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TWO APPLICATIONS OF DIFFERENCE METHODS TO CHAINS OF LINEAR OSCILLATORS

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TWO APPLICATIONS OF DIFFERENCE

METHODS TO CHAINS OF

LINEAR OSCILLATORS

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SUMMARY

The goal of this study is to answer two questions concerning linear spring-mass chains in which all the spring constants are equal but the masses are not all equal and do not vary monotonically along the chain. It is known from earlier work that a solution of the equations of motion of an infinite spring-mass chain can be given in terms of sequences of orthogonal polynomials and that if the mass is non-constant and non-monotone the polynomials are not any of the classical orthogonal polynomials. The procedure used to construct such a solution requires that the weight function and interval of orthogonality of the polynomials be known. If the polynomials satisfied a fourth-order differential equation, then the weight function and interval of orthogonality could be deduced from this equation in the same way they are deduced for the classical orthogonal polynomials from the second-order differential equations which those polynomials satisfy.

Thus the first question is whether there exists a sequence of non-classical orthogonal polynomials satisfying a certain recurrence relation and at the same time satisfying a fourth-order differential equation. Unfortunately from the point of view of solving the equations of motion of an infinite chain of oscillators, it is shown that no such sequence exists.

The second question concerns a finite chain of linear oscillators in which one mass differs from the rest. A transcendental equation satisfied by the characteristic frequencies of the system is derived; and it is shown that as the altered mass approaches zero, exactly one of the characteristic frequencies approaches plus infinity, while all the others remain in a certain bounded interval. It is also shown that if the altered mass is in the middle of the chain, the largest characteristic frequency can be pushed out of this interval by a relatively small change in the mass.

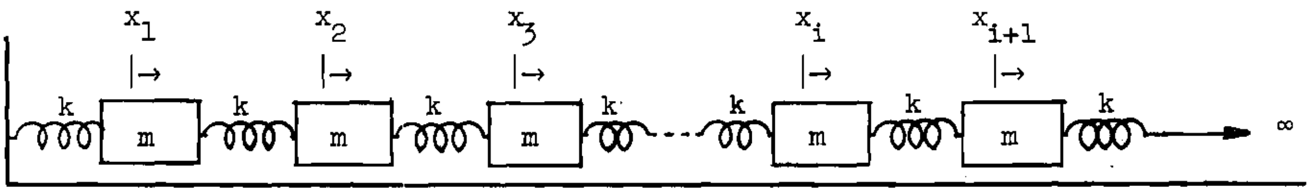
CHAPTER I

INTRODUCTION

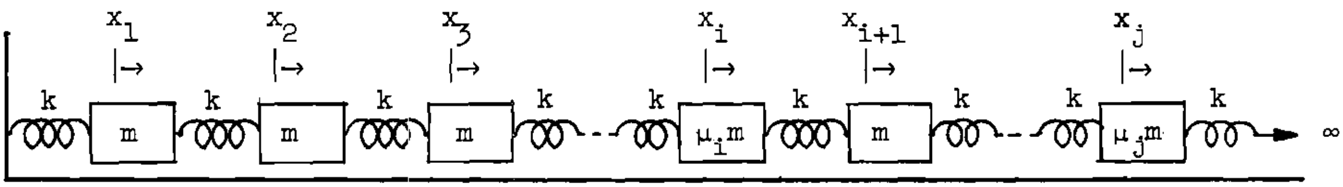
The purpose of this study is to answer two questions related to the impurity problem, two versions of which may be visualized in terms of the physical systems shown in Figure 1. In Figure 1(a) infinitely many similar masses free to slide on a frictionless plane are joined together by identical massless linear springs. Figure 1(b) differs from 1(a) only in that finitely many (p) of the masses, not necessarily consecutive, no longer have the common value m; they have values $\mu_\alpha m$ ($\mu_\alpha > 0, \mu_\alpha \neq 1$; $\alpha = 1,2,\cdots,p$). Figure 1(c) is like 1(a) except that the number of masses is a fixed positive integer N. In Figure 1(d) the j<u>th</u> ($1 \leqq j \leqq N$) mass of Figure 1(c) has been replaced by a mass of value μm ($\mu > 0$, $\mu \neq 1$). The impurity problem consists of comparing some property or properties of the system shown in Figure 1(b) with the corresponding property or properties of the system in Figure 1(a), and similarly for Figures 1(d) and 1(c).

The two questions to be answered may be formulated in the following way.

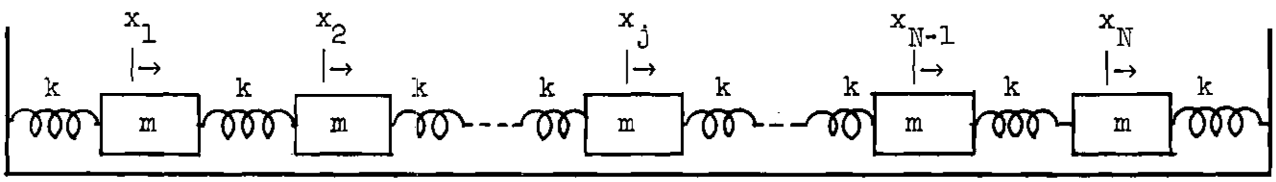
(i) Consider the sequence $\left\{P_n(x)\right\}_{n=0}^{\infty}$ of polynomials generated by the recurrence relation



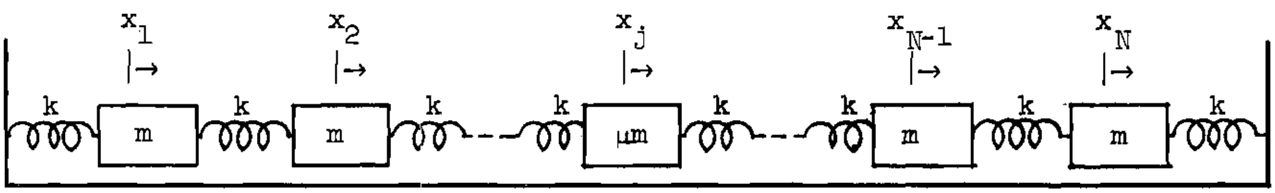
(a) Infinite Chain of Similar Springs and Masses



(b) Infinite Chain in Which Finitely Many Masses Have Values $\mu_\alpha m$ ($\mu_\alpha > 0$, $\mu_\alpha \neq 1$; $\alpha = 1, 2, \cdots, p$)



(c) Finite Chain of Similar Masses and Springs



(d) Finite Chain in Which the j<u>th</u> Mass Has Value μm ($\mu > 0$; $\mu \neq 1$)

Figure 1. Finite and Infinite Chains of Linear Oscillators

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_{n+1}(x) &= \frac{a\,x}{n+1} P_n(x) - \frac{c}{n(n+1)} P_{n-1}(x), \; n \geq 1 \, ,
\end{aligned}
\tag{1}
$$

where a and c are non-zero real constants (independent of x and n). For what values of a and c, if any, do the polynomials $P_n(x)$ (for each n) fail to satisfy a second-order ordinary differential equation and yet satisfy a fourth-order differential equation of the form

$$
a_0(x)y^{IV} + a_1(x)y''' + a_2(x)y'' + a_3(x)y' + \lambda_n a_4(x)y = 0 \, , \tag{2}
$$

where λ_n is a function of n but not of x and is not identically zero in n?

(ii) For the chain of linear oscillators shown in Figure 1(d), is there a simple transcendental equation whose solutions give the natural frequencies of the system as functions of j, N, and μ? If so, what is the equation and is it possible by using this equation to bound the natural frequencies for arbitrary positive μ?

Since the relationship of question (i) to the impurity problem is not obvious, some explanation is needed. The solution of the impurity problem for the infinite chain in which only finitely many masses have values other than m (Figure 1(b)) appears to be very difficult. Consequently, it has been conjectured that some insight into the problem may be obtained by considering spring-mass chains in which infinitely many masses have values other than m but in which m_n (the value of the nth mass) is not a constant nor a monotone function of n. By using suitable sequences of orthogonal polynomials, A. G. Law [1] has given solutions of the equations of motion of a number of infinite spring-mass chains. Results obtained by J. W. Jayne [2] and W. F. Martens [3] show, however, that if all the spring constants in such a chain are equal and if the sequence of polynomials required in Law's solutions is a Sturm-Liouville sequence associated with a second-order ordinary differential equation (all the classical orthogonal polynomials are in this category), then m_n is either constant or monotone in n. Hence, no sequence of classical orthogonal polynomials (or, more generally, no Sturm-Liouville sequence of orthogonal polynomials associated with a second-order ordinary differential equation) is suitable for studying a chain in which the spring constants are equal but m_n is non-constant and non-monotone. On the other hand, Law's solutions require that the interval of orthogonality of the polynomials, the distribution associated with them, and the three-term recurrence relation which they satisfy be known. Favard [4] has given a simple condition sufficient to guarantee that a sequence $\{\phi_n(x)\}$ of polynomials generated by a three-term recurrence relation of the form

$$\begin{aligned} \phi_0(x) &= 1 \\ \phi_1(x) &= A_0 x + B_0 \\ \phi_{n+1}(x) &= (A_n x + B_n)\, \phi_n(x) - C_n\, \phi_{n-1}(x) , \quad n \geq 1 , \end{aligned} \tag{3}$$

be orthogonal with respect to some distribution on some interval. However, the problem of actually finding the distribution and the interval when the recurrence relation (3) is known has been solved only in special cases. These facts suggest that it may be fruitful to identify sequences of non-classical orthogonal polynomials associated with fourth-order (but not second-order) differential equations in the hope that such sequences will permit the investigation of chains in which m_n is neither constant nor monotone. For such sequences the interval of orthogonality and the weight function can be determined from the differential equation. An equation of fourth (rather than third) order seems appropriate because no differential equation of odd order is self-adjoint.

It would be desirable, therefore, to identify all sequences of polynomials generated by a recurrence relation of the form (3) which satisfy a fourth-order differential equation of the form (2). In this study, however, a less ambitious task is undertaken—namely, to consider sequences of polynomials which satisfy a differential equation of the form (2) but are generated by a recurrence relation of the less general form (1). Such a recurrence relation corresponds (when a weight function exists) to an even weight function on an interval symmetric about $x = 0$. In the sequel it is shown, somewhat surprisingly, that all orthogonal polynomials which satisfy a recurrence relation of the form (1) and a differential equation of the form (2) in which λ_n is not identically zero also satisfy a second-order differential equation. Thus, question (i) leads only to polynomials already known to be unsuitable for the study of infinite spring-mass chains in which all the spring constants are equal but m_n is non-constant and non-monotone.

The relationship of question (ii) to the impurity problem for the finite chain is apparent. In the pages which follow, a transcendental equation is derived whose solutions give the natural frequencies of the physical system as functions of j, N, and μ. It is shown that as μ approaches zero through positive values, exactly one of the natural frequencies becomes unboundedly large while the remaining N-1 natural frequencies lie in the interval $0 < w < 2\sqrt{\frac{k}{m}}$, which is the interval in which all N natural frequencies lie when $\mu = 1$.

The attention of the reader is invited to the fact that two versions of the impurity problem are discussed—one for the infinite chain and one for the finite chain. Question (i) is concerned with the effort to obtain information about the impurity problem in the infinite chain. Question (ii) is concerned with the impurity problem in the finite chain, but it also has a connection with the impurity problem in the infinite chain. The connection rests on the following fact: solutions for infinite chains are often obtained as limits (as $N \to \infty$) of solutions for finite chains. The answer to question (ii) is intended, then, to throw some light on a possible limiting process of this kind, in the execution of which it is necessary to know how the zeros of the characteristic polynomials of the Nth-order system distribute themselves as $N \to \infty$. A further discussion of this comment is beyond the scope of the present study, and the reader is referred to Law [1].

CHAPTER II

THE FIRST QUESTION: RECURSIVELY GENERATED POLYNOMIALS SATISFYING A FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATION

The object of this chapter is to answer question (i), page 1, and in doing so to explain why a recurrence relation of the form (1) was chosen for study.

The basic goal was to find an easy way to construct sequences of non-classical orthogonal polynomials about which all the required information would be easily available—recurrence relation, interval of orthogonality, and weight function (or corresponding distribution). In the light of previous work by Jayne [2], the task of identifying all sequences of polynomials generated by a recurrence relation of the form (3) which satisfy a fourth-order differential equation of the form (2) seemed too difficult for a project of the duration intended. So a special case of (3) was sought which still appeared to promise some measure of success in generating sequences of polynomials which were solutions of a fourth-order (but not of a second-order) differential equation. The reasoning which led to the special case (1) was as follows.

If x is a real parameter and if a and c are non-zero real constants (independent of x and n), the initial-value problem

$$y'' - axy' + cy = 0 \qquad (' \sim \frac{d}{dt})$$
$$y(0) = 1 \tag{4}$$
$$y'(0) = x$$

has a solution of the form

$$y(x\,;\,t) = \sum_{n=0}^{\infty} P_n(x)t^n \tag{5}$$

in which $P_n(x)$ is a polynomial of degree n satisfying the three-term recurrence relation

$$\begin{aligned} P_o(x) &= 1 \\ P_1(x) &= x \\ P_{n+1}(x) &= \frac{a\,x}{n+1} P_n(x) - \frac{c}{n(n+1)} P_{n-1}(x)\ ,\ n \geq 1\ . \end{aligned} \tag{6}$$

Furthermore, the usual method of solving linear differential equations with constant coefficients shows that the solution of system (4) is also expressible in the form

$$y(x\,;\,t) = \frac{1}{2}\left[e^{\left(\frac{a\,x + \sqrt{a^2x^2 - 4c}}{2}\right)t} + e^{\left(\frac{a\,x - \sqrt{a^2x^2 - 4c}}{2}\right)t}\right]$$

$$+ \frac{(2 - a)x}{2\sqrt{a^2x^2 - 4c}}\left[e^{\left(\frac{a\,x + \sqrt{a^2x^2 - 4c}}{2}\right)t} - e^{\left(\frac{a\,x - \sqrt{a^2x^2 - 4c}}{2}\right)t}\right]\ , \tag{7}$$

which serves then as the generating function for the polynomials $P_n(x)$. Writing (7) in the form

$$y(x\,;\,t) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1} n!} \left\{ (a\,x + \sqrt{a^2x^2 - 4c})^n + (a\,x - \sqrt{a^2x^2 - 4c})^n \right.$$

$$\left. + \frac{(2 - a)x}{\sqrt{a^2x^2 - 4c}} \left[(a\,x + \sqrt{a^2x^2 - 4c})^n - (a\,x - \sqrt{a^2x^2 - 4c})^n \right] \right\} t^n \qquad (8)$$

and comparing (8) with (5) shows that

$$P_n(x) = \frac{1}{2^{n+1} n!} \left\{ (a\,x + \sqrt{a^2x^2 - 4c})^n + (a\,x - \sqrt{a^2x^2 - 4c})^n \right.$$

$$\left. + \frac{(2 - a)\,x}{\sqrt{a^2x^2 - 4c}} \left[(a\,x + \sqrt{a^2x^2 - 4c})^n - (a\,x - \sqrt{a^2x^2 - 4c})^n \right] \right\},\ n \geq 0\ ; \qquad (9)$$

and hence the second-order difference system (6) involving the parameter x is solved. Thus, an explicit expression for $P_n(x)$ and a generating function for the sequence $\left\{P_n(x)\right\}$ are known.

If now for some choice of the constants a and c the polynomials $P_n(x)$ should prove to be solutions of a fourth-order ordinary differential equation of the form (2), all the information needed about them would be known—the recurrence relation, the interval of orthogonality, the weight function, and even an explicit expression for $P_n(x)$.

So consider the sequence $\left\{P_n(x)\right\}_0^\infty$ of polynomials generated by the recurrence relation (6), where a and c are nonzero real constants. If $P_0(x)$ is a solution of the equation

$$a_0(x)\,y^{(IV)} + a_1(x)\,y''' + a_2(x)\,y'' + a_3(x)\,y' + \lambda_n\,a_4(x)\,y = 0 \qquad (10)$$

for n = 0, then $\lambda_0\,a_4(x) \equiv 0$. But if $a_4(x) \equiv 0$, then the requirement that P_1, P_2, P_3, and P_4 be solutions of (10) would imply that

$a_0(x) \equiv a_1(x) \equiv a_2(x) \equiv a_3(x) \equiv 0$; and equation (10) would become a triviality. Hence, $a_4(x) \not\equiv 0$, and $\lambda_0 = 0$. With $a_4(x) \not\equiv 0$, requiring that P_1, P_2, P_3, and P_4 be solutions of (10) shows that (10) must have the form

$$\left\{(\frac{\lambda_1}{6} - \frac{\lambda_2}{4} + \frac{\lambda_3}{6} - \frac{\lambda_4}{24})x^4 + \frac{c}{a^2}\left[(\frac{a+1}{6})\lambda_1 + (\frac{5a-2}{24})\lambda_2 - (\frac{a+1}{6})\lambda_3 + (\frac{a+2}{24})\lambda_4\right]x^2 + \frac{c^2}{24a^3}\left[(a+1)\lambda_2 - \lambda_4\right]\right\} y^{(IV)} + \left\{\left[-\frac{3(\lambda_1 - \lambda_2) + \lambda_3}{6}\right]x^3 - \frac{c}{6a^2}\left[3a\lambda_2 + (a+1)(\lambda_1 - \lambda_3)\right]x\right\} y''' + \left\{(\lambda_1 - \frac{\lambda_2}{2})x^2 + \frac{c\lambda_2}{2a}\right\}y'' - \lambda_1 xy' + \lambda_n y = 0 \ . \tag{11}$$

To find λ_n, assume that P_n satisfies (11), and differentiate the resulting identity n times. Then

$$\lambda_n = -\lambda_1\left[\frac{n(n-2)(n-3)(n-4)}{6}\right] + \lambda_2\left[\frac{n(n-1)(n-3)(n-4)}{4}\right] - \lambda_3\left[\frac{n(n-1)(n-2)(n-4)}{6}\right] + \lambda_4\left[\frac{n(n-1)(n-2)(n-3)}{24}\right] . \tag{12}$$

Introducing this expression for λ_n in (11) yields the form which the fourth-order equation (10) must have if P_0, P_1, P_2, P_3, and P_4 are solutions.

It will now be shown that if a is neither 1 nor 2, the polynomials $P_n(x)$ generated by (6) do not satisfy (10) for every n (= 0, 1, 2, $\cdots$) unless λ_n is identically zero in n. In particular it will be shown that if P_5 and P_6 as calculated from (6) are assumed to be solutions of (11), in which λ_n has the value (12), and if a is neither 1 nor 2, then $\lambda_n \equiv 0$.

Suppose then that P_5 and P_6 are solutions of (11). The two identities in x which result from this assumption imply that λ_1, λ_2, λ_3, and λ_4 must satisfy the following equations.

$$\lambda_1(15a + 25) + \lambda_2(-5a - 40) + \lambda_3(a_1 + 23) + \lambda_4(-5) = 0 \qquad (13a)$$

$$\lambda_1(a^2 - 8a - 3) + \lambda_2(5a^2 + 30a + 10) + \lambda_3(-a^2 - 24a - 13)$$

$$+ \lambda_4(5a + 5) = 0 \qquad (13b)$$

$$\lambda_1(64a + 136) + \lambda_2(-15a - 210) + \lambda_3(120) + \lambda_4(a - 26) = 0 \qquad (13c)$$

$$\lambda_1(-72a - 72) + \lambda_2(16a^2 + 168a + 140)$$

$$+ \lambda_3(-120a - 120) + \lambda_4(-a^2 + 24a + 37) = 0 \qquad (13d)$$

$$\lambda_1(24) + \lambda_2(-a^2 - 3a - 50) + \lambda_3(40) + \lambda_4(a - 11) = 0 \qquad (13e)$$

However, these equations have only the trivial solution (and hence, $\lambda_n \equiv 0$) unless $a = 1$ or $a = 2$. To reach this conclusion, proceed as follows.

Consider only equations (13a), (13c), (13d), (13e)—a homogeneous system of four linear equations in the four unknowns λ_1, λ_2, λ_3, and λ_4 having a non-trivial solution if and only if the determinant of the coefficients vanishes. Equating this determinant to zero yields the equation

$$(a - 1)^2(a - 2)^2(a^2 - 7a + 20) = 0 \,, \qquad (14)$$

whose only real roots are 1 and 2. Hence, the system (13a), (13c), (13d), (13e)—and consequently the entire system (13) of five equations—has only the trivial solution unless $a = 1$ or $a = 2$.

But Jayne [2] has shown that the polynomials $P_n(x)$ generated by the recurrence relation (6) satisfy a <u>second</u>-order differential equation if $a = 1$ or $a = 2$. Hence, any polynomials generated by (6) which satisfy

(11) with $\lambda_n \neq 0$ are simply some type of classical orthogonal polynomials unsuited to the investigation of the impurity problem in the infinite chain. Thus no new polynomials suitable for that purpose have been found by answering question (i).

CHAPTER III

THE SECOND QUESTION: NATURAL FREQUENCIES OF A LINEAR CHAIN CONTAINING ONE ALTERED MASS

In preparation for the study of question (ii), page 2, it will be helpful to calculate the natural frequencies of the system shown in Figure 1(c), page 2. The displacements indicated there satisfy the system of differential equations

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) , \\ m\ddot{x}_n &= -k(x_n - x_{n-1}) + k(x_{n+1} - x_n) , \; 2 \leqq n \leqq N-1 , \\ m\ddot{x}_N &= -k(x_N - x_{N-1}) - kx_N . \end{aligned} \tag{14}$$

One asks, for what values of the parameter w, if any, does this system have a solution of the form $x_n = A_n \cos wt$? Substituting in (14) leads to the system

$$\begin{aligned} (-mw^2 + 2k)A_1 + (-k)A_2 &= 0 , \\ -kA_{n-1} + (-mw^2 + 2k)A_n + (-k)A_{n+1} &= 0 , \; 2 \leqq n \leqq N-1 , \\ -kA_{N-1} + (-mw^2 + 2k)A_N &= 0 , \end{aligned} \tag{15}$$

which can be written more compactly as

$$\begin{aligned} A_{n+1} + (\frac{mw^2}{k} - 2)A_n + A_{n-1} &= 0 , \; 1 \leqq n \leqq N , \\ A_0 = 0 , \; A_{N+1} &= 0 . \end{aligned} \tag{16}$$

The system (16), which is a second-order difference equation with boundary conditions, determines the amplitudes A_n of the displacements. The solution is found by considering several cases.

To search for values of w in the range $0 \leqq \frac{mw^2}{k} \leqq 4$, the substitution $\frac{mw^2}{k} - 2 = - 2 \cos \alpha$ is made so that (16) becomes

$$A_{n-1} - 2 \cos \alpha \, A_n + A_{n+1} = 0 \; , \; 1 \leqq n \leqq N \; , \qquad (17)$$
$$A_0 = 0 \; , \; A_{N+1} = 0 \; .$$

If $\sin \alpha = 0$, and $\cos \alpha = 1$, the solution of (17) (see Hildebrand [6]) takes the form $A_n = c_1 + c_2 n$, where c_1 and c_2 are arbitrary constants. The conditions $A_0 = 0$, $A_{N+1} = 0$ require that $c_1 = c_2 = 0$ so that (17) has only the trivial solution. If $\sin \alpha = 0$ and $\cos \alpha = -1$, the solution of (17) is $A_n = (-1)^n(d_1 + d_2 n)$. Again the boundary conditions demand that $d_1 = d_2 = 0$ so that only the trivial solution results. If $\sin \alpha \neq 0$, the solution of (17) takes the form $A_n = e_1 \cos n\alpha + e_2 \sin n\alpha$. Now $A_0 = 0$ requires that $e_1 = 0$ and $A_{N+1} = 0$ requires that $e_2 \sin(N + 1)\alpha = 0$. So for a non-trivial solution, it is necessary that $\sin(N + 1)\alpha = 0$; hence, since $\sin \alpha \neq 0$, $\alpha_p = \frac{p\pi}{N + 1}$, $p = 1, 2, \cdots, N$ (though there are a countable infinity of values of p for which $\sin \alpha_p = 0$, it is necessary to consider only the range $1 \leqq p \leqq N$, because these values of p yield all the N natural frequencies which the system has, and additional values of p simply repeat some of them).

The natural (or characteristic) frequencies may be found by using the equation

$$w^2 = \frac{2k}{m}(1 - \cos\alpha) = \frac{4k}{m}\sin^2\frac{\alpha}{2} .$$

$$\text{Thus } w_p = 2\sqrt{\frac{k}{m}}\sin\left(\frac{p}{N+1}\right)\frac{\pi}{2} , \; p = 1, 2, \cdots , N. \tag{18}$$

Notice that for each p, $0 < w_p < 2\sqrt{\frac{k}{m}}$.

If one wishes to search (in this case, unnecessarily) for values of w in the range $\frac{mw^2}{k} > 4$, the substitution $\frac{mw^2}{k} - 2 = 2\cosh\alpha$ is used. Equation (16) then takes the form

$$A_{n+1} + 2(\cosh\alpha)\, A_n + A_{n-1} = 0 , \; 1 \leqq n \leqq N ,$$
$$A_0 = 0 , \; A_{N+1} = 0 . \tag{19}$$

The solution of (19) is $A_n = (-1)^n \left[f_1 \cosh n\alpha + f_2 \sinh n\alpha \right]$. But now $A_0 = 0$ implies $f_1 = 0$, and $A_{N+1} = 0$ implies $f_2 = 0$, so that (19) has only the trivial solution.

In the version of the impurity problem considered here, one of the masses of mass m is replaced by a mass μm, where $0 < \mu < \infty$, $\mu \neq 1$. It is known (see Courant-Hilbert [5]) that if $\mu > 1$, each characteristic frequency will be equal to or less than the corresponding value given by (18), while if $\mu < 1$, the converse will be true. For this reason it is convenient to divide the following study into two cases.

Case I ($\mu > 1$). In this case it is known that for any characteristic frequency w, $0 < w < 2\sqrt{\frac{k}{m}}$. Two subcases are considered depending on whether the altered mass is an end mass or not.

If the altered mass is an end mass (see Figure 1(d) with j = 1), it is easily seen by techniques similar to those used previously that the relevant difference system is

$$
\begin{aligned}
(-\mu m w^2 + 2k)A_1 + (-k)A_2 &= 0 , \\
(-k)A_{n-1} + (-mw^2 + 2k)A_n + (-k)A_{n-1} &= 0 , \quad 2 \leqq n \leqq N - 1 , \qquad (20) \\
(-k)A_{N-1} + (-mw^2 + 2k)A_N &= 0 .
\end{aligned}
$$

With the substitution $\frac{mw^2}{k} - 2 = -2 \cos \alpha$, from which it follows that $\frac{\mu m w^2}{k} - 2 = \mu(\frac{mw^2}{k} - 2) + 2(\mu - 1)$, (20) becomes

$$[2\mu \cos \alpha + 2(1 - \mu)]A_1 - A_2 = 0 , \tag{21a}$$

$$A_{n+1} - (2 \cos \alpha) A_n + A_{n-1} = 0 , \quad 2 \leqq n \leqq N , \tag{21b}$$

$$A_{N+1} = 0 . \tag{21c}$$

There is no need to consider the special cases $\sin \alpha = 0$, $\cos \alpha = 1$ and $\sin \alpha = 0$, $\cos \alpha = -1$, since for any characteristic frequency w, it is known that $0 < w < 2\sqrt{\frac{k}{m}}$. Thus the solution of (21) takes the form

$$A_n = c_1 \cos n\alpha + c_2 \sin n\alpha ,$$

and the boundary conditions (21a), (21c) require that

$$c_1 \cos 2\alpha + c_2 \sin 2\alpha = [2\mu \cos \alpha + 2(1 - \mu)] A_1 , \tag{22}$$

$$c_1 \cos(N + 1) \alpha + c_2 \sin(N + 1) \alpha = 0 . \tag{23}$$

The determinant of coefficients of this pair of linear equations is $\sin(N + 1) \alpha \cos 2\alpha - \cos(N + 1) \alpha \sin 2\alpha$ or $\sin(N - 1) \alpha$. So if $\sin(N - 1) \alpha \neq 0$,

$$c_1 = [2\mu \cos \alpha + 2(1 - \mu)] \frac{\sin(N + 1)\alpha}{\sin(N - 1)\alpha} A_1 ,$$

$$c_2 = - [2\mu \cos \alpha + 2(1 - \mu)] \frac{\sin(N + 1)\alpha}{\sin(N - 1)\alpha} A_1 ,$$

and

$$A_n = [2\mu \cos \alpha + 2(1 - \mu)]A_1 \frac{[\sin(N + 1)\alpha \cos n\alpha - \cos(N + 1)\alpha \sin n\alpha]}{\sin(N - 1)\alpha}$$

or $$A_n = [2\mu \cos \alpha + 2(1 - \mu)]A_1 \frac{\sin(N + 1 - n)\alpha}{\sin(N - 1)\alpha} .$$

Now requiring that A_n reduce to A_1 when $n = 1$ implies that

$$A_1 \sin(N - 1)\alpha = [2\mu \cos\alpha + 2(1 - \mu)]A_1 \sin N \alpha ,$$

or, since $A_1 = 0$ yields only the trivial solution,

$$\sin(N - 1)\alpha = [2\mu \cos \alpha + 2(1 - \mu)] \sin N \alpha . \qquad (24)$$

On the other hand, if $\sin(N - 1)\alpha = 0$, (23) may be rewritten as

$$c_1\cos(N - 1 + 2)\alpha + c_2\sin(N - 1 + 2)\alpha = 0 \qquad \text{or}$$

$$c_1[\cos(N - 1)\alpha \cos 2\alpha - \sin(N - 1)\alpha \sin 2\alpha]$$

$$+ c_2[\sin(N - 1)\alpha \cos 2\alpha + \cos(N - 1)\alpha \sin 2\alpha] = 0 ;$$

and since $\sin(N - 1)\alpha = 0$ and hence $\cos(N - 1)\alpha \neq 0$, this equation reduces to

$$c_1\cos 2\alpha + c_2 \sin 2\alpha = 0 . \qquad (25)$$

But (25) and (22) together require that $[2\mu \cos \alpha + 2(1 - \mu)]A_1 = 0$ or $[2\mu \cos \alpha + 2(1 - \mu)] = 0$, so that (24) is still satisfied.

Now if the altered mass is the <u>jth</u> mass ($1 < j < N$), the relevant difference system is

$$A_{n+1} - (2\cos\alpha)\, A_n + A_{n-1} = 0 \;,\; 1 \leq n \leq j - 1 \;, \tag{26a}$$

$$A_{j+1} + [2(\mu - 1) - 2\mu \cos\alpha] A_j + A_{j-1} = 0 \;, \tag{26b}$$

$$A_{n+1} - (2\cos\alpha)\, A_n + A_{n-1} = 0 \;,\; j + 1 \leq n \leq N \;, \tag{26c}$$

$$A_0 = A_{N+1} = 0 \;. \tag{26d}$$

Again the cases $\sin\alpha = 0$, $\cos\alpha = 1$ and $\sin\alpha = 0$, $\cos\alpha = -1$ cannot occur so that the solution will take the form

$$A_n = \begin{cases} c_1 \cos n\alpha + c_2 \sin n\alpha \;,\; 1 \leq n \leq j \;, \\ d_1 \cos n\alpha + d_2 \sin n\alpha \;,\; j \leq n \leq N \;. \end{cases}$$

Requiring that $A_o = 0$ and that A_n reduce to A_j when $n = j$ yields

$$A_n = \left(\frac{\sin n\alpha}{\sin j\alpha}\right) A_j \;,\; 1 \leq n \leq j \;, \text{ so long as } \sin j\,\alpha \neq 0 \;.$$

Similarly $A_{N+1} = 0$ and $A_n = A_j$ when $n = j$ imply that

$$A_n = \left[\frac{\sin(N + 1 - n)\alpha}{\sin(N + 1 - j)\alpha}\right] A_j \;,\; j \leq n \leq N \;, \text{ if } \sin(N + 1 - j)\alpha \neq 0 \;.$$

Now substituting for A_{j-1} and A_{j+1} in (26b) gives

$$\frac{A_j \sin(N - j)\alpha}{\sin(N + 1 - j)\alpha} + [2(\mu - 1) - 2\mu \cos\alpha]\, A_j + \frac{A_j \sin(j - 1)\alpha}{\sin j\,\alpha} = 0 \;;$$

and since $A_j \neq 0$ for a non-trivial solution, this equation becomes

$$\sin(N - j)\alpha \sin j\,\alpha + [2(\mu - 1) - 2\mu\cos\alpha]\sin(N + 1 - j)\alpha \sin j\,\alpha$$
$$+ \sin(N + 1 - j)\alpha \sin(j - 1)\alpha = 0 \;. \tag{27}$$

It appears that there are three exceptional cases to be considered—namely,

$\sin j\alpha = 0$, $\sin(N + 1 - j)\alpha \neq 0$; $\sin j\alpha \neq 0$, $\sin(N + 1 - j)\alpha = 0$; and $\sin j\alpha = 0$, $\sin(N + 1 - j)\alpha = 0$. But it will now be shown that if (26) is to have a non-trivial solution, then $\sin j\alpha = 0$ if and only if $\sin(N + 1 - j)\alpha = 0$ so that the first two possibilities are eliminated. If $\sin j\alpha = 0$, then $A_j = 0$; and since $A_{N+1} = 0$, the requirement that $A_n = d_1 \cos n\alpha + d_2 \sin n\alpha$ for $j \leqq n \leqq N + 1$ yields

$$d_1 \cos j\alpha + d_2 \sin j\alpha = 0 \qquad (28)$$

and

$$d_1 \cos(N + 1)\alpha + d_2 \sin(N + 1)\alpha = 0 . \qquad (29)$$

But since $\sin j\alpha = 0$, (28) implies $d_1 = 0$; and then for a non-trivial solution (29) implies $\sin(N + 1)\alpha = 0$ (if $d_2 = 0$, then $A_n = 0$ for $j \leqq n \leqq N$; but then (26b) implies $A_{j-1} = 0$ which makes $A_n = 0$ for all n). But if $\sin(N + 1)\alpha = 0$, then $\sin(N + 1 - j)\alpha = \sin(N + 1)\alpha \cos j\alpha + \cos(N + 1)\alpha \sin j\alpha = 0$; so $\sin j\alpha = 0$ implies that $\sin(N + 1 - j)\alpha = 0$. Conversely, if $\sin(N + 1 - j)\alpha = 0$, then $A_{N+1} = 0$ implies that

$$d_1 \cos(N + 1)\alpha + d_2 \sin(N + 1)\alpha = 0 ,$$

or

$$d_1 \cos(N + 1 - j + j)\alpha + d_2 \sin(N + 1 - j + j)\alpha = 0 ,$$

or

$$d_1 \cos(N + 1 - j)\alpha \cos j\alpha + d_2 \cos(N + 1 - j)\alpha \sin j\alpha = 0 .$$

But since $\cos(N + 1 - j)\alpha \neq 0$,

$$d_1 \cos j\alpha + d_2 \sin j\alpha = 0 .$$

That is, $A_j = 0$, or $c_2 \sin j\alpha = 0$, which implies $\sin j\alpha = 0$ for a non-trivial solution. Thus the only exceptional case is $\sin j\alpha = 0$ and $\sin(N + 1 - j)\alpha = 0$. But then equation (27) is still satisfied.

Now note that if $j = 1$, then (27) reduces to (24) since $\sin\alpha \neq 0$. Thus for the case $\mu > 1$, whether the altered mass occupies an end position or an interior position, (27) is the desired transcendental equation satisfied by the characteristic frequencies.

Case II ($\mu < 1$). There is no longer any guarantee that all characteristic frequencies are less than $2\sqrt{\frac{k}{m}}$. The possibility that $2\sqrt{\frac{k}{m}}$ is a characteristic frequency is explored first. This possibility corresponds to $\sin\alpha = 0$, $\cos\alpha = -1$ in the previous analysis. If the altered mass is an end mass, the relevant difference system is

$$[2 - 4\mu]A_1 - A_2 = 0 \,,$$
$$A_{n+1} + 2A_n + A_{n-1} = 0, \; 2 \leq n \leq N \,,$$
$$A_{N+1} = 0 \,.$$

The solution takes the form $A_n = (-1)^n(e_1 + e_2 n)$, and the boundary conditions require that

$$e_1 + 2e_2 = (2 - 4\mu)A_1 \,,$$
$$e_1 + (N + 1)e_2 = 0 \,.$$

If $N = 1$, there is no solution unless $\mu = \frac{1}{2}$; and if $N > 1$, $e_1 = \frac{N+1}{N-1}(2 - 4\mu)A_1$, $e_2 = \frac{4\mu - 2}{N - 1}A_1$, so that $A_n = (-1)^n (2 - 4\mu)\frac{N + 1 - n}{N - 1}A_1$. But A_n should reduce to A_1 when $n = 1$. Hence

$$A_1 = \frac{(-1)(2 - 4\mu)N\,A_1}{N - 1} \quad \text{or} \quad A_1\left[1 + \frac{(2 - 4\mu)N}{N - 1}\right] = 0 \ .$$

Thus there is a non-trivial solution only if $\mu = \frac{3N - 1}{4N}$. If the altered mass is an interior mass, the relevant difference system is

$$A_{n+1} + 2A_n + A_{n-1} = 0 \ , \ 1 \leqq n \leqq j - 1 \ ,$$

$$A_{j+1} + (4\mu - 2)A_j + A_{j-1} = 0 \ ,$$

$$A_{n+1} + 2A_n + A_{n-1} = 0 \ , \ j + 1 \leqq n \leqq N \ ,$$

$$A_0 = A_{N+1} = 0 \ .$$

Now using the boundary conditions and in addition requiring that A_n reduce to A_j when $n = j$ gives

$$A_n = \begin{cases} (-1)^{n-j} \left(\frac{n}{j}\right) A_j \ , & 1 \leqq n \leqq j \ , \\ (-1)^{n-j} \frac{[n - (N+1)]}{j - (N+1)} A_j \ , & j \leqq n \leqq N \ . \end{cases}$$

The interface condition $A_{j+1} + (4\mu - 2)A_j + A_{j-1} = 0$ requires that $A_j\left[-\frac{j - N}{j - (N+1)} + (4\mu - 2) - \frac{j - 1}{j}\right] = 0$; so there is a non-trivial solution only if $\mu = 1 - \frac{N + 1}{4_j(N + 1 - j)}$ (note that if $j = 1$, this expression reduces to $\mu = \frac{3N - 1}{4N}$) .

If there are any characteristic frequencies larger than $2\sqrt{\frac{k}{m}}$, they can be found by using the substitution $\frac{mw^2}{k} - 2 = 2 \cosh \alpha, \ \alpha > 0$. If the altered mass is an end mass, the difference system is

$$[2\mu \cosh \alpha - 2(1-\mu)]A_1 + A_2 = 0 ,$$
$$A_{n+1} + (2 \cosh \alpha)A_n + A_{n-1} = 0 , \; 2 \leqq n \leqq N ,$$
$$A_{N+1} = 0 .$$

The solution takes the form

$$A_n = (-1)^n [c_1 \cosh n\,\alpha + c_2 \sinh n\,\alpha] .$$

The boundary conditions require that

$$c_1 \cosh 2\,\alpha + c_2 \sinh 2\,\alpha = [-2\mu \cosh \alpha + 2(1-\mu)]A_1 ,$$
$$c_1 \cosh(N+1)\,\alpha + c_2 \sinh(N+1)\,\alpha = 0 .$$

If $N > 1$ (the case $N = 1$ is of no interest), these equations imply

$$c_1 = \frac{[-2\mu \cosh \alpha + 2(1-\mu)]A_1 \sinh(N+1)\alpha}{\sinh(N-1)\alpha} \quad ;$$

$$c_2 = \frac{-[-2\mu \cosh \alpha + 2(1-\mu)]A_1 \cosh(N+1)\alpha}{\sinh(N-1)\alpha} \quad ;$$

$$A_n = \frac{(-1)^n [-2\mu \cosh \alpha + 2(1-\mu)]A_1 \sinh(N+1-n)\alpha}{\sinh(N-1)\alpha} .$$

But requiring that $A_n = A_1$ when $n = 1$ yields

$$A_1 = \frac{-[-2\mu \cosh \alpha + 2(1-\mu)]A_1 \sinh N\,\alpha}{\sinh(N-1)\alpha} .$$

And since $A_1 \neq 0$ for a non-trivial solution, this equation implies

$$\sinh(N-1)\alpha = [2\mu \cosh \alpha - 2(1-\mu)]\sinh N\,\alpha . \tag{30}$$

If the altered mass is an interior mass, the difference system is

$$A_{n+1} + (2\cosh\alpha)A_n + A_{n-1} = 0\ ,\ 1 \leqq n \leqq j - 1\ ,$$

$$A_{j+1} + [2(\mu - 1) + 2\mu\cosh\alpha]A_j + A_{j-1} = 0\ ,$$

$$A_{n+1} + (2\cosh\alpha)\,A_n + A_{n-1} = 0\ ,\ j + 1 \leqq n \leqq N\ ,$$

$$A_0 = A_{N+1} = 0\ .$$

The solution takes the form

$$A_n = \begin{cases} (-1)^n\,[c_1\cosh n\,\alpha + c_2\sinh n\,\alpha]\ ,\ 1 \leqq n \leqq j\ , \\ (-1)^n\,[d_1\cosh n\,\alpha + d_2\sinh n\,\alpha]\ ,\ j \leqq n \leqq N\ ; \end{cases}$$

and the satisfaction of the boundary conditions requires that

$$A_n = \begin{cases} (-1)^{n-j}\,A_j\,\dfrac{\sinh n\,\alpha}{\sinh j\,\alpha}\ ,\ 1 \leqq n \leqq j\ , \\[2ex] (-1)^{n-j}\,A_j\,\dfrac{\sinh\,(N + 1 - n)\alpha}{\sinh\,(N + 1 - j)\alpha}\ ,\ j \leqq n \leqq N\ . \end{cases}$$

Satisfying the equation $A_{j+1} + [2(\mu - 1) + 2\mu\cosh\alpha]A_j + A_{j-1} = 0$ implies that $-\dfrac{\sinh\,(N - j)\alpha}{\sinh\,(N + 1 - j)\alpha} + [2(\mu - 1) + 2\mu\cosh\alpha] - \dfrac{\sinh\,(j - 1)\alpha}{\sinh j\,\alpha} = 0$, or $\sinh(N - j)\alpha\,\sinh j\,\alpha + [2(\mu - 1) + 2\mu\cosh\alpha]\,\sinh\,(N + 1 - j)\,\alpha\,\sinh j\,\alpha$

$$+ \sinh\,(N + 1 - j)\,\alpha\,\sinh\,(j - 1)\,\alpha = 0\ . \qquad (31)$$

Note that if $j = 1$, equation (31) reduces to equation (30); so in the case $\mu < 1$, if there are any characteristic frequencies greater than $2\sqrt{\frac{k}{m}}$, they may be found from equation (31). Those characteristic frequencies less than $2\sqrt{\frac{k}{m}}$ may, of course, be found from equation (27).

However, more information may be deduced from equation (31). For this purpose it is convenient to write the equation in the form

$$2(\mu - 1) + 2\mu \cosh \alpha = \frac{\sinh(N - j)\alpha}{\sinh(N + 1 - j)\alpha} + \frac{\sinh(j - 1)\alpha}{\sinh j \alpha} .$$

The left side of this equation is an increasing function of α for $\alpha > 0$, while the right side is the sum of two functions of the type $f(\alpha) = \frac{\sinh k \alpha}{\sinh(k + 1)\alpha}$, where k is a non-negative integer. A function of this type is non-increasing as can be seen by considering the derivative

$$\begin{aligned} f'(\alpha) &= \frac{k \cosh k \alpha \sinh(k + 1)\alpha - (k + 1) \sinh k \alpha \cosh(k + 1)\alpha}{\sinh^2(k + 1)\alpha} \\ &= \frac{k \sinh \alpha - \cosh(k + 1)\alpha \sinh k \alpha}{\sinh^2(k + 1)\alpha} \\ &\leq \frac{k \sinh \alpha - \sinh k \alpha}{\sinh^2(k + 1)\alpha} \leq 0 \quad \text{for} \quad k = 0, 1, 2, \cdots . \end{aligned}$$

For convenience let $g(\alpha) = 2(\mu - 1) + 2\mu \cosh \alpha$ and $h(\alpha) = \frac{\sinh(N - j)\alpha}{\sinh(N + 1 - j)\alpha} + \frac{\sinh(j - 1)\alpha}{\sinh j \alpha}$. Note that $g(0) = 4\mu - 2$ and

$$\lim_{\alpha \to 0} h(\alpha) = \frac{N - j}{N + 1 - j} + \frac{j - 1}{j} = 2 - \frac{1}{N + 1 - j} - \frac{1}{j} = 2 - \frac{N + 1}{j(N + 1 - j)} .$$

Now since $g(\alpha)$ is increasing and $h(\alpha)$ is decreasing (except in the trivial case $N = 1$), there will be no solution to the equation $g(\alpha) = h(\alpha)$ unless $4\mu - 2 \leq 2 - \frac{N + 1}{j(N + 1 - j)}$, which implies $\mu \leq 1 - \frac{N + 1}{4j(N + 1 - j)}$, and only one solution if this condition is satisfied (see Figure 2). Thus if $\mu > 1 - \frac{N + 1}{4j(N + 1 - j)}$, all the characteristic frequencies are less than $2\sqrt{\frac{k}{m}}$; if $\mu = 1 - \frac{N + 1}{4j(N + 1 - j)}$, one characteristic frequency equals $2\sqrt{\frac{k}{m}}$ while the others are less than $2\sqrt{\frac{k}{m}}$; and if $\mu < 1 - \frac{N + 1}{4j(N + 1 - j)}$, one characteristic frequency is greater than $2\sqrt{\frac{k}{m}}$ and the others are less than $2\sqrt{\frac{k}{m}}$. Furthermore, when $\mu < 1 - \frac{N + 1}{4j(N + 1 - j)}$, upper and lower

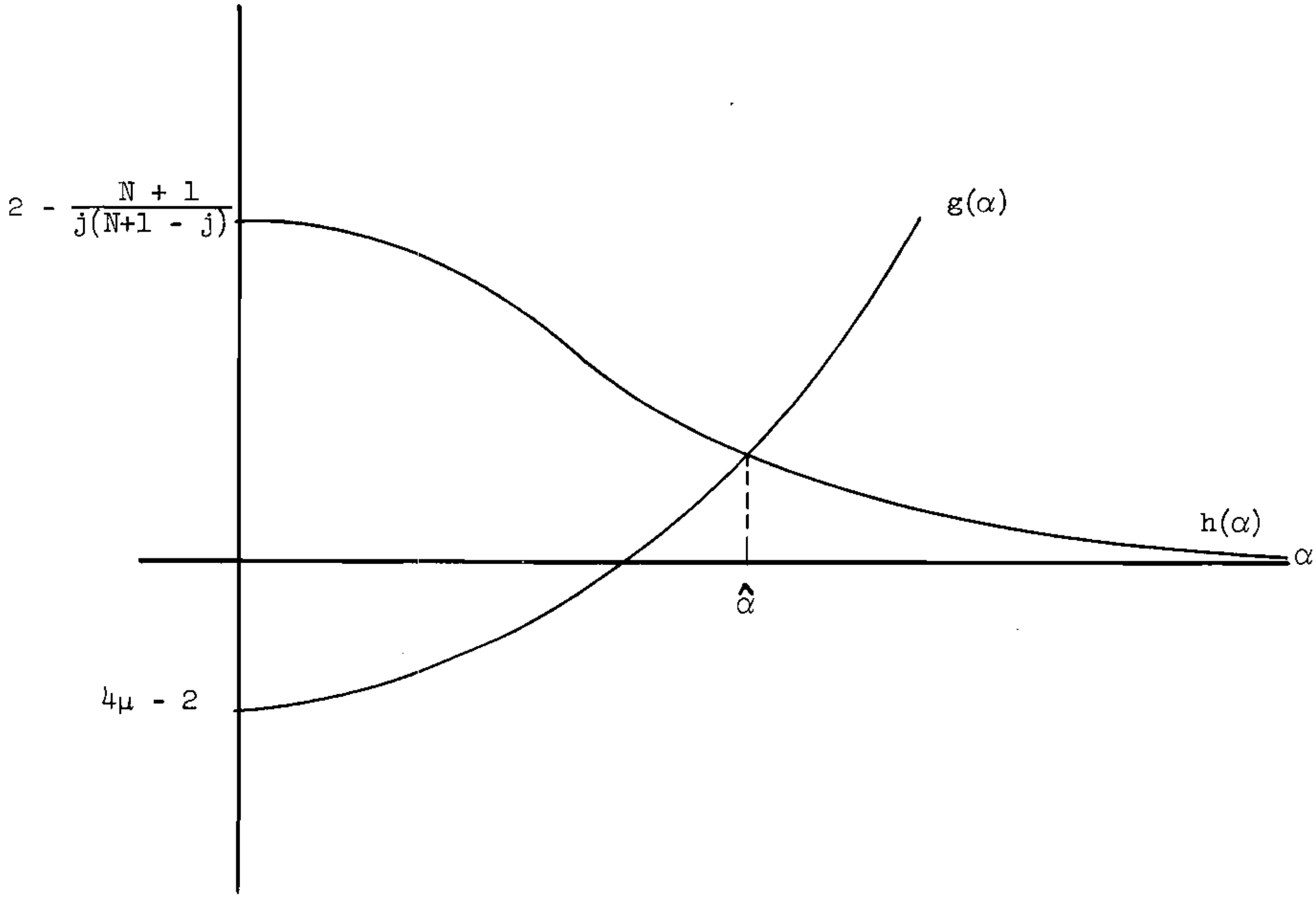


Figure 2. Graph of the Functions $g(\alpha)$ and $h(\alpha)$.

bounds on the largest characteristic frequency may be determined by the following reasoning (see Figure 2). Since $h(\alpha)$ is non-increasing and since $\lim_{\alpha \to \infty} h(\alpha) = 0$, the value $\hat{\alpha}$ at which $g(\alpha) = h(\alpha)$ satisfies the inequality $0 \leq g(\hat{\alpha}) \leq 2 - \frac{N + 1}{j(N + 1 - j)}$ or $0 \leq 2(\mu - 1) + 2\mu \cosh \hat{\alpha} \leq 2 - \frac{N + 1}{j(N + 1 - j)}$. Hence, $\frac{1}{\mu} \leq 1 + \cosh \hat{\alpha} \leq \frac{2}{\mu}\left[1 - \frac{N + 1}{4j(N + 1 - j)}\right]$. And since the corresponding characteristic frequency is computed from the relation $w^2 = \frac{2k}{m}(1 + \cosh \hat{\alpha})$, it follows that

$$\frac{2k}{m\mu} \leq w^2 \leq \frac{4k}{m\mu}\left[1 - \frac{N + 1}{4j(N + 1 - j)}\right] = \frac{4k\mu_c}{m\mu} .$$

In particular, as μ approaches zero, the largest characteristic frequency increases without bound.

In addition, it is interesting to investigate the size of the critical value of μ as a function of j. Since $\mu_c = 1 - \frac{N + 1}{4j(N + 1 - j)}$, $\frac{d\mu_c}{dj} = \frac{(N + 1)[N + 1 - 2j]}{4j^2(N + 1 - j)^2}$; thus $\frac{d\mu_c}{dj} = 0$ implies $j = \frac{N + 1}{2}$. If N + 1 is even, then $\frac{N + 1}{2}$ is an integer, so the largest value of μ_c is $\frac{N}{N + 1}$. If N + 1 is odd, the largest value of μ_c occurs when $j = \frac{N}{2}$ or $j = \frac{N}{2} + 1$. For either of these values of j, $\mu_c = 1 - \frac{(N + 1)}{N(N + 2)}$. Thus it is clear (1) that the largest value of μ_c is attained when the altered mass is as close to the middle of the chain as possible and (2) that when the altered mass occupies that position μ_c is only slightly less than unity for large N.

Finally note that it is not really necessary to use equation (31) to find all the characteristic frequencies even if one of them is larger than $2\sqrt{\frac{k}{m}}$. Consider the original substitution $w^2 = \frac{2k}{m}(1 - \cos \alpha)$, but now no longer require that α be a real variable. Instead let α be confined to the right-angled path in the complex plane described as follows: $\alpha = \alpha_1 + i\,\alpha_2$, where for $0 \leqq \alpha_1 < \pi$, $\alpha_2 = 0$ and for $\alpha_1 = \pi$, $\alpha_2 \geqq 0$. Recalling that $\cos(\alpha_1 + i\,\alpha_2) = \cos\alpha_1 \cosh\alpha_2 - i \sin\alpha_1 \sinh\alpha_2$, it is clear that so long as α is confined to the path described, $\cos\alpha$ is real, and in fact as α moves along the real axis from $\alpha_1 = 0$ to $\alpha_1 = \pi$ and then up along the line $\alpha_2 = \pi$, $\cos\alpha$ decreases continuously from 1 to $-\infty$. Now when $\alpha_1 = \pi$ and $\alpha_2 > 0$, $\cos\alpha = \cos(\pi + i\,\alpha_2) = -\cosh\alpha_2$, so that $\frac{2k}{m}(1 - \cos\alpha) = \frac{2k}{m}(1 + \cosh\alpha_2)$. Also $\sin(\pi + i\,\alpha_2) = \sin\pi \cosh\alpha_2 + i \cos\pi \sinh\alpha_2 = -i \sinh\alpha_2$. If $-\cosh\alpha$ is substituted for $\cos\alpha$ and $-i\sinh\alpha$ is substituted for $\sin\alpha$ in equation (27), it becomes equation (31). Thus in every case all characteristic frequencies can be calculated from equation (27).

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