions f and g. This implies that $[f] \wedge [g] = \text{g.l.b.}([f], [g]) = [f \wedge g]$ and $[f] \vee [g] = \text{l.u.b.}\ ([f], [g]) = [f \vee g]$. Thus any pair of functions in $L(X)$ has a greatest lower bound and a least upper bound in $L(X)$ so that $L(X)$ is a lattice.

That for each $t \varepsilon T$, L_t is a sublattice of $L(X)$ is an immediate consequence of the following properties of characteristic functions of measurable sets

(4.6.4) $$K_{\bigcup_{n=1}^{\infty} B_n} = \underset{n \geq 1}{\text{l.u.b.}} \left\{ K_{B_n} \right\},$$

(4.6.5) $$K_{\bigcap_{n=1}^{\infty} B_n} = \underset{n \geq 1}{\text{g.l.b.}} \left\{ K_{B_n} \right\}.$$

These two identities also imply the validity of (4.6.1) and (4.6.2).

If $B \subsetneq A_t$ then $[K_B] \wedge [K_{A_t - B}] = [K_{B \cap (A_t - B)}] = [K_\emptyset] = 0_t$ and $[K_B] \vee [K_{A_t - B}] = [K_{B \cup (A_t - B)}] = [K_{A_t}] = 1_t$, so that (4.6.3) is valid.

The preceding relations together with the fact that the set intersection and set union relations are distributive with respect to each other imply that L_t is a Boolean algebra for each $t \varepsilon T$. ∎

(4.7) Remark: For each $t \varepsilon T$ the set A_t has finite strictly positive measure, so that $[K_{A_t}] > [0]$ and $[K_{A_t}] \varepsilon L(X)$. This implies that the Boolean algebra L_t contains more than one point. By the Stone representation theorem, the Stone space of L_t exists for each $t \varepsilon T$. The symbol S_t will denote the Stone space of L_t; $\mathcal{B}_t$ will denote the collection of all open-closed subsets of S_t; and $\mathcal{J}_t$ will denote the topology on S_t generated by the open-closed base $\mathcal{B}_t$. For each $[K_{C_t}] \varepsilon L_t$, $B_{[K_{C_t}]}$ will denote the unique set in $\mathcal{B}_t$ corresponding to the function $[K_{C_t}]$ in L_t under the Stone representation theorem. When no confusion can arise symbols such as $B_{K_{C_t}}$ will also be used.

The following properties of the correspondence $[K_{C_t}] \longleftrightarrow B_{K_{C_t}}$ are listed below for future reference. The index t is a fixed but otherwise arbitrary element of T.

(4.7.1) $B_{K_{C_t}} = B_{K_{D_t}}$ if and only if $[K_{C_t}] = [K_{D_t}]$,

(4.7.2) $B_{K_{C_t}} \cap B_{K_{D_t}} = B_{K_{C_t} \wedge K_{D_t}} = B_{K_{C_t \cap D_t}}$,

(4.7.3) $B_{K_{C_t}} \cup B_{K_{D_t}} = B_{K_{C_t} \vee K_{D_t}} = B_{K_{C_t \cup D_t}}$,

(4.7.4) $B_{1_t} = S_t$ and $B_{o_t} = \emptyset$,

(4.7.5) $S_t - B_{K_{C_t}} = B_{K'_{C_t}} = B_{K_{A_t - C_t}}$

(4.7.6) If $[K_{C_t}] \leq [K_{D_t}]$, then $B_{K_{C_t}} \subseteq B_{K_{D_t}}$.

All of these properties are direct consequences of the Stone representation theorem and of Lemma (4.6).

(4.8) Lemma: If $t, t' \in T$ and $t \neq t'$, then $S_t \cap S_{t'} = \emptyset$.

Proof: Suppose that $[K_{C_t}] \in L_t \cap L_{t'}$. Then $[K_{C_t}] \leq [K_{A_t}]$ and $[K_{C_A}] \leq [K_{A_{t'}}]$, so that $[K_{C_t}] \leq [K_{A_t}] \wedge [K_{A_{t'}}] = [K_{A_t} \wedge K_{A_{t'}}] = [K_{A_t \cap A_{t'}}]$. But $\mu(A_t \cap A_{t'}) = 0$, so that $[K_{A_t \cap A_{t'}}] = [0]$, and hence $[K_{C_t}] = [0]$. By the Stone representation theorem, the points in each Stone space S_t are maximal ideals in L_t and hence do not contain $0_t = [0]$. Since $L_t \cap L_{t'} = \{[0]\}$ for $t \neq t'$, it follows that no point in S_t is in $S_{t'}$ and no point in $S_{t'}$ is in S_t. This implies $S_{t'} \cap S_t = \emptyset$. ∎

(4.9) Lemma: Let $S = \bigcup_{t \in T} S_t$, $\mathcal{B} = \bigcup_{t \in T} \mathcal{B}_t$ and

$$\mathcal{J} = \left\{ \bigcup_{G \in \mathcal{K}} G : \mathcal{K} \subseteq \bigcup_{t \in T} \mathcal{J}_t \right\}.$$

Then $\mathcal{J}$ is a topology on S, and the topological space $(S, \mathcal{J})$ is a locally compact, totally disconnected Hausdorff space. Each set in $\mathcal{B}$ is both open and closed and $\mathcal{B}$ is an open base for the topology $\mathcal{J}$ on S.

Proof: $\mathcal{J}$ is a Topology on S.

Clearly $\emptyset \in \mathcal{J}$ and $S \in \mathcal{J}$. By definition, unions of subcollections of $\mathcal{J}$ are in $\mathcal{J}$. If $A, B \in \mathcal{J}$, then $A = \bigcup_{G \in \mathcal{K}_1} G$ and $B = \bigcup_{H \in \mathcal{K}_2} H$, where $\mathcal{K}_1, \mathcal{K}_2 \subseteq \bigcup_{t \in T} \mathcal{J}_t$. This implies that $A \cap B = \bigcup_{G \in \mathcal{K}_1, H \in \mathcal{K}_2} (G \cap H)$. For any $t \in T$ either $G, H \in \mathcal{T}_t$; in which case $G \cap H \in \mathcal{T}_t$; or $G \in \mathcal{T}_t$ and $H \in \mathcal{T}_{t'}$, where $t' \neq t$; in which case $G \cap H \subseteq S_t \cap S_{t'} = \emptyset$. Thus the collection $\{G \cap H : G \in \mathcal{K}_1, H \in \mathcal{K}_2\}$ is a subcollection of $\bigcup_{t \in T} \mathcal{T}_t$ and hence $A \cap B \in \mathcal{T}$. Since $\mathcal{T}$ contains $\emptyset$ and S and is closed under arbitrary unions and finite non-empty intersections, $\mathcal{T}$ is a topology on S.

Each Set in $\mathcal{B}$ Is Both Open and Closed and S Is Totally Disconnected.

For each t_o all members of $\mathcal{B}_{t_o}$ are in $\mathcal{T}_{t_o}$ and hence in $\mathcal{T}$. But if $B_{K_{C_{t_o}}} \in \mathcal{B}_{t_o}$, then $S - B_{K_{C_{t_o}}} = (S_{t_o} - B_{K_{C_{t_o}}}) \cup \left(\bigcup_{t \neq t_o} S_t\right)$. Since $S_{t_o} - B_{K_{C_{t_o}}} = B_{K_{A_{t_\alpha} - C_{t_o}}} \in \mathcal{B}_{t_o}$, $S - B_{C_{t_o}}$ is the union of a collection of open sets and is hence open. This implies that each set in $\mathcal{B}_t$ is both open and closed, for any $t \in T$.

Let $p_1, p_2 \in S$ and $p_1 \neq p_2$. Then either $p_1 \in S_t$ and $p_2 \in S_{t'}$, where $t \neq t'$, or $p_1, p_2 \in S_t$, for some $t \in T$. In the former case $S_t \in \mathcal{B}_t$ is an open-closed set containing p_1 and not containing p_2. In the latter case, by Theorem (3.25) there exists an open-closed set $B_{K_{C_t}} \in \mathcal{B}_t$ such that $p_1 \in B_{K_{C_t}}$ and $p_2 \notin B_{K_{C_t}}$. Thus if p_1 and p_2 are any two distinct points in S, there exists an open-closed set containing one and not containing the other. By Definition (3.22) S is totally disconnected.

S is a Hausdorff Space.

In view of (3.22.1) and the preceding paragraph it suffices to show that S is a T_1 space. That is, it suffices to show that sets in S consisting of only one point are closed. Since all sets in each $\mathcal{B}_t$ are, in particular, closed, the conclusion follows from Lemma (3.7).

$\mathcal{B}$ Is an Open Base for $\mathcal{T}$.

Let $H \varepsilon \mathcal{T}$. Then $H = \bigcup_{G \varepsilon \mathcal{K}} G$, where $\mathcal{K} \subseteq \bigcup_{t \varepsilon T} \mathcal{T}_t$. If $H = \emptyset$, then $H \varepsilon \mathcal{B}$; otherwise, let $x \varepsilon H$; then $x \varepsilon G$, where $G \varepsilon \mathcal{T}_t$, for some $t \varepsilon T$. Since $\mathcal{B}_t$ is an open base for $\mathcal{T}_t$, this implies that there exists a set $B_{K_{C_t}} \varepsilon \mathcal{B}_t \subseteq \mathcal{B}$ such that $x \varepsilon B_{K_{C_t}} \subseteq G \subseteq H$. It follows that $\mathcal{B}$ is an open base for $\mathcal{T}$.

S Is Locally Compact.

It suffices to show that for each $t \varepsilon T$, S_t is compact. Since $\mathcal{B}$ is an open base for $\mathcal{T}$ it suffices to show that for every open cover of S_t by sets in $\mathcal{B}$ there exists a finite subcover which also contains S_t.

Let $\{B_{K_{C_t}}\}$ be an open cover of S_t by sets in $\mathcal{B}$. Discard all sets in the above open cover which contain no points of S_t. What remains is an open cover of S_t by sets in $\mathcal{B}_t$. Since S_t is compact its original topology, for which $\mathcal{B}_t$ is an open base, there exists a finite subcover of $\{B_{K_{C_t}}\}$ which also covers S_t. ∎

(4.10) Lemma: Let $\sigma(\mathcal{B})$ denote the minimal σ-ring containing $\mathcal{B}$. Let the set function $\bar{m}$ be defined on $\mathcal{B}$ by $\bar{m}\,(B_{[K_{C_t}]}) = \mu(C_t)$ for every function $[K_{C_t}] \varepsilon L_t$ and every index $t \varepsilon T$. The set function $\bar{m}$ is a measure on $\mathcal{B}$. There exists a well-defined measure m, defined on $\sigma(\mathcal{B})$, such that $m(B) = \bar{m}(B)$ for every $B \varepsilon \mathcal{B}$.

Proof: If $B \varepsilon \mathcal{B}$ then there exists exactly one index $t \varepsilon T$ and exactly one function $[K_{C_t}] \varepsilon L_t$ such that $B = B_{[K_{C_t}]}$. For all functions $K_{D_t} \varepsilon [K_{C_t}]$ there follows $K_{D_t} = K_{C_t}$ a.e. on X with respect to μ, and hence $\mu(D_t) = \mu(C_t)$. This implies that the set function $\bar{m}$ is well-defined on $\mathcal{B}$.

Let $B_{K_{C_{t_1}}}$ and $B_{K_{D_{t_2}}}$ be sets in $\mathcal{B}$ such that $B_{K_{C_{t_1}}} \cap B_{K_{D_{t_2}}} = \emptyset$ and $B_{K_{C_{t_1}}} \cup B_{K_{D_{t_2}}} = B_{K_{E_{t_3}}} \varepsilon \mathcal{B}$. By definition of $\mathcal{B}$, the latter condition implies that $B_{K_{C_{t_1}}}, B_{K_{D_{t_2}}}, B_{K_{E_{t_3}}} \varepsilon \mathcal{B}_t$ for some $t \varepsilon T$, and hence $t_1 = t_2 = t_3 = t$ and $[K_{C_t}], [K_{D_t}], [K_{E_t}] \varepsilon L_t$. Theorems (3.25) and (4.7) imply that $B_{K_{C_t} \wedge K_{D_t}} = B_{K_{C_t}} \cap B_{K_{D_t}} = \emptyset$ and hence $[K_{C_t \cap D_t}] = [K_{C_t}] \wedge [K_{D_t}] = [0]$. It follows that $\mu(C_t \cap D_t) = 0$.

Thus $B_{K_{C_t} \vee K_{D_t}} = B_{K_{C_t}} \cup B_{K_{D_t}} = B_{K_{E_t}}$, and hence $[K_{C_t \cup D_t}] = [K_{C_t}] \vee [K_{D_t}] = [K_{E_t}]$. It follows from (4.4.1) that $\mu(E_t) = \mu(C_t \cup D_t) = \mu(C_t) + \mu(D_t)$.

The above results imply that

$$\bar{m}(B_{K_{C_t}} \cup B_{K_{D_t}}) = \bar{m}(B_{K_{C_t} \vee K_{D_t}}) = \bar{m}(B_{K_{C_t \cup D_t}}) =$$

$$= \mu(C_t \cup D_t) = \mu(C_t) + \mu(D_t) = \bar{m}(B_{K_{C_t}}) + \bar{m}(B_{K_{D_t}}).$$

Thus $\bar{m}$ is finitely additive on $\mathcal{B}$.

Now suppose that $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$, $B_n \cap B_m = \emptyset$ if $m \neq n$, and

$$\sum_{n=1}^{\infty} B_n = B \varepsilon \mathcal{B} .$$

By definition of $\mathcal{B}$ this implies that $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{B}_t$ for some fixed $t \in T$, and consequently that $B_n \subseteq S_t$ for all $n \geq 1$. By (4.9) S_t is compact and all sets B and B_n for all $n \geq 1$ are both open and closed. It follows that B is compact and $\{B_n\}_{n=1}^{\infty}$ is an open cover for B. This implies that there exists a finite subcover $\{B_{n_j}\}_{n_j=1}^{\infty}$ of $\{B_n\}_{n=1}^{\infty}$ such that $B = \sum_{n'=1}^{N} B_{n'}$. This is impossible unless only finitely many of the sets B_n are non-empty, since the collection $\{B_n\}_{n=1}^{\infty}$ is pairwise disjoint. Hence there exists no countably infinite, pairwise disjoint collection of non-empty sets in $\mathcal{B}$ whose union is also in $\mathcal{B}$. The set function $\bar{m}$ is thus (vacuously) countably additive on $\mathcal{B}$. This implies that $\bar{m}$ is a measure on $\mathcal{B}$.

A simple computation shows that $\mathcal{B}$ is a semi-ring. Hence by the Carathéodory extension theorem[1] for semi-rings there exists a well-defined m, defined on $\sigma(\mathcal{B})$, and agreeing with $\bar{m}$ on $\mathcal{B}$. ∎

(4.10.1) <u>Definition</u>: The symbol $\sigma(\mathcal{B})$ will be used throughout this chapter to denote the minimal σ-ring containing the semi-ring $\mathcal{B}$.

(4.11) Let $\mathcal{B}_m$ be the σ-ring whose elements are subsets of S of the form $C \cup N$, where $C \in \sigma(\mathcal{B})$ and N is a subset of some set $D \in \sigma(\mathcal{B})$ such that $m(D) = 0$. The σ-ring $\mathcal{B}_m$ is called the <u>completion of $\sigma(\mathcal{B})$ for the measure m</u>. The set function $\hat{m}$ defined on $\mathcal{B}_m$ by $\hat{m}(B) = m(C)$, where $B = C \cup N$, is, by elementary measure theory, a well defined

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[1]See Glossary.

measure on $\mathcal{B}_m$. To avoid excessive notation the measure $\hat{m}$, defined above, will be identified with the measure m, so that $m(C \cup N) = m(C)$ whenever $C \in \sigma(\mathcal{B})$ and there exists some $D \in \sigma(\mathcal{B})$ such that $N \subseteq D$ and $\mu(D) = 0$. By elementary measure theory the <u>measure space</u> $(S, \mathcal{B}_m, m)$ is a <u>complete measure space</u>.

(4.12) <u>Definition</u>: Let P denote the collection of all characteristic functions in $L(X)$, and let the mapping Φ from P into $\sigma(\mathcal{B})$ be defined as follows:

If $[K_A] > 0$, let

$$\Phi([K_A]) = \sum_{n=1}^{\infty} B_{[K_{A \cap A_{t_n}}]},$$

where the collection $\{t_n\}_{n=1}^{\infty}$ of indices corresponding to A is defined, as in (4.5), to be the collection $\{t : t \in T \text{ and } \mu(A \cap A_t) > 0\}$.

Let

$$\Phi[0] = \emptyset .$$

(4.13) <u>Lemma</u>: The mapping Φ is well-defined on P into $\sigma(\mathcal{B})$.

<u>Proof</u>: Let $[K_A] = [K_C] \neq [0]$. Then $K_{A \Delta C} = |K_A - K_C| = 0$ a.e. on X and hence $\mu(A \Delta C) = 0$. This implies that $A \overset{\mu}{=} C$. Since $A \overset{\mu}{=} \bigcup_{n=1}^{\infty} (A \cap A_{t_n})$ and $C \overset{\mu}{=} \bigcup_{k=1}^{\infty} (C \cap A_{t'_k})$, the transitivity and symmetry of equality mod μ imply that $\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \overset{\mu}{=} \bigcup_{k=1}^{\infty} (C \cap A_{t'_k})$.

Now suppose that some $t'_{k_0} \notin \{t_n\}_{n=1}^{\infty}$. Then

$$0 < \mu(C \cap A_{t'_{k_0}}) = \mu\left[(C \cap A_{t'_{k_0}}) \cap \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n})\right)\right] +$$

$$+ \mu \left[(C \cap A_{t'_{k_o}}) \cap \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right)^c \right] \leqq$$

$$\leqq \sum_{n=1}^{\infty} \mu \left((C \cap A_{t'_{k_o}}) \cap (A \cap A_{t_n}) \right) + \mu \left[(C \cap A_{t'_{k_o}}) \cap \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right)^c \right]$$

$$\leqq \sum_{n=1}^{\infty} \mu (A_{t'_{k_o}} \cap A_{t_n}) + \mu \left[(C \cap A_{t'_{k_o}}) \cap \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right)^c \right]$$

$$= 0 + \mu \left[(C \cap A_{t'_{k_o}}) \cap \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right)^c \right]$$

$$\leqq \mu \left[\left(\bigcup_{k=1}^{\infty} (C \cap A_{t'_k}) \right) - \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right) \right]$$

$$\leqq \mu \left[\left(\bigcup_{k=1}^{\infty} (C \cap A_{t'_k}) \right) \Delta \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right) \right] .$$

This contradicts the fact that $\bigcup_{k=1}^{\infty} (C \cap A_{t'_k}) \doteq \bigcup_{n=1}^{\infty} (A \cap A_{t_n})$. It follows that $\{t'_k\}_{n=1}^{\infty} \subseteqq \{t_n\}_{n=1}^{\infty}$. The same arguments imply that $\{t_n\}_{n=1}^{\infty} \subseteqq \{t'_k\}_{k=1}^{\infty}$, and hence $\{t'_k\}_{k=1}^{\infty} = \{t_n\}_{n=1}^{\infty}$.

Suppose now that $\mu((C \cap A_{t_{n_o}}) - (A \cap A_{t_{n_o}})) > 0$ for some n_o. Then

$$0 < \mu((C \cap A_{t_{n_o}}) - (A \cap A_{t_{n_o}})) = \mu \left[((C \cap A_{t_{n_o}}) - \right.$$

$$\left. - (A \cap A_{t_{n_o}})) \cap \left(\bigcup_{n \neq n_o}^{\infty} (A \cap A_{t_n}) \right) \right] + \mu \left[((C \cap A_{t_{n_o}}) \right.$$

$$\left. - (A \cap A_{t_{n_o}})) \cap \left(\bigcup_{n \neq n_o}^{\infty} (A \cap A_{t_n}) \right)^c \right] \leqq$$

$$\leq \sum_{n \neq n_o} \mu \left[((C \cap A_{t_{n_o}}) - (A \cap A_{t_{n_o}})) \cap (A \cap A_{t_n}) \right] +$$

$$+ \mu \left[((C \cap A_{t_{n_o}}) \cap (A \cap A_{t_{n_o}})^c \cap \left(\bigcap_{n \neq n_o} (A \quad A_{t_n})^c \right) \right] \leq \sum_{n \neq n_o} \mu (A_{t_{n_o}} \cap A_{t_n}) +$$

$$+ \mu \left[(C \cap A_{t_{n_o}}) \cap \bigcap_{n=1}^{\infty} (A \cap A_{t_n})^c \right] = 0 + \mu \left[(C \cap A_{t_{n_o}}) \cap \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right)^c \right]$$

$$\leq \mu \left[\left(\bigcup_{n=1}^{\infty} (C \cap A_{t_n}) \right) \cap \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right)^c \right]$$

$$\leq \mu \left[\left(\bigcup_{n=1}^{\infty} (C \cap A_{t_n}) \right) \Delta \left(\bigcup_{n=1}^{\infty} (A \cap A_{t_n}) \right) \right] .$$

This contradicts the fact that $\bigcup_{n=1}^{\infty} (C \cap A_{t_n}) \overset{\mu}{=} \bigcup_{n=1}^{\infty} (A \cap A_{t_n})$. It follows that $\mu((C \cap A_{t_n}) - (A \cap A_{t_n})) = 0$ for each n. By the same argument $\mu((A \cap A_{t_n}) - (C \cap A_{t_n})) = 0$ for each n. Thus $C \cap A_{t_n} \overset{\mu}{=} A \cap A_{t_n}$ for each n.

This implies $[K_{A \cap A_{t_n}}] = [K_{C \cap A_{t_n}}]$ for each n and hence $\Phi[K_A] = \Phi[K_C]$.

Since the mapping Φ is obviously well-defined for $[K_A] = [0]$, this concludes the proof. ∎

(4.14) <u>Comment</u>: The preceding lemma actually proves more than its statement might seem to imply. More precisely, if $[K_A] \in P$, than any set C in $\mathcal{R}$ such that $C \overset{\mu}{=} A$ may be used to compute $\Phi[K_A]$. <u>Exactly</u> the same set will result in all cases. This result will henceforth be used without comment.

(4.15) Lemma: The mapping Φ has the following properties:

(4.15.1) If $[K_{A_1}] \leq [K_{A_2}]$, then $\Phi[K_{A_1}] \subseteq \Phi[K_{A_2}]$.

(4.15.2) $\mu(A) = m(\Phi[K_A])$

(4.15.3) If $[K_{A_1}] \wedge [K_{A_2}] = [0]$, then $\Phi[K_{A_1}] \cap \Phi[K_{A_2}] = \emptyset$.

(4.15.4) $\Phi[K_{A_1}] \cap \Phi[K_{A_2}] \overset{m}{=} \Phi([K_{A_1}] \wedge [K_{A_2}])$

(4.15.5) $\Phi[K_{A_1}] \cup \Phi[K_{A_2}] \overset{m}{=} \Phi([K_{A_1}] \vee [K_{A_2}])$

(4.15.6) $\Phi[K_{A_1}] - \Phi[K_{A_2}] \overset{m}{=} \Phi[K_{A_1 \cap A_2{}^c}]$

Proof: It suffices to consider only the case where $[K_A]$, $[K_{A_1}]$, $[K_{A_2}] > [0]$.

(4.15.1) If $[K_{A_1}] \leq [K_{A_2}]$, then for each $t \in T$ $[K_{A_1 \cap A_t}] =$

$= [K_{A_1}] \wedge [K_{A_t}] \leqq [K_{A_2}] \wedge [K_{A_t}] = [K_{A_2 \cap A_t}]$, so that by (4.7.6),

$B_{K_{A_1 \cap A_t}} \subseteq B_{K_{A_2 \cap A_t}}$ for all t, and hence $\Phi[K_{A_1}] \subseteq \Phi[K_{A_2}]$.

(4.15.2) By (4.5) $\mu(A) = \sum_{n=1}^{\infty} (A \cap A_{t_n}) = \sum_{n=1}^{\infty} m(B_{K_{A \cap A_{t_n}}})$

$$= m\left(\sum_{n=1}^{\infty} B_{K_{A \cap A_{t_n}}}\right) = m\, \Phi[K_A]\,.$$

(4.15.3) Let $\Phi[K_{A_1}] = \sum_{n=1}^{\infty} B_{K_{A_1 \cap A_{t_n}}}$ and let $\Phi[K_{A_2}] = \sum_{m=1}^{\infty} B_{K_{A_2 \cap A_{t'_m}}}$.

Then $\Phi[K_{A_1}] \cap \Phi[K_{A_2}] = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_{K_{A_1} \cap A_{t_n}} \cap B_{K_{A_2} \cap A_{t'_m}}$.

If $t'_m \neq t_n$ then $B_{K_{A_1} \cap A_{t_n}} \cap B_{K_{A_2} \cap A_{t'_m}} \subseteq S_{t_n} \cap S_{t'_m} = \emptyset$.

If $t'_m = t_n = t$ then $B_{K_{A_1} \cap A_t} \cap B_{K_{A_2} \cap A_t} = B_{K_{A_1} \cap A_{t_n} \wedge K_{A_2} \cap A_{t_n}} =$

$= B_{0_t} = \emptyset$. Thus if $[K_{A_1}] \wedge [K_{A_2}] = 0$, then $\Phi[K_{A_1}] \cap \Phi[K_{A_2}] = \emptyset$.

(4.15.4) The proof of this identity is analogous to the proofs of the following two identities and is omitted.

(4.15.5) It must be shown that $m((\Phi[K_{A_1}] \cup \Phi[K_{A_2}]) \,\Delta\, \Phi([K_{A_1}] \vee [K_{A_2}])) = 0$. Since (4.15.1) implies that $\Phi([K_{A_1}] \vee [K_{A_2}]) \supseteq \Phi[K_{A_1}] \cup \Phi[K_{A_2}]$ and since, by (4.6.1), $[K_{A_1}] \vee [K_{A_2}] = [K_{A_1 \cup A_2}]$, it suffices to show that

$$m(\Phi[K_{A_1 \cup A_2}] \cap (\Phi[K_{A_1}] \cup \Phi[K_{A_2}])^c) = 0.$$

Let $\Phi[K_{A_1}] = \sum_{n=1}^{\infty} B_{K_{A_1} \cap A_{t'_n}}$, $\Phi[K_{A_2}] = \sum_{n=1}^{\infty} B_{K_{A_2} \cap A_{t''_n}}$ and

$$\Phi[K_{A_1 \cup A_2}] = \sum_{n=1}^{\infty} B_{K_{(A_1 \cup A_2)} \cap A_{t_n}} = \sum_{n=1}^{\infty} B_{(K_{A_1} \cap A_{t_n}) \vee (K_{A_2} \cap A_{t_n})} =$$

$$= \sum_{n=1}^{\infty} (B_{K_{A_1} \cap A_{t_n}}) \cup (B_{K_{A_2} \cap A_{t_n}}),$$

where the collections $\{t_n\}_{n=1}^{\infty}$, $\{t'_n\}_{n=1}^{\infty}$, $\{t''_n\}_{n=1}^{\infty}$ are chosen in the manner specified in (4.12). By the procedure for choosing the above three collections of indices it is clear that $\{t_n\} = \{t'_n\} \cup \{t''_n\}$, and hence that each t_n is a member of either $\{t'_n\}$ or $\{t''_n\}$. If $t_n \notin \{t'_n\}$ then $m(B_{K_{A_1} \cap A_{t_n}}) = \mu(A_1 \cap A_{t_n}) = 0$. If $t_n \notin \{t''_n\}$ then $m(B_{K_{A_2} \cap A_{t_n}}) = \mu(A_2 \cap A_{t_n}) = 0$. This implies that

$$m\left[\left[(B_{K_{A_1} \cap A_{t_n}}) \cup (B_{K_{A_2} \cap A_{t_n}})\right] \cap \left(\bigcap_{n=1}^{\infty} B^c_{K_{A_1} \cap A_{t'_n}}\right) \cap \left(\bigcap_{n=1}^{\infty} B^c_{K_{A_2} \cap A_{t''_n}}\right)\right] = 0$$

for each $n \geq 1$, and consequently that

$$m(\Phi[K_{A_1 \cup A_2}] \cap (\Phi[K_{A_1}] \cup \Phi[K_{A_2}])^c) = 0 .$$

(4.15.6) Let $\Phi[K_{A_1}] = \sum_{n=1}^{\infty} B_{K_{A_1} \cap A_{t'_n}}$, $\Phi[K_{A_2}] = \sum_{n=1}^{\infty} B_{K_{A_2} \cap A_{t''_n}}$, and

$\Phi[K_{A_1 \cap A_2^c}] = \sum_{n=1}^{\infty} B_{K_{A_1} \cap A_2^c \cap A_{t_n}}$, where $\{t_n\}$, $\{t'_n\}$, and $\{t''_n\}$ are chosen as specified in (4.12).

It must be shown that $m((\Phi[K_{A_1}] - \Phi[K_{A_2}]) \triangle \Phi[K_{A_1 \cap A_2^c}]) = 0$.

But

$$(\Phi[K_{A_1}] - \Phi[K_{A_2}]) \triangle \Phi[K_{A_1 \cap A_2^c}] = [(\Phi[K_{A_1}] \cap (\Phi[K_{A_2}])^c \cap (\Phi[K_{A_1 \cap A_2^c}])^c] \cup$$

$$\cup [\Phi[K_{A_1 \cap A_2^c}] \cap (\Phi[K_{A_2}] \cup (\Phi[K_{A_1}])^c)] =$$

$$= [\Phi[K_{A_1}] \cap (\Phi[K_{A_2}])^c \cap (\Phi[K_{A_1 \cap A_2^c}])^c] \cup [\Phi[K_{A_1 \cap A_2^c}] \cap \Phi[K_{A_2}]] ,$$

since $\Phi[K_{A_1}] \supseteq \Phi[K_{A_1 \cap A_2^c}]$.

Let $(I) = \Phi[K_{A_1}] \cap (\Phi[K_{A_2}])^c \cap (\Phi[K_{A_1} \cap A_2^c])^c$ and

$(II) = \Phi[K_{A_1} \cap A_2^c] \cap \Phi[K_{A_2}]$.

Then

$$(II) = \left[\sum_{n=1}^{\infty} (B_{K_{A_1} \cap A_2^c \cap A_{t_n}})\right] \cap \left[\sum_{m=1}^{\infty} (B_{K_{A_2} \cap A_{t''_m}})\right] =$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[(B_{K_{A_1} \cap A_2^c \cap A_{t_n}}) \cap (B_{K_{A_2} \cap A_{t''_m}})\right] .$$

If $t_n \neq t''_m$, then

$$(B_{K_{A_1} \cap A_2^c \cap A_{t_n}}) \cap (B_{K_{A_2} \cap A_{t''_m}}) \subsetneq S_{t_n} \cap S_{t''_m} = \emptyset ,$$

where S_{t_n}, $S_{t''_m}$ are the Stone spaces of L_{t_n} and $L_{t''_m}$, defined in (4.7). If $t_n = t''_m$, then

$$(B_{K_{A_1} \cap A_2^c \cap A_{t_n}}) \cap (B_{K_{A_2} \cap A_{t''_m}}) = B_{K_{A_1} \cap A_2^c \cap A_2 \cap A_{t_n}} =$$

$$= B_{0_{t_n}} = \emptyset .$$

This implies that (II) is empty.

Upon expansion,

$$(I) = \left[\sum_{n=1}^{\infty} B_{K_{A_1} \cap A_{t'_n}}\right] \cap \left[\bigcap_{m=1}^{\infty} B^c_{K_{A_2} \cap A_{t''_m}}\right] \cap \left[\bigcap_{k=1}^{\infty} B^c_{K_{A_1} \cap A_2^c \cap A_{t_k}}\right].$$

For each t'_n, if $t'_n \notin \{t_n\} \cup \{t''_n\}$, then $\mu(A_1 \cap A_2^c \cap A_{t'_n}) = 0 = \mu(A_2 \cap A_{t'_n})$, so that $0 \leq \mu(A_1 \cap A_{t_n}) \leq \mu(A_1 \cap A_2^c \cap A_{t'_n}) + \mu(A_2 \cap A_{t'_n}) = 0$. This implies $[K_{A_1} \cap A_{t'_n}] = 0$ and hence $m(B_{K_{A_1} \cap A_{t'_n}}) = 0$.

If $t'_n \varepsilon \{t_n\} \cup \{t''_n\}$, then one of the following three cases occurs:

(a) $$t'_n \varepsilon \{t_n\} \cap \{t''_n\},$$

(b) $$t'_n \varepsilon \{t_n\} - \{t''_n\},$$

(c) $$t'_n \varepsilon \{t''_n\} - \{t_n\}.$$

In case (a) there are indices t_k and t''_m such that $t'_n = t_k = t''_m$, and

$$\left[B_{K_{A_1} \cap A_{t'_n}}\right] \cap \left[\bigcap_{m=1}^{\infty} B^c_{K_{A_2} \cap A_{t''_m}}\right] \cap \left[\bigcap_{k=1}^{\infty} B^c_{K_{A_1} \cap A_2^c \cap A_{t_k}}\right] \subseteq$$

$$\left[(B_{K_{A_1} \cap A_{t'_n}}) \cap (B^c_{K_{A_2} \cap A_{t''_m}})\right] \cap \left[\bigcap_{k=1}^{\infty} B^c_{K_{A_1} \cap A_2^c \cap A_{t_k}}\right] \subseteq$$

$$(B_{K_{A_1} \cap A_{t'_n}}) \cap (B^c_{K_{A_2} \cap A''_m}) \cap (B^c_{K_{A_1} \cap A_2^c \cap A_{t_k}}) =$$

$$= (B_{K_{A_1} \cap A_{t'_n}}) \cap (B^c_{K_{A_2} \cap A_{t'_n}}) \cap (B^c_{K_{A_1} \cap A_2^c \cap A_{t'_n}}) =$$

$$= (B_{K_{A_1} \cap A_{t'_n}}) \cap (S^c_{t'_n} \cup (S_{t'_n} - B_{K_{A_2} \cap A_{t'_n}})) \cap (S^c_{t_n} \cup (S_{t'_n} -$$

$$- B_{K_{A_1} \cap A_2^c \cap A_{t'_n}}) = (B_{K_{A_1} \cap A_{t'_n}}) \cap (B_{K_{A_{t'_n}}} - (A_2 \cap A_{t'_n}) \cap$$

$$(B_{K_{A_{t'_n}} - (A_1 \cap A_2^c \cap A_{t'_n})}) = B_{\left[K_{A_1 \cap A_{t'_n}} \wedge K_{A_2^c \cap A_{t'_n}} \wedge K_{(A_1 \cap A_2^c)^c \cap A_{t'_n}}\right]}$$

$$= B_{K_{(A_1 \cap A_2^c \cap (A_1 \cap A_2^c)^c) \cap A_{t'_n}}} = B_{o_{t'_n}} = \emptyset .$$

In case (b)

$$\left[B_{K_{A_1 \cap A_{t'_n}}}\right] \cap \left[\bigcap_{m=1}^{\infty} B^c_{K_{A_2 \cap A_{t''_m}}}\right] \cap \left[\bigcap_{k=1}^{\infty} B^c_{K_{A_1 \cap A_2^c \cap A_{t_k}}}\right] \subseteq$$

$$B_{K_{A_1 \cap A_{t'_n}}} \cap B^c_{K_{A_1 \cap A_2^c \cap A_{t'_n}}} = (B_{K_{A_1 \cap A_{t'_n}}}) \cap (B_{K_{A_{t'_n} \cap (A_1 \cap A_2^c)^c}}) =$$

$$= (B_{K_{A_1 \cap A_{t'_n}}}) \cap (B_{K_{(A_1^c \cap A_{t'_n}) \cup (A_2 \cap A_{t'_n})}}) =$$

$$= (B_{K_{A_1 \cap A_{t'_n}}}) \cap \left[(B_{K_{A_1^c \cap A_{t'_n}}}) \cup (B_{K_{A_2 \cap A_{t'_n}}})\right] =$$

$$= B_{o_t} \cup B_{K_{A_1 \cap A_2 \cap A_{t'_n}}} \subseteq B_{K_{A_2 \cap A_{t'_n}}} .$$

But $t'_n \notin \{t''_n\}$ and thus $\mu(A_2 \cap A_{t'_n}) = 0$. This implies that $m(B_{K_{A_2 \cap A_{t'_n}}}) = 0$, and hence that

$$m\left[\left[B_{K_{A_1 \cap A_{t'_n}}}\right] \cap \left[\bigcap_{m=1}^{\infty} B^c_{K_{A_2 \cap A_{t''_m}}}\right] \cap \left[\bigcap_{k=1}^{\infty} B^c_{K_{A_1 \cap A_2^c \cap A_{t_k}}}\right]\right] = 0.$$

In case (c)

$$(B_{K_{A_1}} \cap A_{t'_n}) \cap \left(\bigcap_{m=1}^{\infty} B^c_{K_{A_2}} \cap A_{t''_m}\right) \cap \left(\bigcap_{k=1}^{\infty} B^c_{K_{A_1}} \cap A_2^c \cap A_{t_k}\right) \subseteq$$

$$(B_{K_{A_1}} \cap A_{t'_n}) \cap (B^c_{K_{A_2}} \cap A_{t'_n}) = (B_{K_{A_1}} \cap A_{t'_n}) \cap (S^c_{t'_n} \cup (S_{t'_n} -$$

$$B_{K_{A_2}} \cap A_{t'_n})) = (B_{K_{A_1}} \cap A_{t'_n}) \cap (B_{K_{A_2^c}} \cap A_{t'_n}) = B_{K_{A_1}} \cap A_2^c \cap A_{t'_n}.$$

But $t'_n \notin \{t_n\}$ and hence $\mu(A_1 \cap A_2^c \cap A_{t'_n}) = 0$. It follows that $m(B_{K_{A_1}} \cap A_2^c \cap A_{t'_n}) = 0$ and hence that

$$m\left[\left[B_{K_{A_1}} \cap A_{t'_n}\right] \cap \left[\bigcap_{m=1}^{\infty} B^c_{K_{A_2}} \cap A_{t''_m}\right] \cap \left[\bigcap_{k=1}^{\infty} B^c_{K_{A_1}} \cap A_2^c \cap A_{t_k}\right]\right] = 0.$$

Thus in all possible cases (I) is composed of countably many measurable sets, any one of which is either empty or has measure zero relative to m. This implies that $m(I) = 0$. Since (II) has been shown to be empty, it follows that $m((I) \cup (II)) = 0$. In view of the preceding this implies that $m((\Phi[K_{A_1}] - \Phi[K_{A_2}]) \Delta \Phi[K_{A_1} \cap A_2{}^c]) = 0$. ∎

(4.16) <u>Lemma</u>: If $B \varepsilon \sigma(\mathcal{B})$ and $m(B) < +\infty$, then there exists a characteristic function $[K_A] \varepsilon P$ such that $\Phi[K_A] \overset{m}{=} B$.

<u>Proof</u>: In the Carathéodory extension procedure for semi-rings the measure m is defined on $\sigma(\mathcal{B})$ by:[1]

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[1]See Glossary, page 84.

$$(4.16.1)\quad m(B) = \inf\left\{\sum_{n=1}^{\infty} m(B_n) : B_n \in \mathcal{B},\ B_n \cap B_m = \emptyset \text{ if } m \neq n,\ \sum_{n=1}^{\infty} B_n \supseteq B\right\},$$

where $B \in \sigma(\mathcal{B})$.

If $B \in \sigma(\mathcal{B})$ then for each positive integer k there exists a pairwise disjoint collection $\{B_{k,n}\}_{n=1}^{\infty}$ of sets in $\mathcal{B}$ such that

$$B_k = \sum_{n=1}^{\infty} B_{k,n} \supseteq B \quad \text{and} \quad m(B) \leq \sum_{n=1}^{\infty} m(B_{k,n}) < m(B) + \frac{1}{k}.$$

By the Stone representation theorem, to each set $B_{k,n}$ there corresponds a function $[K_{E_{k,n}}]$ in L such that $B_{k,n} = B_{[K_{E_{k,n}}]}$. Since for each k the collection $\{B_{k,n}\}_{n=1}^{\infty}$ is pairwise disjoint, it follows that $[K_{E_{k,n}}] \wedge [K_{E_{k,m}}] = [K_{E_{k,n} \cap E_{k,m}}] = [0]$ if $m \neq n$ and hence $\mu(E_{k,n} \cap E_{k,m}) = 0$ if $m \neq n$.

Let $E_k = \bigcup_{n=1}^{\infty} E_{k,n}$. The set of all points in E_k which belong to more than one set $E_{k,n}$ is just $\bigcup_{m=1}^{\infty}\left(E_{k,m} \cap \left(\bigcup_{n\neq m}^{\infty} E_{k,n}\right)\right)$. But

$$\mu\left(E_{k,m} \cap \left(\bigcup_{n\neq m}^{\infty} E_{k,n}\right)\right) \leq \sum_{n\neq m}^{\infty} \mu(E_{k,m} \cap E_{k,n}) = 0$$

for each m, and hence $\mu\left(\bigcup_{m=1}^{\infty}\left(E_{k,m} \cap \left(\bigcup_{n\neq m}^{\infty} E_{k,n}\right)\right)\right) = 0$. This implies that $[K_{E_k}] = \left[\sum_{n=1}^{\infty} K_{E_{k,n}}\right]$.

Since $\mu(E_{k,n}) = m(B_{K_{E_{k,n}}})$ for each n, and since $m(B_k) = \sum_{n=1}^{\infty} m(B_{K_{E_{k,n}}})$, it follows from Lemma (4.4) that

$$\mu(E_k) = \sum_{n=1}^{\infty} \mu(E_{k,n}) = \sum_{n=1}^{\infty} m(B_{K_{E_{k,n}}}) = m(B_k) < m(B) + \frac{1}{k} < +\infty .$$

Thus $\mu(E_k) < +\infty$ and $[K_{E_k}] \varepsilon P$, so that $\Phi[K_{E_k}]$ exists.

Since each set $B_{k,n} = B_{K_{E_{k,n}}}$ is in $\mathcal{B}$, each $B_{K_{E_{k,n}}}$ is contained in exactly one $S_{t_{k,n}}$ and each $E_{k,n}$ is contained in exactly one $A_{t_{k,n}}$, where the sets $A_{t_{k,n}}$ are as defined in (4.3). This implies that the only indices $t \varepsilon T$ for which $\mu(E_k \cap A_t) > 0$ could occur are the indices $\{t_{k,n}\}_{n=1}^{\infty} \subseteq T$. It follows that

$$\Phi[K_{E_k}] \subseteq \sum_{n=1}^{\infty} B_{K_{E_{k,n}}}$$

and in fact

$$\Phi[K_{E_k}] \overset{m}{=} \sum_{n=1}^{\infty} B_{K_{E_{k,n}}} ,$$

because the only sets $B_{K_{E_{k,n}}}$ which could fail to appear in the expansion of $\Phi[K_{E_k}]$ are those for which $m(B_{K_{E_{k,n}}}) = 0 = \mu(E_{k,n})$. Hence $\Phi[K_{E_k}] \subseteq B_k$ and $\Phi[K_{E_k}] \overset{m}{=} B_k$.

Thus for each positive integer k, $m(B) \leq m(B_k) = m(\Phi[K_{E_k}]) < m(B) + \frac{1}{k}$, $\bigcap_{k=1}^{\infty} B_k \supseteq B$, and $m(B) \leq m\left(\bigcap_{k=1}^{\infty} B_k\right) < m(B) + \frac{1}{k}$.

This implies that $m(B) = m\left(\bigcap_{k=1}^{\infty} B_k\right)$.

Since

$$\bigcap_{k=1}^{\infty} B_k \supseteq \bigcap_{k=1}^{\infty} \Phi[K_{E_k}], \quad \text{and} \quad \bigcap_{k=1}^{\infty} B_k - \bigcap_{k=1}^{\infty} \Phi[K_{E_k}] \subseteq \bigcup_{k=1}^{\infty} (B_k - \Phi[K_{E_k}]),$$

it follows that

$$m\left(\bigcap_{k=1}^{\infty} B_k - \bigcap_{k=1}^{\infty} \Phi[K_{E_k}]\right) \leqq \sum_{k=1}^{\infty} m(B_k - \Phi[K_{E_k}]) = 0,$$

and hence that $\bigcap_{k=1}^{\infty} B_k \overset{m}{=} \bigcap_{k=1}^{\infty} \Phi[K_{E_k}]$.

Lemma (4.15), together with the fact that $\mu(E_k) = m(\Phi[K_{E_k}]) = m(B_k) < +\infty$ for each positive integer k, implies that for each positive integer n

$$\mu\left(\bigcap_{k=1}^{n} E_k\right) = m\left(\Phi\left[K_{\bigcap_{k=1}^{n} E_k}\right]\right) = m\left(\Phi[\bigwedge_{k=1}^{n}[K_{E_k}]]\right) = m\left(\bigcap_{k=1}^{n} \Phi[K_{E_k}]\right).$$

Thus

$$\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} \mu\left(\bigcap_{k=1}^{n} E_k\right) = \lim_{n\to\infty} m\left(\Phi\left[K_{\bigcap_{k=1}^{n} E_k}\right]\right) =$$

$$= \lim_{n\to\infty} m\left(\bigcap_{k=1}^{n} \Phi\left[K_{E_k}\right]\right) = m\left(\bigcap_{k=1}^{\infty} \Phi[K_{E_k}]\right) = m\left(\bigcap_{k=1}^{\infty} B_k\right) = m(B).$$

This implies that $\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = m(B)$. Since, by (4.15.2), $\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = m\left(\Phi\left[K_{\bigcap_{k=1}^{\infty} E_k}\right]\right)$, it follows that $m(B) = m\left(\Phi\left[K_{\bigcap_{k=1}^{\infty} E_k}\right]\right) =$

$$m\left(\bigcap_{k=1}^{\infty}\Phi[K_{E_k}]\right) = m\left(\bigcap_{k=1}^{\infty}B_k\right).$$

By (4.15.1) $\Phi\left[K_{\bigcap_{k=1}^{\infty}E_k}\right] \subseteq \bigcap_{k=1}^{\infty}\Phi[K_{E_k}]$, and since all sets concerned are of finite measure,

$$m\left(\Phi\left[K_{\bigcap_{k=1}^{\infty}E_k}\right] - B\right) \leqq m\left(\bigcap_{k=1}^{\infty}\Phi[K_{E_k}] - B\right) \leq m\left(\bigcap_{k=1}^{\infty}B_k - B\right) =$$

$$= m\left(\bigcap_{k=1}^{\infty}B_k\right) \; m(B) = 0 .$$

Also

$$m\left(B - \Phi\left[K_{\bigcap_{k=1}^{\infty}E_k}\right]\right) \leq m\left(\bigcap_{k=1}^{\infty}B_k - \Phi\left[K_{\bigcap_{k=1}^{\infty}E_k}\right]\right) = m\left(\bigcap_{k=1}^{\infty}B_k\right) -$$

$$- m\left(\Phi\left[K_{\bigcap_{k=1}^{\infty}E_k}\right]\right) = 0 .$$

This implies that $m\left(B \,\Delta\, \Phi\left[K_{\bigcap_{k=1}^{\infty}E_k}\right]\right) = 0$ and concludes the proof. ∎

(4.17) Definition: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{a}, m)$ be two complete measure spaces. An isometric isomorphism from $L(X)$ onto $L(Y)$ is a one-to-one transformation F from $L(X)$ onto $L(Y)$ such that

(4.17.1) $$F([f] + [g]) = F[f] + F[g] ,$$

(4.17.2) $$\lambda F[f] = F(\lambda[f]) , \quad \text{and}$$

(4.17.3) $$\|F[f]\| = \|[f]\| ,$$

for every $[f]$, $[g] \varepsilon L(X)$, and all real scalars λ.

Two L spaces are said to be isomorphic and isometric if there exists an isometric isomorphism from one space onto the other.

(4.18) Comment: Since the only characteristic functions $[K_A]$ which belong to $L(X)$ are those for which $\mu(A) < +\infty$, one might suspect an isomorphism could be constructed from $L(X)$ onto $L(S)$ in terms of just these functions. More precisely, it would seem plausible that if characteristic functions in $L(X)$ could be associated with characteristic functions in $L(S)$ by means of a measure-preserving transformation which also preserves finite set operations, then, by forming linear combinations on both sides, an isomorphism could be constructed from one space onto the other. The following theorem shows that this is the case. As might be expected, the major difficulty lies in showing that the transformation so defined preserves limits of ascending sequences of non-negative measurable simple functions. This problem is solved by appealing to the well-known theorem that the L space of any complete measure space is a complete normed linear space. The only other difficulty is concerned with showing that the transformation is indeed a transformation from $L(X)$ onto $L(S)$. This difficulty is overcome by use of (4.16). It should be noted that no correspondence of any sort is asserted between all the measurable sets in X and all the measurable sets in S. Only those sets in X which may be represented as countable unions of measurable sets of finite measure appear again in S as images of their characteristic functions.

(4.19) Theorem: The space $L(X)$ is isomorphic and isometric to the space $L(S)$.

Proof: Let the mapping F be defined on all measurable simple functions $[f] \varepsilon L(X)$, where

$$[f] = \sum_{j=1}^{n} a_j[K_{A_j}], \quad \text{and} \quad \mu(A_j \cap A_k) = 0 \quad \text{if } j \neq k \ ,$$

by

$$F\left(\sum_{j=1}^{n} a_j[K_{A_j}]\right) = \sum_{j=1}^{n} a_j[K_{\Phi[K_{A_j}]}] .$$

It will now be shown that F is well defined on all measurable simple functions in $L(X)$. It suffices to show that if $[K_{A_1}]$, $[K_{A_2}]$, $[K_{B_1}]$, and $[K_{B_2}]$ are characteristic functions in $L(X)$ such that $[K_{A_1} + K_{A_2}] = [K_{B_1} + K_{B_2}]$, then $\left[K_{\Phi[K_{A_1}]} + K_{\Phi[K_{A_2}]}\right] = \left[K_{\Phi[K_{B_1}]} + K_{\Phi[K_{B_2}]}\right]$.

Let $[K_{A_1} + K_{A_2}] = [K_{B_1} + K_{B_2}]$. Then $[K_{A_1 \cap A_2}] = [K_{B_1 \cap B_2}]$ and $[K_{A_1 \Delta A_2}] = [K_{B_1 \Delta B_2}]$. It follows from Lemmas (4.6) and (4.15) that

$$\Phi[K_{A_1}] \cap \Phi[K_{A_2}] \overset{m}{=} \Phi[K_{A_1 \cap A_2}] = \Phi[K_{B_1 \cap B_2}] \overset{m}{=} \Phi[K_{B_1}] \cap \Phi[K_{B_2}]$$

and

$$\Phi[K_{A_1}] \Delta \Phi[K_{A_2}] = (\Phi[K_{A_1}] - \Phi[K_{A_2}]) \cup (\Phi[K_{A_2}] - \Phi[K_{A_1}])$$

$$\overset{m}{=} \Phi[K_{A_1 \cap A_2^c}] \cup \Phi[K_{A_2 \cap A_1^c}] \overset{m}{=} \Phi[K_{(A_1 \cap A_2^c) \cup (A_2 \cap A_1^c)}] =$$

$$= \Phi[K_{A_1 \Delta A_2}] = \Phi[K_{B_1 \Delta B_2}] \overset{m}{=} \Phi[K_{B_1}] \Delta \Phi[K_{B_2}] .$$

Thus

$$\left[K_{\Phi[K_{A_1}]} + K_{\Phi[K_{A_2}]}\right] = \left[K_{\Phi[K_{A_1}] \vartriangle \Phi[K_{A_2}]} + 2K_{\Phi[K_{A_1}] \quad \Phi[K_{A_2}]}\right]$$

$$= \left[K_{\Phi[K_{B_1}] \vartriangle \Phi[K_{B_2}]} + 2K_{\Phi[K_{B_1}] \quad \Phi[K_{B_2}]}\right] = \left[K_{\Phi[K_{B_1}]} + K_{\Phi[K_{B_2}]}\right] .$$

It follows that F is well defined on all measurable simple functions in $L(X)$, and by definition

$$(4.19.1) \qquad F(\lambda[f] + \beta[g]) = \lambda F[f] + \beta F[g] ,$$

for all measurable simple functions $[f], [g] \varepsilon L(X)$, and for all real numbers λ and β.

If $[f] = \sum_{j=1}^{n} a_j \, [K_{A_j}]$ is a measurable simple function in $L(X)$ then

$$\|[f]\| = \int f \, d\mu = \sum_{j=1}^{n} |a_j| \, \mu(A_j) =$$

$$= \sum_{j=1}^{n} |a_j| \, m \, (\Phi[K_{A_j}]) = \int F([f]) \, dm =$$

$$= \| F[f] \| .$$ [1]

---

[1]The above integrations are to be taken over the largest sets on which the respective integrands do not vanish. Thus

$$\int f d\mu = \int_{\{x \,:\, x \,\varepsilon\, X,\ f(x) \neq 0\}} f \, d\mu$$

Thus

$$(4.19.2) \qquad \|F[f]\| = \|[f]\| ,$$

for all measurable simple functions $[f] \in L(X)$.

If $[f]$, $[g]$ are measurable simple functions in $L(X)$ and $[f] \leqq [g]$, then by (4.15.1), $F[f] \leq F[g]$.

Now let $[f]$ be a non-negative measurable function in $L(X)$. Then there exists an ascending sequence of non-negative measurable simple functions $[f_n]$ in $L(X)$ such that $\lim_{n\to\infty} \|[f_n] - [f]\| = 0$. Define $F[f]$ by $\lim_{n\to\infty} \|F[f_n] - F[f]\| = 0$. By the preceding paragraph $\{F[f_n]\}$ is an ascending sequence bounded above in norm by $\|[f]\|$. Since $L(S)$ is a complete normed linear space this implies there exists a limit in $L(S)$ for the sequence $\{F[f_n]\}$. Hence the above definition is non-vacuous. The following argument shows that, besides being non-vacuous, the definition of $F[f]$ is also unambiguous.

Suppose $[g_n]$ is an ascending sequence of non-negative measurable simple functions in $L(X)$ such that $\lim_{n\to\infty} \|[g_n] - [f]\| = 0$. Then the sequence $\{[h_n]\}$ of non-negative measurable simple functions defined by $[h_{2n-1}] = F[f_n]$, $[h_{2n}] = F[g_n]$ for $n = 1,2,\ldots,$ is a Cauchy sequence in $L(S)$, since by (4.19.1) and (4.19.2)

$$0 = \lim_{m,n\to\infty} \|[g_n] - [f_m]\| = \lim_{m,n\to\infty} \|F([g_n] - [f_m])\| =$$

$$= \lim_{m,n\to\infty} \|F[g_n] - F[f_m]\| .$$

The completeness of $L(S)$ implies that there exists a function $[h]$ in $L(S)$ such that $\lim_{n-\infty} \|[h_n] - [h]\| = 0$. Since all subsequences of a convergent sequence converge to the same limit, $\{F[f_n]\}$ converges to $[h]$ and so does $\{F[g_n]\}$. The sequence $\{F[f_n]\}$ was shown to be convergent and its limit was denoted $F[f]$. Thus $F[f] = \lim_{n\to\infty} F[f_n] = [h] = \lim_{n\to\infty} F[g_n]$. This implies that the mapping F is well defined on all non-negative measurable functions in $L(X)$, and the following lemma has been established.

(4.19.3) If $[f]$ is a non-negative measurable function in $L(X)$, then there exists a unique non-negative measurable function $F[f]$ in $L(S)$, such that $F[f] = \lim_{n\to\infty} F[f_n]$, where $[f_n]$ is any ascending sequence of non-negative measurable simple functions in $L(X)$ whose limit is $[f]$.

Since addition and scalar multiplication are continuous, (4.19.1) and (4.19.3) imply:

(4.19.4) If $[f]$ and $[g]$ are non-negative measurable functions in $L(X)$, and if λ and β are any two non-negative real numbers, then

$$F(\lambda[f] + \beta[g]) = \lambda F[f] + \beta F[g] .$$

A direct consequence of (4.19.2) and (4.19.3) is:

(4.19.5) If $[f]$ is any non-negative measurable function in $L(X)$, then

$$\|[f]\| = \|F[f]\| .$$

Each function $[f]$ in $L(X)$ is completely determined by some <u>everywhere finite real-valued</u> summable function $f \varepsilon [f]$. Thus if $[f] \varepsilon L(X)$, then $[f] = [f^+ - f^-] = [f^+] - [f^-]$. The definition $F[f] = F[f^+] - F[f^-]$, together with (4.19.4) and (4.19.5), immediately implies

(4.19.6) If $[f]$, $[g] \in L(X)$, and λ and β are real numbers, then

$$F(\lambda[f] + \beta[g]) = \lambda F[f] = \beta F[g]$$

and

$$\|[f]\| = \|F[f]\| .$$

The mapping F so defined from $L(X)$ into $L(S)$ is norm-preserving and hence inherently one-to-one; more specifically,

(4.19.7) If $F[f] = F[g]$ for $[f]$, $[g] \in L(X)$, then $[f] = [g]$.

A proof of (4.19.7) may be constructed as follows:

Suppose that $[f]$, $[g] \in L(X)$ and $F[f] = F[g]$. Then by (4.19.6), $[0] = F[f] - F[g] = F([f] - [g])$, so that $0 = \|F([f] - [g])\| = \|[f] - [g]\|$, and hence $[f] = [g]$.

It remains to show that the mapping F maps $L(X)$ onto $L(S)$. Lemma (4.16) and the definition of F imply that for each characteristic function $[K_B] \in L(S)$ there exists some characteristic function $[K_A] \in L(X)$ such that $F[K_A] = [K_B]$. It follows immediately that if $[h]$ is any measurable simple function in $L(S)$, then there exists a measurable simple function $[f_h] \in L(X)$ such that $F[f_h] = [h]$.

If $[h]$ is any non-negative measurable function in $L(S)$, then there exists an ascending sequence $\{[h_n]\}$ of non-negative measurable simple functions in $L(S)$ such that $\lim_{n\to\infty} \|[h_n] - [h]\| = 0$. To each $[h_n]$ there corresponds a non-negative measurable simple function $[f_n]$ in $L(X)$ such that $F[f_n] = [h_n]$. Since $\max_{1\leq k\leq n} [f_k]$ is a non-negative measurable simple function in $L(X)$ for each $n \geq 1$, it follows that

$$F\left(\max_{1\le k\le n}[f_k]\right) = \max_{1\le k\le n} F[f_k] = \max_{1\le k\le n}[h_k] = [h_n] ,$$

and, by definition of F,

$$F\left(\lim_{n\to\infty}\max_{1\le k\le n} F[f_k]\right) = \lim_{n\to\infty} F\left(\max_{1\le k\le n}[f_k]\right) =$$

$$= \lim_{n\to\infty}\max_{1\le k\le n} F[f_k] = \lim_{n\to\infty}[h_n] = [h] .$$

Thus for each non-negative measurable function $[h] \varepsilon L(S)$ there exists a non-negative measurable function $[f_h] \varepsilon L(X)$, such that $F[f_h] = [h]$.

It is now an immediate consequence of the definition of F that:

(4.19.8) For each $[h] \varepsilon L(S)$, there exists a function

$$[f] \varepsilon L(X) \text{ such that } F[f] = [h]$$

The mapping F is thus an isometric isomorphism from $L(X)$ onto $L(S)$ and the two spaces are, by definition, isometric and isomorphic. ∎

Although the σ-ring, $\sigma(\mathcal{B})$, of the preceding theorem may seem rather pathological, it actually is not so ill-behaved. In fact:

(4.20) <u>Theorem</u>: All bounded continuous real-valued functions on S which vanish at infinity are measurable relative to $\sigma(\mathcal{B})$.

<u>Proof</u>: Let $\mathcal{S}$ denote the class of all measurable simple functions f on S of the form $f = \sum_{i=1}^{N} a_i K_{B_i}$, with $B_i \varepsilon \mathcal{B}$ for $i = 1,\ldots, N$. If G is any open set in the real line, then the inverse image of G under a characteristic function K_{B_i} is one of the four sets S, $S-B_i$, B_i, or $\emptyset$. In each case the inverse image of G is open in S since each

set $B_i \varepsilon \mathcal{B}$ is both open and closed in S. This implies that any characteristic function K_{B_i} with $B_i \varepsilon \mathcal{B}$ is continuous on S and consequently that all functions in $\mathcal{S}$ are continuous.

Since $S = \bigcup_{t\varepsilon T} S_t$ and $S_t \varepsilon \mathcal{B}_t \subseteq \mathcal{B}$ for each $t \varepsilon T$, there exists, for each point x in S, a function f_x in $\mathcal{S}$ such that $f_x(x) \neq 0$.

The collection $\mathcal{S}$ has separation property (II), (cf. Definition 2.2) since if x and y are any two distinct points of S, there exists a set $B \varepsilon \mathcal{B}$ such that $K_B(x) = 1$ and $K_B(y) = 0$.

If $f \varepsilon \mathcal{S}$ then f is of the form

$$f = \sum_{i=1}^{N} a_i K_{B_i}.$$

This implies that $f(x) = 0$ for $x \notin \bigcup_{i=1}^{N} S_i$, where $S_i \supseteq B_i$ for $i = 1 \ldots N$. Since each S_i is compact it follows that $\bigcup_{i=1}^{N} S_i$ is compact, and hence that each function in $\mathcal{S}$ vanishes outside some compact subset of S. In particular this implies that each function in $\mathcal{S}$ vanishes at infinity.

By Corollary (2.23.1), the smallest closed subalgebra of $C_r^o(S)$ containing $\mathcal{S}$ is $C_r^o(S)$ itself. (The symbol $C_r^o(S)$ denotes the space of all bounded, continuous, real-valued functions defined on S which vanish at infinity. See (2.17).)

If K_A and K_B are in $\mathcal{S}$, then $A, B \varepsilon \mathcal{B}$, so that $A \cap B \varepsilon \mathcal{B}$ and $K_A \cdot K_B = K_{A \cap B} \varepsilon \mathcal{S}$. It follows that finite products of characteristic functions in $\mathcal{S}$ are also in $\mathcal{S}$. Since $\mathcal{S}$ is closed under the formation of linear combinations of members of $\mathcal{S}$, this implies that is also closed under the formation of finite products of its members. Thus, by definition, $\mathcal{S}$ is a subalgebra of $C_r^o(S)$.

Since the smallest closed subalgebra of $C_r^o(S)$ containing a subalgebra of $C_r^o(S)$ is the collection of all uniform limits of sequences whose terms are members of the subalgebra in question, it follows that each member of $C_r^o(S)$ is the funiform limit of a sequence of functions in $\mathcal{S}$. However each function in $\mathcal{S}$ is measurable and limits of sequences of measurable functions are measurable. It follows that each function in $C_r^o(S)$ is measurable. ∎

(4.21) Definition: Let X be a locally compact Hausdorff space. The Baire σ-ring in X is defined to be the minimal σ-ring containing all compact subsets of X which may be written as a countable intersection of open sets in X. Sets in the Baire σ-ring are called the Baire sets of X.

(4.22) Theorem: Let X be a locally compact Hausdorff space and let $\mathcal{a}$ be an open base for X. If $\mathcal{R}$ is any σ-ring in X containing $\mathcal{a}$ then $\mathcal{R}$ contains the Baire σ-ring in X.

Proof: If C is any compact subset of an open subset G of X then there exists a collection $\{G_t : t \in T\}$ of sets in $\mathcal{a}$ such that $G = \bigcup_{t \in T} G_t \supseteq C$. By compactness of C there exists a finite subcollection $\{G_k\}_{k=1}^n$ such that $G \supseteq \bigcup_{k=1}^n G_k \supseteq C$. The set $E = \bigcup_{k=1}^n G_k$ is in $\mathcal{R}$ It has been established that if C is a compact subset of some open set G in X then there exists a set E in $\mathcal{R}$ such that $C \subseteq E \subseteq G$.

Now let C be a compact set in X such that $C = \bigcap_{n=1}^{\infty} H_n$ where each set H_n is open. By the preceding, for each $n = 1,2,\ldots,$ there

exists a set E_n in $\mathcal{Q}$ such that $C \subsetneq E_n \subsetneq H_n$ and hence $C = \bigcap_{n=1}^{\infty} E_n \in \mathcal{Q}$.
It follows that $\mathcal{Q}$ contains all compact sets in X which may be written as a countable intersection of open sets in X. By Definition (4.21) $\mathcal{Q}$ contains the Baire σ-ring in X. ∎

(4.23) <u>Theorem</u>: The σ-ring $\sigma(\mathcal{B})$ is the Baire σ-ring in S.

<u>Proof</u>: By Lemma (4.9) $\mathcal{B}$ is an open base for S. Since $\sigma(\mathcal{B})$ contains $\mathcal{B}$, it follows from (4.22) that $\sigma(\mathcal{B})$ contains the Baire σ-ring in S. Lemma (4.9) implies that every set in $\mathcal{B}$ is both open and closed and that every set in $\mathcal{B}$ is contained in some compact set in S. Thus every set in $\mathcal{B}$ is a compact open set in S. It follows that every set in $\mathcal{B}$ is a Baire set of S. Since the Baire σ-ring contains $\mathcal{B}$ it also contains $\sigma(\mathcal{B})$, the minimal σ-ring containing $\mathcal{B}$. ∎

(4.24) <u>Remark</u>: Some authorities assert that the theory of integration is no more general than the theory of integration with respect to a measure defined on the Baire sets in a locally compact Hausdorff space. The source usually quoted to justify this assertion is KAKUTANI [1]. The theorems of this chapter constitute a detailed measure-theoretic proof of the applicable result which Kakutani's more general theorem implies: The L-space of any complete measure space is isomorphic and isometric, as a normed linear space, to the L-space of a complete measure space whose σ-ring of measurable sets is a completion of the Baire σ-ring in some locally compact Hausdorff space.

An especially convincing argument in favor of considering measure spaces whose ground sets are <u>not</u> assumed to be locally compact Hausdorff spaces is contained in HALMOS [2]. A rebuttal appears in DIEUDONNÉ [1].

APPENDIX

Glossary

A partially ordered set is a non-empty set X in which a binary relation "$\leqq$" is defined, which satisfies the following three conditions:

(1) $$x \leqq x \quad \text{for all} \quad x \varepsilon X ,$$

(2) $$\text{If} \quad x \leq y \quad \text{and} \quad y \leq x, \quad \text{for} \quad x, y \varepsilon X, \quad \text{then} \quad x = y,$$

(3) $$\text{If} \quad x \leq y \quad \text{and} \quad y \leq z, \quad \text{then} \quad x \leq z, \quad \text{where} \quad x, y, z \varepsilon X.$$

The relation $x \leqq y$ will also be written $y \geqq x$. The two inequalities are intended to have the same meaning. The relation $x < y$ means $x \leq y$ and $x \neq y$. The relation "$>$" is defined analogously. If A is any non-empty subset of X, an element a in A is said to be a maximal element of A if $b > a$ is true for no $b \varepsilon A$. Minimal elements of non-empty subsets of X are defined analogously. By a lower bound of a non-empty subset A of X is meant an element ℓ in X such that $\ell \leqq a$, for all $a \varepsilon A$. An upper bound of a non-empty subset A of X is an element $x \varepsilon X$ such that $a \leqq x$, for all $a \varepsilon A$. A greatest lower bound or infimum of a non-empty subset A of X is a lower bound of A which is also an upper bound of the set of all lower bounds of A. The designations "g.l.b." and "inf." will be used to denote greatest lower bound. A least upper bound or supremum of a non-empty subset A of X is an upper bound of A which is also a lower bound of the set of all upper bounds of A. The designations "l.u.b." and "sup." will be used to denote least upper bound.

A lattice is a partially ordered set X, any two of whose elements x and y have a greatest lower bound $x \wedge y$ in X, and a least upper bound $x \vee y$ in X. A lattice L is called a distributive lattice if and only if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for all $x, y, z \in L$.

A vector lattice V is a real vector space which is also a lattice whose order relation is related to scalar multiplication by the conditions:

(1) If $x, y \in V$ and λ is any positive real scalar then,

$$x \leqq y \Longleftrightarrow \lambda x \leqq \lambda y .$$

(2) If $x, y \in V$ and α is any negative real scalar then,

$$x \leqq y \Longleftrightarrow \alpha y \leqq \alpha x .$$

An algebra A is a vector space whose vectors can be multiplied in such a way that the following identities are satisfied for all $x, y, z \in A$ and all scalars α.

(1) $$x(y z) = (x y) z$$

(2) $$x(y + z) = x y + x z$$

(3) $$(x + y)z = x z + y z$$

(4) $$\alpha(x y) = (\alpha x)y = x(\alpha y)$$

A commutative algebra A is an algebra whose multiplication satisfies the condition $x\,y = y\,x$ for all $x, y \in A$.

An algebra is called a real (complex) algebra if the scalars are real (complex) numbers.

A subalgebra of an algebra A is vector subspace of A which contains the product of each pair of its elements.

If S is any non-empty set, a collection $\mathcal{J}$ of subsets of S is called a topology on S if it satisfies the following conditions:

(1) The union of any subcollection of $\mathcal{J}$ belongs to $\mathcal{J}$.

(2) The intersection of any non-empty finite subcollection of $\mathcal{J}$ belongs to $\mathcal{J}$.

(3) $S \in \mathcal{J}$ and $\emptyset \in \mathcal{J}$, where $\emptyset$ denotes the empty set.

An ordered pair $(S, \mathcal{J})$ in which the first component, S, is a non-empty set, and the second component, $\mathcal{J}$, is a topology on S is called a topological space. The set S is called its ground set and the elements of S are called its points. A subset of S is called open if and only if it belongs to $\mathcal{J}$. When no confusion can arise topological spaces will be denoted by the symbol of their ground sets. Thus the statement "S is a topological space" means that S is the ground set of some topological space $(S, \mathcal{J})$. Sets in $\mathcal{J}$ are called open sets and sets whose complements are in $\mathcal{J}$ are called closed sets.

If $(S, \mathcal{J})$ is a topological space and X is a non-empty subset of S, the relative topology on X is defined to be the class of all intersections of X with sets in $\mathcal{J}$. It is easily seen that the relative topology on X is indeed a topology as defined above. The space X equipped with its relative topology is called a subspace of $(S, \mathcal{J})$.

A topological space $(S, \mathcal{J})$ is called a T_1-space if for every $x \in S$ the set $\{x\}$ is closed. A topological space $(S, \mathcal{J})$ is called a Hausdorff space if, given any two distinct points $x, y \in S$, there exist disjoint sets G_x and G_y in $\mathcal{J}$ such that $x \in G_x$ and $y \in G_y$.

Let $(S, \mathcal{J})$ be a topological space. A class $\{G_\lambda : \lambda \in I\}$ of sets in $\mathcal{J}$ is called an open cover of S if $\bigcup_{\lambda \in I} G_\lambda = S$. A subclass of an open cover of S which is itself an open cover of S is called an open subcover of S. A topological space S is called compact if every open cover of S has a finite subcover of S. A non-empty subset X of a topological space $(S, \mathcal{J})$ is called a compact subset of S if the subspace X is compact in its relative topology. The empty set $\emptyset$ will be called a compact subset of S by convention.

A topological space $(S, \mathcal{J})$ is called locally compact if every point in X is contained in some open set which in turn is contained in some compact subset of S.

Let $(X, \mathcal{J})$ be a locally compact topological space. Let x_∞ be any object not in X. (It is assumed that such an object always exists.) Form the set $X_\infty = X \cup \{x_\infty\}$ and define a collection $\mathcal{J}_\infty$ by saying that a subset Y of X_∞ shall belong to $\mathcal{J}_\infty$ if either (i) $Y \in \mathcal{J}$ or (ii) $X_\infty - Y$ is a closed compact subset of $(X, \mathcal{J})$. The collection $\mathcal{J}_\infty$ is a topology and $(X_\infty, \mathcal{J}_\infty)$ is a topological space called the one-point compactification of $(X, \mathcal{J})$. The original space $(X, \mathcal{J})$ is a subspace of $(X_\infty, \mathcal{T}_\infty)$ and the topology $\mathcal{T}$ is just the relative topology of X_∞ considered as a subspace of $(X_\infty, \mathcal{T}_\infty)$.

The Carathéodory Extension Procedure

Let X be a non-empty space. A collection $\mathcal{S}$ of subsets of X is called a semiring if it satisfies the following conditions:

(1) $$\emptyset \in \mathcal{S}$$

(2) $$A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$$

(3) If $A, A_1 \in \mathcal{S}$ and $A_1 \subseteq A$ then there exists a finite collection $\{A_k\}_{k=2}^n$ of pairwise disjoint sets in $\mathcal{S}$ such that $A_k \subseteq A - A_1$ for $k = 2, \ldots, n$, and $\bigcup_{k=1}^n A_k = A$.

Let X be a non-empty space, let $\mathcal{S}$ be a semiring in X, and let $\overline{m}$ be a non-negative countably additive set function on $\mathcal{S}$. The set function $\overline{m}$ is called a measure on $\mathcal{S}$. Let $\mathcal{R}(\mathcal{S})$ denote the minimal ring containing $\mathcal{S}$. The ring $\mathcal{R}(\mathcal{S})$ is just the collection of all unions of finite pairwise disjoint collections of sets in $\mathcal{S}$. Let $R_\sigma(\mathcal{S})$ denote the minimal σ-ring containing $\mathcal{S}$, or equivalently the minimal σ-ring containing $\mathcal{R}(\mathcal{S})$. Let $H(\mathcal{S})$ denote the σ-ring of all subsets of countable unions of sets in $\mathcal{S}$.

The measure $\overline{m}$ on $\mathcal{S}$ possesses a unique extension to a measure $\hat{m}$ on $\mathcal{R}(\mathcal{S})$. The measure $\hat{m}$ is defined on $\mathcal{R}(\mathcal{S})$ by

$$\hat{m}\left(\bigcup_{k=1}^n A_k\right) = \sum_{k=1}^n \overline{m}(A_k)$$

whenever $\bigcup_{k=1}^{n} A_k$ is a set in $\mathcal{R}(\mathcal{S})$ and the collection $\{A_k\}_{k=1}^{n}$ is a pairwise disjoint subcollection of $\mathcal{S}$. The definition of $\hat{m}$ on $\mathcal{R}(\mathcal{S})$ is unambiguous and $\hat{m}(A) = \bar{m}(A)$ for every $A \in \mathcal{S}$.

If $A \in H(\mathcal{S})$, define

$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} \bar{m}\ (A_n) : \bigcup_{n=1}^{\infty} A_n \supseteq A\ ,\ \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{S}\right\}.$$

Then m^* is an outer measure on $H(\mathcal{S})$. Define

$$\mathcal{M} = \{A : A \in H(\mathcal{S}) \text{ and } m^*(T) = m^*(T \cap A) + m^*(T-A) \text{ for all } T \in H(\mathcal{S})\}.$$

The class $\mathcal{M}$ is a σ-ring of sets and the inclusions

$$\mathcal{S} \subseteq \mathcal{R}(\mathcal{S}) \subseteq \mathcal{R}_\sigma(\mathcal{S}) \subseteq \mathcal{M} \subseteq H(\mathcal{S})$$

are valid. The restriction m of m^* to sets in $\mathcal{M}$ is a <u>measure</u> on $\mathcal{M}$. This measure is called <u>the Carathéodory extension of</u> $\bar{m}$. If $A \in \mathcal{S}$, then $\bar{m}(A) = \hat{m}(A) = m(A)$ and if $B \in \mathcal{R}(\mathcal{S})$, then $\hat{m}(B) = m(B)$.

If the semiring $\mathcal{S}$ and the measure $\bar{m}$ satisfy the condition

(1) If $A \in \mathcal{R}_\sigma(\mathcal{S})$, then there exists a countable collection $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{S}$ such that $\bar{m}(A_n) < +\infty$ for each $n = 1,2,\ldots$, and $A \subseteq \bigcup_{n=1}^{\infty} A_n$.

the measure $\bar{m}$ is said to be <u>σ-finite</u> and the Carathéodory extension m of $\bar{m}$ is unique in the sense that if m_1 is any measure defined on $\mathcal{R}_\sigma(\mathcal{S})$ such that $m_1(A) = \bar{m}(A)$ for all $A \in \mathcal{S}$, then $m_1(B) = m(B)$ for all $B \in \mathcal{R}_\sigma(\mathcal{S})$.

If the measure $\bar{m}$ is not <u>σ-finite</u> then the Carathéodory extension m of $\bar{m}$ to a measure on $\mathcal{R}_\sigma(\mathcal{S})$ <u>may or may not be unique</u>.

In the proof of Lemma (4.16) the Carathéodory extension of m was defined on $\sigma(\mathcal{B})$ by

$$m(B) = \inf \left\{ \sum_{n=1}^{\infty} m(B_{t_n}) : B_{t_n} \in \mathcal{B},\ \ B_{t_n} \cap B_{t_m} = \emptyset \ \text{ if } \ m \neq n, \right.$$
$$\left. \text{and} \ \sum_{n=1}^{\infty} B_{t_n} \supseteq B \right\} .$$

In this particular case this definition is equivalent to the definition given on the preceding page. The reason why the two definitions are equivalent is that any countable union of sets in the semiring $\mathcal{B}$ may be expressed as a countable union of pairwise disjoint sets in $\mathcal{B}$.

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