On the Controllability of Bimodal Piecewise Linear Systems*

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Abstract. This paper studies controllability of bimodal systems that consist of two linear dynamics on each side of a given hyperplane. We show that the controllability properties of these systems can be inferred from those of linear systems for which the inputs are constrained in a certain way. Inspired by the earlier work on constrained controllability of linear systems, we derive necessary and sufficient conditions for a bimodal piecewise linear system to be controllable.

1 Introduction

One of the most basic concepts in control theory is the notion of controllability. This concept has been studied extensively for linear systems, nonlinear systems, infinite-dimensional systems and so on. The notion of controllability plays a role for instance in stability theory and in realization theory; more recently it has also been used in safety studies where it is important to know whether certain regions of the state space are reachable or not under the influence of an external input. While the algebraic characterization of controllability of finite-dimensional linear systems is among the classical results of systems theory, global controllability results for nonlinear systems have been hard to come by. In this paper we consider global controllability for two related classes of piecewise linear systems, and obtain a complete characterization.

One class of switched linear systems that we consider consists of controlled systems whose dynamics depends on the sign of one of the state variables. Such systems have two modes, and the switching between these modes is determined by the zero crossings of the designated state variable or more generally of some linear function of the state variables. The evolution of the state variables is influenced not only by the internal dynamics, but also by an external input

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which indirectly affects the switching behavior of the system. In the second class of switched systems that we study here, it is the input vector that may switch between two possible values, and the switching is determined directly by the sign of the input variable itself. Models of this type may be used to describe situations where "pushing" and "pulling" have different effects (besides a sign change). It turns out that the controllability problems for these two classes are closely related; we establish this relation by means of a special state representation akin to the strict feedback form that is used in backstepping control design.

The controllability problems that we consider are specified more precisely in the next section, in which we also present the main results of the paper along with some discussion of how these results relate to the existing literature. Most of the proofs are in the Appendix which follows after the conclusions section.

The following notational conventions will be in force throughout the paper. The symbol $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ n-tuples of real numbers, and $\mathbb{R}^{n\times m}$ $n \times m$ real matrices. The set of complex numbers is denoted by $\mathbb{C}$, natural numbers by $\mathbb{N}$. The set of locally integrable functions is denoted by $\mathcal{L}_1^{\text{loc}}$, absolutely continuous functions by $\mathcal{AC}$, and infinitely differentiable functions by $\mathfrak{C}^\infty$. For a matrix $A \in \mathbb{R}^{n\times m}$, A^T stands for its transpose, $\ker A$ for its kernel, i.e. the set $\{x \in \mathbb{R}^m \mid Ax = 0\}$, $\operatorname{im} A$ for its image, i.e. the set $\{y \in \mathbb{R}^n \mid y = Ax \text{ for some } x \in \mathbb{R}^m\}$, $\exp(A)$ for its exponential. If B has also m columns then $\operatorname{col}(A, B)$ denotes the matrix obtained by stacking A over B. If $B \in \mathbb{R}^{p\times q}$ then $\operatorname{blockdiag}(A, B)$ denotes the block diagonal $(n + p) \times (m + q)$ matrix for which the left upper $n \times m$ block is A, the right lower $p \times q$ block is B, and the rest of the entries are zero.

2 Main results

Consider the bimodal piecewise linear system given by

$$\dot{x}(t) = \begin{cases} A_1 x(t) + bu(t) & \text{if } c^T x(t) \leqslant 0, \\ A_2 x(t) + bu(t) & \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{1}$$

where A_1, $A_2 \in \mathbb{R}^{n\times n}$ and b, $c \in \mathbb{R}^{n\times 1}$. We assume that the dynamics is continuous along the hyperplane $\{x \mid c^T x = 0\}$, i.e.

$$c^T x = 0 \Rightarrow A_1 x = A_2 x. \tag{2}$$

As the right hand side of (1) is Lipschitz continuous in the x variable, one can show that for each initial state $x_0 \in \mathbb{R}^n$ and input $u \in \mathcal{L}_1^{\text{loc}}$ there exists a unique absolutely continuous function x satisfying (1) almost everywhere.

The system (1) is a special case of a family of hybrid systems that are called *linear complementarity systems* (LCSs). Lying in the intersection of the mathematical programming and systems theory, LCSs find applications in various engineering fields as well as economical sciences. We refer to [3] and the references therein for an account of the previous work on LCSs. An LCS is a system of the form

$$\dot{x}(t) = Ax(t) + Ez(t) + Bu(t) \tag{3a}$$
$$w(t) = Cx(t) + Dz(t) \tag{3b}$$
$$0 \leqslant z(t) \perp w(t) \geqslant 0. \tag{3c}$$

Here $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$, $D \in \mathbb{R}^{k\times k}$, $E \in \mathbb{R}^{n\times k}$, the inequalities are componentwise, and $z \perp w$ means that $z^T w = 0$. The relation (3c) is known as the *complementarity condition* and the pair (z, w) as *complementarity variables*. Note that the complementarity conditions require, at least, one of the complementarity variables to be zero at a given time instant.

To see that (1) is a type of LCS, note that the condition (2) implies that the difference $A_2 - A_1$ is, at most, of rank one and its kernel contains the kernel of c^T. Therefore, one can find a vector $e \in \mathbb{R}^{n\times 1}$ such that $A_2 - A_1 = ec^T$. Consider the LCS

$$\dot{x}(t) = A_2 x(t) + ez(t) + bu(t) \tag{4a}$$
$$w(t) = c^T x(t) + z(t) \tag{4b}$$
$$0 \leqslant z(t) \perp w(t) \geqslant 0 \tag{4c}$$

where there is only one pair of complementarity variables. As a consequence, the overall system has two 'modes' (i.e. it is bimodal). Indeed, if the variable z is zero on an interval of time, then $c^T x$ is nonnegative on that interval and the system follows the dynamics of $\dot{x} = A_2 x + bu$. Alternatively, if the variable w is zero on an interval then $c^T x$ is nonpositive on that interval and the system follows the dynamics of $\dot{x} = (A_2 - ec^T)x + bu$. Note that $A_2 - ec^T = A_1$ by the construction of the vector e and hence (4) is equivalent to (1) in the obvious sense.

2.1 Controllability of linear systems

From a control theory point of view, one of the very immediate issues is the controllability of the system at hand. More precisely, the question is whether an arbitrary initial state x_0 can be steered to an arbitrary final state x_f. Following the classical literature, we say that the system (1) is *completely controllable* if for any pair of states (x_0, x_f) there exists an input $u \in \mathcal{L}_1^{\text{loc}}$ such that the solution of (1) with $x(0) = x_0$ passes through x_f, i.e. $x(\tau) = x_f$ for some $\tau > 0$.

Before studying the controllability of (1), we want to discuss some of the available results on the controllability of linear systems. Note that the system (1) is nothing but a single-input linear system when $A_1 = A_2 = A$. In this case, (1) can be written as

$$\dot{x} = Ax + bu. \tag{5}$$

Ever since Kalman's seminal work [5] introduced the notion of controllability (and also observability) in the state space framework, it has been one of the central notions in systems and control theory. Tests for controllability were given by Kalman himself and many others (see e.g. [4] for historical details). The following theorem summarizes the classical results on the controllability of linear systems for the single input case.

Theorem 1. *The following statements are equivalent.*

1. *The system* (5) *is completely controllable.*
2. *The matrix* $\begin{bmatrix} b \; Ab \cdots A^{n-1}b \end{bmatrix}$ *is of rank* n.
3. *For any eigenpair* (λ, z) *of* A^T *(i.e.,* $z^T A = \lambda z^T$*),* $z^T b \neq 0$.
4. *The rank of the matrix* $\begin{bmatrix} sI - A \; b \end{bmatrix}$ *is equal to* n *for all* $s \in \mathbb{C}$.

In practice, one may encounter controllability problems for which the input may only take values from a set $\Omega \subset \mathbb{R}$. A typical example of such constrained controllability problems would be a (linear) system that may admit only positive controls. Study of constrained controllability goes back to the sixties (see for instance [6]). Early results consider only restraint sets Ω which contain the origin in their interior. The following theorem can be proven with the help of [6, Thm. 8, p. 92].

Theorem 2. *Consider the system* (5) *for which the input function is constrained as* $u(t) \in \Omega$ *where* Ω *is a compact set which contains zero in its interior. Then,* (5) *is completely controllable if and only if* (A, b) *is controllable and all eigenvalues of* A *lie on the imaginary axis.*

When only positive controls are allowed, the set Ω does not contain the origin in its interior. Saperstone and Yorke [7] were the first to consider such constraint sets. In particular, they considered the case $\Omega = [0, 1]$. More general restraint sets were studied by Brammer [2]. All these results were obtained for the multi-input case. For the single-input case, Brammer's contribution can be stated as follows.

Theorem 3. *Consider the system* (5) *for which the input function is constrained as* $u(t) \in \Omega$ *where the restraint set* Ω *has the following properties.*

i. $0 \in \Omega$,
ii. *convex hull of* Ω *has nonempty interior.*

Then, (5) *is completely controllable if and only if the following conditions hold.*

1. *The pair* (A, b) *is controllable.*
2. *There is no real eigenvector* w *of* A^T *satisfying* $w^T bv \leqslant 0$ *for all* $v \in \Omega$.

As a consequence of the above theorem, necessary and sufficient conditions for the complete controllability of the system (5) with a nonnegative input are i) the pair (A, b) is controllable, and ii) A has no real eigenvalue.

The main goal of the present paper is to investigate controllability properties of a piecewise linear system of the form (1). Although none of the above results are directly applicable, we will see that they will play a crucial role in studying controllability of piecewise linear systems.

2.2 Controllability of bimodal piecewise linear systems

For the moment, we focus on systems of the form

$$\dot{\zeta}(t) = K\zeta(t) + \begin{cases} N\eta(t) & \text{if } \eta(t) \leqslant 0 \\ P\eta(t) & \text{if } \eta(t) \geqslant 0, \end{cases} \tag{6}$$

where $K \in \mathbb{R}^{k\times k}$, $N \in \mathbb{R}^k$, $P \in \mathbb{R}^k$. As we shall see later, controllability of (6) is closely related to that of (1).

For (6), unlike the standard controllability problems, we will consider absolutely continuous inputs η. The following theorem presents necessary and sufficient conditions for the controllability of (6).

Theorem 4. *The following statements are equivalent.*

1. *For each ζ_0, $\zeta_f \in \mathbb{R}^k$ and η_0, $\eta_f \in \mathbb{R}$, there exist a real number $T > 0$ and a solution $(\zeta, \eta) \in \mathcal{AC}^{k+1}$ of (6) such that*

$$\zeta(0) = \zeta_0, \qquad \zeta(T) = \zeta_f \tag{7}$$

$$\eta(0) = \eta_0, \qquad \eta(T) = \eta_f. \tag{8}$$

2. *There exists no nonzero w such that*

$$w^T \exp(Kt)N \leqslant 0 \text{ and } w^T \exp(Kt)P \geqslant 0 \tag{9}$$

   for all $t \geqslant 0$.
3. *$(K, \begin{bmatrix} N & P \end{bmatrix})$ is controllable and $K^T z = \lambda z$, $\lambda \in \mathbb{R}$, $z \neq 0 \Rightarrow (z^T N)(z^T P) > 0$.*

Remark 1. When $N = P$, the system (6) is nothing but a linear system given by $\dot{\zeta} = K\zeta + P\eta$. As $N = P$, the condition $(z^T N)(z^T P) > 0$ is satisfied for any nonzero vector z. Hence, the third condition is equivalent to saying that (K, P) is a controllable pair.

Remark 2. Another special case that is captured by our theorem is the controllability of linear systems with positive controls. Indeed, if we take $N = 0$ controllability properties of the system (6) must be equivalent to those of the system $\dot{\zeta} = K\zeta + P\eta$ where η is restricted to be pointwise nonnegative. In this case, $(z^T N)(z^T P)$ is always zero. Therefore, the third condition of the above theorem is equivalent to saying that (K, P) is a controllable pair and K has no real eigenvalues. In other words, Theorem 3 is a special case of Theorem 4 when Ω is the set of nonnegative real numbers.

Now, we turn to the system (1). Define the transfer functions $G_i(s) = c^T(sI - A_i)^{-1}b$ for $i = 1, 2$. It follows from (2) that $G_1(s) \equiv 0$ if and only if $G_2(s) \equiv 0$. If $G_i(s) \equiv 0$ then the system (1) is not completely controllable. In the rest of the paper, we assume that $G_i(s) \not\equiv 0$ for $i = 1, 2$. Let $\mathcal{V}_i^\star$ be the largest (A_i, b)-controlled invariant subspace that is contained in $\ker c^T$. In other words, $\mathcal{V}_i^\star$ is the largest of the subspaces $\mathcal{V}_i$ such that $(A - bf^T)\mathcal{V}_i \subseteq \mathcal{V}_i$ for some $f \in \mathbb{R}^n$ and

$\mathcal{V}_i \subseteq \ker c^T$. Also let $\mathcal{S}_i^\star$ be the smallest (c^T, A_i)-conditioned invariant subspace that contains $\operatorname{im} b$. Equivalently, $\mathcal{S}_i^\star$ is the smallest of the subspaces $\mathcal{S}_i$ such that $(A - gc^T)\mathcal{S}_i \subseteq \mathcal{S}_i$ for some $g \in \mathbb{R}^n$ and $\operatorname{im} b \subseteq \mathcal{S}_i$. We refer to [1] for a more detailed discussion on the controlled and conditioned invariant subspaces. Since $G_i(s) \not\equiv 0$, it is invertible. As a consequence, a well-known result of the geometric control theory states that $\mathcal{V}_i^\star \oplus \mathcal{S}_i^\star = \mathbb{R}^n$. By using (2), one can show that

1. $\mathcal{V}_1^\star = \mathcal{V}_2^\star =: \mathcal{V}^\star$,
2. $A_1 |_{\mathcal{V}_1^\star} = A_2 |_{\mathcal{V}_2^\star}$,
3. $\mathcal{S}_1^\star = \mathcal{S}_2^\star =: \mathcal{S}^\star$.

This means that we can rewrite (1) as

$$\dot{x} = \begin{cases} \begin{bmatrix} H & g_1 c_2^T \\ b_2 f^T & J_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u & \text{if } c_2^T x_2 \leqslant 0 \\ \begin{bmatrix} H & g_2 c_2^T \\ b_2 f^T & J_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ b_2 \end{bmatrix} u & \text{if } c_2^T x_2 \geqslant 0 \end{cases} \tag{10}$$

by choosing a basis for $\mathbb{R}^n$ which is adopted to $\mathcal{V}^\star$ and $\mathcal{S}^\star$. Here, $b_2 \in \mathbb{R}^{n_2}$, $c_2 \in \mathbb{R}^{n_2}$, $f \in \mathbb{R}^{n_1}$, $g_i \in \mathbb{R}^{n_1}$, $H \in \mathbb{R}^{n_1 \times n_1}$, and $J_i \in \mathbb{R}^{n_2 \times n_2}$ where $n_1 = \dim(\mathcal{V}^\star)$ and $n_2 = \dim(\mathcal{S}^\star)$. Let $e = \operatorname{col}(e_1, e_2)$ where $e_1 \in \mathbb{R}^{n_1}$ and $e_2 \in \mathbb{R}^{n_2}$ in this new coordinates. Note that

$$e_1 = g_2 - g_1. \tag{11}$$

Furthermore, the transfer functions $c_2^T(sI - J_i)^{-1} b_2$ do not have any finite zeros and the pairs (J_i, b_2) are controllable.

At this point, we claim that the system (10) is completely controllable if and only if for each x_0 and x_f there exist a real number $T > 0$ and $x = \operatorname{col}(x_1, x_2) \in \mathcal{AC}^n$ such that

$$\dot{x}_1 = Hx_1 + \begin{cases} g_1 c_2^T x_2 & \text{if } c_2^T x_2 \leqslant 0 \\ g_2 c_2^T x_2 & \text{if } c_2^T x_2 \geqslant 0 \end{cases} \tag{12}$$

with $x(0) = x_0$ and $x(T) = x_f$. The 'only if' part is evident. For the 'if' part, let x_0 and x_f be given arbitrary states. Let T and $x = \operatorname{col}(x_1, x_2)$ be such that (12) is satisfied with $x(0) = x_0$ and $x(T) = x_f$. Note that $c_2^T(sI - J_i)^{-1} b_2$ have polynomial inverses, say $L_i(s)$, as they both have no finite zeros. Now, it can be verified that the input

$$u = -f^T x_1 + \begin{cases} L_1(\frac{d}{dt}) c_2^T x_2 & \text{if } c_2^T x_2 \leqslant 0 \\ L_2(\frac{d}{dt}) c_2^T x_2 & \text{if } c_2^T x_2 \geqslant 0 \end{cases}$$

steers the initial state x_0 of the system (10) to the final state x_f in T units of time.

Hence, in view of Theorem 4, we proved that the system (10) (equivalently (1)) is completely controllable if and only if

1. $(H, \begin{bmatrix} g_1 \; g_2 \end{bmatrix})$ is controllable, and
2. The implication

$$H^T z = \lambda z,\ \lambda \in \mathbb{R},\ z \neq 0 \Rightarrow (z^T g_1)(z^T g_2) > 0 \tag{13}$$

holds.

We claim that $(H, \begin{bmatrix} g_1 \; g_2 \end{bmatrix})$ is controllable if and only if so is $(A_1, \begin{bmatrix} b \; e \end{bmatrix})$. To see this, we will use the Hautus test. Note that

$$\operatorname{rank}(\begin{bmatrix} sI - A_1 \; b \; e \end{bmatrix}) = \operatorname{rank}(\begin{bmatrix} sI - H & -g_1 c_2^T & 0 & e_1 \\ -b_2 f^T & sI - J_1 & b_2 & e_2 \end{bmatrix}). \tag{14}$$

After performing elementary column operations, we obtain

$$\begin{aligned} \operatorname{rank}(\begin{bmatrix} sI - A_1 \; b \; e \end{bmatrix}) &= \operatorname{rank}(\begin{bmatrix} sI - H & -g_1 c_2^T & 0 & e_1 \\ 0 & sI - J_1 & b_2 & e_2 \end{bmatrix}) & (15) \\ &= \operatorname{rank}(\begin{bmatrix} sI - H & -g_1 c_2^T & e_1 \end{bmatrix}) + \operatorname{rank}(\begin{bmatrix} sI - J_1 & b_2 & e_2 \end{bmatrix}). & (16) \end{aligned}$$

As the pair (J_1, b_2) is controllable, the last summand equals to n_2. Note that the first one is equal to $\operatorname{rank}(\begin{bmatrix} sI - H \; g_1 \; g_2 \end{bmatrix})$ in view of (11). Consequently, $(H, \begin{bmatrix} g_1 \; g_2 \end{bmatrix})$ is controllable if and only if $(A_1, \begin{bmatrix} b \; e \end{bmatrix})$ is controllable.

On the other hand, straightforward calculations show that (13) is equivalent to the implication

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} \lambda I - A_i & b \\ c^T & 0 \end{bmatrix} = 0,\ \lambda \in \mathbb{R},\ v \neq 0,\ i = 1, 2 \Rightarrow \mu_1 \mu_2 > 0. \tag{17}$$

Thus, we proved the following theorem.

Theorem 5. *Let e be such that $A_2 - A_1 = ec^T$. The bimodal piecewise linear system* (1) *is completely controllable if and only if the following conditions hold.*

1. *The pair $(A_1, \begin{bmatrix} b \; e \end{bmatrix})$ is controllable.*
2. *The implication*

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} \lambda I - A_i & b \\ c^T & 0 \end{bmatrix} = 0,\ \lambda \in \mathbb{R},\ v \neq 0,\ i = 1, 2 \Rightarrow \mu_1 \mu_2 > 0. \tag{18}$$

holds.

3 Conclusions

We have obtained algebraic characterizations of controllability for two related classes of bimodal piecewise linear systems. These characterizations generalize classical results for single-mode linear systems as well as controllability results for systems subject to positive control. An interesting problem for further research is the characterization of controllability for similar systems with multiple inputs or outputs whose signs determine mode changes. Such systems may have many modes. Another question of interest would be to establish the relation between controllability and stabilizability in the context of the classes of switching linear systems considered here.

Appendix: Proof of Theorem 4

First we need some preparations. The following proposition will simplify the analysis of the controllability properties of (6).

Proposition 1. *The following statements are equivalent.*

1. *For each* ζ_0, $\zeta_f \in \mathbb{R}^k$ *and* η_0, $\eta_f \in \mathbb{R}$, *there exist a real number* $T > 0$ *and a solution* $(\zeta, \eta) \in \mathcal{AC}^{k+1}$ *of* (6) *such that*

$$\zeta(0) = \zeta_0, \qquad \zeta(T) = \zeta_f \tag{19}$$
$$\eta(0) = \eta_0, \qquad \eta(T) = \eta_f. \tag{20}$$

2. *For each* ζ_0, $\zeta_f \in \mathbb{R}^k$, *there exist a real number* $T > 0$ *and a solution* $(\zeta, \eta) \in \mathcal{AC}^{k+1}$ *of* (6) *such that*

$$\zeta(0) = \zeta_0, \quad \zeta(T) = \zeta_f \tag{21}$$
$$\eta(0) = \eta(T) = 0. \tag{22}$$

3. *For each* $\zeta_m \in \mathbb{R}^k$, *there exist real numbers* $T_-, T_+ > 0$ *and two solutions* $(\zeta_-, \eta_-) \in \mathcal{AC}^{k+1}$ *and* $(\zeta_+, \eta_+) \in \mathcal{AC}^{k+1}$ *of* (6) *such that*

$$\zeta_-(0) = \zeta_m, \zeta_-(T_-) = 0 \qquad \zeta_+(0) = 0, \zeta_+(T_+) = \zeta_m \tag{23}$$
$$\eta_-(0) = \eta_-(T_-) = 0 \qquad \eta_+(0) = \eta_+(T_+) = 0. \tag{24}$$

Proof. 1⇒2: Evident.

2⇒3: Evident.

3⇒1: Suppose that the statement 3 holds. We claim that for any ζ_0, $\zeta_f \in \mathbb{R}^k$ and η_0, $\eta_f \in \mathbb{R}$, there exist a real number $T > 0$ and a solution $(\zeta, \eta) \in \mathcal{AC}^{k+1}$ of (6) such that

$$\zeta(0) = \zeta_0, \qquad \zeta(T) = \zeta_f \tag{25a}$$
$$\eta(0) = \eta_0, \qquad \eta(T) = \eta_f. \tag{25b}$$

In what follows we construct such a solution.

i. Let η_{pre} be a $\mathfrak{C}^\infty$-function such that

$$\eta_{\text{pre}}(0) = \eta_0 \text{ and } \eta_{\text{pre}}(1) = 0.$$

Let $(\zeta_{\text{pre}}, \eta_{\text{pre}})$ be the solution of (6) with $\zeta_{\text{pre}}(0) = \zeta_0$. Define $\zeta_0' := \zeta_{\text{pre}}(1)$.

ii. Let η_{post} be a $\mathfrak{C}^\infty$-function such that

$$\eta_{\text{post}}(0) = 0 \text{ and } \eta_{\text{post}}(1) = \eta_f.$$

Let $(\zeta_{\text{post}}, \eta_{\text{post}})$ be the solution of (6) with $\zeta_{\text{post}}(1) = \zeta_f$. Define $\zeta_f' := \zeta_{\text{post}}(0)$.

iii. The statement 3 guarantees the existence of the solutions $(\zeta_-, \eta_-) \in \mathcal{AC}^{k+1}$ and $(\zeta_+, \eta_+) \in \mathcal{AC}^{k+1}$ of (6) such that

$$\zeta_-(0) = \zeta_0', \zeta_-(T_-) = 0 \qquad \zeta_+(0) = 0, \zeta_+(T_+) = \zeta_f' \tag{26}$$
$$\eta_-(0) = \eta_-(T_-) = 0 \qquad \eta_+(0) = \eta_+(T_+) = 0. \tag{27}$$

Consider a $\mathfrak{C}^\infty$-function η satisfying

$$\eta(t) = \begin{cases} \eta_{\text{pre}}(t) & \text{if } 0 \leqslant t \leqslant 1, \\ \eta_-(t-1) & \text{if } 1 \leqslant t \leqslant 1 + T_-, \\ \eta_+(t-1-T_-) & \text{if } 1 + T_- \leqslant t \leqslant 1 + T_- + T_+, \\ \eta_{\text{post}}(t-1-T_- - T_+) & \text{if } 1 + T_- + T_+ \leqslant t \leqslant 2 + T_- + T_+. \end{cases}$$

Let ζ be the concatenation of the functions ζ_{pre}, ζ_-, ζ_+, and ζ_{post} in the same manner. By construction, (ζ, η) is a solution of (6) satisfying (25). ∎

The next lemma provides necessary and sufficient conditions for the system (6) to be controllable from the origin.

Lemma 1. *The following statements are equivalent.*

1. *For each $\zeta_m \in \mathbb{R}^k$, there exist a real number $T > 0$ and a solution $(\zeta, \eta) \in \mathcal{AC}^{k+1}$ of (6) such that*

$$\zeta(0) = 0, \zeta(T) = \zeta_m \tag{28}$$
$$\eta(0) = \eta(T) = 0. \tag{29}$$

2. *There exists no nonzero w such that*

$$w^T \exp(Kt) N \leqslant 0 \text{ and } w^T \exp(Kt) P \geqslant 0 \tag{30}$$

for all $t \geqslant 0$.

Proof. 1⇒2: Suppose that 1 holds but 2 does not. Let w be such that

$$w^T \exp(Kt) N \leqslant 0 \text{ and } w^T \exp(Kt) P \geqslant 0 \tag{31}$$

for all $t \geqslant 0$. Then, for any $\eta \in \mathcal{AC}$ the solution of (6) with $\zeta(0) = 0$ satisfies

$$w^T \zeta(T) = w^T \int_0^T \exp(K(T-s))(-N\eta^-(s) + P\eta^+(s))\, ds \geqslant 0. \tag{32}$$

In other words, 1 fails for any ζ_m with $w^T \zeta_m < 0$. Contradiction!

2⇒1: Consider for each $\varDelta > 0$ a nonnegative valued $\mathfrak{C}^\infty$-function $\eta^\varDelta$ with $\text{supp}(\eta^\varDelta) \subseteq (\frac{\varDelta}{4}, 3\frac{\varDelta}{4})$ and

$$\int_{\varDelta/4}^{3\varDelta/4} \eta^\varDelta(t)\, dt = 1.$$

It is a standard fact from distribution theory that η^{Δ} converges to a Dirac impulse as Δ tends to zero. Now, consider the input

$$\eta(t) = a_0\eta^{\Delta}(t) - a_1\eta^{\Delta}(t-\Delta) + - \cdots - a_{2q-1}\eta^{\Delta}(t-(2q-1)\Delta). \tag{33}$$

where $0 \leqslant t \leqslant 2q\Delta$ and all a_is are nonnegative. Note that η is a $\mathfrak{C}^{\infty}$-function. Obviously, $\eta \in \mathcal{AC}$ for $T = 2q\Delta$. Let $M(\Delta)$ be defined as the integral

$$\int_0^{\Delta} \exp(K(\Delta - s))\eta^{\Delta}(s)\, ds.$$

Note that $M(\Delta)$ commutes with K and hence with $\exp(K\cdot)$. The input given by (33) steers the origin to the state

$$\zeta(T) = M(\Delta) \sum_{i=1}^{2q-1} \exp(K(2q-1-i)\Delta)L_i a_i \tag{34}$$

under the dynamics of (6). Here $L_i = P$ if i is even and $L_i = -N$ if i is odd. Therefore, if ζ_m is a nonnegative linear combination of the columns of a matrix of the form

$$Q(\Delta, q) := M(\Delta)\begin{bmatrix}-N \exp(K\Delta)P \cdots \exp(K(2q-1)\Delta)P\end{bmatrix} \tag{35}$$

then there exists a solution of (6) which satisfies the properties (28). Now, suppose that 2 holds but 1 does not. Then, there should exist a ζ_m such that it cannot be written as a nonnegative linear combination of the columns of a matrix $Q(\Delta, q)$ for any pair (Δ, q). It follows from Farkas' lemma that for each Δ and q there exists $w_{\Delta,q}$ such that

$$w_{\Delta,q}^T \zeta_m < 0 \tag{36a}$$

$$w_{\Delta,q}^T Q(\Delta, q) \geqslant 0. \tag{36b}$$

Obviously, we can take $\| w_{\Delta,q} \| = 1$ without loss of generality. Take a sequence of real numbers Δ_i that converges to zero. Choose a positive real number T. Let q_i be the smallest integer such that $T \leqslant 2q_i\Delta_i$. As w_{Δ_i,q_i} is bounded, it admits a convergent subsequence due to the well-known Bolzano-Weierstrass theorem. Therefore, we can assume, without loss of generality, that the sequence w_{Δ_i,q_i} itself is convergent. Let w_T denote its limit. Note that, in view of (36b), one has

$$w_{\Delta_i,q_i}^T M(\Delta_i)\exp(K(2j)\Delta_i)N \leqslant 0 \tag{37a}$$

$$w_{\Delta_i,q_i}^T M(\Delta_i)\exp(K(2j+1)\Delta_i)P \geqslant 0 \tag{37b}$$

for all $j = 0, 1, \ldots, q_i - 1$. Let j_i be the smallest integer such that $t \leqslant (2j_i + 1)\Delta_i$ for a fixed $t \in [0, T]$. Obviously, $2j_i\Delta_i$ and $(2j_i + 1)\Delta_i$ both converge to t. Note that $M(\Delta)$ converges to the identity matrix as Δ tends to zero. By taking the limit of (37), one has

$$w_T^T \exp(Kt)N \leqslant 0 \tag{38a}$$

$$w_T^T \exp(Kt)P \geqslant 0 \tag{38b}$$

for all $t \in [0, T]$ since $M(\Delta)$ converges to the identity matrix as Δ tends to zero. Note that $\| w_T \|= 1$. The Bolzano-Weierstrass theorem asserts that there exists a convergent subsequence within the set $\{w_T \mid T \in \mathbb{N}\}$, say w_{T_i}. Let w denote the limit of w_{T_i} as T_i tends to infinity. We claim that

$$w^T \exp(Kt)N \leqslant 0 \tag{39a}$$

$$w^T \exp(Kt)P \geqslant 0 \tag{39b}$$

for all $t \geqslant 0$. To show this, suppose that $w^T \exp(Kt')N > 0$ for some t'. Then, for some sufficiently large T', one has $w_{T'}^T \exp(Kt')N > 0$ and $t' < T'$. However, this cannot happen due to (38a). In a similar fashion, one can conclude that (39b) holds. As $w \neq 0$, (39) contradicts the statement 2. ∎

The condition (30) is existential in nature and as such it cannot be verified easily. Our next aim is to provide an alternative characterization of (30). First, we focus on the case for which K has no real eigenvalues. The following lemma can be found in [2, proof of Theorem 1.4].

Lemma 2. *Let $K \in \mathbb{R}^{k\times k}$ and $R \in \mathbb{R}^{n\times m}$. If K has no real eigenvalues and (K, R) is controllable then there exists no nonzero w such that $w^T \exp(Kt)R \leqslant 0$ for all $t \geqslant 0$.*

When (K, R) is not controllable, a similar result can be stated as follows.

Lemma 3. *Let $K \in \mathbb{R}^{k\times k}$ and $R \in \mathbb{R}^{n\times m}$. If K has no real eigenvalues then the implication*

$$w^T \exp(Kt)R \leqslant 0 \textit{ for all } t \geqslant 0 \Rightarrow w^T \exp(Kt)R = 0 \textit{ for all } t = 0 \tag{40}$$

holds.

Proof. With no loss of generality, one may assume that the pair (K, R) is in the following canonical form

$$K = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix}, \quad R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \tag{41}$$

where (K_{11}, R_1) is controllable. Note that

$$\exp(Kt) = \begin{bmatrix} \exp(K_{11}t) & * \\ 0 & \exp(K_{22}) \end{bmatrix}. \tag{42}$$

Hence, $w^T \exp(Kt)R = w_1^T \exp(K_{11}t)R_1$ for any w with a partition $w = \mathrm{col}(w_1, w_2)$ that conforms to the partition (41). Let w be such that $w^T \exp(Kt)R \leqslant 0$

for all $t \geqslant 0$. This would mean that $w_1^T \exp(K_{11}t)R_1 \leqslant 0$ for all $t \geqslant 0$. As (K_{11}, R_1) is controllable, however, Lemma 2 implies that $w_1 = 0$. Consequently, $w^T \exp(Kt)R = 0$ for all $t \geqslant 0$. ∎

At the other extreme, the case for which K has only real eigenvalues stands. The following lemma presents an alternative characterization of the condition (30) for this case.

Lemma 4. *Let $K \in \mathbb{R}^{k\times k}$, $N \in \mathbb{R}^k$, and $P \in \mathbb{R}^k$. Suppose that K has only real eigenvalues. Then, the following conditions are equivalent.*

1. *There exists no nonzero w such that $w^T \exp(Kt)N \leqslant 0$ and $w^T \exp(Kt)P \geqslant 0$ for all $t \geqslant 0$.*
2. *Any eigenvector z of K^T satisfies $(z^T N)(z^T P) > 0$.*

Proof. 1⇒2: Suppose that 1 holds but 2 does not. Then, for an eigenvector of z of K^T one has $z^T N \leqslant 0$ and $z^T P \geqslant 0$. Obviously, $z^T \exp(Kt)N \leqslant 0$ and $z^T \exp(Kt)P \geqslant 0$ for all $t \geqslant 0$. Contradiction!

2⇒1: Suppose that 2 holds but 1 does not. Let $w \neq 0$ be such that

$$w^T \exp(Kt)N \leqslant 0 \text{ and } w^T \exp(Kt)P \geqslant 0 \text{ for all } t \geqslant 0.$$

It follows from [2, Lemma 2.4] that

$$w^T \exp(Kt) = \sum_{i=1}^{q} t^{j_i} \exp(\lambda_i t)[z_i^T + f_i^T(t)] \tag{43}$$

where

i. $\lambda_1 > \lambda_2 > \cdots > \lambda_q$ are the q distinct eigenvalues of the matrix K,
ii. $K^T z_i = \lambda_i z_i$,
iii. if $z_i = 0$ then $f_i^T(t) \equiv 0$,
iv. j_is are nonnegative integers, and
v. the functions f_i vanish as t tends to infinity.

Let q' be the smallest integer such that $z_{q'} \neq 0$. Note that the sign of $w^T \exp(Kt)N$ for all sufficiently large t is the same as the sign of $z_{q'}^T N$. Similarly, the sign of $w^T \exp(Kt)P$ for all sufficiently large t is the same as the sign of $z_{q'}^T P$. Therefore, $(w^T \exp(Kt)N)(w^T \exp(Kt)P) > 0$ for all sufficiently large t. Contradiction! ∎

The above proof has the following side result that will be used later.

Corollary 1. *Let $K \in \mathbb{R}^{k\times k}$, $N \in \mathbb{R}^k$, and $P \in \mathbb{R}^k$. Suppose that K has only real eigenvalues and for any eigenvector z of K^T there holds that $(z^T N)(z^T P) > 0$. Then, for any vector w*

$$(w^T \exp(Kt)N)(w^T \exp(Kt)P) > 0 \tag{44}$$

for all sufficiently large t.

Lemma 5. *Let $K \in \mathbb{R}^{k\times k}$, $N \in \mathbb{R}^k$, and $P \in \mathbb{R}^k$. The following statements are equivalent.*

1. *There exists no nonzero w such that $w^T \exp(Kt)N \leqslant 0$ and $w^T \exp(Kt)P \geqslant 0$ for all $t \geqslant 0$.*
2. *The pair $(K, \begin{bmatrix} N & P \end{bmatrix})$ is controllable and $(z^T N)(z^T P) > 0$ for any real eigenvector z of K^T.*

Proof. 1⇒2: Suppose that $(K, \begin{bmatrix} N & P \end{bmatrix})$ is not controllable. Then, the matrix $\begin{bmatrix} s'I - K & N & P \end{bmatrix}$ is not of full row rank for some $s' \in \mathbb{C}$, i.e. there should exists a nonzero complex vector v such that $v^* \begin{bmatrix} s'I - K & N & P \end{bmatrix} = 0$. Let $v = v_1 + iv_2$ where v_1 and v_2 are real vectors, and also let $s' = \sigma + i\omega$ where σ and ω are real numbers. Clearly, $v_i^T N = v_i^T P = 0$ for $i = 1, 2$. Note that

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} K = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}. \tag{45}$$

This would result in

$$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} \exp(Kt) = \exp(\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t) \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}. \tag{46}$$

Therefore, we have $w^T \exp(Kt)N = w^T \exp(Kt)P = 0$ for any linear combination w of the vectors v_1 and v_2. We reach a contradiction. Consequently, the matrix $\begin{bmatrix} sI - K & N & P \end{bmatrix}$ must have full row rank for all $s \in \mathbb{C}$. Suppose, now, that there exists a real eigenvector of K^T such that $(z^T N)(z^T P) \leqslant 0$. Without loss of generality, we can assume that $z^T N \leqslant 0$ and $z^T P \geqslant 0$. This, however, would mean that $z^T \exp(Kt)N \leqslant 0$ and $z^T \exp(Kt)P \geqslant 0$ for all $t \geqslant 0$. Contradiction! Therefore, $(z^T N)(z^T P)$ must be positive for any real eigenvector of K^T.

2⇒1: Suppose that 2 holds but 1 does not. Let the nonzero vector w satisfy $w^T \exp(Kt)N \leqslant 0$ and $w^T \exp(Kt)P \geqslant 0$ for all $t \geqslant 0$. We can assume that the matrix K has the form $K = \mathrm{blockdiag}(K_1, K_2)$ (with possibly empty blocks) where K_1 has only real eigenvectors and K_2 has no real eigenvectors. Clearly, $\exp(Kt) = \mathrm{blockdiag}(\exp(K_1 t), \exp(K_2 t))$. Let the partitions $N = \mathrm{col}(N_1, N_2)$, $P = \mathrm{col}(P_1, P_2)$, and $w = \mathrm{col}(w_1, w_2)$ conform to the above partition of K. Then, we have

$$w_1^T \exp(K_1 t)N_1 + w_2^T \exp(K_2 t)N_2 \leqslant 0 \tag{47a}$$

$$w_1^T \exp(K_1 t)P_1 + w_2^T \exp(K_2 t)P_2 \geqslant 0 \tag{47b}$$

for all $t \geqslant 0$. It follows from Corollary 1 that $w_1^T \exp(K_1 t)N_1$ and $w_1^T \exp(K_1 t)P_1$ have the same sign for all sufficiently large t as every real eigenvector z of K^T satisfies $(z^T N)(z^T P) > 0$. Then, in order the relations (47) to hold, either

$$w_2^T \exp(K_2 t)N_2 \leqslant 0 \tag{48}$$

or

$$w_2^T \exp(K_2 t)P_2 \geqslant 0 \tag{49}$$

should be satisfied for all $t \geqslant t_0 \geqslant 0$ for some t_0. Therefore, either

$$\tilde{w}_2^T \exp(K_2 t) N_2 \leqslant 0 \tag{50}$$

or

$$\tilde{w}_2^T \exp(K_2 t) P_2 \geqslant 0 \tag{51}$$

is satisfied for all $t \geqslant 0$ where $\tilde{w}_2 := \exp(K^T t_0) w_2$. This means that either (50) or (51) should be satisfied as equality for all $t \geqslant 0$ in view of Lemma 3. We claim that in fact both are satisfied as equality. To see this, first suppose that (50) is satisfied as equality. From (48) and (47a), we get $w_1^T \exp(K_1 t) N_1 \leqslant 0$ for all $t \geqslant 0$. As a consequence of Corollary 1 and (47b), we get $w_2^T \exp(K_2 t) P_2 \geqslant 0$ for all $t \geqslant 0$. Due to Lemma 2 this would mean that (51) is also satisfied as equality for all $t \geqslant 0$. Now, suppose that (51) is satisfied as equality. Similar analysis as above would show that (50) should be satisfied as equality in this case. Since both (50) and (51) are satisfied as equality, the vector $\tilde{w}_2$ should lie in the intersection of the uncontrollable spaces of the pairs (K_2, N_2) and (K_2, P_2). By hypothesis, therefore, $\tilde{w}_2 = w_2 = 0$. From (47) and Lemma 4, we conclude that $w_1 = 0$. Hence, $w = 0$. Contradiction! ∎

After all these preparations, we are in a position to prove Theorem 6. Lemma 5 proves the equivalence of the second and third statements. Note that the conditions in 3 are satisfied by a triple (K, N, P) if and only if they are satisfied by $(-K, -N, -P)$. Therefore, the third statement is equivalent to the third statement in Proposition 1 due to Lemma 1. This concludes the proof.

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