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# Infinite-dimensional symmetries of two-dimensional generalized Burgers equations

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The conditions for a class of generalized Burgers equations which *a priori* involve nine arbitrary functions of one or two variables to allow an infinite-dimensional symmetry algebra are determined. Although this algebra can involve up to two arbitrary functions of time, it does not allow a Virasoro subalgebra. This result reconfirms a long-standing fact that variable coefficient generalizations of a nonintegrable equation should be expected to remain as such. © *2010 American Institute of Physics.* [doi:10.1063/1.3456061]

## I. INTRODUCTION

The two-dimensional Burgers equation in normalized (appropriately scaled) form,

$$u_{xt} + (uu_x)_x - u_{yy} = 0, \tag{1.1}$$

is a well-known model for describing the propagation of confined sound beams in slightly nonlinear media without dissipation. This equation which is often referred to as the Zabolotskaya–Khoklov (ZK) equation was derived in 1969 in a paper by Zabolotskaya and Khoklov.[27] A generalization of (1.1) taking dissipation into account was proposed by Kuznetsov.[18] It has the form

$$(u_t + uu_x - u_{xx})_x - u_{yy} = 0. \tag{1.2}$$

Equation (1.2) can be called Zabolotskaya–Khoklov–Kuznetsov (ZKK) equation.

A number of works have been devoted to the construction of exact solutions, in particular, traveling wave solutions of (1.1) and (1.2) by different approaches. For example, a Painlevé analysis of (1.2) was carried out in Ref. 25, where the authors obtained a restricted Bäcklund transformation that maps a subclass of solutions onto a linear heat equation. The first attempt toward conservation laws and Lie symmetries of Eq. (1.1) appeared in Ref. 3. However, the correct form of the symmetry algebra of (1.1) was provided in Ref. 23. Similarity reductions of Eq. (1.2) can be found in Ref. 24. Recently, albeit defects, a more general study of the group-invariant solutions of (1.1) based on a list of inequivalent subalgebras of the maximal invariance algebra under the adjoint symmetry transformations is done in a series of papers.[19,20]

The Lie symmetry algebra $L$ of the ZK equation is known to be an infinite-dimensional algebra. A suitable basis for $L$ (Ref. 23) is given by

$$T(f) = f\partial_t + \frac{1}{6}(2xf' + y^2f'')\partial_x + \frac{2y}{3}f'\partial_y + \frac{1}{6}(-4uf' - 2xf'' - y^2f''')\partial_u, \tag{1.3a}$$

$$X(g) = g\partial_x - g'\partial_u, \tag{1.3b}$$

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$$Y(h) = \tfrac{1}{2}yh'\partial_x + h\partial_y - \tfrac{1}{2}yh''\partial_u, \tag{1.3c}$$

$$D = 2x\partial_x + y\partial_y + 2u\partial_u, \tag{1.3d}$$

where $f, g, h$ are arbitrary smooth functions of the time variable $t$ and the prime denotes derivative with respect to $t$. The commutation relations are

$$[D, Y(g)] = -2X(g), \qquad [X(g), Y(h)] = 0, \tag{1.4a}$$

$$[D, Y(h)] = -Y(h), \qquad [X(g), T(f)] = X(f'g/3 - fg'), \tag{1.4b}$$

$$[D, T(f)] = 0, \qquad [Y(h), T(f)] = Y\left(\tfrac{2}{3}f'h - fh'\right), \tag{1.4c}$$

$$[X(g_1), X(g_2)] = 0, \qquad [Y(h_1), Y(h_2)] = X(h_1h_2' - h_1'h_2)/2, \tag{1.4d}$$

$$[T(f_1), T(f_2)] = T(f_1f_2' - f_1'f_2). \tag{1.4e}$$

The ZK symmetry algebra $L$ can be written as a semidirect sum Lie algebra (Levi decomposition), $L = S \uplus R$, where $S = \{T(f)\}$ is the semisimple part, also called Levi factor of $L$ and $R = \{X(g), Y(h), D\}$ is its radical. $S$ is a simple Lie algebra, i.e., it has no nontrivial ideal. This follows easily from the Lie algebra isomorphism,

$$\rho: J(\mathbb{R}) \to S, \quad f(t)\partial_t \mapsto T(f(t)),$$

where the Lie algebra $J(\mathbb{R}) = \{f(t)\partial_t : f \in C^\infty(\mathbb{R})\}$ of smooth vector fields on $\mathbb{R}$ is a simple algebra in the Cartan's classification. The radical $R$ (maximal solvable ideal) is actually nonnilpotent. The algebra $S$ can be identified as a centerless Virasoro algebra. The algebra $R$ is a subalgebra of a centerless Kac–Moody algebra. The subalgebra $\{D\}$ corresponds to the invariance of ZK equation under dilations.

The presence of this special type of symmetry algebra immediately suggests that the equation under study can have a very good chance of being integrable (in any sense of the word). This is indeed the case for the ZK equation. ZK equation is known to be linearizable by the generalized hodograph transformation.[9]

Symmetry properties and exact solutions of the variable coefficient variant of Eq. (1.2), that we call the two-dimensional generalized Burgers (2DGB) equation,[10,11]

$$(u_t + uu_x - u_{xx})_x + \sigma(t)u_{yy} = 0, \tag{1.5}$$

were investigated by the author. The Lie symmetry algebra of the 2DGB was shown to have a non-Abelian Kac–Moody structure, and for arbitrary $\sigma$ it is realized by

$$\hat{\mathbf{V}} = X(f) + Y(g), \tag{1.6}$$

$$X(f) = f(t)\partial_x + \dot{f}(t)\partial_u, \tag{1.7a}$$

$$Y(g) = g(t)\partial_y - \frac{\dot{g}(t)}{2\sigma(t)}y\partial_x - \frac{d}{dt}\left(\frac{\dot{g}(t)}{2\sigma(t)}\right)y\partial_u, \tag{1.7b}$$

where $f(t)$ and $g(t)$ are arbitrary smooth functions and the dots denote time derivatives. In the constant coefficient case, when $\sigma$=constant, the symmetry algebra has two additional generators,

$$D = x\partial_x + \tfrac{3}{2}y\partial_y + 2t\partial_t - u\partial_u, \quad T = \partial_t, \tag{1.8}$$

reflecting the invariance of the equation under appropriate dilations and time translations. The algebra admits several extensions for several special forms of $\sigma(t)$ (see Refs. 11 and 10, for the details).

A remark here is in order. While the symmetry algebras of both Eqs. (1.1) and (1.2) are infinite dimensional, their Lie-algebraic structures are very different in nature. The first equation admits a symmetry algebra having the Kac–Moody–Virasoro structure which is enjoyed by almost every integrable equation in 2+1-dimensions (so far there is only one exception). On the other hand, a Virasoro subalgebra is not contained in the symmetry algebra of the latter. This structure is typical for nonintegrable equations.

In this article we extend further (1.5) to include additional spatial derivatives together with time and $y$-dependent coefficients and consider the generalized 2+1-dimensional Burgers equations,

$$\begin{aligned}(u_t + p(t)uu_x + q(t)u_{xx})_x + \sigma(y,t)u_{yy} + a(y,t)u_y \\ + b(y,t)u_{xy} + c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0,\end{aligned} \tag{1.9}$$

where we assume that in some neighborhood we have

$$p(t) \neq 0, \quad q(t) \neq 0, \quad \sigma(y,t) \neq 0. \tag{1.10}$$

The other functions in (1.9) are arbitrary. We note that these equations specialize to (1.5) when

$$p = 1, \quad q = -1, \quad \sigma(y,t) = \sigma(t), \quad a = b = c = e = f = h = 0.$$

The purpose of this article is to study the symmetry properties of (1.9). We intend to determine the cases when (1.9) has an infinite-dimensional symmetry group. More important, we would like to look at the possibility of whether it can have a Kac–Moody–Virasoro structure. As already stated above, the main reason for this quest is that the presence of a Virasoro subalgebra in the Lie symmetry algebra may exhibit a strong indication of the integrability of the equation. For a more detailed discussion of these issues, the reader is referred to Ref. 4–6, 26, 12, and 14.

Even when the algebra has no structure of a Virasoro algebra which is a common property of integrable equations in 2+1 dimensions, the existence of an infinite-dimensional symmetry group can be used to obtain large classes of solutions by the tools of Lie group theory.

Painlevé test, *inter alia*, is always at one's disposal to extract some information about integrability or partial integrability of variable coefficient partial differential equations. While computer algebra packages are developed to serve this purpose, the computations involved in such equations with many variable coefficients as the one under study usually turn up to be unmanageably lengthy and complex. Therefore, we prefer to take a symmetry approach instead.

In Sec. II we introduce allowed transformations that take equations of form (1.9) into other equations of the same class. That is, they may change the unspecified functions in (1.9), but not introduce other terms, or extra dependence on other variables. These transformations are used to simplify (1.9) and transform them into (2.6) that we call the "canonical generalized Burgers" (CGB) equations. In Sec. III we determine the general form of the symmetry algebra of the CGB equations and obtain the determining equations for the symmetries. In Sec. IV we look at the possibility if the CGB equations can be invariant under arbitrary reparametrization of time at all. Section V is devoted to the case when the CGB equation is invariant under a Kac–Moody algebra. Some conclusions are presented in Sec. VI.

## II. ALLOWED TRANSFORMATIONS AND CGB EQUATIONS

We want to map (1.9) to some simple (canonical) form. The main tool we shall use is the allowed transformations which are defined to be invertible smooth point transformations,

$$\tilde{x}=X(x,y,t,u), \quad \tilde{y}=Y(x,y,t,u), \quad \tilde{t}=T(x,y,t,u), \quad \tilde{u}=U(x,y,t,u), \tag{2.1}$$

taking equations of form (1.9) into another equations of the same form, but possibly with different coefficient functions. More precisely, the transformed equations will be the same as those Eqs. (1.9), but the arbitrary functions can change. The typical features of the equations are that the new functions $\tilde{p}(\tilde{t})$ and $\tilde{q}(\tilde{t})$ depend on $\tilde{t}$ alone, the others on $\tilde{y}$ and $\tilde{t}$, but no $\tilde{x}$ dependence is introduced. The only $\tilde{t}$-derivative is $\tilde{u}_{\tilde{x}\tilde{t}}$, the only nonlinear term is $\tilde{p}(\tilde{t})(\tilde{u}\tilde{u}_{\tilde{x}})_{\tilde{x}}$, and the only derivative higher than a second order one is $\tilde{q}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}$. These form-preserving conditions restrict (2.1) to the form (the so-called local fiber-preserving transformations)

$$\begin{gathered} u(x,y,t)=R(t)\tilde{u}(\tilde{x},\tilde{y},\tilde{t})-\frac{\dot{\alpha}}{\alpha p}x+S(y,t), \\ \tilde{x}=\alpha(t)x+\beta(y,t), \quad \tilde{y}=Y(y,t), \quad \tilde{t}=T(t), \\ \alpha\neq 0, \quad R\neq 0 \quad Y_y\neq 0, \quad \dot{T}\neq 0, \quad \dot{\alpha}f(y,t)=0. \end{gathered} \tag{2.2}$$

We note that the constraint $\dot{\alpha}f(y,t)=0$ should be imposed for the new coefficient $\tilde{h}$ to have no dependence on $x$. The new coefficients in the transformed equation satisfy

$$\begin{aligned} &\tilde{p}(\tilde{t})=p(t)\frac{R\alpha}{\dot{T}}, \quad \tilde{q}(\tilde{t})=q(t)\frac{\alpha^2}{\dot{T}}, \\ &\tilde{\sigma}(y,t)=\sigma(y,t)\frac{Y_y^2}{\alpha\dot{T}}, \\ &\tilde{a}(\tilde{y},\tilde{t})=\frac{1}{\alpha\dot{T}}\{aY_y+\sigma Y_{yy}\}, \\ &\tilde{b}(\tilde{y},\tilde{t})=\frac{1}{\alpha\dot{T}}\{(b\alpha+2\sigma\beta_y)Y_y+\alpha Y_t\}, \\ &\tilde{c}(\tilde{y},\tilde{t})=\frac{1}{\alpha\dot{T}}\{c\alpha^2+\beta_t\alpha+pS\alpha^2+\sigma\beta_y^2+b\alpha\beta_y\}, \\ &\tilde{e}(\tilde{y},\tilde{t})=\frac{1}{\alpha R\dot{T}}\{R\alpha e-R\dot{\alpha}+\dot{R}\alpha+aR\beta_y+\sigma R\beta_{yy}\}, \\ &\tilde{f}(\tilde{y},\tilde{t})=\frac{1}{\alpha\dot{T}}f, \\ &\tilde{h}(\tilde{y},\tilde{t})=\frac{1}{\alpha R\dot{T}}\left\{h-\frac{d}{dt}\left(\frac{\dot{\alpha}}{\alpha p}\right)+\frac{1}{p}\left(\frac{\dot{\alpha}}{\alpha}\right)^2+\sigma S_{yy}+aS_y+fS-e\frac{\dot{\alpha}}{\alpha p}\right\}. \end{aligned} \tag{2.3}$$

We now choose the functions $R(t)$, $T(t)$, and $Y(y,t)$ in Eq. (2.2) to satisfy

$$\dot{T}(t) = q(t)\alpha^2(t), \quad R(t) = \frac{q}{p}\alpha,$$
$$Y_y = \alpha^{3/2}\sqrt{\left|\frac{q(t)}{\sigma(y,t)}\right|} \tag{2.4}$$

and thus normalize

$$\tilde{p}(\tilde{t}) = 1, \quad \tilde{q}(\tilde{t}) = 1, \quad \tilde{\sigma}(\tilde{y},\tilde{t}) = \varepsilon = \mp 1. \tag{2.5}$$

By an appropriate choice of the functions $\beta(y,t)$ and $S(y,t)$, we can arrange to have

$$\tilde{e}(\tilde{y},\tilde{t}) = \tilde{h}(\tilde{y},\tilde{t}) = 0.$$

Finally, Eqs. (1.9) are reduced to their canonical form,

$$(u_t + uu_x + u_{xx})_x + \varepsilon u_{yy} + a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + f(y,t)u = 0, \quad \varepsilon = \pm 1. \tag{2.6}$$

With no loss of generality we can restrict our study to symmetries of Eqs. (2.6). All results obtained for Eqs. (2.6) can be transformed into results for Eqs. (1.9), using transformations (2.2). We shall call Eqs. (2.6) the "CGB equations."

These types of transformations were found for a class of generalized one-dimensional Burgers equations in Ref. 16. We recall that the terms "allowed transformations" and "form-preserving transformations" were first used within the framework of group analysis of differential equations in Refs. 8 and 7. The task of determining form-preserving point transformations for a general class of partial differential equations first appeared in Ref. 17. The same approach has been used in a number of papers (for instance, see Refs. 2 and 28). A similar problem for a general evolution equation where both point and contact transformations were involved has been attacked from a different point of view, namely, via differential forms in Ref. 1. The formalized and rigorous version of the notion of a form-preserving transformation is called an admissible transformation. This formalization is presented, e.g., in Ref. 22.

We mention that Lie point transformations are particular cases of allowed transformations. When the form of the coefficients is preserved, allowed transformations coincide with symmetry transformations of the equation.

Meanwhile, it is the notion of equivalence group that makes it possible to solve a general symmetry classification problem in a systematic way in that it is used to find canonical forms for vector fields which are admitted as symmetry generators for the given equation.[1]

The equivalence group $\mathcal{E}$ of (2.6) consists of transformations (2.2) with $\alpha$=const. Of course, by setting $\alpha$=1, formulas (2.4) can be further simplified. Also, using a subgroup of $\mathcal{E}$ consisting of the point transformations,

$$u(x,y,t) = \tilde{u}(\tilde{x},\tilde{y},\tilde{t}) + \left(\frac{\varepsilon}{2}\dot{b}(t) - c_1(t)\right)y - c_0(t) + \frac{\varepsilon}{4}b^2(t),$$
$$\tilde{x} = x - \frac{\varepsilon}{2}b(t)y, \quad \tilde{y} = y, \quad \tilde{t} = t, \tag{2.7}$$

one can see that among the most general CGB equations (2.6), those that can be mapped to the ZKK equation,

$$(u_t + uu_x + u_{xx})_x + \varepsilon u_{yy} = 0$$

should have the particular form

$$(u_t + uu_x + u_{xx})_x + \varepsilon u_{yy} + b(t)u_{xy} + (c_0(t) + c_1(t)y)u_{xx} = 0.$$

## III. DETERMINING EQUATIONS FOR THE SYMMETRIES

We restrict ourselves to Lie point symmetries. The Lie algebra of the symmetry group is realized by vector fields of the form

$$\hat{\mathbf{V}} = \xi\partial_x + \eta\partial_y + \tau\partial_t + \phi\partial_u, \tag{3.1}$$

where $\xi$, $\eta$, $\tau$, and $\phi$ are functions of $x$, $y$, $t$, and $u$. To determine the form of $\hat{\mathbf{V}}$ we apply the standard infinitesimal symmetry algorithm (see, for instance, Olver's book[21]) which is basically tantamount to requiring that the third prolongation $\mathrm{pr}^{(3)}\,\hat{\mathbf{V}}$ of the vector field on the third order jet space $J^3$ having appropriate local coordinates should annihilate the equation on its solution manifold. This requirement produces an overdetermined set of linear partial differential equations for the coefficients $\xi$, $\eta$, $\tau$, and $\phi$ in (3.1).

For (2.6) these equations which do not involve the functions $a$, $b$, $c$, and $f$ can be solved, and we find that the general element of the symmetry algebra has the form

$$\hat{\mathbf{V}} = \tau(t)\partial_t + \left(\tfrac{1}{2}\dot{\tau}x + \xi_0(y,t)\right)\partial_x + \left(\tfrac{3}{4}\dot{\tau}y + \eta_0(t)\right)\partial_y + \left(-\tfrac{1}{2}\dot{\tau}u + \tfrac{1}{2}\ddot{\tau}x + S(y,t)\right)\partial_u, \tag{3.2}$$

where

$$S(y,t) = -\tau c_t - \left(\tfrac{3}{4}\dot{\tau}y + \eta_0\right)c_y + \xi_{0,t} + b\xi_{0,y} - \tfrac{1}{2}c\dot{\tau}. \tag{3.3}$$

The remaining determining equations for $\tau(t)$, $\eta(t)$, and $\xi_0(y,t)$ are

$$4\tau a_t + (3\dot{\tau}y + 4\eta_0)a_y + 3a\dot{\tau} = 0, \tag{3.4}$$

$$-4\dot{\eta}_0 - 3y\ddot{\tau} + 4\tau b_t + (3\dot{\tau}y + 4\eta_0)b_y + b\dot{\tau} - 8\varepsilon\xi_{0,y} = 0, \tag{3.5}$$

$$a\xi_{0,y} + \varepsilon\xi_{0,yy} = 0, \tag{3.6}$$

$$f\ddot{\tau} = 0, \tag{3.7}$$

$$6f\dot{\tau} + 4f_t\tau + f_y(3\dot{\tau}y + 4\eta_0) = 0, \tag{3.8}$$

$$2\dddot{\tau} + 4fS + 4aS_y + 4\varepsilon S_{yy} = 0. \tag{3.9}$$

At this juncture, there are different directions to go for dealing with determining equations. One is to perform a complete symmetry analysis of Eqs. (3.3)–(3.9) for arbitrary (given) functions $a$, $b$, $c$, and $f$. Of course, one can well proceed to determine the coefficients given that the equation is invariant under low-dimensional Lie algebras. Works in this direction for equations having dependence on several arbitrary functions of both independent and dependent variables and their derivatives exist in the literature (see, for example, Refs. 29, 2, 28, 15, 13, and 1). This approach requires the knowledge of structural results on the classical Lie algebras. Here we shall take another approach and determine the conditions on these functions that permit the symmetry algebra to be infinite dimensional. This will happen when at least one of the functions $\tau(t)$, $\eta_0(t)$, and $\xi_0(y,t)$ remains an arbitrary function of at least one variable.

## IV. SEARCH FOR THE VIRASORO SYMMETRIES OF THE CGB EQUATIONS

We are looking for conditions on the coefficients $a$, $b$, $c$, and $f$ that allow Eqs. (3.4)–(3.9) to be solved without imposing any conditions on $\tau(t)$. Below we shall see that this cannot be realized for any possible choice of the coefficients.

From Eq. (3.7) we see that $\tau$ is linear in $t$, unless we have $f(y,t)\equiv 0$. Once this condition is imposed, Eqs. (3.7) and (3.8) are solved identically. Equation (3.4) leaves $\tau(t)$ free if either we have $a=0$ or $a=a_0(y+\lambda(t))^{-1}$, where $a_0\neq 0$ is a constant and $\lambda(t)$ is some function of $t$. We investigate the two cases separately. First let us assume

$$a=\frac{a_0}{y+\lambda(t)}, \quad a_0\neq 0. \tag{4.1}$$

Then we view Eq. (3.4) as an equation for $\eta_0(t)$ and obtain

$$\eta_0(t)=\tfrac{1}{3}(2\lambda\dot{\tau}-3\dot{\lambda}\tau). \tag{4.2}$$

From Eq. (3.6) we see $\xi_0(y,t)$ may be an arbitrary function of $t$, but never of $y$ (we have $\varepsilon=\pm 1$). Two possibilities for $\xi_0(y,t)$ occur.

(1) $a_0\varepsilon\neq 1$.

$$\xi_0=\frac{1}{1-a_0\varepsilon}\mu_1(t)(y+\lambda)^{-a_0\varepsilon+1}+\mu_0(t). \tag{4.3}$$

(2) $a_0\varepsilon=1$.

$$\xi_0=\mu_1(t)\ln(y+\lambda)+\mu_0(t). \tag{4.4}$$

We must now put $\xi_0$ of (4.3) or (4.4) into Eq. (3.5) and solve the obtained equation for $\mu_1(t)$. The expression for $\mu_1(t)$ must be independent of $y$ for all values of $\tau$. Moreover, for $\tau(t)$ to remain free, there must be no relation between $b(y,t)$ and $\tau(t)$. These conditions cannot be satisfied for any value of $a_0\varepsilon$. Hence, if $a(y,t)$ is as in Eq. (4.1) the generalized Burgers equations (2.6) do not allow a Virasoro algebra.

The other case to consider is $a=0$ (in addition to $f=0$). Equation (3.6) is easily solved in this case, and we obtain

$$\xi_0(y,t)=\mu_1(t)y+\mu_0(t) \tag{4.5}$$

with $\mu_1(t)$ and $\mu_0(t)$ arbitrary. We insert $\xi_0(y,t)$ into Eq. (3.5) and try to solve for $\mu_1(t)$. This is possible if and only if we have $b=b_1(t)y+b_0(t)$. On the other hand, the $y$ independent coefficient of (3.5) restricts the form of $\tau$ which implies that no Virasoro algebra can exist at all. In the following analysis we shall see that in that case there can exist at most two arbitrary functions.

**Theorem 1:** *The CGB equations (2.6) can never allow the Virasoro algebra as a symmetry algebra for any choice of the coefficients.*

## V. KAC–MOODY SYMMETRIES OF THE CGB EQUATIONS

In Sec. IV we have shown that the symmetry algebra of the CGB equations cannot contain a Virasoro algebra. In this section we will determine the conditions on the functions $a(y,t)$, $b(y,t)$, $c(y,t)$, and $f(y,t)$ under which the CGB equations only allow a Kac–Moody algebra. Thus, the function $\tau(t)$ will not be free, but $\eta_0(t)$ of Eq. (3.2) will be free, or $\xi_0(y,t)$ will involve at least one free function of $t$.

### A. The function $\eta_0(t)$ free

Equation (3.4) will relate $\eta_0$ and $a(y,t)$ unless we have $a_y=0$. Hence we put $a_y=0$. For $a=a(t)\neq 0$ Eq. (3.4) implies $\tau(t)=\tau_0 a^{-4/3}$. Equation (3.6) yields

$$\xi_0(y,t) = \xi_1(t)e^{-a\varepsilon y} + \xi_0(t).$$

Equation (3.5) then provides a relation between $\eta_0(t)$ and $b(y,t)$. Hence $\eta_0(t)$ is not free. Thus, if $\eta_0(t)$ is to be a free function, we must have $a(y,t)=0$. Equation (3.4) is satisfied identically. From Eq. (3.6) we have

$$\xi_0(y,t) = \rho(t)y + \sigma(t). \tag{5.1}$$

Equation (3.5) will leave $\eta_0$ free only if we have

$$b(y,t) = b_1(t)y + b_0(t), \tag{5.2}$$

$$\rho(t) = \frac{\varepsilon}{8}(-4\dot{\eta}_0 + 4\tau\dot{b}_0 + 4\eta_0 b_1 + b_0\dot{\tau}), \tag{5.3}$$

$$3\ddot{\tau} - 4(\tau b_1)^{\cdot} = 0. \tag{5.4}$$

For $f \neq 0$ we have $\ddot{\tau}=0$ and Eq. (3.9) will relate $\eta_0(t)$ to $c(y,t)$, $b_1$, and $b_0$. Thus, for $\eta_0(t)$ to be free, we must have $f(y,t)=0$. Equation (3.9) reduces to

$$-2\varepsilon\dddot{\tau} + \dot{\tau}(8c_{yy} + 3yc_{yyy}) + 4(\tau c_{yyt} + \eta_0 c_{yyy}) = 0.$$

$\eta_0(t)$ is free if we have

$$c(y,t) = c_2(t)y^2 + c_1(t)y + c_0(t), \tag{5.5}$$

$$-2\varepsilon\dddot{\tau} + (8\tau\dot{c}_2 + 16\dot{\tau}c_2) = 0. \tag{5.6}$$

The only equation that remains to be solved is Eq. (5.6). Both functions $\eta_0(t)$ and $\sigma(t)$ remain free. The most general CGB equations allowing $\eta_0(t)$ to be a free function is obtained if Eq. (5.6) is solved identically by putting $\tau=0$. Then $\eta_0(t)$ and $\sigma(t)$ are arbitrary. On the other hand, from (5.6) we see that $\tau(t)$ cannot remain free. This again implies that the symmetry algebra can by no means be Virasoro type. Using Eq. (3.2) and the above results with the identification $\eta=\eta_0$, $\xi=\sigma$ we obtain the following theorem.

**Theorem 2:** *The equations*

$$(u_t + uu_x + u_{xx})_x + \varepsilon u_{yy} + (b_1(t)y + b_0(t))u_{xy} + (c_2(t)y^2 + c_1(t)y + c_0(t))u_{xx} = 0, \tag{5.7}$$

*where $\varepsilon = \pm 1$ and $b_0, b_1, c_0, c_1, c_2$ are arbitrary functions of t, are the most general CGB equations, invariant under an infinite-dimensional Lie point symmetry group depending on two arbitrary functions. Its Lie algebra has a Kac–Moody structure and is realized by vector fields of the form*

$$\hat{\mathbf{V}} = X(\xi) + Y(\eta), \tag{5.8}$$

*where $\xi(t)$ and $\eta(t)$ are arbitrary smooth functions of time and*

$$X(\xi) = \xi\partial_x + \dot{\xi}\partial_u, \tag{5.9}$$

$$Y(\eta) = \eta\partial_y + \frac{\varepsilon}{2}y(-\dot{\eta} + b_1\eta)\partial_x + \left\{\left[-2c_2\eta + \frac{\varepsilon}{2}(-\ddot{\eta} + \dot{b}_1\eta + b_1^2\eta)\right]y - c_1\eta + \frac{\varepsilon}{2}b_0(-\dot{\eta} + b_1\eta)\right\}\partial_u. \tag{5.10}$$

*Remarks:*

(1) Equations (5.7) can be further simplified by allowed transformations. Indeed, let us restrict transformation (2.2) to

$$u(x,y,t) = \tilde{u}(\tilde{x},\tilde{y},\tilde{t}) + S_1(t)y + S_0(t),$$

$$\tilde{x} = x + \beta_1(t)y + \beta_0(t), \quad \tilde{y} = y + \gamma(t), \quad \tilde{t} = t. \tag{5.11}$$

For any functions $b_1(t)$ and $c_2(t)$ we can choose $S_1$, $S_0$, $\beta_0$, $\beta_1$, and $\gamma$ to set $b_0$, $c_1$, and $c_0$ equal to zero. Thus, with no loss of generality, we can set

$$b_0(t) = c_1(t) = c_0(t) = 0 \tag{5.12}$$

in Eqs. (5.7) and (5.10).

(2) Let us now consider the cases when Eqs. (5.7) admit an additional symmetry. To do this we should solve Eqs. (5.4) and (5.6).

*Case 1:* $b_1=0$, $c_2\neq 0$.

We assume (5.12) is already satisfied. From (5.4) we have $\tau=\tau_1 t+\tau_0$ and from (5.6)

$$c_2 = k\tau^{-2} = k(\tau_1 t + \tau_0)^{-2},$$

where $\tau_1$, $\tau_0$, $k$ are constants. The additional symmetry is

$$T = (\tau_1 t + \tau_0)\partial_t + \tfrac{1}{2}\tau_1 x\partial_x + \tfrac{3}{4}\tau_1 y\partial_y - \tfrac{1}{2}\tau_1 u\partial_u. \tag{5.13}$$

Under translation of $t$, it is equivalent to the dilatational symmetry,

$$D = t\partial_t + \tfrac{1}{2}x\partial_x + \tfrac{3}{4}y\partial_y - \tfrac{1}{2}u\partial_u.$$

*Case 2:* $b_1\neq 0$, $b_0=c_1=c_0=0$.

Equation (5.4) can be integrated to give a first order linear equation for $\tau$ in terms of $b_1$ and Eq. (5.6) provides the constraint between $b_1$ and $c_2$,

$$\frac{d}{dt}(\tau^2 c_2) = \frac{\varepsilon}{4}\tau\frac{d^2}{dt^2}(\tau b_1). \tag{5.14}$$

The additional element of the symmetry algebra in this case is

$$T = \tau\partial_t + \tfrac{1}{2}\dot{\tau}x\partial_x + \tfrac{3}{4}\dot{\tau}y\partial_y + \left[\tfrac{1}{2}\ddot{\tau}x - (\tau\dot{c}_2 + 2c_2\dot{\tau})y^2 - \tfrac{1}{2}\dot{\tau}u\right]\partial_u \tag{5.15}$$

with $\tau$ being a solution of

$$\dot{\tau} - \tfrac{4}{3}b_1\tau = k.$$

## B. One free function in symmetry algebra

We have established that if $\tau(t)$ is free in Eq. (3.2), then there are three free functions. If $\tau$ is not free, but $\eta_0(t)$ is, then there are two free functions. Now let $\tau(t)$ and $\eta_0(t)$ be constrained by the determining equations, but let some freedom remain in the function $\xi_0(y,t)$.

First of all we note that if we put

$$\tau = 0, \quad \eta_0 = 0, \quad \xi_0(y,t) = \xi(t) \tag{5.16}$$

in Eq. (3.2) then Eqs. (3.4)–(3.8) are satisfied identically and Eq. (3.9) reduces to

$$f\dot{\xi} = 0. \tag{5.17}$$

Hence

$$X(\xi) = \xi(t)\partial_x + \dot{\xi}(t)\partial_u, \tag{5.18}$$

with $\xi(t)$ arbitrary, generates Lie point symmetries of the CGB equation for $f(y,t)=0$ and any functions $a(y,t)$, $b(y,t)$, and $c(y,t)$.

For $f\neq 0$ we have $\tau=\tau_1 t+\tau_0$ from Eq. (3.7). Equation (3.6) then determines the $y$ dependence of $\xi_0$.

We skip the details here and just state that the remaining equations, (3.5), (3.8), and (3.9), do not allow any solutions with free functions.

We state this result as a theorem.

**Theorem 3:** *CGB equations (2.6) are invariant under an infinite-dimensional Abelian group generated by vector field (5.18) for* $f(y,t)=0$ *and* $a,b,c$ *arbitrary.*

Theorems 2 and 3 sum up all cases when the symmetry algebra of the CGB equation is infinite dimensional.

## VI. APPLICATIONS AND CONCLUSIONS

We have identified all cases when the generalized Burgers equations have an infinite-dimensional symmetry group. Let us now discuss the implications of this result.

### A. Equations with non-Abelian Kac–Moody symmetry algebra

Symmetry algebra (5.8) of Eqs. (5.7) is infinite dimensional and non-Abelian. Indeed, we have

$$[Y(\eta_1),Y(\eta_2)] = X(\xi), \quad \xi = -\frac{\varepsilon}{2}(\eta_1\dot{\eta}_2 - \dot{\eta}_1\eta_2). \tag{6.1}$$

We can apply the method of symmetry reduction to obtain particular solutions. The operator $X(\xi)$ defined by Eq. (5.9) generates the transformations

$$\tilde{x} = x + \lambda\xi(t), \quad \tilde{y} = y, \quad \tilde{t} = t, \quad \tilde{u}(\tilde{x},\tilde{y},\tilde{t}) = u(x,y,t) + \lambda\dot{\xi}(t), \tag{6.2}$$

where $\lambda$ is a group parameter. We see that (6.2) is a transformation to a frame moving with an arbitrary acceleration in the $x$ direction. For $\xi$ constant this is a translation, and for $\xi$ linear in $t$ this is a Galilei transformation. An invariant solution will have the form

$$u = \frac{\dot{\xi}}{\xi}x + F(y,t). \tag{6.3}$$

Substituting into Eq. (5.7) we obtain the family of solutions,

$$u = \frac{\dot{\xi}}{\xi}x - \frac{\varepsilon}{2}\frac{\ddot{\xi}}{\xi}y^2 + \rho(t)y + \sigma(t), \tag{6.4}$$

with $\rho(t)$ and $\sigma(t)$ arbitrary.

The transformation corresponding to the general element $Y(\eta)+X(\xi)$ with $\eta\neq 0$ is easy to obtain, but more difficult to interpret. An invariant solution will have the form

$$u = \left[ -c + \frac{\varepsilon}{4}\left( \dot{b} + b^2 - \frac{\ddot{\eta}}{\eta} \right) \right] y^2 + \frac{\dot{\xi}}{\eta} y + F(z,t),$$
$$z = x + \frac{\varepsilon}{4}\left( -b + \frac{\dot{\eta}}{\eta} \right) y^2 - \frac{\xi}{\eta} y. \tag{6.5}$$

We have put $b_1=b$, $c_2=c$, $b_0=c_1=c_0=0$, which can be done with no loss of generality, cf. remark after Theorem 1. We now put $u$ of Eq. (6.5) into Eq. (5.7) (for $c_1=c_0=b_0=0$) and obtain the reduced equation

$$(F_t + FF_z + F_{zz})_z + \varepsilon \frac{\xi^2}{\eta^2} F_{zz} + \frac{1}{2}\left( \frac{\dot{\eta}}{\eta} - b \right) F_z - 2c\varepsilon + \frac{1}{2}\left( \dot{b} + b^2 - \frac{\ddot{\eta}}{\eta} \right) = 0. \tag{6.6}$$

Putting

$$F(z,t) = \tilde{F}(\tilde{z},\tilde{t}), \quad \tilde{z} = z + \beta(t), \quad \tilde{t} = t,$$
$$\dot{\beta}(t) = -\varepsilon \frac{\xi^2}{\eta^2}, \tag{6.7}$$

we eliminate the $F_{zz}$ term. Choosing $\dot{\eta}/\eta = b(t)$ we obtain the equation

$$(F_t + FF_z + F_{zz})_z = 2\varepsilon c(t), \tag{6.8}$$

which integrates to

$$F_t + FF_z + F_{zz} = 2\varepsilon c(t) z + h(t) \tag{6.9}$$

for an arbitrary function $h$. We note that for $c=0$, $h=0$ ($h$ can be set to zero by a time dependent translation of $F$), (6.9) reduces to the one-dimensional Burgers equation. Its linearizability by the famous Hopf–Cole transformation mapping its solutions to the positive solutions of the linear heat equation is a well-known fact.[21]

## B. Comments

By the results of this paper we have shown that neither 2+1-dimensional Burgers equations nor their generalizations of form (1.9) can allow a Virasoro type symmetry group. The largest infinite-dimensional symmetry allowed can be Kac–Moody type. In addition to this, for specific choice of the coefficients it has one more symmetry. It should also be worthwhile stressing the fact that the Kac–Moody type symmetries admitted by generalized KP (Ref. 14) and Burgers equations agree while the remarkable Virasoro structure inherent in the KP equation and its variable coefficient extensions which is possible only for special choices of the coefficients does not survive in the latter.

The most ubiquitous symmetry of the generalized Burgers equations is transformation (6.2) to an arbitrary frame moving in the $x$ direction. Its presence only requires the coefficient $f(y,t)$ in Eq. (1.9) for $p=1$ or in (2.6) to be $f(y,t) \equiv 0$. Invariance of a solution under such a general transformation is very restrictive and leads to solutions that are at most linear in the variable $x$ and have a prescribed $y$ dependence [see solutions (6.4)].

The transformations generated by $Y(\eta)$ leave a more restricted class of generalized Burgers equations invariant, those of Eq. (5.7). The invariant solutions have form (6.5). They are obtained by solving reduced equation (6.6) with $F_{zz}$ transformed away or (6.8). For $c(t)=0$ this is just the usual Burgers equation, for arbitrary $b(t)$, as long as we choose $\dot{\eta}/\eta = b(t)$. Any solution of the Burgers equation or the linear heat equation will, via Eq. (6.5), provide $y$ dependent solutions of the corresponding generalized Burgers equation.

We finish by a final remark. One can imbed one-dimensional additional subalgebras into Kac–Moody subalgebras to form two-dimensional subalgebras (in canonical form). Invariance under them will lead to reductions to ordinary differential equations (see Ref. 11). We leave a systematic study of reductions and their possible integrations to construct new exact solutions to a separate work.

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