

On The Critical Point Structure of Eigenfunctions Belonging to the First Nonzero Eigenvalue of A Genus Two Closed Hyperbolic Surface

Carlos A. Cadavid¹, Maria C. Osorno², Oscar E. Ruiz³
CAD/CAM/CAE Laboratory, Universidad EAFIT, Medellin, Colombia

Accepted 10 May, 2012

Abstract- We develop a method based on spectral graph theory to approximate the eigenvalues and eigenfunctions of the Laplace-Beltrami operator of a compact riemannian manifold. The method is applied to a closed hyperbolic surface of genus two. The results obtained agree with the ones obtained by other authors by different methods, and they serve as experimental evidence supporting the conjectured fact that the generic eigenfunctions belonging to the first nonzero eigenvalue of a closed hyperbolic surface of arbitrary genus are Morse functions having the least possible total number of critical points among all Morse functions admitted by such manifolds.

Keywords: Closed hyperbolic surface, Laplacian, Eigenvalue, Eigenfunction, Critical point, Spectral graph theory

1. Introduction

The determination of eigenfunctions of the Laplacian operator on a Manifold has been conducted by several authors and with different methods. These eigenfunctions are relevant in Astrophysics, Geometric Modeling, Topology, Differential Geometry, Quantum Mechanics, among others. In (Aurich, 1989) the low lying eigenvalues of a closed hyperbolic surface of genus two, that in the sequel will also be referred to as a double doughnut, are calculated using finite element methods. In (Aurich, 1993) the same calculation is performed but using boundary element methods. Works (Aurich, 1989) and (Aurich, 1993) were motivated by questions in the area of quantum chaos. In (Cornish, 1998) the authors also calculate low lying eigenvalues and the corresponding eigenfunctions of a double doughnut, using a finite difference discretization of a wave equation. Their interest in this problem arose mainly from their study of the relation between the present distribution of the cosmic microwave radiation of the universe and its topological structure. Article (Bachelot-Motet) also finds approximations to a number of

eigenvalues and corresponding eigenfunctions of a double doughnut by using finite element methods. In the recent paper (Strohmaier, 2011) a rigorous method, based on the method of particular solutions, is formulated for solving the same problem for closed hyperbolic surfaces of all genera. It is important to point out that in all these works the motivation was entirely different from ours.

1.1 Motivation.

The main motivation for the present article is the evaluation of the generality of an observation made by one of the authors concerning the fact that in various simple (e.g. locally homogeneous) compact riemannian manifolds (e.g. flat tori and round spheres of arbitrary dimension) the generic eigenfunction of the Laplace-Beltrami operator belonging to the first nonzero eigenvalue are minimal Morse functions in the sense of having the least possible total number of critical points among all Morse functions admitted by the manifold. In this paper we take the next obvious step, namely we try to determine the presence of the same phenomenon in the case of compact hyperbolic surfaces. Actually, we only consider a particular genus-2 hyperbolic surface. In the flat tori and round sphere cases a closed form for the eigenfunctions of all eigenvalues is known. Unfortunately, the eigenfunctions do not admit such nice closed form in the hyperbolic case. This fact forces one to look instead for approximations of the eigenfunctions obtained by some numerical method.

2. Methodology

2.1. Representing closed hyperbolic surfaces

Let (M, g) be a riemannian 2-manifold having empty boundary. If M is compact, connected and orientable, and the gaussian curvature of g at each point of M is -1 , then (M, g) is called a *closed hyperbolic surface*. The only

¹ ccadavid@eafit.edu.co

² mosornot@eafit.edu.co

³ oruiz@eafit.edu.co

topological invariant of these objects is their genus. This is a nonnegative integer, which by an immediate application of the Gauss-Bonnet theorem, has value two or more. The classification up to isometry of closed hyperbolic surfaces of fixed genus is a vastly more complex issue. There is a standard procedure to construct all possible closed hyperbolic surfaces. The disk

$D = \{(x, y) \in \frac{R^2}{x^2 + y^2} < 1\}$ endowed with the riemannian metric given in Cartesian coordinates by $g_D = \frac{dx^2 + dy^2}{(1-x^2-y^2)^2}$ is known as Poincaré's disk model (of

2-dimensional hyperbolic geometry). We denote by d_D the resulting notion of distance between points of D . We will sometimes think of D as the open unit disk of the complex plane \mathbb{C} . Let us denote by $Isom^+$ the group formed by all orientation preserving isometries of (D, g_D) and let $\Gamma \subset Isom^+$ be a discrete and cocompact subgroup. These conditions guarantee that the quotient manifold D/Γ together with the metric induced by g_D in the obvious way, is a closed hyperbolic surface. Accordingly, it is sufficient to have a method to produce all possible discrete co-compact subgroups of $Isom^+$ in order to construct all possible closed hyperbolic surfaces. Such method was discovered by Poincaré and is known as Poincaré's theorem. It roughly says that in order to construct any such subgroup it is enough to give a polygon $K \subset D$ whose closure in D is compact and an indexed family of isometries $\{\gamma_i \in Isom^+ / C \text{ a side of } K\}$, satisfying a number of properties (see (Waldschmidt, 1989.)). The subgroup is the one generated by these γ_i 's. In the present article we will study eigenvalues and eigenfunctions of the Laplace-Beltrami operator of a particular genus two closed hyperbolic surface, that we describe next. Let C_k be the circle of radius $\sqrt{\frac{\sqrt{2}-1}{2}}$ and center $-\sqrt{\frac{1+\sqrt{2}}{2}} * e^{\frac{-(k-1)\pi i}{4}}$ for $K = 1, \dots, 8$, and let D_k be the corresponding open disks. Let K be the octagon $D - \cup D_k$ (see Figure 1). The vertices

of K are the points P_k contained in D and belonging to $C_k \cap C_{k-1}$ for $K = 1, \dots, 7$, plus the point P_8 contained in D and belonging to $C_8 \cap C_1$. The oriented sides of K will be denoted by $P_i P_j$ where P_i and P_j are its endpoints. For $l = 0, \dots, 4$ let g_l be the isometry of D defined by:

$$g_l(z) = \frac{(1 + \sqrt{2})z + \sqrt{2 + 2\sqrt{2}}e^{\frac{il\pi}{4}}}{\sqrt{2 + 2\sqrt{2}}e^{-\frac{il\pi}{4}}z + (1 + \sqrt{2})}$$

It can be directly verified that $g_0(P_1 P_8) = P_4 P_5$, $g_1(P_8 P_7) = P_3 P_4$, $g_2(P_7 P_6) = P_2 P_3$, $g_3(P_6 P_5) = P_1 P_2$.

If one takes $\gamma_{P_1 P_2} = g_3, \gamma_{P_2 P_3} = g_2, \gamma_{P_3 P_4} = g_1, \gamma_{P_4 P_5} = g_0, \gamma_{P_5 P_6} = g_3^{-1}, \gamma_{P_6 P_7} = g_2^{-1}, \gamma_{P_7 P_8} = g_1^{-1}, \gamma_{P_8 P_1} = g_0^{-1}$, then all conditions of Poincaré's theorem are satisfied and the isometries $\{g_0, g_1, g_2, g_3\}$ generate a discrete compact subgroup Γ of $Isom^+$. D/Γ is therefore a closed hyperbolic surface whose genus turns out to be 2. There is an alternative and more intuitive description of the riemannian manifold D/Γ . The polygon K can be regarded as a riemannian manifold with corners, and the action of Γ restricted to K as a rule for gluing ∂K with itself. General theorems guarantee that the quotient K/Γ inherits a smooth structure and a riemannian structure from the corresponding structures carried by K . It can be proved that the riemannian manifold K/Γ so defined is isometric to the riemannian manifold D/Γ . Topologically, the manifold K/Γ is obtained by taking K and gluing the side $P_6 P_5$ with the side $P_1 P_2$ according to g_3 , the side $P_7 P_6$ with the side $P_2 P_3$ according to g_2 , the side $P_8 P_7$ with the side $P_3 P_4$ according to g_1 , and the side $P_1 P_8$ with the side $P_4 P_5$ according to g_0 . It can be easily verified that if $\bar{P}, \bar{P}' \in K$, then the riemannian distance between the points \bar{P}, \bar{P}' in K/Γ is $d_{K/\Gamma}(\bar{P}, \bar{P}') = \min\{d_D(P, P'), d_D(P, g_0(\bar{P}')), \dots, d_D(P, g_l(\bar{P}'))\}$.

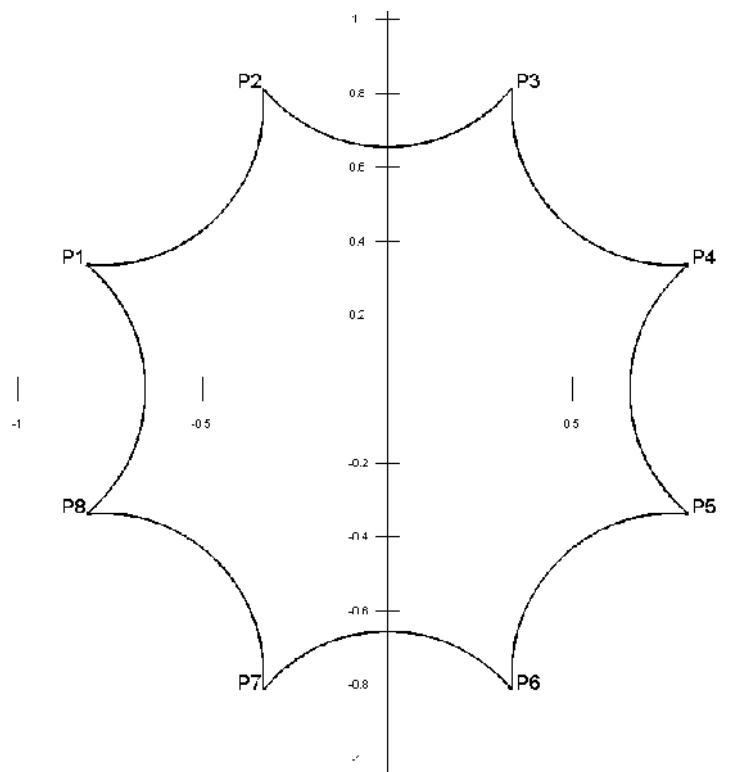


Figure 1: Polygon K

2.2. Discretization

In order to obtain approximations to the eigenvalues and eigenfunctions of the Laplace-Beltrami operator of the riemannian manifold D/Γ it is necessary to construct a hyperbolic mesh for this object. This is done in the following way. We first fix a positive integer n and then divide each side $P_j P_{j+1}$ of K by taking points $Q_{0,j}, \dots, Q_{n,j} \in P_j P_{j+1}$ such that $Q_{0,j} = P_j$, $Q_{n,j} = P_{j+1}$ and $d_D(Q_{i,j}, Q_{i+1,j})$ is independent of i . (Here $P_j P_{j+1}$ is meant to be $\overline{P_j P_{j+1}}$) Let Q denote the collection formed by all $Q_{i,j}$'s. Since the polygon K is regular, the numbers $d_D(Q_{i,j}, Q_{i+1,j})$ are also independent of j . Let us denote by ι this common number. Then we randomly pick a large number of points out of the interior of K , taking into account the notion of area induced by g_D , i.e. we pick points near a point $(x, y) \in K$ with probability $\left(\frac{\iota}{(1-x^2-y^2)^2}\right) / A(K)$ where $A(K) = \int_K \frac{\iota}{(1-x^2-y^2)^2} dx dy$ is the total hyperbolic area of K . Let Q' be the collection so obtained. Then a subcollection \mathcal{P} of $Q \cup Q'$ is chosen

satisfying i) $Q \subset \mathcal{P}$ and ii) maximality respect to the property that no two points in it are at hyperbolic distance less than ι , i.e. for each point in Q' that is not in \mathcal{P} there is a point in \mathcal{P} so that their hyperbolic distance is less than or equal to ι . Next a triangulation for K with vertex set \mathcal{P} is built satisfying the demand that all arcs $Q_{i,j} Q_{i+1,j}$ are sides of triangles. The algorithm used for building this triangulation is based on Delaunay's method. The underlying graph of the triangulation so obtained is usually nearly regular, and the hyperbolic length of its edges is nearly constant. Since the geodesic arc joining two points in D is unique, all that is needed in order to specify the triangulation is the set of pairs of points of \mathcal{P} corresponding to sides of triangles. Now, due to the way the points of Q were chosen, this triangulation of K descends to a triangulation of K/Γ . The underlying graph of this triangulation is the one having vertex set \mathcal{P}/Γ and edge set $\{(\overline{P}, \overline{P}') / \{P, P'\} \text{ is a side of a triangle in } K\}$. We make this into a weighted graph by attaching weight $1/\iota$ to all edges. (The notions of graph and weighted graph are defined in the next section.) This weighted graph will be

regarded as an approximation of the riemannian manifold K/Γ .

2.3 Approximating eigenvalues and corresponding eigenfunctions of K/Γ

In this section we present the method used to get a feel for the shape of the eigenfunctions of the Laplace-Beltrami operator on K/Γ corresponding to the first nonzero eigenvalue. In particular we will be interested in the critical point structure of such functions.

2.3.1. Spectral graph theory method

The *spectral graph theory method* consists in approximating a riemannian manifold by a *weighted graph*, the real valued functions on the manifold by the real valued functions defined on the set of vertices of the graph, and the Laplace-Beltrami operator by certain *discrete Laplacian operator* of the weighted graph. The eigenvalues and eigenvectors of the latter operator are approximations for the eigenvalues and eigenfunctions of the Laplace-Beltrami operator of the riemannian manifold, whose accuracy depends on how well the weighted graph approximates the riemannian manifold. As a matter of fact, we prefer to calculate the eigenvalues and eigenfunctions of another operator that is closely related to the discrete Laplacian operator just mentioned.

We now recall the basic concepts of spectral graph theory. A *graph* G is a pair (V, E) consisting of a set V that we assume finite, and a set E whose elements are two-element subsets of V . The elements of V are called *vertices* and those of E are called *edges*. The fact that $\{x, y\}$ in E is abbreviated by $x \sim y$. A *weighted graph* is a graph together with a rule that assigns a nonnegative real number to each edge. We denote by ω_{xy} the nonnegative number attached to edge $\{x, y\}$. Notice that $\omega_{xy} = \omega_{yx}$. There are several matrices associated to this object. The purpose of spectral graph theory is to study graphs by looking at the similarity classes of the associated matrices. The most basic matrix is the *adjacency matrix* A defined as the matrix whose (x, y) entry is ω_{xy} if $\{x, y\} \in E$ and zero otherwise. The degree of a vertex x of a weighted

graph is $d(x)$. The matrix $\mathcal{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$ is also very important. \mathcal{L} is called the *normalized Laplacian matrix* of the weighted graph. An analogue of the Laplace-Beltrami operator for a weighted graph is the one sending each function $f: V \rightarrow \mathbb{R}$ to the function $\Delta f: V \rightarrow \mathbb{R}$ defined as

$$\Delta_{\mathcal{G}} f(x) = \frac{2}{m(x)} \sum_{y \sim x} \omega_{yx} (f(x) - f(y)) \quad (1)$$

where $m(x)$ denotes $\sum_{y \sim x} \frac{1}{\omega_{xy}}$. We observe that in case

\tilde{G} is k -regular (i.e. each vertex of \tilde{G} is connected to exactly k other vertices) and all weights are equal to $\frac{1}{k}$ for a fixed $k > 0$ it can be easily verified that

$$\Delta_{\tilde{G}} f = \frac{2}{k^2} \bar{L} \bar{D}^{-1} f \quad (2)$$

Where $\bar{L} = \bar{D} - \bar{A}$ and \bar{A}, \bar{D} denote the adjacency and degree matrices for the unweighted graph underlying \tilde{G} , i.e. the one obtained from \tilde{G} by replacing all of its weights by 1.

Let (M, g) be a compact riemannian manifold without boundary. Now we present the setting in (Fujiwara, 1995). Let ϵ be a positive real number. A subset W of M is said to be ϵ -separated if the riemannian distance between any two points of W is not less than ϵ . Suppose that V is an ϵ -separated set that is maximal, i.e. no superset of V is ϵ -separated. Now consider the weighted graph having vertex set V , edge set formed by those pairs of points in V whose riemannian distance does not exceed 3ϵ , and ω_{xy} the riemannian distance l_{xy} between x and y . Any weighted graph obtained in this way is termed an ϵ -net of (M, g) . Paper (Fujiwara, 1995) describes how the eigenvalues of the operator $\Delta_{\mathcal{G}}$ defined in equation 1 on an ϵ -net \tilde{G} for (M, g) approach the eigenvalues of the Laplace-Beltrami operator of (M, g) . Our approach differs from Fujiwara's in two ways, namely (i) we use the operator defined by the right hand side of equation 2 and (ii) a Delaunay's method based algorithm is used for

declaring pairs of distinct points to be edges of the approximating graph.

3. Results

In this section we present the results obtained by the application of the spectral graph theory method discussed in the previous sections. As explained in section 2.2, the first step is to divide each side of the octagon K into segments of the same hyperbolic length. Several segment sizes were tried and very good results were obtained by dividing each side into $n = 80$ segments, each of hyperbolic length $l = 0.0453$. Then a large number of

points are randomly selected out of the interior of K and a straightforward algorithm is applied to this set of points in order to extract a maximal l -separated subset from it. The latter set turned out to consist of 3827 points. Then a Delaunay method based algorithm is applied to this set of points producing a triangulation of K . For visualization purposes,

Figure 2 displays the result obtained by using $n = 40$. It has 1480 points.

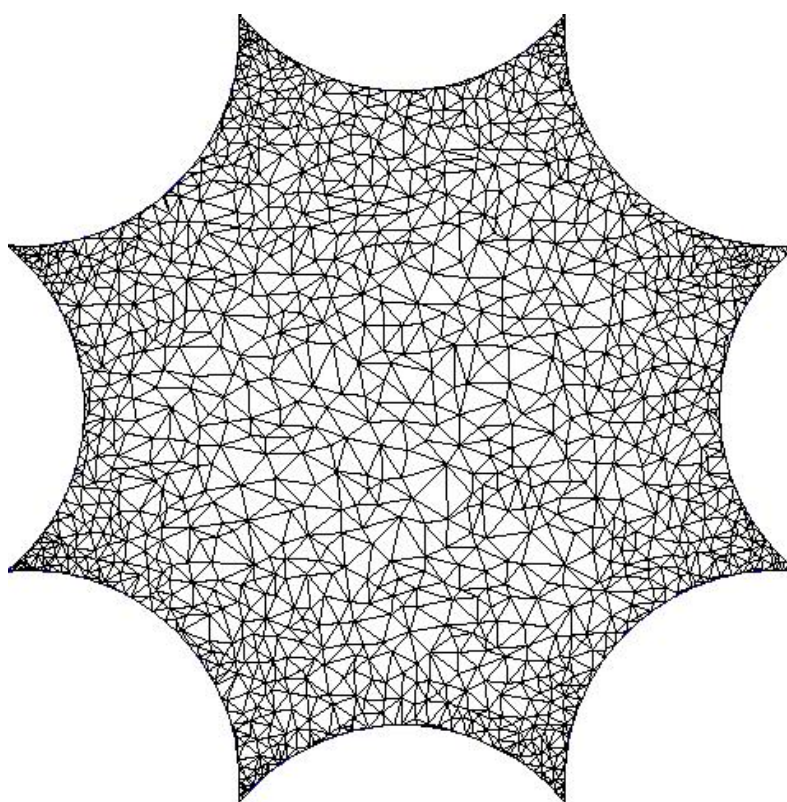


Figure 2. Triangulation of K with 1480 vertices obtained starting with $n=40$

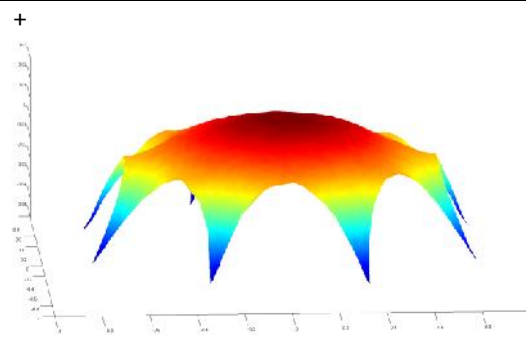
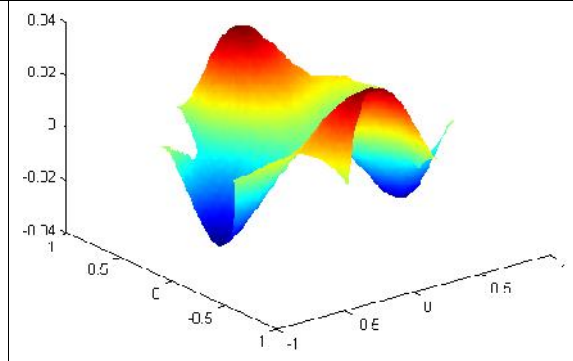
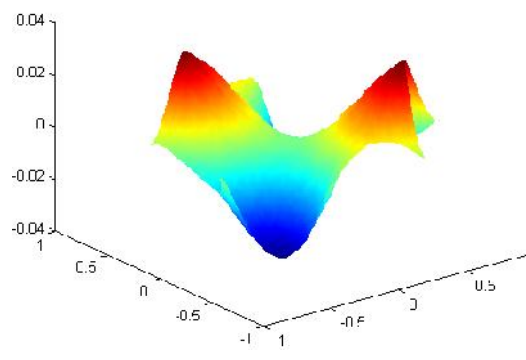
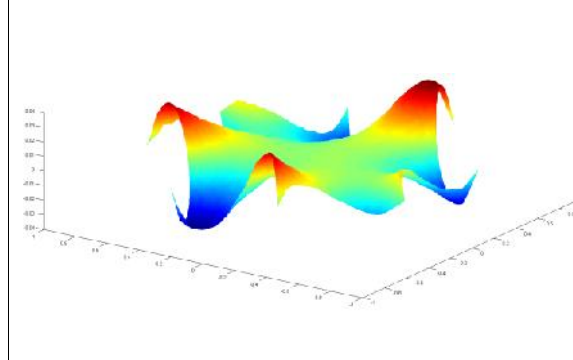
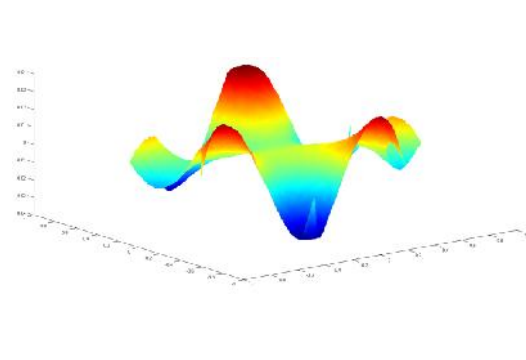
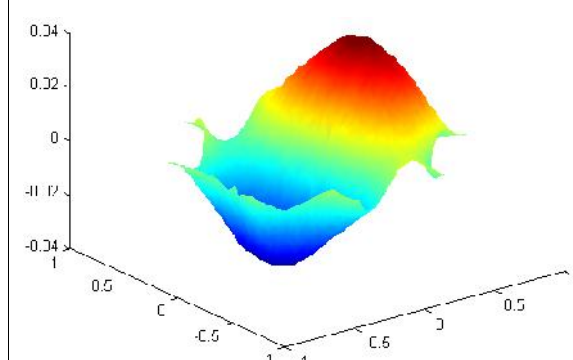
The next step is to produce the adjacency matrix of the unweighted graph underlying the triangulation. This information is further processed in order to obtain the adjacency matrix \bar{A} of the unweighted graph obtained from the previous unweighted graph by identifying the vertices lying on the boundary of K according to the glueing maps g_0, \dots, g_3 . Then the matrix $\frac{2}{0.0453^2} \bar{L} \bar{D}^{-1}$ is

calculated and its eigenvalues and eigenvectors are obtained using MATLAB.

Table 1 shows the first eight eigenvalues together with the graphs of their corresponding eigenfunctions. It is important to remark that all eigenvalues turned out to have multiplicity one, which is the case for a generic matrix. We now compare this result with those obtained by Bachelot-Motet in (Bachelot-Motet). The

eigenfunctions we obtained corresponding to the eigenvalues $\lambda = 3.0511, 3.8844, 3.9136$ are similar to the three eigenfunctions of the eigenvalue 3.388 shown in Figure 4.4 of (Bachelot-Motet). This is reflecting the fact that the first nonzero eigenvalue of the closed hyperbolic surface of genus 2 has multiplicity three, and that as the mesh gets finer the first three nonzero eigenvalues come together, becoming in the limit a single eigenvalue with multiplicity three. It is interesting to notice that these three functions seem to be related by a symmetry of the closed hyperbolic surface of genus two. If we take the average of 3.0511, 3.8844, 3.9136 we obtain 3.6287, a value that seems close to the value 3.8388 obtained by Bachelot-Motet. The next two eigenfunctions we obtained,

namely those corresponding to $\lambda = 4.9358, 5.1111$ are similar to the two eigenfunctions of the eigenvalue 5.353 depicted in Figure 4.5 of (Bachelot-Motet). As before, these two functions seem to be related by a symmetry of the closed hyperbolic surface of genus two. Now the average of 4.9358, 5.1111 is 5.0274, a number that seems close to the value 5.307 obtained by Bachelot-Motet. If we keep doing this, but now grouping eigenfunctions by their similarity up to some symmetry of the closed hyperbolic surface of genus two, and averaging their eigenvalues, we obtain the results shown in Table 2. The corresponding values obtained by Bachelot-Motet are also shown in that table.

λ	Eigenfunction	λ	Eigenfunction
3.0511		3.8844	
3.9136		4.9358	
5.1111		6.1275	

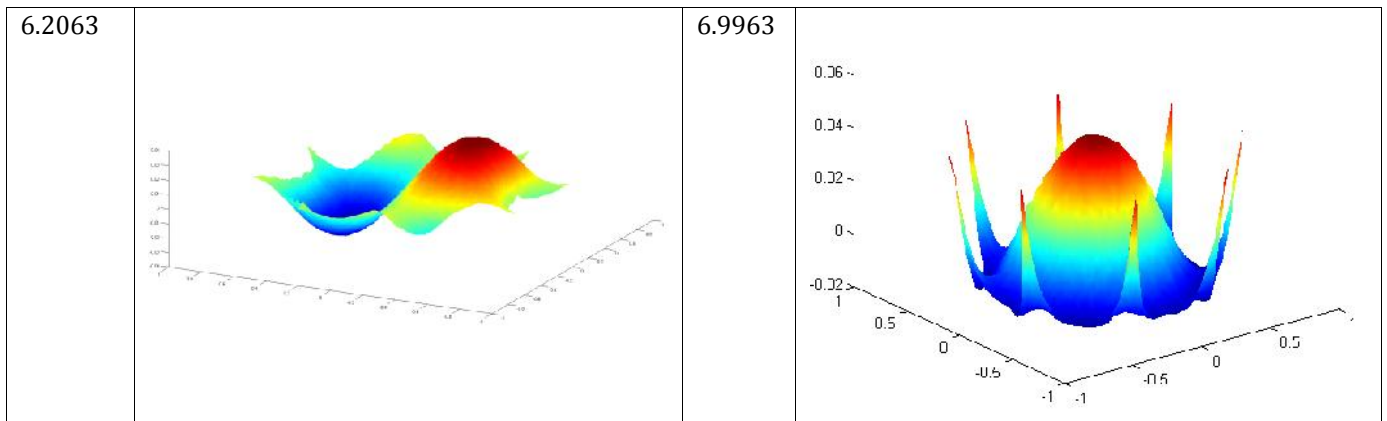


Table 1: Eigenvalues and eigenfunctions.

λ	λ Bachelot-Motet
3.6287	3.8388
5.0274	5.307
8.0065	8.193
14.4819	14.743
15.329	15.284
18.5908	18.735
20.955	20.592
23.3647	23.204
28.2561	28.151
31.766	31.193

Table 2: Comparison between the first ten nonzero eigenvalues obtained by the spectral graph theory method (first column) and the Bachelot-Motet method (second column).

Now let us take a look at the critical points structure of the eigenfunction corresponding to $\lambda = 3.0511$ displayed in Table 1. We note that this function has one local maximum (at the center of octagon), four saddle points (at the middle points of the sides of the octagon), and one local minimum (at each of the eight vertices of the octagon). So this function has 6 critical points, and, for topological reasons, this is the least number of critical points a Morse function defined on a closed surface of genus two admits.

4. Discussion of Results

The Literature Review conducted in our research indicates that the method here implemented (spectral graph theory) to estimate the eigenfunctions of the Laplacian of a 2-

manifold is new in this field. We based our method on the convergence properties of the Laplacian of graphs approximating the riemannian manifold. The results that we obtained show a good agreement with the ones obtained in the works mentioned in the Introduction. The agreement between the shape of the eigenfunctions found by our method and other methods is particularly good. Our results are mainly comparable with the ones obtained by Bachelot-Motet (Bachelot-Motet). It should be mentioned that the method developed in the present article can be applied to arbitrary compact riemannian manifolds and seems to be a good alternative due to its apparently low computational cost. Our results serve as experimental support for the hypothesis that eigenfunctions corresponding to the first nonzero eigenvalue in a closed hyperbolic surface of any genus have the least possible total number of critical points. For

topological reasons, the total number of critical points for a Morse function in the genus-2 case must be at least 6. The (approximate) eigenfunctions that we obtained have 6 critical points: a local maximum, a local minimum, and four saddle points. Finally, we remark that we did not attempt to estimate the numerical error incurred, and therefore the results obtained at this time are only of illustrative value.

5. Conclusions and Future Work

A method for obtaining approximations to the eigenvalues and eigenfunctions of a closed hyperbolic genus two surface based on ideas from spectral graph theory has been developed. Although several very good methods for the same purpose have been previously developed, the present approach seems to be a reasonable alternative due to its apparently low computational cost and its generality. The method was applied and the results provide experimental evidence regarding the fact that the generic eigenfunction of the first nonzero eigenvalue is a Morse function having the least possible total number of critical points among all Morse functions admitted by that manifold.

The method should be tried in higher genus closed hyperbolic surfaces and also in the case of geometric 3-dimensional manifolds, i.e. the ones obtained as quotients of any of Thurston's eight geometries. Once the eigenfunctions of the first nonzero eigenvalue are obtained, it would be very interesting to evaluate the simplicity of their critical point structure.

References

Aurich R. and Steiner, F. Periodic-orbit sum rules for the Hadamard-Gutzwiller model [Journal] // *Physica D: Nonlinear Phenomena*. - 1989. - pp. 169-193.

Aurich R. and Steiner, F. Statistical properties of highly excited quantum eigenstates of a strongly chaotic system. [Journal] // *Physica D: Nonlinear Phenomena*. - 1993. - pp. 185-214.

Bachelot-Motet A. Wave computation on the hyperbolic double doughnut [Journal] // *Journal of Computational Mathematics*. - 2010. - pp. 790-806.

Cornish N.J. and Turok, N.G. Ringing the eigenmodes from compact manifolds [Publicación periódica] // *Classical and Quantum Gravity*. - 1998. - Vol. 15. - pág. 2699.

Fujiwara K. Eigenvalues of laplacians on a closed riemannian manifold and its nets [Journal] // *Proceedings of the American Mathematical Society*. - 1995. - Vol. 123. - pp. 2585--2594.

Strohmaier A. and Uski, V. A Rigorous Algorithm for the Computation of Eigenvalues, Spectral Zeta Functions and Zeta-Determinants on Hyperbolic Surfaces. [Journal] // Arxiv preprint arXiv:1110.2150. - 2011.

Waldschmidt M. and Moussa, P. and Luck, JM and Itzykson, C. From Number theory to Physics. [Journal] // *Les Houches*. - 1989.