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On a Semi Symmetric Metric Connection with a Special Condition on a Riemannian Manifold

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Abstract. In this study, we consider a manifold equipped with semi symmetric metric connection whose the torsion tensor satisfies a special condition. We investigate some properties of the Ricci tensor and the curvature tensor of this manifold . We obtain a necessary and sufficient condition for the mixed generalized quasi-constant curvature of this manifold. Finally, we prove that if the manifold mentioned above is conformally flat, then it is a mixed generalized quasi- Einstein manifold and we prove that if the sectional curvature of a Riemannian manifold with a semi symmetric metric connection whose the special torsion tensor is independent from orientation chosen, then this manifold is of a mixed generalized quasi constant curvature.

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1. Introduction

The notion of a generalized quasi- Einstein manifold was introduced by De and Ghosh [5]. A non-flat Riemannian manifold M is called a generalized quasi Einstein manifold if its Ricci tensor R_{kj} is not identically zero and satisfies the condition

$$R_{kj} = \alpha g_{kj} + \beta u_k u_j + \gamma v_k v_j$$

where α, β, γ are non-zero scalars and u_k and v_k are covariant vectors such that u_k and v_k are orthogonal to each other vector fields on M. The mixed generalized quasi Einstein manifold was defined by Bhattacharyya and De [1]. A non-flat Riemannian manifold M is called a

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mixed generalized quasi Einstein manifold if its Ricci tensor R_{kj} is non-zero and satisfies the condition

$$R_{kj} = \alpha g_{kj} + \beta a_k a_j + \gamma b_k b_j + \vartheta \left[a_k b_j + b_k a_j \right] \tag{1}$$

where $\alpha, \beta, \gamma, \vartheta$ are non-zero scalars and a_k and b_k are covariant vectors such that a_k and b_k are orthogonal unit vector fields on M. Moreover, it is stated that a Riemannian manifold is of a mixed generalized quasi constant curvature if the curvature tensor of this manifold satisfies the condition

$$\begin{aligned} R_{ikjm} = & \, p \left[g_{kj} g_{im} - g_{ij} g_{km} \right] \\ & + q \left[g_{im} a_k a_j - g_{km} a_i a_j + g_{kj} a_i a_m - g_{ij} a_k a_m \right] \\ & + s \left[g_{im} b_k b_j - g_{km} b_i b_j + g_{kj} b_i b_m - g_{ij} b_k b_m \right] \\ & + t \left[\left\{ a_k b_j + b_k a_j \right\} g_{im} - \left\{ a_i b_j + b_i a_j \right\} g_{km} \right. \\ & \left. + \left\{ a_i b_m + b_i a_m \right\} g_{kj} - \left\{ a_k b_m + b_k a_m \right\} g_{ij} \right] \end{aligned} \tag{2}$$

where p, q, r, s, t are non-zero scalars and a_k and b_k are covariant vectors such that a_k and b_k are orthonormal unit vector fields on M [1].

Let $\overline{\nabla}$ be a linear connection on M . The torsion tensor is given by,

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]$$

The connection $\overline{\nabla}$ is symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. If there is a Riemannian metric g in M such that

$$\overline{\nabla} g = 0 \tag{3}$$

then the connection $\overline{\nabla}$ is a metric connection, otherwise it is non-metric [12]. A linear connection is said to be a semi symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = w(Y)X - w(X)Y \tag{4}$$

where $w(X) = g(X,U)$ and U is a vector field. In [9], Pak showed that a Hayden connection with the torsion tensor of the form (4) is a semi symmetric metric connection. In [11], Yano proved that in order that a Riemannian manifold admits a semi symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian manifold be conformally flat, for some properties of Riemannian manifolds with a semi symmetric metric connection, see also [4, 6, 8, 10]

The components of semi symmetric metric connection are given by

$$\Gamma^l_{ik} = \left\{ \begin{matrix} l \\ ik \end{matrix} \right\} + \delta^l_i w_k - g_{ik} w^l \tag{5}$$

where w_t and $w^l = w_t g^{tl}$ are covariant and contravariant components of a vector field, respectively and

$$\overline{\nabla}_k w_j = \nabla_k w_j - w_k w_j + w g_{kj}, w = w_t w^t \tag{6}$$

By using (5), we obtain,

$$\overline{R}_{ikjm} = R_{ikjm} - g_{im}\pi_{kj} + g_{km}\pi_{ij} - g_{kj}\pi_{im} + g_{ij}\pi_{km} \tag{7}$$

where $\overline{R}_{ikjm}$ and R_{ikjm} are the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively [11]. And π is a tensor field of type $(0,2)$ defined by

$$\pi_{kj} = \nabla_k w_j - w_k w_j + \frac{1}{2} g_{kj} w \tag{8}$$

Transvecting the equation (7) with g^{im}, we get

$$\overline{R}_{kj} = R_{kj} - (n-2)\pi_{kj} - \pi g_{kj} \tag{9}$$

where $\overline{R}_{kj}$ and R_{kj} are the Ricci tensors for the connections $\overline{\nabla}$ and ∇, respectively and $\pi = \pi_{im} g^{im}$.

Multiplying (9) by g^{kj}, we obtain

$$\overline{R} = R - 2(n-1)\pi \tag{10}$$

where $\overline{R}$ and R are the scalar curvatures of semi symmetric metric connection and the Levi-Civita connection, respectively.

2. A Riemannian Manifold Admitting a Special Semi Symmetric Metric Connection

De and Sengupta considered a semi symmetric metric connection and studied some properties of an almost contact manifold of a semi symmetric metric connection whose the torsion tensor satisfies a special condition different from the following condition [2]. In this section, we consider a manifold equipped with a semi symmetric metric connection whose the torsion T satisfies the following condition

$$\overline{\nabla}_j T^l_{ik} = a_j T^l_{ik} + b_j b^l g_{ik} + \delta^l_j b_i a_k \tag{11}$$

where $b^l = b_t g^{tl}$. The equation (4) can be written in the following form

$$T^l_{ik} = \delta^l_i w_k - \delta^l_k w_i$$

Contracting on l and i in the last equation, we get

$$T^l_{lk} = (n-1) w_k \tag{12}$$

Thus, we can find

$$\overline{\nabla}_j T^l_{lk} = (n-1)\overline{\nabla}_j w_k \tag{13}$$

Moreover, by using (11), we obtain

$$\overline{\nabla}_j T^l_{lk} = a_j T^l_{lk} + b_j b_k + b_j a_k \tag{14}$$

From (12)-(14), it is found that

$$\overline{\nabla}_j w_k = a_j w_k + \frac{1}{n-1} b_j b_k + \frac{1}{n-1} b_j a_k \tag{15}$$

After that, from the covariant derivative of w_k with respect to $\overline{\nabla}$, we get the following

$$\nabla_j w_k = \overline{\nabla}_j w_k + w_k w_j - g_{jk} w \tag{16}$$

Substituting (16) in (8), we find

$$\pi_{kj} = \overline{\nabla}_k w_j - \frac{1}{2} g_{kj} w \tag{17}$$

Again, using (15) and (17) , we obtain

$$\pi_{kj} = a_k w_j + \frac{1}{n-1} b_k b_j + \frac{1}{n-1} b_k a_j - \frac{1}{2} g_{kj} w \tag{18}$$

Then, if we substitute (18) in (7), we get

$$\begin{aligned}\overline{R}_{ikjm} = {} & R_{ikjm} \\ & + w\left(g_{im} g_{kj} - g_{km} g_{ij}\right) \\ & - g_{im}\left(a_k w_j + \frac{1}{n-1} b_k b_j + \frac{1}{n-1} b_k a_j\right) \\ & + g_{km}\left(a_i w_j + \frac{1}{n-1} b_i b_j + \frac{1}{n-1} b_i a_j\right) \\ & - g_{kj}\left(a_i w_m + \frac{1}{n-1} b_i b_m + \frac{1}{n-1} b_i a_m\right) \\ & + g_{ij}\left(a_k w_m + \frac{1}{n-1} b_k b_m + \frac{1}{n-1} b_k a_m\right)\end{aligned} \tag{19}$$

From (19), we have the following theorem:

Theorem 1. *The curvature tensor of a Riemannian manifold admitting a semi symmetric metric connection whose the torsion tensor satisfies the condition (11) is of the form (19).*

Now, we recall some theorems which will be used in this section:

Theorem 2. *[3] The Ricci tensor $S(X,Y)$ of a semi symmetric metric connection $\overline{\nabla}$ with the associated 1-form w will be symmetric if and only if w is closed.*

Theorem 3. *[3] A necessary and sufficient condition that the Ricci tensor of the semi symmetric metric connection $\overline{\nabla}$ to be symmetric is that the curvature tensor $\overline{R}$ of $(0,4)$ type with respect to the connection $\overline{\nabla}$ satisfies one of the following two conditions:*

i $\overline{R}_{ikjm} = \overline{R}_{jmik}$

ii $\overline{R}_{ikjm}+\overline{R}_{kjim}+\overline{R}_{jikm}=0.$

From (19), we can write

$$\begin{aligned}\overline{R}_{jmik}=&R_{jmik}\\&+w\left(g_{jk}g_{im}-g_{mk}g_{ji}\right)\\&-g_{jk}\left(a_m w_i+\frac{1}{n-1}b_m b_i+\frac{1}{n-1}b_m a_i\right)\\&+g_{mk}\left(a_j w_i+\frac{1}{n-1}b_j b_i+\frac{1}{n-1}b_j a_i\right)\\&-g_{mi}\left(a_j w_k+\frac{1}{n-1}b_j b_k+\frac{1}{n-1}b_j a_k\right)\\&+g_{ji}\left(a_m w_k+\frac{1}{n-1}b_m b_k+\frac{1}{n-1}b_m a_k\right)\end{aligned} \tag{20}$$

we assume that the associated 1-form w of a Riemannian manifold admitting a semi symmetric metric connection whose the torsion tensor satisfies the condition (11) is closed. In virtue of Theorem 2, the Ricci tensor of a Riemannian manifold with a semi symmetric metric connection is symmetric. Thus, due to Theorem 3, we get

$$\overline{R}_{jmik}=\overline{R}_{ikjm} \tag{21}$$

In case the equation (21) is satisfied, we find

$$\begin{aligned}0=&g_{im}\left[a_j\left(w_k-\frac{1}{n-1}b_k\right)-a_k\left(w_j-\frac{1}{n-1}b_j\right)\right]\\&+g_{km}\left[a_i\left(w_j-\frac{1}{n-1}b_j\right)-a_j\left(w_i-\frac{1}{n-1}b_i\right)\right]\\&+g_{kj}\left[a_m\left(w_i-\frac{1}{n-1}b_i\right)-a_i\left(w_m-\frac{1}{n-1}b_m\right)\right]\\&+g_{ij}\left[a_k\left(w_m-\frac{1}{n-1}b_m\right)-a_m\left(w_k-\frac{1}{n-1}b_k\right)\right]\end{aligned} \tag{22}$$

Transvecting (22) with g^{im} , we get

$$(2-n)\left[a_k\left(w_j-\frac{1}{n-1}b_j\right)-a_j\left(w_k-\frac{1}{n-1}b_k\right)\right]=0 \tag{23}$$

Since $n>2$, we get

$$a_k\left(w_j-\frac{1}{n-1}b_j\right)=a_j\left(w_k-\frac{1}{n-1}b_k\right) \tag{24}$$

Now, permutating the indices and adding the three equations side by side, we obtain

$$\overline{R}_{ikjm}+\overline{R}_{kjim}+\overline{R}_{jikm} \tag{25}$$

$$
\begin{aligned}
&= g_{im}\left[a_j\left(w_k-\frac{1}{n-1}b_k\right)-a_k\left(w_j-\frac{1}{n-1}b_j\right)\right]\\
&+ g_{km}\left[a_i\left(w_j-\frac{1}{n-1}b_j\right)-a_j\left(w_i-\frac{1}{n-1}b_i\right)\right]\\
&+ g_{jm}\left[a_k\left(w_i-\frac{1}{n-1}b_i\right)-a_i\left(w_k-\frac{1}{n-1}b_k\right)\right]
\end{aligned}
$$

Conversely, let us assume that (24) is satisfied. Then, the expression on the right side of (25) vanishes. It means that the curvature tensor of the connection $\overline{\nabla}$ satisfies the first Bianchi Identity. Due to Theorem 3, the Ricci tensor with respect to the connection $\overline{\nabla}$ is symmetric. Because of Theorem 2, the associated 1-form w of a Riemannian manifold with a semi symmetric metric connection is closed. Hence, we can establish the following theorem:

Theorem 4. *A necessary and sufficient condition that the associated 1-form w of a Riemannian manifold with a semi symmetric metric connection whose the torsion tensor satisfies the condition (11) to be closed is that the condition (24) is satisfied.*

Suppose that w is closed. Substituting (15) in (16), we get

$$\nabla_j w_k = a_j w_k + \frac{1}{n-1}b_j b_k + \frac{1}{n-1}b_j a_k + w_k w_j - g_{jk}w \tag{26}$$

Subtracting the corresponding equation found by interchanging k and j in (26) from (26), we get the equation (24). Thus, by using Theorem 2, Theorem 3 and Theorem 4, we have the following Theorem:

Theorem 5. *In a Riemannian manifold with a semi symmetric metric connection whose the torsion tensor satisfies the condition (11), a necessary and sufficient condition that the condition (24) to be satisfied is that it is satisfied any one of the following properties:*

i The curvature tensor with respect to the connection $\overline{\nabla}$ of this manifold has the properity of block symmetry,

ii The curvature tensor with respect to the connection $\overline{\nabla}$ of this manifold satisfies the first Bianchi Identity,

iii The Ricci tensor of this manifold is symmetric.

3. Conformally Flat Manifolds with Semi Symmetric Metric Connection Satisfying some Special Condition

In this section, we shall investigate a Riemannian manifold M admitting a semi symmetric metric connection whose the torsion tensor satisfies a special condition in the case of conformally flat. Firstly, we consider the condition (11). Then,

$$\overline{\nabla}_j T^l_{ik} = a_j T^l_{ik} + b_j b^l g_{ik} + \delta^l_j b_i a_k \tag{27}$$

where a_k and b_k be orthogonal to each other. The conformal curvature tensor is given by

$$C_{ikjm} = R_{ikjm} - \frac{1}{n-2}\left[R_{im}g_{kj} - R_{km}g_{ij} + R_{kj}g_{im} - R_{ij}g_{km}\right] + \frac{R}{(n-1)(n-2)}\left[g_{im}g_{kj} - g_{km}g_{ij}\right] \tag{28}$$

Now, we remember that it is well known the following Theorem:

Theorem 6. *[11] In order that a Riemannian manifold admits a semi symmetric metric connection curvature tensor vanishes, it is necessary and sufficient condition that the Riemannian manifold be conformally flat.*

Suppose that this manifold is conformally flat. Hence, we can write

$$\overline{R}_{ikjm} = 0 \tag{29}$$

Therefore, due to (7) and (29), we obtain

$$R_{ikjm} = g_{im}\pi_{kj} - g_{km}\pi_{ij} + g_{kj}\pi_{im} - g_{ij}\pi_{km} \tag{30}$$

Multiplying (29) by g^{im}, we get the corresponding identity

$$\overline{R}_{kj} = 0 \tag{31}$$

Transvecting (19) with g^{im} and using (31), we have

$$R_{kj} = \left[(1-n)w + \left(a^m w_m + \frac{1}{n-1}b + \frac{1}{n-1}b^m a_m\right)\right] g_{kj} + (n-2)\left(a_k w_j + \frac{1}{n-1}b_k b_j + \frac{1}{n-1}b_k a_j\right) \tag{32}$$

where $a^m = a_i g^{im}, b = b_m b^m \neq 0$. Since a_k and b_k are the orthogonal vector fields, it can be written

$$R_{kj} = \left[(1-n)w + \phi + \frac{1}{n-1}b\right] g_{kj} + (n-2)\left(a_k w_j + \frac{1}{n-1}b_k b_j + \frac{1}{n-1}b_k a_j\right) \tag{33}$$

where $a^m w_m = \phi$ is a non-zero scalar function. Subtracting (33) from the corresponding equation found by interchanging k and j in (33), we get (24). Transvecting (24) with $a^j b^k$, we find

$$b^k w_k = \frac{ab}{n-1} \tag{34}$$

where $a^m a_m = a \neq 0$. From (34), it is seen that b_k can not be orthogonal to w_k . Again, multiplying (24) by a^k, we get

$$w_j = \theta a_j + \frac{1}{n-1}b_j \tag{35}$$

where $\theta = \frac{\phi}{a} \neq 0$. By using (35), we find that a_j is not orthogonal to w_j. Substituting (29) and (35) in (19), we obtain

$$\begin{aligned} R_{ikjm} = {} & w\left(g_{km}g_{ij} - g_{im}g_{kj}\right) \\ & + \theta\left[g_{im}a_k a_j - g_{km}a_i a_j + g_{kj}a_i a_m - g_{ij}a_k a_m\right] \\ & + \frac{1}{n-1}\left[g_{im}b_k b_j - g_{km}b_i b_j + g_{kj}b_i b_m - g_{ij}b_k b_m\right] \\ & + \frac{1}{n-1}\left[g_{im}\left(a_k b_j + b_k a_j\right) - g_{km}\left(a_i b_j + b_i a_j\right)\right. \\ & \left. + g_{kj}\left(a_i b_m + b_i a_m\right) - g_{ij}\left(a_k b_m + b_k a_m\right)\right] \end{aligned} \tag{36}$$

If $w = \theta\phi + \frac{ab}{(n-1)^2} \neq 0$, and since a_k and b_k are the orthogonal vector fields, the equation (36) is equivalent to (2). This implies that such a manifold is of a mixed generalized quasi constant curvature.

Multiplying (36) by g^{im},we obtain

$$R_{kj} = \mu g_{kj} + (n-2)\theta a_k a_j + \left(\frac{n-2}{n-1}\right)\left(b_k b_j + a_k b_j + b_k a_j\right) \tag{37}$$

where

$$\mu = (1-n)w + \theta a + \frac{b}{n-1} \tag{38}$$

Suppose that $\mu \neq 0$.

Conversely, suppose that this manifold is of a mixed generalized quasi constant curvature. Multiplying (2) by g^{im} , we obtain

$$\begin{aligned} R_{kj} = {} & \left[p(n-1) + qa + bs\right] g_{kj} + q(n-2)a_k a_j \\ & + s(n-2)b_k b_j + t(n-2)\left(a_k b_j + b_k a_j\right) \end{aligned} \tag{39}$$

Transvecting (39) with g^{kj} , we find

$$R = (n-1)\left[np + 2qa + 2sb\right] \tag{40}$$

Let us substitute (2) , (39) and (40) in (28).Then, if $w = -p, \theta = q$ and $t = s = \frac{1}{n-1}$, we get

$$C_{ikjm} = 0$$

We may now establish the following theorem:

Theorem 7. *In a Riemannian manifold with a semi symmetric metric connection whose the torsion tensor satisfies the condition (27), a necessary and sufficient condition that this manifold to be of a mixed generalized quasi constant curvature is that it is conformally flat.*

When we compare (37) with (1), if $p(n-1) + qa + bs \neq 0$, we can say that this manifold is a mixed generalized quasi Einstein manifold. Thus, we can state the following theorem:

Theorem 8. *A conformal flat Riemannian manifold with a semi symmetric metric connection whose the torsion tensor satisfies the condition (27) is a mixed generalized quasi Einstein manifold.*

Theorem 9. *[13] If a Riemannian manifold admits a semi symmetric metric connection with constant sectional curvature, then this manifold is conformally flat.*

Thus, in virtue of Theorem 7, Theorem 8 and Theorem 9, we can establish the following theorems:

Theorem 10. *If the sectional curvature of a Riemannian manifold with a semi symmetric metric connection whose the torsion tensor satisfies the condition (27) is independent from the orientation chosen, then*

i It is of a mixed generalized quasi constant curvature,

ii It is a mixed generalized quasi Einstein manifold.

Theorem 11. *If the sectional curvature of a Riemannian manifold with a semi symmetric metric connection whose the torsion tensor satisfies the condition (27) is independent from the orientation chosen, then the condition (24) is satisfied.*

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