JOURNAL OF MATHEMATICAL PHYSICS **52**, 093702 (2011)

Variable coefficient nonlinear Schrödinger equations with four-dimensional symmetry groups and analysis of their solutions

C. Özemir[1,a)] and F. Güngör[2,b)]

[1]*Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, 34469 Istanbul, Turkey*

[2]*Department of Mathematics, Faculty of Arts and Sciences, Doğuş University, 34722 Istanbul, Turkey*

(Received 18 February 2011; accepted 15 August 2011; published online 8 September 2011)

Analytical solutions of variable coefficient nonlinear Schrödinger equations having four-dimensional symmetry groups, which are in fact the next closest to the integrable ones occurring only when the Lie symmetry group is five-dimensional, are obtained using two different tools. The first tool is to use one-dimensional subgroups of the full symmetry group to generate solutions from those of the reduced ordinary differential equations, namely, group invariant solutions. The other is by truncation in their Painlevé expansions. © *2011 American Institute of Physics*. [doi:10.1063/1.3634005]

I. INTRODUCTION

The purpose of this paper is to classify solutions of a general class of variable coefficient nonlinear Schrödinger equations (VCNLS) of the form

$$\begin{gathered} i\psi_t + f(x,t)\psi_{xx} + g(x,t)|\psi|^2\psi + h(x,t)\psi = 0, \\ f = f_1 + if_2, \quad g = g_1 + ig_2, \quad h = h_1 + ih_2, \\ f_j, g_j, h_j \in \mathbb{R}, \quad j = 1,2, \quad f_1 \neq 0, \quad g_1 \neq 0. \end{gathered} \tag{1.1}$$

with the property that they are invariant under four-dimensional Lie symmetry groups. This class of equations models various nonlinear phenomena, for instance, see Ref. 1 and the references therein. Symmetry classes of (1.1) are obtained in Ref. 2 and canonical equations admitting Lie symmetry algebras L of dimension $1 \leq \dim L \leq 5$ are presented there. A suitable basis for the maximal algebra L ($\dim L = 5$) is

$$T = \partial_t, \quad P = \partial_x, \quad W = \partial_\omega, \quad B = t\partial_x + \frac{1}{2}x\partial_\omega, \quad D = t\partial_t + \frac{1}{2}x\partial_x - \frac{1}{2}\rho\partial_\rho, \tag{1.2}$$

which is isomorphic to the one-dimensional extended Galilei similitude algebra gs(1). Here, $\psi \in \mathbb{C}$ is expressed in terms of the modulus and the phase of the wave function

$$\psi(x,t) = \rho(x,t)e^{i\omega(x,t)}. \tag{1.3}$$

An equation of class (1.1) admits this algebra as long as the coefficients f, g, and h can be mapped into

$$f = 1, \quad g = \epsilon + ig_2, \quad \epsilon = \pm 1, \quad g_2 = \text{const.}, \quad h = 0, \tag{1.4}$$

by point transformations. This is nothing but the standard cubic nonlinear Schrödinger equation (NLSE). For the form of the coefficients obeying the constraints imposed by the Painlevé test, we

---

[a)]Electronic mail: ozemir@itu.edu.tr.
[b)]Electronic mail: fgungor@dogus.edu.tr.

TABLE I. Four-dimensional symmetry algebras and the coefficients in (1.1).

No	Algebra	f	g	h	Conditions
L_1	$\{T, D_1, C_1, W\}$	1	$(\epsilon + i\gamma)\frac{1}{x}$	$(h_1 + ih_2)\frac{1}{x^2}$	
L_2	$\{T, P, B, W\}$	1	$\epsilon + i\gamma$	ih_2	$h_2 \neq 0$
L_3	$\{T, P, D_2, W\}$	$1 + if_2$	$\epsilon + i\gamma$	0	$f_2 \neq 0$
L_4	$\{P, B, D_2, W\}$	1	$\epsilon + i\gamma$	$i\frac{h_2}{t}$	$h_2 \neq 0$
L_5	$\{P, B, C_2, W\}$	1	$\frac{\epsilon+i\gamma}{1+t^2}$	$\frac{2h_1+i(2h_2-t)}{2(1+t^2)}$	

had been able to transform (1.1) to the usual NLSE. In two recent papers Refs. 3 and 4, the conditions imposed by the Painlevé test were shown to be equivalent to those having a Lax pair. Therefore, these conditions are also necessary for integrability.

We intend to present a detailed analysis of solutions to VCNLS equations in the absence of integrability using two different approaches. We focus on the canonical equations which are representatives of the equations from class (1.1) having four-dimensional Lie algebras. A list of four-dimensional symmetry algebras and the corresponding coefficients for the canonical equations is given in Table I (We hereby correct an error in Ref. 2 on the basis element C_2 of L_5 and the form of the invariant equation).

Here, γ, h_1, h_2 are constants and $\epsilon = \pm 1$. With the wave function written in the polar form (1.3), the basis elements for the symmetry algebras are given by

$$T = \partial_t, \quad P = \partial_x, \quad W = \partial_\omega, \quad B = t\partial_x + \frac{1}{2}x\partial_\omega \tag{1.5}$$

and

$$C_1 = t^2\partial_t + xt\partial_x - \frac{1}{2}t\rho\partial_\rho + \frac{1}{4}x^2\partial_\omega, \quad D_1 = 2t\partial_t + x\partial_x - \frac{1}{2}\rho\partial_\rho$$

for L_1, $D_2 = t\partial_t + \frac{1}{2}x\partial_x - \frac{1}{2}\rho\partial_\rho$ for L_3 and L_4, and $C_2 = (1 + t^2)\partial_t + xt\partial_x + \frac{1}{4}x^2\partial_\omega$ for the algebra L_5. We note that L_1 is a non-solvable and L_2 is a nilpotent algebra and the other three are solvable and non-nilpotent. In addition, L_1 and L_3 are decomposable, whereas the others are not.

These canonical equations do not pass the Painlevé test for Partial Differential Equations (PDEs); therefore, they are not integrable and will be the main subject of this study. We are going to apply two different methods: Symmetry reduction and truncated Painlevé expansions. The first is to make use of the one-dimensional subalgebras of the four-dimensional algebras given in Table I and the second is to find a valid truncated series solution to the equation.

The paper is organized as follows. In Sec. II, we find the group-invariant equations for the canonical equations having four-dimensional symmetry algebras. Section III is devoted to the analysis of the reduced systems and completes the study of the invariant solutions. In Sec. IV, we apply the method of truncated Painlevé expansions to the canonical equations to obtain exact solutions.

II. ONE-DIMENSIONAL SUBALGEBRAS AND REDUCTIONS TO ODES

As we are interested in group invariant solutions, we only need one-dimensional subalgebras. This is the case because we restrict ourselves to subgroups of the symmetry group having generic orbits of codimension 3 in the space $\{x, t\} \times \{\rho, \omega\}$.

The classification of one-dimensional subalgebras under the action of the group of inner automorphisms of the four-dimensional symmetry groups is a standard one. We do not provide the calculations leading to the conjugacy inequivalent list of subalgebras. The classification method can be found for example in Refs. 5–7.

The main result is that every one-dimensional subalgebra of the symmetry algebra is conjugate to precisely one of the subalgebras given in the Table II.

Using these subalgebras, we perform the reductions leading to the ordinary differential equations (ODEs). We exclude the subalgebras whenever the invertibility requirement is violated. This is the

TABLE II. One-dimensional subalgebras of four-dimensional algebras under the adjoint action of the full symmetry group.

Algebra		Subalgebra	$a, b, c \in \mathbb{R}, \epsilon_1 = \mp 1$
L_1	$L_{1.1} = \{T + C_1 + aW\}$	$L_{1.2} = \{D_1 + bW\}$	$L_{1.3} = \{T + cW\}$
L_2	$L_{2.1} = \{P\}$	$L_{2.2} = \{T + aW\}$	$L_{2.3} = \{B + bT\}$
	$L_{2.4} = \{W\}$		
L_3	$L_{3.1} = \{T\}$	$L_{3.2} = \{P\}$	$L_{3.3} = \{T + \epsilon_1 W\}$
	$L_{3.4} = \{P + \epsilon_1 W\}$	$L_{3.5} = \{D_2 + aW\}$	$L_{3.6} = \{T + \epsilon_1 P + bW\}$
	$L_{3.7} = \{W\}$		
L_4	$L_{4.1} = \{P\}$	$L_{4.2} = \{B\}$	$L_{4.3} = \{P + \epsilon_1 B\}$
	$L_{4.4} = \{D_2 + aW\}$	$L_{4.5} = \{W\}$	
L_5	$L_{5.1} = \{B\}$	$L_{5.2} = \{C_2 + aW\}$	$L_{5.3} = \{W\}$

only case for the gauge symmetry W and it does not lead to any group-invariant solutions. We first write the wave function in the form (1.3) and obtain (1.1) as a system of two real second order nonlinear PDEs, given by

$$-\rho\,\omega_t + f_1(\rho_{xx} - \rho\omega_x^2) - f_2(2\rho_x\omega_x + \rho\omega_{xx}) + g_1\rho^3 + h_1\rho = 0, \tag{2.1a}$$

$$\rho_t + f_2(\rho_{xx} - \rho\omega_x^2) + f_1(2\rho_x\omega_x + \rho\omega_{xx}) + g_2\rho^3 + h_2\rho = 0. \tag{2.1b}$$

In this system, coefficient functions with indices 1, 2 are real and imaginary parts of f, g, and h. They are all functions of x and t. For example, if we would like to see the system for the algebra L_1, looking at the Table I we simply replace $h_1(x, t)$ of (2.1a) by $\frac{h_1}{x^2}$ and for the algebra L_5 by $\frac{h_1}{1+t^2}$, this time h_1 being a constant.

Invariant surface condition for a specific subalgebra gives the similarity variable for the functions ρ and ω. Use of this variable in (2.1), therefore, reduces the number of independent variables in the system from two to one, converting it to a system of ODEs. These nonlinear systems of ODEs arise as first or second order nonlinear equations. First order systems are usually solved by standard methods so that we avoid presenting their explicit solutions. We also did not include the reduced ODEs in a few intractable cases. The task of solving coupled nonlinear second order ODEs is indeed a challenge. Luckily, for specific values of the constants appearing in the reduced equations, we have been able to succeed in decoupling the systems and left the search for solutions to the Sec. III. For full details, we refer to the arXiv version[8] of the paper.

A. Non-solvable algebra $L_1 = \{T, D_1, C_1, W\}$

Commutators for the basis elements of the four-dimensional algebra L_1 satisfy

$$[T, D_1] = 2T, \quad [T, C_1] = D_1, \quad [D_1, C_1] = 2C_1 \tag{2.2}$$

with W being the center element, that is, commuting with all the other elements. The algebra has the direct sum structure

$$L_1 = \mathrm{sl}(2, \mathbb{R}) \oplus \{W\}. \tag{2.3}$$

Representative equation of the algebra is

$$i\psi_t + \psi_{xx} + (\epsilon + i\gamma)\frac{1}{x}|\psi|^2\psi + (h_1 + ih_2)\frac{1}{x^2}\psi = 0 \tag{2.4}$$

with the real constants $\epsilon = \mp 1, \gamma, h_1, h_2$.

1. Subalgebra $L_{1.1} = \{T + C_1 + aW\}$

Invariance under the subalgebra $L_{1.1}$ implies that the solution has the form

$$\psi(x,t) = \frac{M(\xi)}{\sqrt{x}} \exp\left[i\left(a \arctan t + \frac{x^2 t}{4(1+t^2)} + P(\xi)\right)\right], \quad \xi = \frac{x^2}{1+t^2} \tag{2.5}$$

and the reduced system of equations satisfied by $M(\xi)$ and $P(\xi)$ are

$$\epsilon M^3 + \left(\frac{3-\xi^2}{4} - a\xi + h_1\right)M - 4\xi^2 M P'^2 + 4\xi^2 M'' = 0, \tag{2.6a}$$

$$\gamma M^3 + h_2 M + 8\xi^2 M' P' + 4\xi^2 M P'' = 0. \tag{2.6b}$$

We first need to decouple these equations to solve for the functions M and P. If (2.6b) is multiplied by M and written as

$$\gamma M^4 + h_2 M^2 + 4\xi^2 (M^2 P')' = 0, \tag{2.7}$$

it is seen that an integral of (2.7) can be obtained for two different cases of the constants.
(i) The case $\gamma = 0$, $h_2 \neq 0$.

It can be shown that the system (2.6) amounts to integrating a third order nonlinear ordinary differential equation from (2.6a),

$$Y'Y''' - \frac{1}{2}Y''^2 + \frac{2}{\xi}Y'Y'' + \frac{1}{2\xi^2}\left(\frac{3-\xi^2}{4} - a\xi + h_1\right)Y'^2 - \frac{2\epsilon}{h_2}Y'^3 - \frac{h_2^2}{8\xi^4}(Y+C)^2 = 0, \tag{2.8}$$

where the functions M, P of (2.5) are related to $Y(\xi)$ by the relations

$$M(\xi) = 2\xi\sqrt{-\frac{1}{h_2}Y'}, \qquad P(\xi) = -\frac{h_2}{4}\int \frac{Y+C}{\xi^2 Y'}\, d\xi. \tag{2.9}$$

(ii) The case $\gamma = h_2 = 0$.

In this case, we can easily decouple the reduced system of equations. Integration of (2.7) gives

$$M^2 P' = C, \qquad P(\xi) = \int \frac{C}{M^2}\, d\xi, \quad C = \text{const.} \tag{2.10}$$

and from (2.6a) we obtain the equation for M,

$$M'' = -\frac{\epsilon}{4\xi^2}M^3 + \frac{1}{4\xi^2}\left(\frac{\xi^2-3}{4} + a\xi - h_1\right)M + C^2 M^{-3}. \tag{2.11}$$

2. Subalgebra $L_{1.2} = \{D_1 + bW\}$

Group-invariant solutions for subalgebra $L_{1.2}$ will have the form

$$\psi(x,t) = \frac{M(\xi)}{\sqrt{x}} \exp\left[i\left(b \ln x + P(\xi)\right)\right], \quad \xi = \frac{x^2}{t}. \tag{2.12}$$

It is straightforward to see that $M(\xi)$ and $P(\xi)$ must satisfy

$$\epsilon M^3 + \left(\frac{3}{4} - b^2 + h_1\right)M + \xi(\xi - 4b)MP' - 4\xi^2 M P'^2 + 4\xi^2 M'' = 0, \tag{2.13a}$$

$$\gamma M^3 + (h_2 - 2b)M + \xi(4b - \xi)M' + 8\xi^2 M' P' + 4\xi^2 M P'' = 0. \tag{2.13b}$$

If we multiply (2.13a) by M, we can write it in the form

$$\gamma M^4 + h_2 M^2 + \xi^2\left[M^2\left(\frac{2b}{\xi} - \frac{1}{2} + 4P'\right)\right]' = 0. \tag{2.14}$$

Integration of (2.14) is possible in two different cases.
(i) The case $\gamma = 0$, $h_2 \neq 0$.

M and P are found from the relations

$$P' = \frac{1}{8} - \frac{b}{2\xi} - \frac{h_2}{4\xi^2}\frac{Y+C}{Y'}, \qquad M^2 = -\frac{1}{h_2}\xi^2 Y', \tag{2.15}$$

where Y is a solution of

$$Y'Y''' - \frac{1}{2}Y''^2 + \frac{2}{\xi}Y'Y'' + \frac{1}{8}\Big(\frac{3+4h_1}{\xi^2} - \frac{2b}{\xi} + \frac{1}{4}\Big)Y'^2 - \frac{\epsilon}{2h_2}Y'^3 - \frac{h_2^2}{8\xi^4}(Y+C)^2 = 0. \tag{2.16}$$

(ii) The case $\gamma = h_2 = 0$.

If (2.14) is integrated once and substituted into (2.13a) we find that M satisfies the second order equation

$$M'' = C^2 M^{-3} - \frac{\epsilon}{4\xi^2}M^3 - \frac{1}{16\xi^2}(3 + 4h_1 - 2b\xi + \frac{1}{4}\xi^2)M. \tag{2.17}$$

3. Subalgebra $L_{1.3} = \{T + cW\}$

Invariance under the subalgebra $L_{1.3}$ implies that the solution will have the form

$$\psi(x,t) = M(x)\exp\Big[i\big(ct + P(x)\big)\Big] \tag{2.18}$$

and here $M(x)$, $P(x)$ satisfy the system

$$\epsilon x M^3 + (h_1 - cx^2)M - x^2 M P'^2 + x^2 M'' = 0, \tag{2.19a}$$

$$\gamma x M^3 + h_2 M + 2x^2 M'P' + x^2 M P'' = 0. \tag{2.19b}$$

Similarly, (1.2) can be arranged as

$$\gamma x M^4 + h_2 M^2 + x^2\left(M^2 P'\right)' = 0 \tag{2.20}$$

and with arguments similar to the preceding algebras we obtain the following results.
(i) The case $\gamma = 0$, $h_2 \neq 0$.

$M(x)$ and $P(x)$ are found from

$$M(x) = \left(-\frac{1}{h_2}x^2 Y'\right)^{1/2}, \qquad P(x) = -h_2\int\frac{Y+C}{x^2 Y'}\,dx. \tag{2.21}$$

Here, $Y(x)$ satisfies a third order equation

$$Y'Y''' - \frac{1}{2}Y''^2 + \frac{2}{x}Y'Y'' + 2\Big(\frac{h_1}{x^2} - c\Big)Y'^2 - \frac{2\epsilon x}{h_2}Y'^3 - \frac{2h_2^2}{x^4}(Y+C)^2 = 0. \tag{2.22}$$

(ii) The case $\gamma = h_2 = 0$.

$M(x)$ is the solution of the equation

$$M'' = C^2 M^{-3} + \Big(c - \frac{h_1}{x^2}\Big)M - \frac{\epsilon}{x}M^3 \tag{2.23}$$

and $P(x)$ is going to be found from

$$P(x) = \int\frac{C}{M^2}\,dx. \tag{2.24}$$

B. Nilpotent algebra $L_2 = \{T, P, B, W\}$

Nonzero commutation relation is $[P, B] = \frac{1}{2}W$. The algebra contains the three-dimensional abelian ideal $\{T, P, W\}$. The action of B on this ideal can be represented by the nilpotent matrix N,

$$\begin{pmatrix} [P, B] \\ [T, B] \\ [W, B] \end{pmatrix} = N \begin{pmatrix} P \\ T \\ W \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, the canonical equation has the form

$$i\psi_t + \psi_{xx} + (\epsilon + i\gamma)|\psi|^2\psi + ih_2\psi = 0 \tag{2.25}$$

with the real constants $\epsilon = \mp 1$, $h_2 \neq 0$, and γ.

1. Subalgebra $L_{2.1} = \{P\}$

The group-invariant solution of $L_{2.1}$ has the form

$$\psi(x, t) = M(t) \exp\big(iP(t)\big) \tag{2.26}$$

and M, P must satisfy

$$\epsilon M^2 - P' = 0, \tag{2.27a}$$

$$\gamma M^3 + h_2 M + M' = 0. \tag{2.27b}$$

We immediately integrate these equations and find

$$M(t) = \left(M_1 \exp(2h_2 t) - \frac{\gamma}{h_2}\right)^{-1/2}, \tag{2.28a}$$

$$P(t) = \begin{cases} \frac{\epsilon}{2\gamma} \ln\left(M_1 - \frac{\gamma}{h_2}\exp(-2h_2 t)\right) + P_1, & \gamma \neq 0, \\ P_1 - \frac{\epsilon}{2h_2 M_1}\exp(2h_2 t), & \gamma = 0 \end{cases} \tag{2.28b}$$

with arbitrary constants M_1, P_1.

2. Subalgebra $L_{2.2} = \{T + aW\}$

A solution invariant under the algebra $L_{2.2}$ must be in the following form:

$$\psi(x, t) = M(x) \exp\Big[i\big(at + P(x)\big)\Big]. \tag{2.29}$$

Here, M, P have to satisfy

$$\epsilon M^3 - aM - MP'^2 + M'' = 0, \tag{2.30a}$$

$$\gamma M^3 + h_2 M + 2M'P' + MP'' = 0. \tag{2.30b}$$

For $\gamma = 0$, decoupling is immediate and we have

$$P(x) = -h_2 \int \frac{Y + C}{Y'}\, dx, \qquad M(x) = (-\frac{1}{h_2}Y')^{1/2}, \tag{2.31}$$

where

$$Y'Y''' - \frac{1}{2}Y''^2 - \frac{2\epsilon}{h_2}Y'^3 - 2aY'^2 - 2h_2^2(Y + C)^2 = 0. \tag{2.32}$$

3. Subalgebra $L_{2.3} = \{B + bT\}$

(i) The case $b \neq 0$. An invariant solution of $L_{2.3}$ is obtained in the form

$$\psi(x,t) = M(\xi)\exp\left[i\left(\frac{1}{2b}xt - \frac{1}{6b^2}t^3 + P(\xi)\right)\right], \qquad \xi = bx - \frac{t^2}{2}. \tag{2.33}$$

Functions M and P are solutions to the system

$$\frac{\epsilon}{b^2}M^3 - \frac{\xi}{2b^4}M - MP'^2 + M'' = 0, \tag{2.34a}$$

$$\gamma M^3 + h_2 M + 2b^2 M'P' + b^2 MP'' = 0. \tag{2.34b}$$

We can arrange (2.34b) as

$$\gamma M^4 + h_2 M^2 + b^2(M^2P')' = 0 \tag{2.35}$$

and for $\gamma = 0$ define $Y(\xi)$, such that

$$(M^2P')' = -\frac{h_2}{b^2}M^2 = Y', \tag{2.36}$$

from which we get

$$P(\xi) = -\frac{h_2}{b^2}\int \frac{Y+C}{Y'}\,d\xi. \tag{2.37}$$

Hence, we have decoupled (2.34a) in the form

$$Y'Y''' - \frac{1}{2}Y''^2 - \frac{2\epsilon}{h_2}Y'^3 - \frac{x}{b^4}Y'^2 - \frac{2h_2^2}{b^4}(Y+C)^2 = 0. \tag{2.38}$$

Here, P is obtained from (2.37) and M is given by the formula

$$M(\xi) = (-\frac{b^2}{h_2}Y')^{1/2}. \tag{2.39}$$

(ii) The case $b = 0$,

$$\psi(x,t) = M(t)\exp\left[i\left(\frac{x^2}{4t} + P(t)\right)\right] \tag{2.40}$$

is the form of the group-invariant solution and the reduced system of equations is

$$\epsilon M^2 - P' = 0, \qquad \gamma M^3 + (h_2 + \frac{1}{2t})M + M' = 0. \tag{2.41}$$

This system is readily solved by standard methods.

C. Solvable algebra $L_3 = \{T, P, D_2, W\}$

The algebra has the abelian ideal $\{T, P, W\}$. Nonzero commutation relations are

$$[D_2, T] = T, \quad [D_2, P] = \frac{1}{2}P. \tag{2.42}$$

The algebra has a decomposable structure

$$L_3 = \{T, P, D_2\} \oplus \{W\}. \tag{2.43}$$

We note the canonical equation

$$i\psi_t + (1 + if_2)\psi_{xx} + (\epsilon + i\gamma)|\psi|^2\psi = 0 \tag{2.44}$$

with the constants $\epsilon = \mp 1$, $f_2 \neq 0$, γ and proceed to find the reduced ODEs.

1. Subalgebra $L_{3.1} = \{T\}$

A solution of VCNLS invariant under the algebra $L_{3.1}$ must have the form

$$\psi(x,t) = M(x)\exp\big(iP(x)\big). \tag{2.45}$$

Here, M and P are found from the following reduced system:

$$\frac{\epsilon + \gamma f_2}{1+f_2^2} M^3 - MP'^2 + M'' = 0, \tag{2.46a}$$

$$\frac{\gamma - \epsilon f_2}{1+f_2^2} M^4 + (M^2P')' = 0. \tag{2.46b}$$

Similar to the preceding calculations, we were able to achieve decoupling for $\gamma = \epsilon f_2$. In this case, $M(x)$ is a solution to the equation

$$M'' = \frac{C^2}{M^3} - \epsilon M^3 \tag{2.47}$$

and $P(x)$ is given by

$$P(x) = \int \frac{C}{M^2} dx. \tag{2.48}$$

2. Subalgebra $L_{3.2} = \{P\}$

Modulus and phase for the group-invariant solution corresponding to the algebra $L_{3.2}$, which is in the form $\psi(x,t) = M(t)\exp\big(iP(t)\big)$, are found from the system

$$\epsilon M^2 - P' = 0, \qquad \gamma M^3 + M' = 0 \tag{2.49}$$

as

$$M(t) = \sqrt{2\gamma t + M_1}, \qquad P(t) = \epsilon(\gamma t^2 + M_1 t) + P_1. \tag{2.50}$$

3. Subalgebra $L_{3.3} = \{T + \epsilon_1 W\}$

The solution in this case should have the form

$$\psi(x,t) = M(x)\exp\Big[i\big(\epsilon_1 t + P(x)\big)\Big], \tag{2.51}$$

where $M,\ P$ satisfy

$$\epsilon M^3 - 2f_2M'P' - (\epsilon_1 + P'^2 + f_2P'')M + M'' = 0, \tag{2.52a}$$

$$\gamma M^3 + 2M'P' + (-f_2P'^2 + P'')M + f_2M'' = 0. \tag{2.52b}$$

In order to decouple these equations we can arrange them as

$$\frac{\epsilon + \gamma f_2}{1+f_2^2} M^3 - MP'^2 + M'' = 0, \tag{2.53a}$$

$$(\gamma - \epsilon f_2)M^4 + \epsilon_1 f_2 M^2 + (1+f_2^2)(M^2P')' = 0. \tag{2.53b}$$

Since $f_2 \neq 0$, a first integral of (2.53b) can be obtained if $\gamma = \epsilon f_2$. Let $Y(x)$ be defined as

$$(M^2P')' = -\frac{\epsilon_1 f_2}{1+f_2^2} M^2 = Y'. \tag{2.54}$$

Then we have

$$M^2 = -\frac{1+f_2^2}{\epsilon_1 f_2} Y', \quad P' = -\frac{\epsilon_1 f_2}{1+f_2^2}\frac{Y+C}{Y'} \tag{2.55}$$

and thus (2.53a) is transformed to an equation in terms of $Y(x)$,

$$Y'Y''' - \frac{1}{2}Y''^2 - \frac{2\epsilon(1+f_2^2)}{\epsilon_1 f_2}Y'^3 - \frac{2f_2^2}{1+f_2^2}(Y+C)^2 = 0. \tag{2.56}$$

4. Subalgebra $L_{3.4} = \{P + \epsilon_1 W\}$

In this case, we have

$$\psi(x,t) = M(t)\exp\left[i\left(\epsilon_1 x + P(t)\right)\right], \tag{2.57}$$

where

$$1 - \epsilon M^2 + P' = 0, \qquad \gamma M^3 - f_2 M + M' = 0. \tag{2.58}$$

Integration of the system is elementary

$$M(t) = \left(M_1 \exp(-2f_2 t) + \frac{\gamma}{f_2}\right)^{-1/2}, \tag{2.59a}$$

$$P(t) = \begin{cases} \frac{\epsilon}{2\gamma}\ln\left(M_1 + \frac{\gamma}{f_2}\exp(2f_2 t)\right) - t + P_1, & \gamma \neq 0, \\ \frac{\epsilon}{2f_2 M_1}\exp(2f_2 t) - t + P_1, & \gamma = 0. \end{cases} \tag{2.59b}$$

5. Subalgebra $L_{3.5} = \{D + aW\}$

We will look for the solution in the form

$$\psi(x,t) = \frac{1}{x}M(\xi)\exp\left[i\left(2a\ln x + P(\xi)\right)\right], \quad \xi = \frac{x^2}{t}. \tag{2.60}$$

The corresponding reduced system for M, P contains second order derivatives in terms of only M or P. Again, as in the previous algebras we have not been able to proceed further.

6. Subalgebra $L_{3.6} = T + \epsilon_1 P + bW$

The modulus M and the phase P of the group-invariant solution

$$\psi(x,t) = M(\xi)\exp\left[i\left(bt + P(\xi)\right)\right], \quad \xi = x - \epsilon_1 t \tag{2.61}$$

satisfy the system

$$(\epsilon + \gamma f_2)M^3 - bM - \epsilon_1 f_2 M' + \epsilon_1 M P' - (1+f_2^2)MP'^2 + (1+f_2^2)M'' = 0, \tag{2.62}$$

$$(\gamma - \epsilon f_2)M^4 + bf_2M^2 - \frac{\epsilon_1}{2}(M^2)' - \epsilon_1 f_2 M^2 P' + (1+f_2^2)(M^2P')' = 0. \tag{2.63}$$

Arranging (2.63) in terms M^2P' and M^2 we can write

$$(M^2P')' - \frac{\epsilon_1 f_2}{1+f_2^2}(M^2P') = \frac{1}{1+f_2^2}\left(\frac{\epsilon_1}{2}(M^2)' - bf_2M^2 + (\epsilon f_2 - \gamma)M^4\right). \tag{2.64}$$

We can reduce this equation to a quadrature by the introduction of an auxiliary function $Y(\xi)$,

$$\left[\exp\left(\frac{-\epsilon_1 f_2}{1+f_2^2}\,\xi\right)M^2P'\right]' = \frac{\exp\left(\frac{-\epsilon_1 f_2}{1+f_2^2}\,\xi\right)}{1+f_2^2}\left(\frac{\epsilon_1}{2}(M^2)' - bf_2M^2 + (\epsilon f_2 - \gamma)M^4\right) = Y' \tag{2.65}$$

having the first integral

$$M^2P' = \exp\big(\frac{\epsilon_1 f_2}{1+f_2^2}\,\xi\big)(Y+C). \tag{2.66}$$

On the other hand, we need to solve for M^2 from

$$(M^2)' - 2\epsilon_1 b f_2 M^2 + 2\epsilon_1(\epsilon f_2 - \gamma)(M^2)^2 = 2\epsilon_1(1+f_2^2)\exp\big(\frac{\epsilon_1 f_2}{1+f_2^2}\,\xi\big)\,Y'. \tag{2.67}$$

Though this equation is of Riccati-type in M^2 which can be linearized through the well-known Jacobi transformation, it does not look promising at all to provide us with any nontrivial solution except for some special case. In fact, the special choice $\gamma = \epsilon f_2$ turns (2.67) into an exact equation

$$\Big(\exp\big(-2\epsilon_1 b f_2\xi\big)\,M^2\Big)' = 2\epsilon_1(1+f_2^2)\exp\Big(\epsilon_1 f_2\big(\frac{1}{1+f_2^2} - 2b\big)\xi\Big)\,Y'. \tag{2.68}$$

If there is a further relation $b = \frac{1}{2(1+f_2^2)}$, then an integration gives

$$M^2(\xi) = 2\epsilon_1(1+f_2^2)\exp\big(\frac{\epsilon_1 f_2}{1+f_2^2}\,\xi\big)\Big(Y(\xi)+C\Big). \tag{2.69}$$

By the relation (2.66),

$$P' = \frac{1}{2\epsilon_1(1+f_2^2)}. \tag{2.70}$$

Therefore, we end up with a decoupled equation for M from (2.62),

$$M'' = \frac{\epsilon_1 f_2}{1+f_2^2}M' + \frac{1}{4(1+f_2^2)^2}M - \epsilon M^3. \tag{2.71}$$

On the other hand, if (2.63) is arranged with the condition $\gamma = \epsilon f_2$ as

$$\Big((1+f_2^2)M^2P' - \frac{\epsilon_1}{2}M^2\Big)' = \epsilon_1 f_2 M^2 P' - b f_2 M^2, \tag{2.72}$$

the choice $b = \frac{1}{2(1+f_2^2)}$ even makes it possible to write this equation in the simpler form $U' = \lambda U$, where $U(\xi) = (1+f_2^2)M^2P' - \frac{\epsilon_1}{2}M^2$, $\lambda = \frac{\epsilon_1 f_2}{1+f_2^2}$. By the solution $U(\xi) = \lambda_0\exp(\lambda\xi)$ with some constant λ_0, we are led to

$$P' = \frac{1}{2\epsilon_1(1+f_2^2)} + \frac{\lambda_0\exp(\lambda\xi)}{1+f_2^2}M^{-2}. \tag{2.73}$$

Substitution of this relation in (2.62) gives the decoupled equation for M,

$$M'' = \frac{\epsilon_1 f_2}{1+f_2^2}M' + \frac{1}{4(1+f_2^2)^2}M - \epsilon M^3 + \frac{\lambda_0^2\exp(2\lambda\xi)}{(1+f_2^2)^2}\,M^{-3}. \tag{2.74}$$

This equation reduces to (2.71) for $\lambda_0 = 0$.

D. Solvable algebra $L_4 = \{P, B, D_2, W\}$

This solvable non-nilpotent algebra is the extension of the nilpotent three-dimensional Lie algebra $\{W, P, B\}$. We represent the action of D_2 on this ideal by a matrix M,

$$\begin{pmatrix}[W, D_2]\\ [P, D_2]\\ [B, D_2]\end{pmatrix} = M\begin{pmatrix}W\\ P\\ B\end{pmatrix}, \qquad M = \begin{pmatrix}0 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & -1/2\end{pmatrix}.$$

We note that the algebra is not decomposable and the representative equation from Table I is

$$i\psi_t + \psi_{xx} + (\epsilon + i\gamma)|\psi|^2\psi + i\frac{h_2}{t}\psi = 0, \tag{2.75}$$

where $\epsilon = \mp 1$, $h_2 \neq 0$, and γ are real constants.

1. Subalgebra $L_{4.1} = \{P\}$

Group-invariant solution is of the form

$$\psi(x,t) = M(t)\exp\big(iP(t)\big) \tag{2.76}$$

and we find

$$M(t) = (2\gamma t \ln t + M_1 t)^{-1/2}, \tag{2.77a}$$

$$P(t) = \begin{cases} \frac{\epsilon}{2\gamma}\ln(2\gamma\ln t + M_1) + P_1, & \gamma \neq 0, \\ \frac{\epsilon}{M_1}\ln t + P_1, & \gamma = 0 \end{cases} \tag{2.77b}$$

for $h_2 = 1/2$, whereas

$$M(t) = \Big(\frac{2\gamma}{1-2h_2}\,t + M_1 t^{2h_2}\Big)^{-1/2}, \tag{2.78a}$$

$$P(t) = \begin{cases} \frac{\epsilon}{2\gamma}\ln(M_1 + \frac{2\gamma}{1-2h_2}\,t^{1-2h_2}) + P_1, & \gamma \neq 0, \\ \frac{\epsilon}{M_1(1-2h_2)}\,t^{1-2h_2} + P_1, & \gamma = 0 \end{cases} \tag{2.78b}$$

for $h_2 \neq 1/2$.

2. Subalgebra $L_{4.2} = \{B\}$

The solution invariant under B has the form

$$\psi(x,t) = M(t)\exp\Big[i\Big(\frac{x^2}{4t} + P(t)\Big)\Big], \tag{2.79}$$

$$M(t) = (M_1\, t^{1+2h_2} - \frac{\gamma}{h_2}\,t)^{-1/2}, \tag{2.80a}$$

$$P(t) = \begin{cases} \frac{\epsilon}{2\gamma}\ln(M_1 - \frac{\gamma}{h_2}t^{-2h_2}) + P_1, & \gamma \neq 0, \\ -\frac{\epsilon}{2h_2 M_1}t^{-2h_2} + P_1, & \gamma = 0. \end{cases} \tag{2.80b}$$

3. Subalgebra $L_{4.3} = \{P + \epsilon_1 B\}$

The corresponding invariant solution is given by

$$\psi(x,t) = M(t)\exp\Big[i\Big(\frac{\epsilon_1 x^2}{4(1+\epsilon_1 t)} + P(t)\Big)\Big]. \tag{2.81}$$

The reduced system becomes

$$\epsilon M^2 - P' = 0, \qquad M' + \Big(\frac{h_2}{t} + \frac{\epsilon_1}{2(1+\epsilon_1 t)}\Big)M + \gamma M^3 = 0. \tag{2.82}$$

For the special case $h_2 = 1/2$, the system is easily integrated by elementary functions. If $h_2 \neq 1/2$, the solutions are expressible in terms of the Gauss' hypergeometric function ${}_2F_1(a,b,c;t)$.

4. Subalgebra $L_{4.4} = \{D + aW\}$

A group-invariant solution invariant under the subalgebra $L_{4.4}$ will be of the form

$$\psi(x,t) = \frac{1}{x} M(\xi) \exp\Big[i\big(2a \ln x + P(\xi)\big)\Big], \quad \xi = \frac{x^2}{t}. \tag{2.83}$$

Functions M, P will be solutions of the system

$$\epsilon M^3 + 2(1-2a^2)M - 2\xi M' + (\xi^2 - 8a\xi)MP' - 4\xi^2 MP'^2 + 4\xi^2 M'' = 0, \tag{2.84a}$$

$$\gamma M^4 + (h_2\xi - 6a)M^2 + \big(4a\xi - \frac{\xi^2}{2}\big)(M^2)' - 2\xi M^2 P' + 4\xi^2(M^2P')' = 0. \tag{2.84b}$$

For $\gamma = 0$, we introduce the function $Y(\xi)$ in (2.84b) such that

$$4\xi^2(M^2P')' - 2\xi M^2 P' = \big(\frac{\xi^2}{2} - 4a\xi\big)(M^2)' + (6a - h_2\xi)M^2 = Y'. \tag{2.85}$$

From these relations, we find

$$M^2P' = \frac{\xi^{1/2}}{4}\Big(\int \xi^{-5/2} Y' d\xi + C\Big) \tag{2.86}$$

and for $h_2 = 1/4$,

$$M^2 = \frac{2\xi^{3/2}}{\xi - 8a}\Big(\int \xi^{-5/2} Y' d\xi + C\Big). \tag{2.87}$$

Thus if $h_2 = 1/4$ we can obtain from (2.86) and (2.87) that

$$P(\xi) = \frac{\xi}{8} - a\ln\xi + P_1. \tag{2.88}$$

This special form of $P(\xi)$ is readily seen to satisfy (2.84b), whereas (2.84a) is decoupled to determine M from

$$M'' = \frac{1}{2\xi}M' - \big(\frac{1}{64} - \frac{a}{4\xi} + \frac{1}{2\xi^2}\big)M - \frac{\epsilon}{4\xi^2}M^3. \tag{2.89}$$

E. Solvable algebra $L_5 = \{B, C_2, W\}$

Here, L_5 is another canonical extension of the nilpotent algebra $\{W, P, B\}$ to a solvable non-nilpotent indecomposable four-dimensional algebra. The element C_2 acts on the ideal $\{W, P, B\}$ by the matrix M as

$$\begin{pmatrix} [W, C_2] \\ [P, C_2] \\ [B, C_2] \end{pmatrix} = M \begin{pmatrix} W \\ P \\ B \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Thus the last canonical equation under investigation will be

$$i\psi_t + \psi_{xx} + \frac{\epsilon + i\gamma}{1+t^2}|\psi|^2\psi + \frac{2h_1 + i(2h_2 - t)}{2(1+t^2)}\psi = 0, \tag{2.90}$$

with the constants $\epsilon = \mp 1$, h_1, h_2, and γ.

1. Subalgebra $L_{5.1} = \{B\}$

The solution has the form

$$\psi(x,t) = M(t)\exp\Big[i\big(\frac{x^2}{4t} + P(t)\big)\Big],$$

TABLE III. Equations under study.

Order	Equation Number
1	(2.27), (2.41), (2.49), (2.58), (2.82), (2.91)
2	(2.11), (2.17), (2.23), (2.47), (2.74), (2.89), (2.96)
3	(2.8), (2.16), (2.22), (2.32), (2.38), (2.56), (2.94)

with functions M, P determined from the system

$$\frac{\epsilon}{1+t^2}M^2+\frac{h_1}{1+t^2}-P'=0,\qquad M'+\Big(\frac{2h_2-t}{2(1+t^2)}+\frac{1}{2t}\Big)M+\frac{\gamma}{1+t^2}M^3=0. \tag{2.91}$$

Again, it is straightforward to integrate this system.

2. Subalgebra $L_{5.2}=\{C_2+aW\}$

Group-invariant solution must have the form

$$\psi(x,t)=M(\xi)\exp\Big[i\Big(a\arctan t+\frac{x^2t}{4(1+t^2)}+P(\xi)\Big)\Big],\qquad \xi=\frac{x}{\sqrt{1+t^2}}. \tag{2.92}$$

Substitution of this solution into the original equation ends up with the system

$$\epsilon M^3+\big(h_1-a-\frac{\xi^2}{4}\big)M-MP'^2+M''=0,\qquad \gamma M^4+h_2M^2+(M^2P')'=0. \tag{2.93}$$

In the case $\gamma=0,\ h_2\neq 0$ decoupling of these equations is possible

$$Y'Y'''-\frac{1}{2}Y''^2-\frac{2\epsilon}{h_2}Y'^3+\big(2h_1-2a-\frac{\xi^2}{2}\big)Y'^2-2h_2^2(Y+C)^2=0, \tag{2.94}$$

$$M=(-\frac{1}{h_2}Y')^{1/2},\quad P'=-h_2\frac{Y+C}{Y'}. \tag{2.95}$$

If $\gamma=h_2=0$, then we have the following ODEs for M and P,

$$M''=C^2M^{-3}+(\frac{\xi^2}{4}+a-h_1)M-\epsilon M^3,\qquad P'=CM^{-2}. \tag{2.96}$$

III. ANALYSIS OF THE REDUCED EQUATIONS

In Table III, we refer to the numbers of the reduced system of equations of first order besides the second and third order equations obtained through the decoupling task for some special values of the arbitrary parameters. We have expressed the solutions of first order equations in Sec. II. This part of the work will be devoted to the study of solutions of the second and third order equations.

A. Third order equations

None of the seven third order equations summarized in Table III passes the Painlevé test for PDEs. Since (2.32) and (2.56) do not contain the independent variable, we can directly lower their order by one, if we set $Y'=W(Y)$. We obtain the following second order equations:

(i) For equation (2.32) with $\dot W=\frac{dW}{dY}$,

$$\ddot W=-\frac{1}{2W}\dot W^2+\frac{2\epsilon}{h_2}+\frac{2a}{W}+\frac{2h_2^2(Y+C)^2}{W^3}. \tag{3.1}$$

(ii) For equation (2.56),

$$\ddot{W} = -\frac{1}{2W}\dot{W}^2 + \frac{2\epsilon(1+f_2^2)}{\epsilon_1 f_2} + \frac{2f_2^2}{1+f_2^2}\frac{(Y+C)^2}{W^3}. \tag{3.2}$$

Equations (3.1) and (3.2) cannot have the Painlevé property unless $h_2 = 0$ and $f_2 = 0$ which are not allowed.

For all the third order equations satisfied by $Y = Y(\xi)$ (including (2.32) and (2.56)) we suggested a first integral of the form

$$A(\xi, Y, Y')\,Y''^2 + B(\xi, Y, Y')\,Y'' + F(\xi, Y, Y') = I \tag{3.3}$$

with some functions A, B, F, and a constant I. It turned out that a first integral of this particular form can exist only possible for (2.22) for the special values of the constants $c = 0$, $h_1 = (5 + 81h_2^2)/36$. The first integral has the form

$$Y''^2 + \frac{10}{3x}Y'Y'' - \frac{2\epsilon x}{h_2}Y'^3 + \frac{25+81h_2^2}{9x^2}Y'^2 + \frac{12h_2^2}{x^3}YY'$$
$$+(\frac{12h_2^2C}{x^3} - \frac{I}{x^{7/3}})Y' + \frac{4h_2^2}{x^4}(Y+C)^2 = 0. \tag{3.4}$$

B. Second order equations

Among the second order equations successfully decoupled from the reduced systems, (2.23) passes the P-test for $h_1 = 5/36$ and so does (2.47) without any condition on the parameters. Before proceeding to the solutions of second order equations passing the P-test, we note that Eq. (2.74) does not contain the independent variable if we choose $\lambda_0 = 0$, which means a reduction in order. Indeed, if we set $M' = a_1 W(M)$ with $a_1 = \frac{\epsilon_1 f_2}{1+f_2^2}$, an Abel equation of the second kind is obtained

$$W(\dot{W} - 1) = \frac{a_2}{a_1^2}M - \frac{\epsilon}{a_1}M^3, \tag{3.5}$$

where $a_2 = \frac{1}{4(1+f_2^2)}$. For $n = \frac{\epsilon_2|f_2|}{\sqrt{1+2f_2^2}} - 3$ and $\epsilon_2 = \mp 1$, $w = w(z)$ a transformation in the parametric form

$$M = z^{\frac{n+2}{2}}w, \qquad W = \frac{1}{n+3}z^{\frac{n+2}{2}}(zw' + \frac{n+2}{2}w) \tag{3.6}$$

converts this equation with $A = -\frac{\epsilon\epsilon_1 f_2(1+f_2^2)}{1+2f_2^2}$ to an equation of Emden-Fowler-type,[9]

$$w'' = Az^n w^3, \tag{3.7}$$

which drove the final nail in the coffin.

We close this Section with the analysis of equations passing the P-test. Painlevé and his successors classified second order differential equations that have at most pole-type singularities in all their solutions and determined such 50 equivalence classes together with their representative equations. For details the interested reader is referred to Ref. 10. Since we have two second order equations passing the P-test, we are going to try to find the equivalence class to which they may belong.

Transformation of (2.47) to the equations numbered PXVIII and PXXXIII in the Painlevé classification of second order nonlinear ODEs, their first integrals and hence solutions in terms of elementary and elliptic functions in various cases were done in Ref. 11. Since a simple substitution and a careful account of the different cases depending on the constants will suffice to find the results for (2.47), we do not reproduce them here and refer the interested reader to that work.

There remains the treatment of Eq. (2.23). If we make a change of the dependent variable as $M = \sqrt{H(x)}$, $H(x) > 0$, we have

$$H'' = \frac{1}{2H}H'^2 + 2(c - \frac{h_1}{x^2})H - \frac{2\epsilon}{x}H^2 + \frac{2C^2}{H}. \tag{3.8}$$

We apply a further transformation $H(x) = \lambda(x)W(\eta(x))$ and find

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 - \frac{1}{\dot{\eta}}\left(\frac{\ddot{\eta}}{\dot{\eta}} + \frac{\dot{\lambda}}{\lambda}\right)\dot{W} + \frac{1}{\dot{\eta}^2}\left(2(c - \frac{h_1}{x^2}) + \frac{\dot{\lambda}^2}{2\lambda^2} - \frac{\ddot{\lambda}}{\lambda}\right)W - \frac{2\epsilon\lambda}{\dot{\eta}^2 x}W^2 + \frac{2C^2}{\lambda^2\dot{\eta}^2}W^{-1}. \tag{3.9}$$

We will determine λ, η such that this equation is of Painlevé-type.

(i) The case $C \neq 0$. If λ, η and other constants are chosen as

$$\eta = \eta_0 x^{2/3}, \quad \lambda = \lambda_0 x^{1/3}, \quad c < 0, \quad \eta_0 = -\left(\frac{9c}{2}\right)^{1/3}, \quad h_1 = \frac{5}{36} \tag{3.10}$$

an equation quite similar to PXXXIV is obtained

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + 4\alpha W^2 - \eta W + 2\delta^2 W^{-1}, \tag{3.11}$$

where $\alpha = -\left(\frac{9}{2c^2}\right)^{1/3}\frac{\epsilon\lambda_0}{4}$, $\delta = \frac{3C}{2\lambda_0}\left(\frac{-2}{9c}\right)^{1/6}$ and λ_0 is arbitrary. By a final transformation

$$2\alpha W = \dot{V} + V^2 + \frac{\eta}{2}, \tag{3.12}$$

we see that V satisfies the equation

$$\ddot{V} = 2V^3 + \eta V + k, \qquad k = -\frac{1}{2} \pm 4\alpha\delta i, \tag{3.13}$$

which is the second Painlevé transcendent so that we can express $V = P_{II}(\eta_0 x^{2/3})$. Since W is complex-valued λ_0 has to be chosen so that the product λW is real.

(ii) The case $C = 0$. If we choose

$$\eta = \eta_0 x^{2/3}, \quad \lambda = \lambda_0 x^{1/3}, \quad \eta_0 = \left(\frac{9c}{4}\right)^{1/3}, \quad \lambda_0 = -\epsilon\left(\frac{32c^2}{9}\right)^{1/3}, \quad h_1 = \frac{5}{36}, \tag{3.14}$$

we arrive at the equation PXX,

$$\ddot{W} = \frac{1}{2W}\dot{W}^2 + 4W^2 + 2\eta W. \tag{3.15}$$

Setting $U^2 = W$ leads to P_{II} again

$$\ddot{U} = 2U^3 + \eta U.$$

We can explicitly give the solution

$$\psi = \lambda_0^{1/2} x^{1/6} P_{II}(\eta_0 x^{2/3}) \exp(i(ct + P_0)). \tag{3.16}$$

IV. SOLUTIONS BY TRUNCATION

In order to investigate Painlevé property for (1.1) in Ref. 1 we wrote the equation with its complex conjugate as the system

$$\begin{aligned} &iu_t + f(x,t)u_{xx} + g(x,t)u^2 v + h(x,t)u = 0, \\ &-iv_t + p(x,t)v_{xx} + q(x,t)uv^2 + r(x,t)v = 0 \end{aligned} \tag{4.1}$$

and expanded u, v as

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)\Phi^{\alpha+j}(x,t), \quad v(x,t) = \sum_{j=0}^{\infty} v_j(x,t)\Phi^{\beta+j}(x,t). \tag{4.2}$$

We obtained the conditions on f, g, h so that all solutions to VCNLS are in this form. Coefficients for canonical equations of four-dimensional subalgebras do not satisfy the compatibility conditions of the P-test; therefore, they do not have the Painlevé property. Since the conditions obtained from the P-test are equivalent to those for having a Lax pair, they are not integrable.[3,4] In this case, if the series (4.2) is truncated at an order $j = N$ and plugged in the equation, a system of equations for $u_j,\ \ j \leq N$, and Φ has to be satisfied. An exact solution is obtained once this system can be solved in a consistent way.

For the Painlevé test to be successful it is required that resonance coefficients u_j corresponding to the resonance indices $j = -1, 0, 3, 4$ are arbitrary. This is true if the compatibility conditions at resonance levels hold. As we already mentioned, this is not the case for our canonical equations, and we first checked whether resonance equations are satisfied at all for some special form of u_j's and Φ. The results were not so promising, since either no condition for Φ or conditions being equivalent to the integrable case can arise. When we were lucky to obtain a specific form for Φ, conditions other than resonance levels did not hold. Therefore, we could not obtain an exact solution and a Bäcklund transformation by the truncation approach. However, when we applied the method as it was done in Ref. 12, we were able to obtain nontrivial exact solutions.

As the first step of the Painlevé test, the leading orders α and β are determined by substitution of $u \sim u_0\Phi^\alpha$ and $v \sim v_0\Phi^\beta$ in (4.1). Balancing the terms of smallest order requires that

$$\alpha + \beta = -2 \tag{4.3}$$

and

$$u_0 v_0 = -\alpha(\alpha - 1)\frac{f}{g}\Phi_x^2 = -\beta(\beta - 1)\frac{p}{q}\Phi_x^2 \tag{4.4}$$

hold. Since the leading orders should be negative integers for the equation to have the Painlevé property, (4.3) implies that $\alpha = -1$ and $\beta = -1$. Since, we are interested in a case in which the equations do not have the P-property, we weaken the condition that α and β are integers and determine the leading orders by solving Eqs. (4.3) and (4.4) simultaneously. This will indeed lead to finding exact solutions by truncation approach.

We successfully applied this approach to the canonical equations of algebras L_1, L_3, L_4. Overdetermined system of equations for L_2 and L_5 algebras are not compatible and the method fails to apply. Below we shall only present the final results and refer to Ref. 8 for further details.

A. Truncation method for the algebra L_1

The coefficients for the algebra L_1 are $f = 1$, $g = (\epsilon + i\gamma)\frac{1}{x}$, $h = (h_1 + ih_2)\frac{1}{x^2}$. We apply the truncation method to the slightly more general coefficients

$$f = 1, \quad g = (\epsilon + i\gamma)\frac{1}{x^a}, \quad h = (h_1 + ih_2)\frac{1}{x^b}, \quad a, b \in \mathbb{R}. \tag{4.5}$$

If $a \neq 1$, $b = 2$, the equation is invariant under the three-dimensional solvable algebra with a basis

$$T = \partial_t, \quad D = t\partial_t + \frac{x}{2}\partial_x + \frac{a-2}{4}\rho\partial_\rho, \quad W = \partial_\omega.$$

The algebra is extended for $a = 1$.

When we solve (4.3) and (4.4) together, for $\gamma \neq 0$ we have

$$\alpha = -1 - i\delta, \quad \beta = -1 + i\delta, \qquad \delta = \frac{-3\epsilon \pm \sqrt{8\gamma^2 + 9}}{2\gamma} \tag{4.6}$$

and (4.4) simplifies as

$$u_0 v_0 = -\frac{3\delta}{\gamma}\, x^a\, \Phi_x^2. \tag{4.7}$$

We truncate the Painlevé expansion at the first order ($j = 0$) and suggest that solution has the form

$$u(x,t) = u_0(x,t)\Phi(x,t)^{-1-i\delta}, \quad v(x,t) = v_0(x,t)\Phi(x,t)^{-1+i\delta}. \tag{4.8}$$

Putting these expressions in (4.1), the terms $\Phi^{-3\pm i\delta}$, $\Phi^{-2\pm i\delta}$, $\Phi^{-1\pm i\delta}$ will appear. We choose the coefficients of these terms equal to zero to obtain a system of three equations for u_0, v_0, and Φ each of which consists of two equations.

The constants δ and γ must satisfy $\frac{\delta}{\gamma} < 0$. This means that we have to choose the negative sign for the formula of δ in (4.6),

$$\delta = \frac{-3\epsilon - \sqrt{8\gamma^2 + 9}}{2\gamma}. \tag{4.9}$$

We solve the system of equations we obtained in various cases depending on the constants a and b. The constants k_0, k_1, k_2, k_3 which will appear in the solutions are arbitrary real numbers.

(1) The case $a = 3$. We solve the equations at order $\Phi^{-1\pm i\delta}$ and find $b = 2$, $h_1 = 1/4$, $h_2 = 0$. As a result, we have

$$u_0(x,t) = c\sqrt{x}, \quad \Phi(x,t) = k_0 \ln|x| + k_1, \tag{4.10}$$

where $c \in \mathbb{C}$ with $|c| = k_0\sqrt{-\frac{3\delta}{\gamma}}$. We write the solution explicitly as

$$u(x,t) = \frac{c\sqrt{x}}{k_0 \ln|x| + k_1}\ \exp\big(-i\delta \ln(k_0\ln|x| + k_1)\big). \tag{4.11}$$

(2) The case $a = -3$. If $b = 2$, $h_2^2 = 3 + 4h_1$ the solution is given by

$$u(x,t) = \frac{c}{x^{1/2}\,(k_0x^2 - 2h_2k_0t + k_1)}\ \exp\left(i \ln \frac{x^{h_2/2}}{(k_0x^2 - 2h_2k_0t + k_1)^{\delta}}\right), \tag{4.12}$$

where $c \in \mathbb{C}$ with the modulus $|c| = 2k_0\sqrt{-\frac{3\delta}{\gamma}}$.

(3) The case $a \neq \mp 3$. There are several different values of the constants to be considered.

(3.i) The case $0 = b = -1 + \frac{a}{3} + b$. We require $a = 3$. This case is not possible since we had been able to find an exact solution in (1.) for $b = 2$.

(3.ii) The case $0 \neq b = -1 + \frac{a}{3} + b$. Since we must have $a = 3$, this case corresponds to (1).

(3.iii) The case $0 = b \neq -1 + \frac{a}{3} + b$.

(A) The case $a \neq 0$. For $a = 6$ and $h_2 = 0$, we have

$$u(x,t) = k_0\sqrt{-\frac{3\delta}{\gamma}}\frac{x^2}{k_0 + k_1x}\ \exp\Big(i\big(h_1t - \delta\ln(k_0x^{-1} + k_1) + k_2\big)\Big). \tag{4.13}$$

(B) The case $a = 0$. We have the explicit solution for $h_2 = 0$ as

$$u(x,t) = \frac{k_0\sqrt{-\frac{3\delta}{\gamma}}}{k_0x + k_1t + k_2}\exp\Big[i\Big((h_1 - \frac{k_1^2}{4k_0^2})t - \frac{k_1}{2k_0}x - \delta\ln(k_0x + k_1t + k_2) + k_3\Big)\Big]. \tag{4.14}$$

(3.iv) The case $b \neq 0 = -1 + \frac{a}{3} + b$. There is no solution in the truncated expansion form.

(3.v) The case $b \neq 0,\ -1 + \frac{a}{3} + b \neq 0,\ b \neq -1 + \frac{a}{3} + b$.

(A) The case $a = 0\,(b \neq \{0, 1\})$. For $h_1 = h_2 = 0$, $|c|^2 = -\frac{3\delta}{\gamma}k_0^2$, $c \in \mathbb{C}$ we obtained the exact solution

$$u(x,t) = \frac{c}{k_0x + k_1t + k_2}\ \exp\Big[-i\Big(\frac{k_1}{2k_0}x + \frac{k_1^2}{4k_0^2}t + \delta\ln(k_0x + k_1t + k_2)\Big)\Big]. \tag{4.15}$$

(B) The case $a \neq \{0, 1\}$. We found for $h_1 = h_2 = 0$, $a = 6$, $|c|^2 = -\frac{3\delta}{\gamma}k_0^2$,

$$u(x,t) = \frac{cx^2}{k_0 + k_1x}\ \exp\Big(-i\delta\ln(k_0x^{-1} + k_1)\Big). \tag{4.16}$$

(C) The case $a = 1\,(b \neq 2/3)$. This last situation is going to give us the exact solution for the canonical equation of the algebra L_1, namely, the case $a = 1$ and $b = 2$.

For $h_2 = 0$, $b = 2$, $h_1 = 5/36$, and $c \in \mathbb{C}$, $|c|^2 = -\frac{4\delta}{3\gamma}$ the solution is found to be

$$u(x,t) = \frac{c\, x^{1/6}}{x^{2/3} + k_1(k_0 t + k_2)^{2/3}} \exp\Big[i\Big(\frac{k_0 x^2}{4(k_0 t + k_2)} - \delta \ln\big(\frac{x^{2/3}}{(k_0 t + k_2)^{2/3}} + k_1\big)\Big)\Big]. \quad (4.17)$$

Remark: Since the canonical equation of L_1 is invariant under the action of the group of transformations $SL(2,\mathbb{R})$, which is the composed action of translation generated by T, scaling D_1, and the conformal transformation of C_1, the solution (4.17) is transformed into a new solution of the canonical equation under this action. Owing to the invariance property under C_1, this transformed solution has a finite time singularity and, therefore, was fruitful to study blow-up profiles. It is exactly this blow-up character that was used in Ref. 13 to establish the existence of singular behaviours of solutions in the sense of L_p and L_∞ norms and in the distributional sense as well.

B. Truncation method for the algebra L_3

Coefficient functions for the algebra L_3 are

$$f = 1 + i f_2, \quad g = (\epsilon + i\gamma), \quad h = 0. \quad (4.18)$$

In fact, this constant coefficient case is included in Ref. 12 but we could not deduce our results from theirs. Differing from the previous algebra, f contains imaginary part and there will be a slight modification in the above construction. If we solve (4.3) and (4.4) together we find the leading orders to be

$$\alpha = -1 - i\delta, \quad \beta = -1 + i\delta, \qquad \delta = \frac{-3(\epsilon + \gamma f_2) \pm \sqrt{9(\epsilon + \gamma f_2)^2 + 8(\gamma - \epsilon f_2)^2}}{2(\gamma - \epsilon f_2)} \quad (4.19)$$

for $\gamma \neq \epsilon f_2$ and (4.4) is equivalent to the condition

$$u_0 v_0 = -\frac{3(1 + f_2^2)}{\gamma - \epsilon f_2}\, \delta\, \Phi_x^2. \quad (4.20)$$

We truncate the Painlevé expansion at the first term and propose a solution of the form (4.8). The system of equations, which appears when these *ansätze* for u and v are put in (4.1) can be solved for Φ and u_0 and the exact solution will be

$$u(x,t) = \frac{c}{k_0 x + k_1} \exp\Big(-i\delta \ln(k_0 x + k_1)\Big) \quad (4.21)$$

for $c \in \mathbb{C}$, $|c|^2 = -\frac{3(1+f_2^2)}{\gamma - \epsilon f_2}\,\delta\, k_0^2$. It is necessary to choose the negative sign for the square root in the formula δ of (4.19).

C. Truncation method for the algebra L_4

We repeat the arguments which worked for algebra L_1 for the potential $h(x,t) = \frac{h_2}{t}$ of algebra L_4 and find

$$u(x,t) = \frac{c}{x + k_0 k_1 t} \exp\Big[i\Big(\frac{x^2}{4t} - \delta \ln\big(\frac{x}{k_0 t} + k_1\big)\Big)\Big], \quad (4.22)$$

if $h_2 = 1/2$. Here, the constant $c \in \mathbb{C}$ must satisfy $|c|^2 = -\frac{3\delta}{\gamma}$, in addition, we have $\delta = \frac{-3\epsilon - \sqrt{8\gamma^2 + 9}}{2\gamma}$.

[1] C. Özemir and F. Güngör, e-print arXiv:1004.0852.
[2] L. Gagnon and P. Winternitz, J. Phys. A **26**, 7061 (1993).
[3] U. Al Khawaja, J. Math. Phys. **51**, 053506 (2010).
[4] M. Sciacca and T. Brugarino, J. Math. Phys. **51**, 093503 (2010).
[5] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York, 1991).
[6] L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Academic, New York, 1982).

[7] P. Winternitz, "Lie groups and solutions of nonlinear partial differential equations," in *Integrable Systems, Quantum Fields, and Quantum Field Theories*, edited by L. A. Ibort and M. A. Rodriguez (Kluwer Academic Publishers, The Netherlands, 1992).
[8] C. Özemir and F. Güngör, e-print arXiv:1102.3814.
[9] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations* (CRC Press, Inc. Boca Raton, Florida, 1995).
[10] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).
[11] C. Özemir and F. Güngör, J. Phys. A **39**, 2973 (2006).
[12] E. Yomba and T. C. Kofané, Phys. Scr. **54**, 576 (1996).
[13] F. Güngör, M. Hasanov, and C. Özemir, e-print arXiv:1101.2307v2.