THE DIMENSION OF A TOPOLOGICAL SPACE

A THESIS

Presented to

The Faculty of the Graduate Division

by

James Risen Gard

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematics

Georgia Institute of Technology

September, 1965

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Approved:

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Thesis Advisor

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Date approved by Advisor: Sept. 8, 1965

ACKNOWLEDGMENTS

I wish to thank my thesis advisor, Dr. R. D. Johnson, for suggesting the topic of this paper and for his many contributions to the development of the results contained herein. I also wish to thank Dr. R. H. Kasriel and Dr. R. M. Ahrens for reading the thesis and making several helpful suggestions for improvement.

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CHAPTER ONE

INTRODUCTION

Since its beginnings in the work of Brouwer, Menger, and Urysohn in the early twentieth century, the theory of dimension in separable metric spaces has been richly developed. It is presented in some detail in Hurewicz and Wallman's work, Dimension Theory (7).

The purpose of this paper is to present some results in dimension theory which apply to more general spaces than separable metric speces. In (7), Hurewicz and Wallman give a definition generally known as small inductive dimension (def. 2.7) for the dimension of a topological space. They prove many propositions about the dimension of separable metric spaces: for example the dimension of E_n is n, and the dimension of a subspace of X is no greater than the dimension of X. In the course of their development, they present several other properties which are shown to characterize the dimension of separable metric spaces.

Since the inductive dimension and these other properties characterize the dimension of Euclidean spaces, there is some justification for using each of these properties to define the dimension of more general spaces. It is natural to try to discover whether these definitions are still equivalent, and what dimension type properties they retain in more general spaces. Most of the results of this paper are concerned with the former question, although a few facts are presented which relate to the latter.

In Chapter II, the several definitions of dimension are presented and some theorems are proved which apply to arbitrary topological spaces. There are some results which describe dimension-type properties, and some facts are proved which are designed to simplify the proofs of some of the later theorems. One theorem is proved which gives a characterization of one of the dimension functions for arbitrary topological spaces. In the second section of Chapter II, zero dimensional spaces are considered. Most of the results of that section are facts which have not been generalized to n-dimensional spaces for positive integers, n.

Chapter III is devoted to the dimension of normal spaces. Following a brief account of some needed facts about simplicial complexes, there is a proof of the equivalence of three of the definitions of dimension for normal spaces. It is also shown that one of the properties introduced earlier can be reformulated in simplier terms for normal spaces, and one theorem about a typical dimension-type property is proved.

Tychonoff spaces are taken up in Chapter IV. One of the dimension definitions introduced in Chapter II is almost meaningless in non-normal spaces, and is only briefly mentioned in Chapter IV. Another of the definitions is modified in a way which is shown not to affect the results for normal spaces, but to be more suitable to working in non-normal spaces. The principal results of the chapter are obtained by comparing the dimension of a Tychonoff space with the dimension of its Stone-Čech Compactification, which is a normal Hausdorff space. At the end of the chapter, an example is presented which partially answers questions about improving the results which have been proved.

CHAPTER II

PRELIMINARY RESULTS AND DIMENSION ZERO

The first section of this chapter is devoted to definitions and preliminary results designed to simplify the proofs of some of the theorems to be encountered in later chapters. There are also some theorems at the end of the section which describe dimension properties for arbitrary topological spaces. The second section gives a description of the properties of zero dimensional spaces.

1. Definitions and Preliminary Results

Definition 2.1 The n-cell, I^n, is the set of points in E_n with norm less than or equal to one.

Definition 2.2 The n-1 sphere, S^{n-1}, is the set of points in E_n with norm equal to one.

Definition 2.3 If U is an open cover of a topological space, X, then a refinement of U is a collection of open sets, V, such that V covers X, and for each v in V there is a set u in U such that v is contained in u.

The principal subject of this paper is the relationships between the three definitions of dimension which follow. The definitions will be stated in a form applicable to arbitrary topological spaces. The term mapping (or map) will mean continuous function. X will always denote a topological space.

<u>Definition 2.4</u> The <u>covering dimension of X</u>, cov X, is defined to be minus one if and only if X is empty. If X is non-empty, and n is a non-negative integer, then cov $X \leq n$ means that every finite open cover of X has a finite refinement of order $\leq n$, where the order of a cover is the maximum integer, k, so that some k+1 sets in the cover have non-empty intersection.

<u>Definition 2.5</u> The <u>stability dimension of X</u>, St X, is defined to be minus one if and only if X is empty. If X is non-empty, and n is a non-negative integer, then St $X \leq n$ means that for each map f on X into I^{n+1}, for each point, y, in I^{n+1}, and for each $\varepsilon > 0$, there is a map g on X into I^{n+1}, such that $||f-g|| < \varepsilon$, and y is not in the range of g, where $||f-g|| = \sup\{||f(x)-g(x)||: x \text{ is in } X\}$.

<u>Definition 2.6</u> The <u>extension dimension of X</u>, Ext X, is defined to be minus one if and only if X is empty. If X is non-empty, and n is a non-negative integer, then Ext $X \leq n$ means that for each closed subset C of X and for each mapping f on C into S^n, there is a mapping g on X into S^n such that $g|C = f$, that is, f can be extended over all of X.

For each of the definitions, the dimension of X is said to be equal to n if it is true that dimension $X \leq n$, but false that dimension $X \leq n-1$. Dimension $X = \infty$, means for each n, it is false that dimension $X \leq n$. A word of caution is in order. It is obvious that cov $X \leq n$ implies cov $X \leq n+1$, however, the corresponding proposition about St X and Ext X is not so evident. This point will be clarified in the theorems of this chapter and the next.

In proving that St $X \leq n$, it is sufficient to show that for each map f on X into I^{n+1}, and for each $\varepsilon > 0$, there is a map g on X into

I^{n+1} such that $||f-g|| < \varepsilon$ and zero is not in the range of g. To see this, suppose y is in I^{n+1}. If $||y|| = 1$, then $g = (1-\varepsilon)f$ is an ε-approximation to f which misses y. If $||y|| < 1$, let h be a homeomorphism of I^{n+1} onto I^{n+1} which takes y onto zero. Then since h^{-1} is uniformly continuous, by constructing some δ-approximation to hf which missed zero, one could obtain an ε-approximation to f which missed y.

The definition of extension dimension is stated in a form which is apparently applicable to arbitrary topological spaces, however, that appearance is somewhat misleading. The usefulness of the extension dimension concept is closely related to the hypothesis of normality. Suppose C and D are disjoint closed subsets of X, and let n be a non-negative integer. Let p and q be any two distinct points in S^n. Define a function f on $C \cup D$ by letting $f(x) = p$ for all x in C, and letting $f(x) = q$ for all x in D. Then f is a continuous function on $C \cup D$ into S^n. If f could be extended over all of X, and if U and V were disjoint neighborhoods of p and q, respectively, then $f^{-1}\{U\}$ and $f^{-1}\{V\}$ would be disjoint neighborhoods of C and D respectively. We thus have the following:

Remark: If n is any non-negative integer, and $\text{Ext } X \leq n$, then X is normal.

There is another standard definition of dimension which will be mentioned here. It has been used by Hurewicz and Wallman (7) in their study of the dimension of separable metric spaces. They show that this latter definition is equivalent to the three stated previously for separable metric spaces, and by using all four properties, and others, they obtain a much richer theory than can be proved under the hypotheses to be employed in most of this paper.

Definition 2.7 The inductive dimension of X, dim X, is minus one if and only if X is empty. If X is non-empty, and n is a non-negative integer, then assuming that dim X $\leq$ k has been defined for $-1 \leq k \leq n-1$, dim X $\leq$ n means that for each x in X, and for each neighborhood U of x, there is a neighborhood V of x, such that dim(boundary V) $\leq n-1$, and $V \subset U$.

The principal results of this paper are directed towards showing the equivalence of the definitions under weaker hypotheses than separable metric. Most of the results deal with with normal and Tychonoff spaces, and can not be extended to include inductive dimension. C. H. Dowker, (3) has presented an example of a normal Hausdorff space for which dim X = 0, and cov X = 1. Dowker also refers to an example of a compact normal Hausdorff space, given by O. V. Lokucievskii, with cov X=1 and dim X=2. Prabir Roy (9) has constructed an example of a metric space for which dim X $\neq$ cov X. There will, however, be some results about inductive dimension presented in the second section of this chapter, where zero dimensional spaces are considered. The remainder of this section is devoted to some dimension properties which apply to arbitrary topological spaces.

Theorem 2.8 For arbitrary topological spaces, St X $\leq$ n implies St X $\leq$ n+1, and cov X $\leq$ n implies cov X $\leq$ n+1.

Proof: For covering dimension, the result is obvious from the definition. Let f map X into I^{n+2}, let y be in I^{n+2}, and let $\varepsilon > 0$ be given. $I^{n+2} = I^1 \times I^{n+1}$. Let p_2 be the projection from I^{n+2} into I^{n+1}, then

$p_2 \circ f$ is a continuous function on X into I^{n+1}. Since St X $\leq$ n, there exists a mapping g on X into I^{n+1}, such that $||p_2 \circ f - g|| < \varepsilon$, and $p_2(y)$ is not in the range of g. Define a mapping h on X into I^{n+2} by letting the first coordinate of h agree with the first coordinate of f, and letting the 2^{nd} through $n+2^{nd}$ coordinates of h be the coordinates of g. Then h is continuous, y is not in the range of h, and $||h - f|| = ||p_2 \circ f - g|| < \varepsilon$. ▯

Suppose that U is a finite open cover of X by n sets, $u_1, \ldots, u_n$, that has a finite open refinement W. If the order of W is less than or equal to m, then a refinement $V = \{v_1, \ldots, v_n\}$ can be obtained with the order of $V \leq m$, and $v_i \subset u_i$ for $1 \leq i \leq n$. For $1 \leq k \leq n$ let v_k be the union of all members of W which are subsets of u_k but not subsets of u_j for any $j < k$. Then $V = \{v_1, \ldots, v_n\}$ is an open cover of X and the order of V is $\leq$ m. Henceforth, a refinement of a cover U will always be assumed to have the same number of elements as U, except that the empty set may be counted more than once in the refinement. The next theorem is designed to simplify many of the proofs involving covering dimension (or in Chapter IV, Z-covering dimension).

Theorem 2.9 If X is an arbitrary topological space, n and k are non-negative integers, $k \geq n+2$, and every open cover of X by k sets has a refinement of order less than or equal to n, then every open cover of X by k+1 sets has a refinement of order less than or equal to n.

Proof: Let $\mathcal{U}^{k+1} = \{U_1^{k+1}, \ldots, U_{k+1}^{k+1}\}$ be an open cover of X. Let $\mathcal{U}^{k+1'} = \{U_1^{k+1}, \ldots, U_{k-1}^{k+1}, U_k^{k+1} \quad U_{k+1}^{k+1}\}$. $\mathcal{U}^{k+1'}$ is an open cover of X by k sets, and thus has a refinement of order $\leq$ n. Let $\mathcal{V} = \{V_1, \ldots, V_k\}$

be such a refinement, and suppose $V_i \subset U_i^{k+1}$ for $1 \leq i \leq k-1$, and $V_k \subset U_{k+1}^{k+1}$. Define an open cover, $\mathcal{U}^k = \{U_i^k : 1 \leq i \leq k+1\}$ by letting $U_i^k = V_i$ for $1 \leq i \leq k-1$, $U_k^k = V_k \cap U_k^{k+1}$, and $U_{k+1}^k = V_k \cap U_{k+1}^{k+1}$. Because order $\mathcal{V} \leq n$, if a point is in n+2 members of $\mathcal{U}^k$, it must be in $V_k \cap U_k^{k+1}$ and also in $V_k \cap U_{k+1}^{k+1}$. Assume a collection of finite open covers of X have been defined as $\mathcal{U}^i$, for $j \leq i \leq k$, such that $\mathcal{U}^i$ refines $\mathcal{U}^{i+1}$, each $\mathcal{U}^i$ is a cover of X by k+1 open sets $\{U_1^i, \ldots, U_{k+1}^i\}$, and if any point is common to n+2 members of $\mathcal{U}^i$ and also is in U_i^i, then it is in $\bigcap_{m=i}^{k} V_m$. Let $\mathcal{U}^{j'} = \{U_1^j, \ldots, U_{j-2}^j, U_{j-1}^j \cup U_j^j, U_{j+1}^j, \ldots, U_{k+1}^j\}$. $\mathcal{U}^{j'}$ is an open cover of X by k sets and thus has a refinement, $\mathcal{V}^j$, of order $\leq n$, assume $\mathcal{V}^j = \{V_1^j, \ldots, V_{j-1}^j, V_{j+1}^j, \ldots, V_{k+1}^j\}$ with $V_i^j \subset U_i^j$ for $i \neq j, j-1$, and $V_{j-1}^j \subsetneq U_{j-1}^j \cup U_j^j$. Let $U_i^{j-1} = V_i^j$ for $i \neq j, j-1$, let $U_{j-1}^{j-1} = V_{j-1}^j \cap U_{j-1}^j$ and $U_j^{j-1} = V_{j-1}^j \cap U_j^j$. Let $\mathcal{U}^{j-1} = \{U_i^{j-1} : 1 \leq i \leq k+1\}$. Then $\mathcal{U}^{j-1}$ refines $\mathcal{U}^j$. Because the order of $\mathcal{V}^j$ is $\leq n$, it is true that if a point is in n+2 members of $\mathcal{U}^{j-1}$, then it must be in U_{j-1}^{j-1} and also in U_j^{j-1}. Now, $U_{j-1}^{j-1} \subset U_{j-1}^j \subset U_{j-1}^k = V_{j-1}$, and $U_j^{j-1} \subset U_j^j$, thus if a point is in n+2 members of $\mathcal{U}^{j-1}$, then it is in the corresponding n+2 members of $\mathcal{U}^j$, and therefore in $\bigcap_{i=j-1}^{k} V_i$. Thus the definition of the covers $\mathcal{U}^i$ can be extended inductively for $1 \le i \le k$ if it is true that each refinement $\mathcal{U}^i$, has order greater than n. If that were the case, then a cover $\mathcal{U}^1$ would be determined which refines $\mathcal{U}^{k+1}$, with the property that if a point is in n+2 members of $\mathcal{U}^1$, then it would be in $\bigcap_{i=1}^{k} V_i$, but $k \ge n+2$ and order $\mathcal{V} \le n$. This is a contradiction, hence for some j, $1 \le j \le k$, it must be true that the order of $\mathcal{U}^j$ is less than or equal to n. □

<u>Theorem 2.10</u> If X is any topological space, then cov $X \le n$ if and only if every open cover of X by n+2 sets has a refinement of order $\le n$.

Proof: Follows from the preceding. □

<u>Theorem 2.11</u> If X is an arbitrary topological space and C is a closed subset of X, then Ext $X \le n$ implies Ext $C \le n$, and cov $X \le n$ implies cov $C \le n$. (The corresponding statement about stability dimension is proved in the next chapter under the hypothesis of normality.)

Proof: For extension dimension the proposition is obvious . Suppose cov $X \le n$. Let $\mathcal{U} = \{U_i : 1 \le i \le k\}$ be an open cover of C. For each U_i there is an open set in X, V_i, so that $U_i = V_i \cap C$. The sets X–C, V_1,

..., V_k form an open cover of X for which there is a refinement of order $\leq n$. Let W_0, ..., W_k be the sets in the refinement, with $W_0 \subset X-C$, and $W_i \subset V_i$ for $1 \leq i \leq k$. The sets $W_i \cap C$, $1 \leq i \leq k$ form an open cover of C of order $\leq n$ which refines $\mathcal{U}$. $\square$

The definitions and theorem which come next give an alternative characterization of stability dimension in arbitrary spaces. The terminology used here is the same as that used by Gillman and Jerison (4). The property P_n described below is a generalization of another property which is proved in Chapter III to be equivalent to cov $X \leq n$, Ext $X \leq n$, and St $X \leq n$ for normal spaces. (theorem 3.27, definition 3.26) The property in Chapter III is shown by Hurewicz and Wallman to characterize the dimension of separable metric spaces.

Definition 2.12 If $A \subset X$, then A is a zero set (of X) if and only if there is a continuous real-valued function f defined on X, such that $A = f^{-1}\{0\}$.

Definition 2.13 Two subsets, B_1 and B_2 are said to be completely separated in X if and only if there is a continuous real-valued function f mapping X into $[0,1]$, with $B_1 \subset f^{-1}\{0\}$ and $B_2 \subset f^{-1}\{1\}$.

In the definition of a zero set, $A = f^{-1}\{0\}$ could be replaced by $A = f^{-1}[C]$ for any closed interval C. In the definition of completely separated sets, the numbers 0 and 1 could be replaced by any two distinct real numbers.

Lemma 2.14 Two sets, B_1 and B_2, are completely separated if and only if they are contained in disjoint zero sets.

Proof: Suppose f maps X into $[0,1]$ with $B_1 \subset f^{-1}\{0\}$ and $B_2 \subset f^{-1}\{1\}$.

Let $g(x) = 1-f(x)$, then $B_1 \subset f^{-1}\, 0$, $B_2 \subset g^{-1}\, 0$, and $f^{-1}\{0\} \cap g^{-1}\{0\} = \emptyset$.

Suppose $B_1 \subset f^{-1}\{0\}$, $B_2 \subset g^{-1}\{0\}$, and $f^{-1}\{0\} \cap g^{-1}\{0\} = \emptyset$. Let $h(x) = f(x)/(|f(x)|+|g(x)|)$, and let $k(x) = (1 \wedge h(x) \vee (0)$. Then $k^{-1}\{1\} \supset g^{-1}\{0\}$ and $f^{-1}\{0\} \subset k^{-1}\{0\}$. □

It is proved in Chapter I of (4) that a finite union of zero sets is a zero set, and a countable intersection of zero sets is again a zero set.

<u>Definition 2.15</u> If X is a topological space, then X has property P_n, n a non-negative integer, if and only if for any n+1 pairs of completely separated sets, B_i and B_i', $1 \leq i \leq n+1$, there exist n+1 zero sets C_i, $1 \leq i \leq n+1$, such that B_i and B_i' are separated in $X-C_i$, and $\bigcap\{C_i : 1 \leq i \leq n+1\} = \emptyset$.

<u>Theorem 2.16</u> If X is any non-empty topological space, then St $X \leq n$ if and only if X has property P_n.

Proof: Suppose St $X \leq n$. Let B_i, B_i' be n+1 pairs of completely separated sets. For each i, $1 \leq i \leq n+1$, there exist a mapping f_i on X into $[-1,1]$ such that $B_i' \subset f_i^{-1}\{-1\}$ and $B_i \subset f_i^{-1}\{1\}$. Let $f: X \rightarrow I^{n+1}$ be defined by: $f(x) = (f_1(x), \ldots, f_{n+1}(x))$. Since St $X \leq n$, there is a mapping, g, on X into I^{n+1} with $||f-g|| < 1/4$ and zero not in the range of g. Suppose $g(x) = (g_1(x), \ldots, g_{n+1}(x))$, then each g_i is continuous. Let $C_i = g_i^{-1}\{0\}$ for $1 \leq i \leq n+1$. B_i and B_i' are separated in $X-C_i$, and

$\bigcap\{C_i : 1 \leq i \leq n+1\} = \emptyset$.

Suppose X has property P_n. Let f be a mapping on X into I^{n+1}, let $\varepsilon_0 > 0$ be given. An ε_0-approximation to f will be constructed which misses zero. Let $\varepsilon = \dfrac{\varepsilon_0}{2(n+1)}$. Let $B_i = f^{-1}\{x\varepsilon I^{n+1} : x_i \geq \varepsilon\}$, $B_i' = f^{-1}\{x\varepsilon I^{n+1} : x_i \leq -\varepsilon\}$, then B_i and B_i' are completely separated in X for $1 \leq i \leq n+1$. Since X has property P_n, there exist n+1 zero sets of X, C_i, such that B_i and B_i' are separated in $X-C_i$, and $\bigcap\{C_i : 1 \leq i \leq n+1\} = \emptyset$. There exist n+1 pairs of disjoint open sets, U_i and U_i', such that $B_i \subset U_i$, $B_i' \subset U_i'$, and $U_i \cup U_i' = X-C_i$. $B_1 \cup B_1' \cup \left(\bigcap_{i=2}^{n+1} C_i\right)$ is a zero set disjoint from C_1, hence there exists a map h_1 on X into $[0,\varepsilon]$ with $C_1 \subset h_1^{-1}\{0\}$ and $B_1 \cup B_1' \cup \left(\bigcap_{i=2}^{n+1} C_i\right) \subset h_1^{-1}\{\varepsilon\}$. Define $g_1 : X \to [-\varepsilon,\varepsilon]$ by $g_1|U_1 = h_1|U_1$, $g_1|U_1' = -h_1|U_1'$ and $g_1|C_1 = h_1|C_1 = 0$. Then g_1 is continuous on X, for if $p\varepsilon U_1$, or U_1', there is a neighborhood of p contained in U_1 or U_1' respectively, and if $p\varepsilon C$ then there is a neighborhood of p such that inside of that neighborhood, $|g_1| = h_1$ is arbitrarily close to zero $= h_1(p) = g_1(p)$. Let $E_1 = h_1^{-1}\{0\}$, then E_1 is a zero set of X, B_1 and B_1' are separated in $X-E_1$, and $E_1 \cap \bigcap_{i=2}^{n+1} C_i = \emptyset$. Suppose that $g_1, \ldots, g_k$ and $E_1, \ldots, E_k$ have been defined for $1 \leq k \leq n$ with g_j mapping X into $[-\varepsilon,\varepsilon]$,

$B_j \subset g_j^{-1}(0,\varepsilon]$, $B_j' \subset g_j^{-1}[-\varepsilon,0)$, $E_j = g_j^{-1}\{0\}$, and $\bigcap_{j=1}^{k} E_j \cap \bigcap_{i=k+1}^{n+1} C_i = \emptyset$. Then $B_{k+1} \cup B_{k+1}' \cup \left(\bigcap_{j=1}^{k} E_j \cap \bigcap_{i=k+2}^{n+1} C_i\right)$ is a zero set disjoint from C_{k+1}. There exists a map h_{k+1} on X into $[0,\varepsilon]$ with $C_{k+1} \subset h_{k+1}^{-1}\{0\}$ and $B_{k+1} \cup B_{k+1}' \cup \left(\bigcap_{j=1}^{k} E_j \cap \bigcap_{i=k+2}^{n+1} C_i\right) \subset h_{k+1}^{-1}\{\varepsilon\}$. Define g_{k+1} on X by: $g_{k+1}|U_{k+1} = h_{K+1}|U_{k+1}$, $g_{k+1}|U_{k+1}' = -h_{k+1}|U_{k+1}'$ and $g_{k+1}|C_{k+1} = 0$. Then g_{k+1} is continuous and has the properties listed above for each g_j. Let $E_{k+1} = g_{k+1}^{-1}\{0\}$ then E_{k+1} has the properties listed for each E_j. Assume that g_j and E_j have been defined inductively in this fashion for $1 \leq j \leq n+1$. Define g mapping X into I^{n+1} by letting the i^{th} component of g(x) equal the i^{th} component of f(x) if the absolute value of the i^{th} component of f(x) is $\geq \varepsilon$, and letting the i^{th} component of g(x) equal $g_i(x)$ if the absolute value of the i^{th} component of f(x) is $\leq \varepsilon$. Then g is continuous on X into I^{n+1}, $||f-g|| < 2(n+1)\varepsilon = \varepsilon_0$, and zero is not in the range of g, since $\bigcap_{i=1}^{n+1} E_i = \emptyset$. □

2. Dimension Zero

Theorem 2.17 For an arbitrary topological space, X, the following four properties are equivalent.

1. Ext X = 0
2. cov X = 0

3. Any two disjoint closed subsets are separated in X.

4. St X = 0, and X is normal.

Proof: 1 implies 2. Suppose O_1 and O_2 are open and $O_1 \cup O_2 = X$. Let $C_1 = X-O_1$, $C_2 = X-O_2$, then $C_1 \cup C_2$ is closed and if h has value 1 everywhere on C_1, and value -1 everywhere on C_2, then h is continuous on $C_1 \cup C_2$. If Ext X = 0, then h can be extended to a continuous function on X into S^0. Let the extension be f, then the pair of open and closed sets, $X - f^{-1}\{1\}$, and $X - f^{-1}\{-1\}$, covers X and is a refinement of order zero. Thus cov X = 0.

2 implies 3. Let C_1 and C_2 be disjoint closed sets in X. $X-C_1$, $X-C_2$, form an open cover of X. Let $V_1 \subset X - C_1$, $V_2 \subset X - C_2$, be a refinement of order zero, then $C_1 \subset V_2$, and $C_2 \subset V_1$, so C_1 and C_2 are separated.

3 implies 4. It is clear that 3 implies normality. Let f be a map of X into I^1, let $\varepsilon > 0$ be given. Let $C_1 = \{x \text{ in } X : f(x) \geq \frac{\varepsilon}{2}\}$, let $C_2 = \{x \text{ in } X : f(x) \leq -\frac{\varepsilon}{2}\}$. C_1 and C_2 are disjoint closed subsets of X, and hence can be separated. Suppose U and V are disjoint open sets with $C_1 \subset U$, and $C_2 \subset V$ and $U \cup V = X$. Let $g(x) = f(x) \vee \frac{\varepsilon}{2}$ if x is in U, $g(x) = f(x) \wedge -\frac{\varepsilon}{2}$ if x is in V, then g is continuous, $||f-g|| < \varepsilon$, and zero is not in the range of g.

4 implies 1. Suppose C is a closed subset of X, and f maps C into S^0, then $D = f^{-1}\{-1\}$, and $E = f^{-1}\{1\}$ are disjoint closed sets. Since X is normal, there exists a continuous function, h, mapping X into $[-1,1]$ with $D \subset h^{-1}\{-1\}$, and $E \subset h^{-1}\{1\}$. Since St X = 0, there exists a continuous function g mapping X into $[-1,1]$ with $||h-g|| < \frac{1}{2}$, and zero not in the range of g. Let $F(x) = -1$, if $g(x) < 0$, and $F(x) = 1$

if $g(x) > 0$. Then F is a continuous extension of f over X into S^0. □

Theorem 2.18 If X is T_1, and satisfies any of the conditions of theorem 2.17, then dim X = 0.

Proof: If X satisfies any of the conditions of theorem 2.17, then cov X = 0. Let x be any point in X, and let U be a neighborhood of x. Then U and $X - \{x\}$ form an open cover of X. Let V, W be a refinement of order zero, with $V \subset U$, and $W \subset X - \{x\}$. Then V is both open and closed, thus V has empty boundary, dim(boundary V) = -1, and x is in V. □

Definition 2.19 A topological space is said to be a Lindelöf space if every open cover has a countable subcover.

Theorem 2.20 If X is a Lindelof space, and dim X = 0, then X satisfies all the properties in theorem 2.17.

Proof: It is convenient to show that cov X = 0. Let U and V be open sets which cover X. Since dim X = 0, for each x in X there is a neighborhood of x with empty boundary contained in one of the sets U or V. Since X is a Lindelöf space, a countable subcollection of these neighborhoods cover X. Suppose $\{O_i : i=1, 2, \ldots\}$ covers X, each O_i is contained in either U or V, and each O_i has empty boundary, hence is both open and closed. Let $W_1 = O_1$, and for $n > 1$, let $W_n = O_n - \bigcap_{i=1}^{n-1} O_i$, then each W_n is both open and closed, and the sets W_n are pairwise disjoint. Let $U' = \bigcup \{W_n : W_n \subset U\}$, and let $V' = \bigcup \{W_n : W_n \subset V$ and $W_n \not\subset U\}$. U'and V' form a refinement of order zero of the cover by

U and V, hence cov X = 0. □

<u>Definition 2.21</u> A topological space, X, is said to be <u>completely regular</u> if for each x in X, and for each neighborhood U of x there is a continuous function f on X into [0,1] such that f(x) = 0, and f has value 1 everywhere on the complement of U.

<u>Theorem 2.22</u> If X is completely regular, and St X = 0, then dim X = 0.

Proof: Let x be any point of X, and let U be a neighborhood of x. Since X is completely regular, there exists a continuous function f mapping X into [0,1] such that f(x) = 0, and f has value one on the complement of U. Since St X = 0, there exists a continuous function g on X into [-1,1] such that $||f-g|| < \frac{1}{4}$, and $\frac{1}{2}$ is not in the range of g. Let V = $g^{-1}[-1,\frac{1}{2}]$, then V has empty boundary and is a neighborhood of x contained in U. □

<u>Theorem 2.23</u> If X is a topological space with only a countable number of points, then St X = 0.

Proof: Let f be a mapping of X into [-1,1]. Let $\varepsilon > 0$ be given and suppose y is in [-1,1]. The range of f is countable, so there is a point, c, in I^1 such that $|y-c| < \varepsilon$, and c is not in the range of f. Define g, mapping X into I^1 as follows:

if $-1 \leq f(x) \leq y - \varepsilon$, let $g(x) = f(x)$;

if $y - \varepsilon \leq f(x) \leq y+\varepsilon$, let $g(x) = (y-\varepsilon) \vee (-1)$ if $c > f(x)$,

let $g(x) = (y+\varepsilon) \wedge (1)$ if $c < f(x)$;

if $y + \varepsilon \leq f(x) \leq 1$, let $g(x) = f(x)$. Then g is continuous, $||f-g|| < 2\varepsilon$, and y is not in the range of g. □

The corresponding statement about covering dimension, inductive dimension, and extension dimension is false, as is seen in the following example.

Example 2.24 A countable Hausdorff space with cov X = dim X = 1, Ext X $= \infty$.

Let $I = \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots, 0\}$. Let X = I x I. Let $X - \{(0,0)\}$ have the relativized topology of the plane. Let a base for the neighborhood system of (0,0) be the family of sets of the form $\{(0,0)\} \cup V_{n,m}$, where $V_{nm} = X \cap ((0,1/n] \times [0,1/m])$. X is then a Hausdorff, non-regular space. To see that X is not regular, note that every closed neighborhood of (0,0) contains points of the form (0,1/p), which can not be in any of the sets which form the base for the neighborhood system of (0,0). Each point in X different from (0,0) has "arbitrarily small" neighborhoods with empty boundary. If U is a neighborhood of (0,0), there is a neighborhood of (0,0), V, of the form $\{(0,0)\} \cup V_{n,m}$ contained in U. Then the boundary of V is the set of points $\{(0, \frac{1}{p}): p \geq m\}$. Relative to the boundary of V, each point in boundary of V has arbitrarily small neighborhoods with empty boundary, thus dim(boundary V) = 0, and hence dim X = 1.

To see that cov X = 1, let $\{U_1, U_2, U_3\}$, be a finite open cover of X. Assume $(0,0) \varepsilon U_1$, then there is a set V, as described above, with $(0,0) \varepsilon V \subset U_1$, and $\bar{V} = X \cap ([0, \frac{1}{n}] \times [0, \frac{1}{m}])$ is both open and closed in X. Suppose $x \varepsilon X$ is a point of the form $(0, \frac{1}{k})$ or $(\frac{1}{k}, 0)$, for each such point, there is an "interval," I_x, of the form $\{\frac{1}{p}, \ldots, 0\} \times \{\frac{1}{k}\}$ with $p > n$ or else of the form $\{\frac{1}{k}\} \times \{\frac{1}{p}, \ldots, 0\}$ with $p > m$. which is an open neighborhood of x, and can be chosen to lie inside of any given neighborhood of x.

Let $W_1 = V \cup (\bigcup\{I_x : x \text{ is } U_1, I_x \subset U_1\})$

let $W_2 = \bigcup\{I_x : x \text{ is in } U_2 - U_1, I_x \subset U_2\}$

let $W_3 = \bigcup\{I_x : x \text{ is in } U_3 - (U_1 \cup U_2), I_x \subset U_3\}$

Then $W_i \subset U_i$ for i = 1, 2, 3, and $W_2 \cap W_3 = \emptyset$, so the order of $\{W_1, W_2, W_3\}$ is less than or equal to one. $X - (W_1 \cup W_2 \cup W_3)$ is finite, and for each x not in $W_1 \cup W_2 \cup W_3$, $\{x\}$ is both open and closed. Let those singletons be labled $W_4, \ldots, W_k$. Then the refinement $\{W_1, \ldots, W_k\}$ covers X and has order less than or equal to one. Thus cov $X \leq 1$.

Since X is Hausdorff, but not regular, it follows that X is not normal, hence Ext $X = \infty$, and it is false that cov $X \leq 0$, so cov $X = 1$.

CHAPTER III

NORMAL SPACES

1. Preliminary Definitions and Theorems

Definition 3.1 A subset of X is a cozero set in X if and only if its complement is a zero set.

Theorem 3.2 If X is normal and $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a finite open cover of X, then there is a refinement of $\mathcal{U}$, $\mathcal{V} = \{V_1, \ldots, V_n\}$, such that $\bar{V}_i \subset U_i$ for $1 \leq i \leq n$.

Proof: Let $W_1 = X - \bigcup_{i=2}^{n} U_i$, $W_1' = X - U_1$, then W_1 and W_1' are disjoint closed sets. Since X is normal, there exist disjoint open sets, O_1 and O_1' with $W_1 \subset O_1$, and $W_1' \subset O_1'$. Let $V_1 = O_1$, then $\bar{V}_1 \subset \bar{O}_1 \subset X - O_1' \subset X - W_1' = U_1$, and $X - V_1 = X - O_1 \subset X - W_1 = \bigcup_{i=2}^{n} U_i$, so $\{V_1, U_2, \ldots, U_n\}$ covers X.

Assume $V_1, \ldots, V_k$ have defined with $\bar{V}_i \subset U_i$ and so that $\{V_1, \ldots, V_k, U_{k+1}, \ldots, U_n\}$ is an open cover of X. Let $W_{k+1} = X - [(\bigcup_{i=1}^{k} V_i) \bigcup (\bigcup_{i=k+2}^{n} U_i)]$, $W_{k+1}' = X - U_{k+1}$, then W_{k+1} and W_{k+1}' are disjoint closed sets, thus there exist disjoint open sets O_{k+1} and O_{k+1}' with $W_{k+1} \subset O_{k+1}$ and $W_{k+1}' \subset O_{k+1}'$. Let $V_{k+1} = O_{k+1}$, then

$\bar{V}_{k+1} \subset X - O'_{k+1} \subset X - W'_{k+1} = U_{k+1}$, and $X - V_{k+1} = X - O_{k+1} \subset X - W_{k+1}$ $= [(\bigcup_{i=1}^{k} V_i) \bigcup (\bigcup_{i=k+2}^{k} U_i)]$, so the sets $V_1, \ldots, V_{k+1}, U_{k+2}, \ldots, U_n$ cover X. If this process is continued for $1 \leq k \leq n-1$, the desired refinement is obtained. □

Theorem 3.3 The sets V_i in the preceding theorem can be taken as co-zero sets.

Proof: Let $\{U_1, \ldots, U_n\}$ be an open cover of X, suppose the cover $\{W_1, \ldots, W_n\}$ refines the first cover, and the cover $\{O1, \ldots, O_n\}$ refines $\{W_1, \ldots, W_n\}$ with $\bar{O}_i \subset W_i$ and $\bar{W}_i \subset U_i$ for $1 \leq i \leq n$. $\bar{O}_i$ and $X - W_i$ are disjoint closed subsets of X so by Urysohn's Lemma, there is a map, f_i, on X into $[0,1]$ with $\bar{O}_i \subset f_i^{-1}\{1\}$, and $X - W_i \subset f_i^{-1}\{0\}$. Let $V_i = X - f_i^{-1}\{0\}$. Then $X = \bigcup_{i=1}^{n} O_i \subset \bigcup_{i=1}^{n} V_i$, so $V_1, \ldots, V_n$ is a cover by cozero sets, and $\bar{V}_i \subset \bar{W}_i \subset U_i$. □

Theorem 3.4 (Tietze) If X is a normal space, C is a closed subset of X, and f is a continuous function on C into $[-1,1]$, then there exists a continuous function g on X into $[-1,1]$ so that $g|C=f$.

The proof is omitted. Hurewicz and Wallman (7), Chapter VI, give a proof of this theorem for separable metric spaces which can be modified using Urysohn's lemma, to apply to normal spaces, (see also (5) or (8)).

Theorem 3.5 If X is a normal space, C is a closed subset of X, and f is a continuous function on C into I^n, then there exists a continuous function g on X into I^n so that $g|C=f$.

A proof is obtained by applying theorem 3.4 to each component of f.

Theorem 3.6 If X is a normal space, C is a closed subset of X, and f is a continuous function on C into S^n, then there exists a closed neighborhood D of C, and a continuous function g on D into S^n, so that $g|C=f$.

Proof: S^n is a subset of I^{n+1}, so by the preceding theorem, there is a continuous extension h of f, mapping X into I^{n+1} with $h|C=f$. Let D = $\{y \text{ in } I^{n+1}: ||y|| \geq \frac{1}{2}\}$, then D is a closed neighborhood of C. Define g on D by: $g(x)=h(x)/||h(x)||$, then g maps D into S^n. For all x in C, $h(x)=f(x)\varepsilon S^n$, and $||h(x)||=1$, hence, for all x in C, $g(x)=f(x)$. □

2. Simplexes and Simplicial Complexes

In the theorems on the dimension of normal spaces, it is sometimes convenient to employ arguments involving simplexes and simplicial complexes. Only a brief description of the facts to be used in this paper will be offered here. The reader is referred to Eilenberg and Steenrod (4), Chapter II, or Aleksandroff (1) for a detailed explanation of these topics.

Definition 3.7 A set of k+1 points in E_n,$\{x_0, \ldots, x_k\}$, $1 \leq k \leq n$, is said to be in general position if the k vectors, $x_i - x_0 : 1 \leq i \leq k$ are linearly independent.

Definition 3.8 A k-simplex, σ^k, is the convex hull of a set of k+1 points in E_n which are in general position, that is, the intersection of all convex sets containing those k+1 points. The points $x_o, \ldots, x_k$ which determine σ^k are the vertices of σ^k.

Remarks: Any subset of a set in general position is also in general position. If σ^k is a k-simplex with vertices $x_o, \ldots, x_k$, then each point in σ^k has a unique representation of the form

$$x = \sum_{i=0}^{k} t_i x_i \text{ with } t_i \geq 0, \text{ and } \sum_{i=0}^{k} t_i = 1.$$

Definition 3.9 If σ^k is a k-simplex, then a face of σ^k is any simplex whose vertices are a subset of the vertices of σ^k.

Definition 3.10 A simplicial complex in E_n is a finite collection of simplexes, K, such that each face of a simplex in K is also in K, and the intersection of any two simplexes in K is either empty or is a face of both of them.

Definition 3.11 If K is a simplicial complex, then the polyhedron of K, $|K|$, is the union of all the simplexes in K.

Definition 3.12 If K is a simplicial complex, and x is a point in $|K|$, then the carrier of x is the intersection of all simplexes in K which contain x.

Definition 3.13 If K is a simplicial complex, and p is a vertex of a simplex in K, then the star of p, St(p), is the union of all $\{x\}$ in $|K|$ such that p is a vertex of the carrier of x.

Remarks: If K is a complex in E_n, then $|K|$ is a compact subset of E_n, and for each vertex of K, St(p) is an open subset of $|K|$. If s is a simplex in K with vertices $p_o, \ldots, p_n$, and x is a point in s, with representation $x = \sum_{i=0}^{n} t_i p_i$, where each $t_i \geq 0$, and $\sum_{i=0}^{n} t_i = 1$, then x is in $St(p_j)$ if and only if $t_j \neq 0$. The collection consisting of a simplex and all of its faces is a simplicial complex. It will be important in the proof of theorem 3.25 to see that if $p_o, \ldots, p_m$ are vertices in K, then there is a simplex in K with $p_o, \ldots, p_m$ as its vertices if and only if $\bigcap_{i=0}^{m} St(p_i) \neq \emptyset$.

Definition 3.14 Let σ^k be a k-simplex with vertices $x_o, \ldots, x_k$. The barycenter of σ^k is the point in σ^k, $\hat{\sigma}^k = \sum_{i=0}^{k} x_i/k+1$.

Notation: If σ^k is a simplex, and s_1 and s_2 are faces of σ^k, then $s_1 < s_2$ means that s_1 is a proper face of s_2.

Definition 3.15 If K is a simplicial complex in E_n, then the first barycentric subdivision of K, $Sd^1(K)$, is the simplicial complex defined as follows: let a simplex, t, be in $Sd^1(K)$ if and only if there exist $s_o, \ldots, s_q$ in K with $s_o < s_1 < \ldots < s_q$, such that the vertices of t are $\hat{s}_o, \ldots, \hat{s}_q$.

Remark: $Sd^1(K)$ is a simplicial complex, and $|Sd^1(K)| = |K|$.

Definition 3.16 If σ^k is a simplex in E_n, then the diameter of σ^k = $\sup\{||x-y|| : x, y \in \sigma^k\}$.

Definition 3.17 If K is a simplicial complex in E_n, then the mesh of K, $mesh(K) = \max\{\text{diameter of } s : s \text{ is a simplex in } K\}$.

Definition 3.18 If K is a simplicial complex in E_n, with $Sd^1(K)$ as defined above, inductively define $Sd^m(K)=Sd^1(Sd^{m-1}(K))$ for integers $m > 1$.

Remark: If K is a simplicial complex, and $\varepsilon > 0$ is given, there exists a positive integer, n, such that mesh $(Sd^n(K)) < \varepsilon$.

Let K^n represent a simplicial complex which consists of an n-simplex and its faces. Let B^{n-1} be the union of all m-simplexes in K^n with $m < n$. There exists a homeomorphism from I^n onto $|K^n|$ which carries S^{n-1} onto B^{n-1}, and the definitions of extension dimension and stability dimension can be restated in terms of mapping into B^n, or K^{n+1}.

Definition 3.19 A basic cover is a cover consisting of cozero sets.

Theorem 3.20 If X is a normal space, $\mathcal{U} = \{U_0, \ldots, U_n\}$ is a basic cover of X, and σ^n is an n-simplex with vertices $p_0, \ldots, p_n$, then there is a continuous function f on X into σ^n with the property that x is in U_i if and only if f(x) is in $St(p_i)$.

Proof: Since each U_i is a cozero set, there is a continuous function h_i on X into [0,1] so that $h_i^{-1}\{0\}=X-U_i$. For each x in X at least one $h_i(x) \neq 0$, thus $\sum_{i=0}^{n} h_i(x) \neq 0$. Define f on X into σ^n by:

$$f(x) = \frac{\sum_{i=0}^{n} h_i(x)p_i}{\sum_{i=0}^{n} h_i(x)},$$

then f is continuous, and f(x) is in $St(p_i)$ if and only if $h_i(x) \neq 0$, which is the case if and only if x is in U_i. □

3. Equivalence of Dimension Concepts in Normal Spaces

Theorem 3.21 If X is normal and Ext X $\leq$ n, then cov X $\leq$ n.

Proof: Let $\mathcal{U} = \{U_0, \ldots, U_{n+1}\}$ be an open cover of X. By theorem 2.10, it suffices to consider a cover with just n+2 sets. By theorem 3.3, it suffices to assume that the members of $\mathcal{U}$ are cozero sets. Let K be an n+1 simplex, and let B be the union of the proper faces of K. Let $p_o, \ldots, p_{n+1}$ be the vertices of K. By theorem 3.20, there is a continuous function f on X into K, such that a point x in X, is in U_i if and only if f(x) is in the star of p_i. Let $W = \bigcap_{i=0}^{n+1} U_i$, then X-W is a closed subset of X, and f|(X-W) is a continuous function on X-W into B. Because Ext X $\leq$ n, there is a continuous function F on X into B such that F|(X-W)=f|(X-W). Let $V_i = F^{-1}(St(p_i))$ for $0 \leq i \leq n+1$. Then $\{V_o, \ldots, V_{n+1}\}$ is a finite open cover of X. Suppose x is in V_i, then if x is in W, x must be in U_i, if x is not in W, then $F(x)=f(x) \varepsilon St(p_i)$, and so x is in U_i because of the way f was constructed, thus $V_i \subset U_i$ for $0 \leq i \leq n+1$. $F(\bigcap_{i=0}^{n+1} V_i) \subset B \cap (\bigcap_{i=0}^{n+1} St(p_i)) = \emptyset$, hence $\bigcap_{i=0}^{n+1} V_i = \emptyset$, so the order of the refinement is $\leq n$. ∎

Theorem 3.22 If X is normal and St $X \leq n$, then Ext $X \leq n$.

Proof: Let C be a closed subset of X, and let f be a continuous function on C into S^n. $S^n \subset I^{n+1}$ so by theorem 3.5, there is a continuous function h on X into I^{n+1} so that $h|C=f$. Because St $X \leq n$, there is a continuous function g on X into I^{n+1} so that $||h-g|| < \frac{1}{4}$ and zero is not in the range of g. Let $C_1 = g^{-1}\{y\varepsilon I^{n+1}: ||y|| \geq \frac{3}{4}\}$, $C_2 = g^{-1}\{y\varepsilon I^{n+1}: ||y|| \leq \frac{1}{2}\}$, then C_1 and C_2 are disjoint closed sets, and $C \subset C_1$. Since X is normal, there is a continuous function k on X into $[0,1]$ such that $k(x)=1$ for all x in C_1, and $k(x)=0$ for all x in C_2. Let f' be defined on X into I^{n+1} by $f'(x)=k(x)h(x)+(1-k(x))g(x)$. Then f' is continuous, $f'|C=f$, and zero is not in the range of f'. If x is in X, let $f''(x)=f'(x)/||f'(x)||$, then f" is continuous on X into S^n, and for each x in C, $f''(x)=f(x)$. ∎

For the definition of stability dimension in terms of mappings into an n+1 simplex K, one would say that for each map f, for each $\varepsilon > 0$, and for each point y in K, there was an ε-approximation of f which missed y. The proof of the next theorem can be simplified by observing that it is sufficient to take y as the barycenter of K, indeed, if y were in some proper face of K then the approximation could be accomplished by simply shrinking K into itself, and if y were an interior point of K, then K could be mapped homeomorphically onto a smaller simplex, K', contained in K, with y as the barycenter of K'. The homeomorphism, and its inverse would be uniformly continuous. If the homeomorphism were h, then by constructing some δ-approximation to hf which missed y, one could obtain an ε-approximation to f which missed y.

The two lemmas which follow are used in the next theorem. The proofs are easy, and will be omitted.

Lemma 3.23 If K is an n+1 simplex, r is a positive number less than one, and q is the barycenter of K, then the mapping h_1 defined on K by $h_1(x)=q+r(x-q)$, is a homeomorphism, h_1 shrinks K onto an n+1 simplex L which is in the interior of K, $h_1(q)=q$, h_1 maps the vertices of K onto the vertices of L, and q is the barycenter of L. The distance from L to the union of the proper faces of K is positive.

Lemma 3.24 If L is an n+1 simplex, q is the barycenter of L, and r is a positive number, then the translation h_2 defined on L by $h_2(x) = x+rv$, where v is any vector in E_{n+1}, is a homeomorphism, h_2 maps L onto an n+1 simplex L', and takes the vertices of L onto the vertices of L', and the barycenter of L onto the barycenter of L'. For any positive integer m, h_2 takes the simplexes in $Sd^m(L)$ onto the simplexes in $Sd^m(L')$, and the interiors of the former onto the corresponding interiors of the latter.

Theorem 3.25 If X is normal and cov $X \leq n$, then St $X \leq n$.

Proof: Let f be a continuous function on X into an n+1 simplex, K. Let $\varepsilon > 0$ be given, and let q be the barycenter of K. An ε-approximation to f will be constructed which misses q. Let B be the union of the proper faces of K, and let d be the diameter of K. Assume $d > \varepsilon$. Let h_1 be a mapping as described in Lemma 3.23, with $r=1-\varepsilon/3d$. Let L be the range of h_1. Choose a positive integer m such that mesh $(Sd^m(L)) < \varepsilon/3$. Choose an n+1 simplexs in $Sd^m(L)$ which has q as one of its vertices (it is easy to prove by induction that there exists such a simplex), let p be the barycenter of s. Construct a translation h_2 on L as in Lemma 3.24 with $v = \frac{q-p}{||q-p||}$, and $r=\min\{||q-p||$, distance from L to B$\}$, then L', the range of h_2 is a subset of K, and for each point x in K, $||h_2h_1(x)-x||$ is less than $2\varepsilon/3$. Let s' denote the image of s under h_2, then q is in the interior of s', thus q is not in any n-simplex of Sd^mL'. Let $\{p_i : 0 \leq i \leq k\}$ be the vertices of the simplexes in $Sd^m(L')$. For $0 \leq i \leq k$, let $V_i = f^{-1}oh_1^{-1}oh_2^{-1}(St(p_i))$, then $\mathcal{V} = \{V_0, \ldots, V_k\}$ is a finite open cover of X, and thus has a refinement of order $\leq$ n. Let $W_0, \ldots, W_k$ be the sets in the refinement with $W_i \subset V_i$. By theorem 3.3, it may be assumed that the sets $W_0, \ldots, W_k$ are cozero sets. For $0 \leq i \leq k$ let g_i be a continuous function on X into [0,1] such that $g_i^{-1}\{0\} = X-W_i$.

Define g on X into L' by:

$$g(x) = \sum_{i=0}^{k} g_i(x)p_i / \sum_{i=0}^{k} g_i(x).$$

then g is continuous, and since the order of $\{W_0, \ldots, W_k\}$ is less than or equal to n, for each x in X, $g_i(x) \neq 0$ for at most n+1 of the indexes i. If $g_{ij}(x) \neq 0$ for j=1, ..., t, then $\bigcap_{j=1}^{t} St(p_j) \neq \emptyset$ (see remarks following def. 3.13), so g takes each point of X into an n-simplex of $Sd^m(L')$, thus q is not in the range of g. If for any point x in X, g(x) is in the star of p_i, then $g_i(x) \neq 0$, which implies x is in $W_i \subset V_i$, and thus h_2oh_1o f(x) is in the star of p_i. That implies that $||g-h_2oh_1of|| \leq \text{mesh}(Sd^mL')$

$= \text{mesh}(Sd^m L) < \varepsilon/3$. But, as noted above, for each x in K, $||h_2h_1(x) - x|| < \frac{2\varepsilon}{3}$, thus $||g(x)-f(x)|| < \varepsilon$. □

Theorems 3.21, 3.22, 3.25 prove that for normal spaces, cov X = n, Ext X = n, and St X = n are equivalent statements. In Chapter II, it was observed that any space with finite extension dimension is normal, combining this with the results of this chapter, we have, for any space with finite extension dimension the covering dimension and stability dimension are the same as the extension dimension; furthermore, in light of theorem 2.8, for any space, X, Ext X $\leq$ n implies Ext X $\leq$ n+1. The example of C. H. Dowker referred to in Chapter II indicates that inductive dimension can not be included in these equivalences.

In theorem 2.16, an alternative characterization of stability dimension was proved: for arbitrary spaces, St X $\leq$ n if and only if X satisfied property P_n. It follows that for normal spaces property P_n also characterizes cov X and Ext X. There is another property which is similar to P_n that can be used to describe the dimension of normal spaces. It is used by Hurewicz and Wallman (7) in characterizing the dimension of separable metric spaces.

<u>Definition 3.26</u> A topological space is said to have property P_n' if and only if for any n+1 pairs of disjoint closed sets, B_i and B_i', $1 \leq i \leq n+1$, there exist n+1 closed sets C_i, $1 \leq i \leq n+1$, such that $\bigcap_{i=1}^{n+1} C_i = \emptyset$, and B_i and B_i' are separated in $X - C_i$.

<u>Theorem 3.27</u> If X is normal, then X has property P_n if and only if X has property P_n', n a non-negative integer.

Proof: Suppose X satisfies property P_n. Let B_i, B_i', $1 \leq i \leq n+1$ be pairs of disjoint closed sets. By Urysohn's Lemma, in a normal space disjoint closed sets are completely separated, thus since X satisfies property P_n, there are n+1 zero sets C_i, $1 \leq i \leq n+1$, such that $\bigcap_{i=1}^{n+1} C_i = \emptyset$ and for each i, B_i and B_i' are separated in $X-C_i$. Zero sets are always closed, hence X satisfies P_n'.

Suppose X satisfies P_n'. Let B_i, B_i', $1 \leq i \leq n+1$ be pairs of completely separated subsets of X. By Lemma 2.14, there exist n+1 pairs of disjoint zero sets (hence disjoint closed sets) V_i, V_i', $1 \leq i \leq n+1$, such that $B_i \subset V_i$, and $B_i' \subset V_i'$. Since X satisfies property P_n', there exist n+1 closed sets E_i, $1 \leq i \leq n+1$, such that $\bigcap_{i=1}^{n+1} E_i = \emptyset$, and V_i, V_i' are separated in $X-E_i$. Let U_i, U_i' be disjoint open sets for $1 \leq i \leq n+1$, with $V_i \subset U_i$, $V_i' \subset U_i'$, and $U_i \bigcup U_i' = X-E_i$. Let $D_1 = V_1 \bigcup V_1' \bigcup [\bigcap_{i=2}^{n+1} E_i]$, then D_1 and E_1 are disjoint closed sets, and hence, by Urysohn's Lemma, are contained in disjoint zero sets. Let C_1 be a zero set disjoint from D_1 which contains E_1, then $B_1 \subset V_1 \subset U_1 \bigcap (X-C_1)$, and $B_1' \subset V_1' \subset U_1' \bigcap (X-C_1)$, so B_1 and B_1' are separated in $X-C_1$, and $C_1 \bigcap (\bigcap_{i=2}^{n+1} E_i) = \emptyset$. Suppose $C_1, \ldots, C_k$ have been chosen so that each C_i

is a zero set, B_i and B_i' are separated in $X-C_i$, and $(\bigcap_{i=1}^{k} C_i) \bigcap (\bigcap_{i=k+1}^{n+1} E_i)$ $= \emptyset$. Let $D_{k+1}=V_{k+1} \bigcup V_{k+1}' \bigcup [(\bigcap_{i=1}^{k} C_i) \bigcap (\bigcap_{i=k+2}^{n+1} E_i)]$. D_{k+1} and E_{k+1} are disjoint closed sets, so by Urysohn's Lemma, there is a zero set C_{k+1} which contains E_{k+1} and is disjoint from D_{k+1}, then $(\bigcap_{i=1}^{k+1} C_i) \bigcap (\bigcap_{i=k+2}^{n+1} E_i)=\emptyset$, and $B_{k+1} \subset V_{k+1} \subset U_{k+1} \bigcap (X-C_{k+1})$, If this process is continued, each E_i can be replaced by a zero set C_i so that the conditions of property P_n are satisfied. □

Theorem 3.28 If X is normal, and $\{C_i\}$ is a countable collection of closed subsets of X such that $\bigcup_{i=1}^{\infty} C_i=X$, and for each i, Ext $C_i \leq n$, then Ext $X \leq n$.

Proof: Let C be a closed subset of X, and let f be a continuous function on C into S^n. By theorem 3.6, there is a closed neighborhood D_o of C and a continuous extension g_o of f mapping D_o into S_n. Let $f_1 = g_o|(D_o \bigcap C_1)$, then f_1 is a continuous function on a closed subset of C_1 into S^n. Since Ext $C_1 \leq n$, there is a continuous f_1' mapping C_1 into S^n so that $f_1'|(D_o \bigcap C_1)=f_1$. Define a mapping f_1'' on $D_o \bigcup C_1$ into S^n by letting $f_1''(x)=f_1'(x)$ if x is in C_1, and let $f_1''(x)=g_o(x)$ if x is in D_o. Since C_1 and D_o are closed, and f_1' agrees with g_o on $D_o \bigcap C_1$,

f_1'' is then continuous on $C_1 \cup D_o$. By theorem 3.6, there is a closed neighborhood D_1 of $C_1 \cup D_o$ and a continuous g_1, mapping D_1 into S^n such that $g_1|(C_1 \cup D_o)=f_1''$. We continue by induction. Assume that f has been extended continuously over D_k which is a closed neighborhood of $C \cup (\bigcup_{i=1}^{k} C_i)$, the extension is g_k, for each $i \leq k$, D_i is a closed neighborhood of D_{i-1}, and $g_i|D_{i-1}=g_{i-1}$. Let $f_{k+1}=g_k|(D_k \cap C_{k+1})$, then because Ext $C_{k+1} \leq n$, there is a continuous function f_{k+1}' mapping C_{k+1} into S^n, so that $f_{k+1}'|(D_k \cap C_{k+1})=f_{k+1}$. Define f_{k+1}'' on $C_{k+1} \cup D_k$ by letting $f_{k+1}''(x)=f_{k+1}'(x)$ for all x in C_{k+1}, and letting $f_{k+1}''(x)=g_k(x)$ for all x in D_k. Because D_k and C_{k+1} are closed and f_{k+1}' and g_k agree on their intersection, f_{k+1}'' will be continuous on the union. By theorem 3.6, there is a closed neighborhood D_{k+1} of $C_{k+1} \cup D_k$ and a continuous extension g_{k+1} of f_{k+1}'' mapping D_{k+1} into S^n. Then D_{k+1} is a closed neighborhood of D_k, and $g_{k+1}|D_k=g_k$. Let a sequence of functions, $\{g_k\}$, be so defined.

For each x in X, let $n(x)=\min\{k: x \text{ is in } C_k\}$. Let $F(x)=g_{n(x)}(x)$, then F is a function on X into S^n, and $F|C=f$. It remains to show that F is continuous. Let x be any point in X, and let V be a neighborhood of F(x). By the construction of the sequence $\{g_k\}$, $F|D_{n(x)}=g_{n(x)}$, and x is in the interior of $D_{n(x)}$. Since $g_{n(x)}$ is continuous on $D_{n(x)}$, there is a neighborhood U of x in $D_{n(x)}$ so that $g_{n(x)}[U] \subset V$. Since $D_{n(x)}$ is a neighborhood of $C_{n(x)}$, U must also be a neighborhood of x in X, and $F[U] \subset V$, so F is continuous. $\square$

This theorem was proved by A. D. Wallace in (11), using a similar proof.

CHAPTER IV

TYCHONOFF SPACES

1. Equivalence of Dimension Concepts

Definition 4.1 Atopological space is completely regular if and only if for each point x in the space, and for each open set U containing x, there is a continuous function f mapping the space into [0,1] such that f(x)=0 and f has value 1 everywhere on the complement of U.

Definition 4.2 A topological space is a Tychonoff space if and only if it is T_1 and completely regular.

In discussing the dimension of Tychonoff spaces, it is convenient to work with an associated compact Hausdorff space, the Stone-Čech Compactification. The construction of the space and proof of the properties of the space can be described in terms of zero sets and the convergence of ultra-filters. The compactification theorem will be stated below without proof. The reader is referred to (5), (Chapter VI) for a detailed development of the topic.

Definition 4.3 A topological space is said to be compact if for every open cover of the space there is a finite subcover.

Definition 4.4 A compactification of a topological space X is a compact topological space Y such that X is a dense subspace of Y.

Theorem 4.5 (Stone-Čech Compactification Theorem) Every Tychonoff space X has a compactification βX with the following properties:

(1) Every continuous function on X into any compact Hausdorff space Y has a continuous extension on βX into Y.

(2) Every bounded continuous real valued function on X has an extension to a continuous real-valued function on βX.

(3) Any two disjoint zero sets in X have disjoint closures in βX.

(4) For any two zero sets, Z_1 and Z_2, of X, $cl_{\beta X}(Z_1 \cap Z_2) = cl_{\beta X}Z_1 \cap cl_{\beta X}Z_2$. (where $cl_{\beta X}A$ is the closure in βX of A)

Furthermore, βX is essentially unique, in the sense that if a compactification T of X satisfies any one of the listed properties, then it satisfied all of them, and there exists a homeomorphism from βX onto T which leaves X pointwise fixed.

At the end of this chapter, an example is given of a Tychonoff space for which St X=0, cov X=1, Ext X= ∞, so for non-normal spaces, the three dimension functions with which we have been working need not agree. It has also been observed that for arbitrary spaces, cov X=0 implies that X is normal. These facts suggest that the previously considered dimension functions might not be very well suited to working in non-normal spaces. For our work in Tychonoff spaces, the definition of covering dimension will be restated in terms of cozero sets. This will yield a definition of dimension in terms of coverings which will be shown to agree with stability dimension for Tychonoff spaces. It will also be shown that the new statement of the covering dimension definition agrees with the original in normal spaces. The definition of $cov_z X$ (below) is found in (5), where it is called dim X.

<u>Definition 4.6</u> If X is a nonempty topological space, and n is a non-

negative integer, then the Z-covering dimension of X is less than or equal to n, $cov_z X \leq n$, if and only if every finite basic cover of X has a finite basic refinement of order less than or equal to n. (see definition 2.19)

In theorem 2.10 it was proved that for arbitrary spaces, X, the covering dimension is less than or equal to n if and only if every open cover by n+2 sets has a refinement of order less than or equal to n. It may be seen that the proof of theorem 2.9 is almost entirely set-theoretic in nature. The only facts of a topological nature involved are that the intersection or union of two open sets is open and, as these statements apply also to cozero sets, the proof of theorem 2.10 could be modified to yield the corresponding theorem about Z-covering dimension:

Theorem 4.7 If X is any topological space, then $cov_z X \leq n$ if and only if every basic cover by n+2 sets has a finite basic refinement of order $\leq n$.

Theorem 4.8 If X is normal, then $cov X \leq n$ if and only if $cov_z X \leq n$.

Proof: Suppose $cov_z X \leq n$. Let $\mathcal{U}$ be a finite open cover of X. By theorem 3.3 $\mathcal{U}$ has a finite basic refinement, $\mathcal{V}$. Since $cov_z X \leq n$, $\mathcal{V}$ has a finite basic refinement of order $\leq n$. Since cozero sets are open, this yields a finite open refinement of $\mathcal{U}$ of order $\leq n$. Thus $cov X \leq n$.

Suppose $cov X \leq n$. Let $\mathcal{U}$ be a finite basic cover of X. Then $\mathcal{U}$ is also an open cover and thus has a finite open refinement of order $\leq n$. By theorem 3.3 this open refinement has a finite basic refinement of order $\leq n$, hence $\mathcal{U}$ has a finite basic refinement of order $\leq n$. Thus $cov_z X \leq n$.

The Stone-Čech compactification of any Tychonoff space is a compact Hausdorff space (hence normal), and therefore, if X is Tychonoff, cov $\beta X = \text{cov}_z \beta X$.

<u>Lemma 4.9</u> If $\mathcal{U}$ is a finite basic cover of X, then

$$\mathcal{U}^{\beta} = \{\beta X - cl_{\beta X}(X - U) : U \varepsilon \mathcal{U}\}$$

is an open cover of βX of the same order as $\mathcal{U}$.

Proof: Suppose $\mathcal{U} = \{U_1, \ldots, U_k\}$, then for each i, $Z_i = X - U_i$ is a zero set of X. Since $\mathcal{U}$ is a cover of X, $\bigcap_{i=1}^{k} Z_i = \emptyset$, and by statement (4) in theorem 4.5, this implies that $\bigcap_{i=1}^{k} (cl_{\beta X} Z_i) = \emptyset$. Thus $\mathcal{U}^{\beta}$ covers βX. Since there are only a finite number of the Z_i, for any subcollection, $\{Z_{i'}\}$, $\bigcup (cl_{\beta X} Z_{i'}) = cl_{\beta X}(\bigcup Z_{i'})$, and it follows that

$$\bigcap (\beta X - cl_{\beta X}(X - U_{i'})) = \beta X - cl_{\beta X}(X - \bigcap U_{i'}),$$

and thus, the orders of $\mathcal{U}$ and $\mathcal{U}^{\beta}$ are the same.

<u>Theorem 4.10</u> If X is a Tychonoff space, then $\text{cov}_z X \le n$ if and only if $\text{cov}_z \beta X \le n$, or cov $\beta X \le n$.

Proof: Suppose cov $\beta X \le n$. Let $\mathcal{U}$ be a finite basic cover of X. By Lemma 4.9, $\mathcal{U}^{\beta}$ covers βX. By theorem 3.3, $\mathcal{U}^{\beta}$ has a finite basic refinement, and since $\text{cov}_z \beta X \le n$, the refinement can be assumed of order $\le n$. The intersection of the members of this refinement with X gives a finite basic refinement of $\mathcal{U}$ of order $\le n$. Hence $\text{cov}_z X \le n$.

Suppose $cov_z X \le n$. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be an open cover of βX. By theorem 3.3, $\mathcal{U}$ has a finite basic refinement, $\mathcal{W} = \{W_1, \ldots, W_k\}$ such that for $1 \le i \le k$, $\bar{W}_i \subset U_i$. Then $\mathcal{W}' = \{W_i \cap X: 1 \le i \le k\}$ is a finite basic cover of X. Since $cov_z X \le n$, $\mathcal{W}'$ has a finite basic refinement $\mathcal{V}$ of order $\le n$. By the lemma $\mathcal{V}^{\beta}$ is an open cover of βX and has order $\le n$. Since $\mathcal{W}$ is a closure refinement of $\mathcal{U}$, $\mathcal{V}^{\beta}$ will be a refinement of $\mathcal{U}$. Thus $cov\ \beta X \le n$. Since βX is normal, $cov\ \beta X = cov_z \beta X$.

<u>Theorem 4.11</u> If X is a Tychonoff space, and $St\ X \le n$, then $St\ \beta X \le n$.

Proof: Suppose f is continuous on βX into I^{n+1}, and let $\varepsilon > 0$ be given. It is sufficient to construct an ε-approximation to f which misses zero (see remarks following theorem 3.22). Assume $\varepsilon < 1$. Let $f' = f|X$. Since $St\ X \le n$, there is a continuous function g' on X into I^{n+1} such that $||f'-g'|| < \varepsilon/2$, and zero is not in the range of g'. Let $V_1 = \{x \varepsilon X: ||g'(x)|| \le \varepsilon/2\}$, and $V_2 = \{x \varepsilon X: ||g'(x)|| \ge \varepsilon/2\}$. Let $h_1(X) = (\varepsilon/2)g'(x)/||g'(x)||$ for all x in V_1, then h_1 is continuous on V_1. Let $h_2 = g'|V_2$, then h_2 is continuous on V_2, and h_1 and h_2 agree on $V_1 \cap V_2$. Let h map X into $\{y \varepsilon I^{n+1}: ||y|| \ge \varepsilon/2\}$, with $h|V_1 = h_1$, and $h|V_2 = h_2$, then h is continuous on X, and $||f'-h|| < \varepsilon$. By part (1) of theorem 4.5, there is a continuous extension g of h mapping βX into $\{y \varepsilon I^{n+1}: ||y|| \ge \varepsilon/2\}$. Zero is not in the range of g, and since X is dense in βX and $g|X = h$, it follows that $||f-g|| \le \varepsilon$.

<u>Theorem 4.12</u> If X is a Tychonoff space and $St\ \beta X \le n$, then $St\ X \le n$.

Proof: Let f map X into I^{n+1}, let $\varepsilon > 0$ be given, and let y be a point in I^{n+1}. By part (1) of theorem 4.5, there is a continuous function f', on βX into I^{n+1}, such that $f'|X=f$. Since St $\beta X \leq n$, there is a continuous function g' on βX into I^{n+1}, so that $||f'-g'|| < \varepsilon$, and y is not the range of g'. Let $g=g'|X$, then $||f-g|| < \varepsilon$, and y is not in the range of g, hence St $X \leq n$.

We have proved that for Tychonoff spaces Z-covering dimension and stability dimension are equivalent concepts. As a consequence of the preceding and earlier theorems, we also have that if X is a Tychonoff space and for some integer, n, Ext $\leq n$, then X is normal, and Ext X=cov X =St X=cov βX=St βX=Ext βX.

2. The Tychonoff Plank

The remainder of this chapter is devoted to a look at the dimension characteristics of the well-known Tychonoff Plank. There are several reasons for considering this space. It provides concrete examples of a few phenomena which have been referred to but not verified in the preceding discussion. The proofs of some of the dimension properties of this space point out the usefulness of theorems on equivalence of the several definitions. It provides answers to some of the questions which might be raised concerning whether or not the previous results can be sharpened, to wit:

question 1: Can it be proved for Tychonoff spaces that Ext X = Ext βX ? The answer is no.

question 2: Can it be proved for arbitrary subsets A of a normal space X that cov A $\leq$ cov X (St A $\leq$ St X, Ext A $\leq$ Ext X)? The answer to all three is no.

question 3: Can it be proved for Tychonoff spaces that cov X= cov_zX? The answer is no.

There are many other questions of similar nature which might be asked, and an interesting subset of them are answered here.

Example: The Tychonoff Plank (c.f. (5) or (7))

Let Ω be the first uncountable ordinal, and let $[0,\Omega]$ be the set of ordinals less than or equal to Ω. Define "intervals" in $[0,\Omega]$ by letting

$$(\alpha_1,\alpha_2) = \{\alpha : \alpha_1 < \alpha < \alpha_2\}$$

$$[\alpha_1,\alpha_2] = \{\alpha : \alpha_1 \leq \alpha \leq \alpha_2\}$$

etc.,

for any two ordinals, α_1 and α_2, less than or equal to Ω. A topology is given for $[0,\Omega]$ by letting the collection of intervals of the form $[0,\alpha)$, (α_1,α_2), or $(\alpha_2,\Omega]$ be a base for the open sets. This topology determines a Hausdorff space. If $\alpha=0$, or if α has an immediate predecessor in $[0,\Omega]$, then $\{\alpha\}$ is an open set. It is important to note that any countable set in $[0,\Omega)$ has a supremum which is less than Ω.

Proposition 1: $[0,\Omega]$ is compact.

Proof: Let $\mathcal{U} = \{U_a : a\varepsilon A\}$ be an open cover of $[0,\Omega]$. Suppose $\Omega\varepsilon U_0$, then there is an ordinal $\alpha < \Omega$ such that the interval $(\alpha,\Omega]$ is contained in U_0. The set $[0,\alpha]$ is countable, thus there is a countable subcollection $\{U_i : i=1, 2, \ldots\}$ of $\mathcal{U}$ which covers $[0,\Omega]$. Assume there is no finite subcover. For each $k=1, 2, \ldots$, let $x_k = \inf\{x < \Omega : [0,x]$ is not covered by $\bigcup_{i=1}^{k} U_i\}$, then the sequence $\{x_k\}$ is an increasing sequence in $[0,\Omega]$ and has a limit point $x < \Omega$. Suppose $x\varepsilon U_n$, then there is an integer

$m > n$, such that $x_m \varepsilon U_n$. By the definition of the sequence $\{x_k\}$, this implies that $[0,x_m]$ is contained in $\bigcup_{k=1}^{m} Uk$. This is a contradiction. Hence $[0,\Omega]$ is compact.

The space $[0,\Omega]$ is then normal. Let ω be the first infinite ordinal, then $[0,\omega]$ is a closed (hence compact Hausdorff) subspace of $[0,\Omega]$. Let $X = [0,\omega] \times [0,\Omega]$, then X is a compact Hausdorff space. The space X is commonly called the Tychonoff Plank. For $\alpha < \Omega$, let α' denote the successor of α.

Proposition 2: $\dim X = 0$.

Proof: Suppose (n,α) is a point in X with $0 < n < \omega$, $0 < \alpha < \Omega$. Let U be an open set containing $\{(n, \alpha)\}$, then there are ordinals n_1, $n_2 < \omega$, and α_1, $\alpha_2 < \Omega$, such that $(n,\alpha) \varepsilon (n_1, n_2) \times (\alpha_1, \alpha_2) \subset U$. If n_2 has an immediate predecessor, let $m=n_2$, if n_2 does not have an immediate predecessor, choose $m \varepsilon (n, n_2)$. If α_2 has an immediate predecessor, let $\mu=\alpha_2$, if α_2 does not have an immediate predecessor, choose $\mu \varepsilon (\alpha,\alpha_2)$. Then $[n_1', m] \times [\alpha_1', \mu]$ is an open and closed subset of U, so its boundary is empty, and it contains $\{(n,\alpha)\}$. A similar proof can be given if $n = \omega$, $\alpha = \Omega$, $n = 0$, or $\alpha = 0$.

Let $T=X-\{(\omega,\Omega)\}$. T is a Tychonoff space, by virtue of being a subspace of a compact Hausdorff space.

Proposition 3: X is the Stone-Čech Compactification of T.

Proof: Every neighborhood of (ω, Ω) meets T, hence T is dense in X. Then X is a compactification of T. Property (2) of theorem 4.5 may be used to prove that $X=\beta T$. If every bounded, continuous, real-valued function defined on T can be extended continuously over X, then $X=\beta T$.

Let f be a bounded, continuous, real valued function defined on T. Let r_1=lim inf f((n,Ω)), and let r_2=lim sup f((n,Ω)), $n < \omega$. There are two sequences of integers, $\{n_j\}$, and $\{m_i\}$, such that lim $f((n_j,\Omega))=r_1$, and lim $f((m_i, \Omega))=r_2$, and both sequences increase to ω. Since f is continuous, for each $n < \omega$, there exists an $\alpha_n < \Omega$, so that $\alpha > \alpha_n$ implies $|f((n,\alpha))-f((n,\Omega))| < 1/n$. Let $\lambda=\sup\{\alpha_n : n\varepsilon\{n_j\} \cup \{m_i\}\}$, then $\lambda < \Omega$, and

$$r_1 = \lim f((n_j, \lambda)) = f((\omega,\lambda)) = \lim f((m_i, \lambda)) = r_2,$$

therefore the limit as $n \longrightarrow \omega$ exists. Let $f((\omega,\Omega))=\lim f((n,\Omega))$. By the above process, it is clear that for each $m < \omega$, $n > m$ and $\alpha > \lambda$ implies $|f((n,\alpha))-f((\omega,\Omega))| < 1/m + |f((n,\Omega))- f((\omega,\Omega))|$, so the extension of f is continuous, which implies X=βT.

Proposition 4. T is not normal, hence for each $n < \omega$, it is false that Ext $T \leq n$.

Proof: Let $L = \{\omega\} \times [0,\Omega)$. Let $M=[0,\omega) \times \{\Omega\}$. L and M are disjoint closed subsets of T. Suppose V and W were disjoint open sets with $L \subset V$, and $M \subset W$. Choose $\alpha_0 \geq 0$, with $(0, \omega_0)\varepsilon W$, and inductively, choose $\alpha_n > \alpha_{n-1}$, with $\alpha_n < \Omega$, such that $(n, \alpha_n)\varepsilon W$. Let $\lambda=\sup\{\alpha_n : n= 0, 1, \ldots\}$, then the sequence (n, α_n) converges to $(\omega, \lambda)\varepsilon V$, but each point of the sequence is in T-V. This is a contradiction since T-V is closed.

By the results in chapter one for zero dimensional spaces, since X is Lindelöf, and dim X=0, it follows that cov X=Ext X=St X=$\mathfrak{a}$. By the results of this chapter, since X=βT, it follows that cov_zT=St T=0.

Since T is not normal, it must be true that cov $T > 0$.

Proposition 5. cov $T = 1$.

Proof: We already have cov $T > 0$. It remains to show that cov $T \leq 1$. Let $\{U_0, U_1, U_2\}$, be an open cover of T. The proof will proceed by several parts.

Part (a) For some $i = 0, 1$, or 2, there exists an ordinal $\lambda < \Omega$ so that for all $\alpha > \lambda$, $(\omega,\alpha)\varepsilon U_i$

Proof: Assume false. Let L be as in proposition 4, and choose $x_1 \varepsilon L-U_0$, choose $x_2 > x_1$ with $x_2 \varepsilon L-U_1$, choose $x_3 > x_2$, so that $x_3 \varepsilon L-U_2$. Continue this process inductivly to obtain an increasing sequence in L, $\{x_i\}$, so that there is a cofinal subsequence in each of $L-U_0$, $L-U_1$, and $L-U_2$. Let $x^*=\sup\{x_i\}$, then x^* is in L, and must be in one of the U_i. Suppose $x^* \varepsilon U_0$, then the sequence $\{x_i\}$ converges to x^*, and there must be a final segment of the sequence in U_0. This contradicts the construction of the sequence, so the assertion of part (a) is true.

Part (b) Let λ be as in part (a), so $\alpha > \lambda$ implies $(\omega,\alpha)\varepsilon U_o$. For some integer, $N < \omega$, the open set, $K=[N,\omega] \times [\lambda', \Omega)$ is contained in U_o.

Proof: Suppose not. Then, for each integer $n < \omega$, there is an ordinal $\alpha_n > \lambda'$, with $\alpha_n < \Omega$, such that $(n, \alpha_n)\varepsilon T-U_o$. For $0 \leq n < \omega$, let $\mu_n=\sup\{\alpha_k : k \geq n\}$ and let $\mu=\inf\{\mu_n\}$, then (ω,μ) is in U_o and is a limit point of the set $\{(n,\alpha_n): n \geq 0\}$, but this is a contradiction, since $T-U_o$ is closed.

Part (c) Let $K'=K \cup ([N,\omega) \times \{\Omega\})$. For $i=0, 1, 2$, let $V_i=U_i \cap K'$. There is a refinement of V_o, V_1, V_2 which covers K', and is of order less than or equal to one.

Proof: For each integer $n \geq N$, let $i(n)$ be the first index i for which the point (n,Ω) is in V_i. For each such n, there is an ordinal, α_n, for which the open set, $I_n=\{n\} \times (\alpha_n,\Omega] \subset U_{i(n)}$. Let $W_o=V_o \cup \{ I_n : i(n)=0\}$. For j=1 or 2, let $W_j= \cup \{ I_n : i(n)=j\}$. Then W_o, W_1, W_2 are open (in T) subsets of K', $W_o \cup W_1 \cup W_2=K'$, $W_i \subset V_i$ for i=0, 1, 2, and $W_1 \cap W_2 = \emptyset$ so the order of $\{W_o, W_1, W_2\}$ is less than or equal to one.

Part (d) There is a refinement of U_o, U_1, U_2 of order less than or equal to one.

Proof: $T-K'$ is an open and closed subset of T and of X. Since cov X=0, by theorem 2.11, $\text{cov}(T-K')=0$, hence there is an open refinement of $(T-K') \cap \{ U_o,U_1,U_2\}$ which covers $T-K'$, is of order zero, and the sets of that refinement will also be open in T. The sets in that last refinement, together with W_0, W_1, W_2 give an open refinement of U_0, U_1, U_2 of order less than or equal to one.

This completes the proof of proposition 5, as theorem 2.10 shows it sufficient to consider a three element cover.

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