A Riemannian Geometry in the q-Exponential Banach Manifold Induced by q-Divergences

Gabriel I. Loaiza O.
Hector R. Quiceno

Universidad EAFIT

August 28, 2013

Table of Contents

1. Objectives
2. q-exponential statistical Banach manifold
3. Induced Geometry manifold

Objectives

Objective

Give new mathematical developments to characterize the geometry induced by the q-exponential Banach manifold, by using q-divergence functionals, such that this geometry turns out to be a generalization of the geometry given by Fisher information metric and Levi-Civita connections.

Specific objectives

1. Introduce the metric.
2. Introduce the connections induced by the q-divergence functional, using the Eguchi relations.
3. Show the zero curvature of the manifold.

Objectives

Objective

Give new mathematical developments to characterize the geometry induced by the q-exponential Banach manifold, by using q-divergence functionals, such that this geometry turns out to be a generalization of the geometry given by Fisher information metric and Levi-Civita connections.

Specific objectives

1. Introduce the metric.
2. Introduce the connections induced by the q-divergence functional, using the Eguchi relations.
3. Show the zero curvature of the manifold.

Objectives

Objective

Give new mathematical developments to characterize the geometry induced by the q-exponential Banach manifold, by using q-divergence functionals, such that this geometry turns out to be a generalization of the geometry given by Fisher information metric and Levi-Civita connections.

Specific objectives

1. Introduce the metric.
2. Introduce the connections induced by the q-divergence functional, using the Eguchi relations.
3. Show the zero curvature of the manifold.

q-exponential statistical Banach manifold

Let q a real number such that $0 < q < 1$.

1. We consider the q-deformed exponential and logarithmic functions which are respectively defined by

$$e_q^x = (1 + (1-q)x)^{1/(1-q)}, \quad \text{if } \frac{-1}{1-q} \leq x \quad \text{and} \quad \ln_q(x) = \frac{x^{1-q} - 1}{1-q}, \quad \text{if } x > 0.$$

2. We consider the operations defined for real numbers x and y by

$$x \oplus_q y := x + y + (1-q)xy \quad \text{and} \quad x \ominus_q y := \frac{x - y}{1 + (1-q)y}, \quad \text{for } y \neq \frac{1}{q-1}.$$

3. It holds that $e_q^{(x_1 \ominus_q x_2)} = \frac{e_q^x}{e_q^{x_2}}$ and $e_q^{(x_1 \ominus_q x_2)} = \frac{e_q^x}{e_q^{x_2}}$.

q-exponential statistical Banach manifold

Let q a real number such that $0 < q < 1$.

1. We consider the q-deformed exponential and logarithmic functions which are respectively defined by

$$e_q^x = (1 + (1-q)x)^{1/(1-q)}, \quad \text{if } \frac{-1}{1-q} \leq x \quad \text{and} \quad \ln_q(x) = \frac{x^{1-q} - 1}{1-q}, \quad \text{if } x > 0.$$

2. We consider the operations defined for real numbers x and y by

$$x \oplus_q y := x + y + (1-q)xy \quad \text{and} \quad x \ominus_q y := \frac{x - y}{1 + (1-q)y}, \quad \text{for } y \neq \frac{1}{q-1}.$$

3. It holds that $e_q^{(x_1 \ominus_q x_2)} = \frac{e_q^{x}}{e_q^{x_2}}$ and $e_q^{(x_1 \ominus_q x_2)} = \frac{e_q^{x}}{e_q^{x_2}}$.

q-exponential statistical Banach manifold

Let (Ω, Σ, μ) be a probability space. Denote by $\mathfrak{M}_\mu$ the set of strictly positive probability densities μ-a.e.
For each $p \in \mathfrak{M}_\mu$ consider the probability space $(\Omega, \Sigma, p \cdot \mu)$, where $(p \cdot \mu)(A) = \int_A p d\mu$. Denote $\| \cdot \|_{p,\infty}$ the norm in $L^\infty(p \cdot \mu)$, with which

$$B_p := \{u \in L^\infty(p \cdot \mu) : \ E_p[u] = 0\},$$

is a closed normed subspace so is a Banach space.

The probability densities $p, z \in \mathfrak{M}_\mu$ are connected by a one-dimensional q-exponential model if there exist $r \in \mathfrak{M}_\mu$, $u \in L^\infty(r \cdot \mu)$, a real function of real variable ψ and $\delta > 0$ such that for all $t \in (-\delta, \delta)$, the function f defined by

$$f(t) = e_q^{tu \ominus_q \psi(t)} r,$$

satisfies that there are $t_0, t_1 \in (-\delta, \delta)$ for which $p = f(t_0)$ and $z = f(t_1)$. The function f is called one-dimensional q-exponential model.

q-exponential statistical Banach manifold

We define the mapping M_p given by

$$M_p(u) = E_p[e_q^{(u)}],$$

denoting its domain by $D_{M_p} \subset L^\infty(p \cdot \mu)$. Also we define the mapping

$$K_p : \; B_{p,\infty}(0,1) \to [0,\infty],$$

for each $u \in B_{p,\infty}(0,1)$, by

$$K_p(u) = \ln_q[M_p(u)].$$

If restricting M_p to $B_{p,\infty}(0,1)$, this function is analytic and infinitely Fréchet differentiable.

q-exponential statistical Banach manifold

Let (Ω, Σ, μ) be a probability space and q a real number with $0 < q < 1$. Let be $V_p := \{u \in B_p : \|u\|_{p,\infty} < 1\}$, for each $p \in M_\mu$. We define the maps $e_{q,p} : \mathcal{V}_p \to \mathfrak{M}_\mu$ by

$$e_{q,p}(u) := e_q^{(u \ominus_q K_p(u))} p,$$

which are injective and their ranges are denoted by $\mathcal{U}_p$. For each $p \in \mathfrak{M}_\mu$ the map $s_{q,p} : \mathcal{U}_p \to \mathcal{V}_p$ given by

$$s_{q,p}(z) := \ln_q\left(\frac{z}{p}\right) \ominus_q E_p\left[\ln_q\left(\frac{z}{p}\right)\right],$$

is precisely the inverse map of $e_{q,p}$. Maps $s_{q,p}$ are the coordinate maps for the manifold and the family of pairs $(U_p, s_{q,p})_{p \in M_\mu}$ define an atlas on M_μ; and the transition maps, for each $u \in s_{q,p_1}(\mathcal{U}_{p_1} \bigcap \mathcal{U}_{p_2})$, are given by

$$s_{p_2}(e_{p_1}(u)) = \frac{u \oplus_q \ln_q(\frac{p_1}{p_2}) - E_{p_2}[u \oplus_q \ln_q(\frac{p_1}{p_2})]}{1 + (1-q)E_{p_2}[u \oplus_q \ln_q(\frac{p_1}{p_2})]}.$$

q-exponential statistical Banach manifold

Given $u \in s_{q,p_1}(U_{q,p_1} \bigcap U_{q,p_2})$, we have that $D(s_{q,p_2} \circ s_{q,p_1}^{-1})(u) \cdot v = A(u) - B(u)E_{p_2}[A(u)]$, where $A(u)$, $B(u)$ are functions depending on u.

The collection of pairs $\{(\mathcal{U}_p, s_{q,p})\}_{p \in \mathfrak{M}_\mu}$ is a C^∞-atlas modeled on B_p, and the corresponding manifold is called *$q-$exponential statistical Banach manifold.*

Finally, the tangent bundle of the manifold, is characterized, (Proposition 15), by regular curves on the manifold, where the charts (trivializing mappings) are given by

$$(g, u) \in \mathcal{T}(\mathcal{U}_p) \to (s_{q,p}(g), A(u) - B(u)E_p[A(u)]),$$

defined in the collection of open subsets $U_p \times V_p$ of $M_\mu \times L^\infty(p \cdot \mu)$.

Induced Geometry manifold

The q-divergence functional is given as follow. Let f be a function, defined for all $t \neq 0$ and $0 < q < 1$, by $f(t) = -t \ \ln_q\left(\frac{1}{t}\right)$ and for $p, z \in \mathfrak{M}_\mu$. The q-divergence of z with respect to p is given by

$$I^{(q)}(z||p) := \int_\Omega p \ f\left(\frac{z}{p}\right) \ d\mu = \frac{1}{1-q}\left[1 - \int_\Omega z^q p^{1-q} \ d\mu\right], \tag{1}$$

which is the Tsallis divergence functional.

We have that the manifold is related with the q-divergence functional as
$$s_{q,p}(z) = \left(\frac{1}{1+(q-1)I^{(q)}(p||z)}\right)\left(\ln_q\left(\frac{z}{p}\right) + I^{(q)}(p||z)\right).$$

Proposition

Let $p, z \in M_\mu$ then $(d_u)_z \ I^{(q)}(z||p)|_{z=p} = (d_v)_p \ I^{(q)}(z||p)|_{z=p} = 0$, where the subscript p, z means that the directional derivative is taken with respect to the first and the second arguments in $I^{(q)}(z||p)$, respectively, along the direction $u \in T_z(M_\mu)$ or $v \in T_p(M_\mu)$.

Induced Geometry manifold

The q-divergence functional is given as follow. Let f be a function, defined for all $t \neq 0$ and $0 < q < 1$, by $f(t) = -t \ \ln_q\left(\frac{1}{t}\right)$ and for $p, z \in \mathfrak{M}_\mu$. The q-divergence of z with respect to p is given by

$$I^{(q)}(z||p) := \int_\Omega p \ f\left(\frac{z}{p}\right) \ d\mu = \frac{1}{1-q}\left[1 - \int_\Omega z^q p^{1-q} \ d\mu\right], \quad (1)$$

which is the Tsallis divergence functional.

We have that the manifold is related with the q-divergence functional as
$$s_{q,p}(z) = \left(\frac{1}{1+(q-1)I^{(q)}(p||z)}\right)\left(\ln_q\left(\frac{z}{p}\right) + I^{(q)}(p||z)\right).$$

Proposition

Let $p, z \in M_\mu$ then $(d_u)_z \ I^{(q)}(z||p)|_{z=p} = (d_v)_p \ I^{(q)}(z||p)|_{z=p} = 0$, where the subscript p, z means that the directional derivative is taken with respect to the first and the second arguments in $I^{(q)}(z||p)$, respectively, along the direction $u \in T_z(M_\mu)$ or $v \in T_p(M_\mu)$.

Induced Geometry manifold

According to Proposition (16), the functional $I^{(q)}(z||p)$ is bounded, since:
$I^{(q)}(z||p) \geq 0$ and equality holds iff $p = z$ and $I^{(q)}(z||p) \leq \int_{\Omega} (z-p)\ f'\left(\frac{z}{p}\right)\ d\mu.$
Then, together with previous proposition, the q-divergence functional induces a Riemannian metric g and a pair of connections, see Eguchi, given by:

$$g(u,v) = -(d_u)_z\ (d_v)_p\ I^{(q)}(z||p)|_{z=p} \quad (2)$$

$$\langle \nabla_w u, v\rangle = -(d_w)_z\ (d_u)_z\ (d_v)_p\ I^{(q)}(z||p)|_{z=p}, \quad (3)$$

where $v \in T_p(M_\mu)$, $u \in T_p(M_\mu)$ and w is a vector field.

Induced Geometry manifold

Denote $\Sigma(\mathfrak{M}_\mu)$ the set of vector fields $u : \mathcal{U}_p \rightarrow T_p(\mathcal{U}_p)$, and $F(M_\mu)$ the set of C^∞ functions $f : U_p \rightarrow R$. The following result establish the metric.
By direct calculation over $I^{(q)}(z||p)$, we obtain
$(d_v)_p I^{(q)}(z||p) = \frac{1}{1-q} \int_\Omega \left[(1-q) - (1-q)p^{(-q)} z^{(q)}\right] \, v d\mu$ and
$(d_u)_z \, (d_v)_p \, I^{(q)}(z||p) = -q \int_\Omega \left[p^{(-q)} z^{(q-1)}\right] \, uv d\mu$, so by (2), it follows
$g(u,v) = q \int_\Omega \frac{uv}{p} \, d\mu$. Ten we have the follow result.

Proposition

Let $p, z \in M_\mu$ and v, u vector fields, the metric tensor (field) $g : \Sigma(\mathfrak{M}_\mu) \times \Sigma(\mathfrak{M}_\mu) \rightarrow F(M_\mu)$ is given by

$$g(u,v) = q \int_\Omega \frac{uv}{p} d\mu.$$

Induced Geometry manifold

Denote $\Sigma(\mathfrak{M}_\mu)$ the set of vector fields $u : \mathcal{U}_p \to T_p(\mathcal{U}_p)$, and $F(M_\mu)$ the set of C^∞ functions $f : U_p \to R$. The following result establish the metric.
By direct calculation over $I^{(q)}(z||p)$, we obtain
$(d_v)_p I^{(q)}(z||p) = \frac{1}{1-q} \int_\Omega \left[(1-q) - (1-q)p^{(-q)} z^{(q)}\right] \; v d\mu$ and
$(d_u)_z \; (d_v)_p \; I^{(q)}(z||p) = -q \int_\Omega \left[p^{(-q)} z^{(q-1)}\right] \; uv d\mu$, so by (2), it follows
$g(u,v) = q \int_\Omega \frac{uv}{p} \; d\mu$. Ten we have the follow result.

Proposition

Let $p, z \in M_\mu$ and v, u vector fields, the metric tensor (field) $g : \Sigma(\mathfrak{M}_\mu) \times \Sigma(\mathfrak{M}_\mu) \to F(M_\mu)$ is given by

$$g(u,v) = q \int_\Omega \frac{uv}{p} d\mu.$$

Induced Geometry manifold

Proposition

The connections are characterized as follows. The family of covariant derivatives (connections) $\bigtriangledown_w^{(q)} u : \Sigma(\mathfrak{M}_\mu) \times \Sigma(\mathfrak{M}_\mu) \to \Sigma(\mathfrak{M}_\mu)$, is given as

$$\bigtriangledown_w^{(q)} u = d_w u - \left(\frac{1-q}{p}\right) uw.$$

It is easy to prove that the associated conjugate connection is given by $\bigtriangledown_w^{*(q)} u = d_w u - \frac{q}{p} uw$. Notice that taking $q = \frac{1-\alpha}{2}$ yields to the Amaris's one-parameter family of $\alpha-$connections in the form

$$\bigtriangledown_w^{(\alpha)} u = d_w u - \left(\frac{1+\alpha}{2p}\right) uw;$$

and taking $q = \frac{1}{2}$ the Levi-Civita connection results.

Induced Geometry manifold

Proposition

The connections are characterized as follows. The family of covariant derivatives (connections) $\nabla_w^{(q)} u : \Sigma(\mathfrak{M}_\mu) \times \Sigma(\mathfrak{M}_\mu) \to \Sigma(\mathfrak{M}_\mu)$, is given as

$$\nabla_w^{(q)} u = d_w u - \left(\frac{1-q}{p} \right) \; uw.$$

It is easy to prove that the associated conjugate connection is given by $\nabla_w^{*(q)} u = d_w u - \frac{q}{p} uw$. Notice that taking $q = \frac{1-\alpha}{2}$ yields to the Amaris's one-parameter family of $\alpha-$connections in the form

$$\nabla_w^{(\alpha)} u = d_w u - \left(\frac{1+\alpha}{2p} \right) \; uw;$$

and taking $q = \frac{1}{2}$ the Levi-Civita connection results.

Induced Geometry manifold

Proposition

Finally, we characterize this geometry by calculating the curvature and torsion tensors, for which it will be proved that equals zero, i.e, for the q-exponential manifold and the connection given in the previous proposition, the curvature tensor and the torsion tensor satisfy $R(u, v, w) = 0$ and $T(u, v) = 0$.

References

- Amari, S.: *Differential-geometrical methods in statistics*. Springer, New York (1985)
- Amari, S., Nagaoka, H.: *Methods of information Geometry*. RI: American Mathematical Society. Translated from the 1993 Japanese original by Daishi Harada, Providence (2000)
- Amari, S. Ohara, A.: *Geometry of q-exponential family ofprobability distributions*. Entropy. 13, 1170-1185 (2011)
- Borges, E.P.: *Manifestaões dinâmicas e termodinâmicas de sistemas não-extensivos*. Tese de Dutorado, Centro Brasileiro de Pesquisas Fisicas, Rio de Janeiro (2004).
- Cena, A., Pistone, G.: *Exponential statistical manifold*. Annals of the Institute of Statistical Mathematics. 59, 27-56 (2006)
- Dawid, A.P: *On the conceptsof sufficiency and ancillarity in the presence of nuisance parameters*. Journal of the Royal Statistical Society B. 37, 248-258 (1975)
- Efron, B.: *Defining the curvature of a statistical problem (with applications to second order efficiency)*. Annals of Statistics. 3, 1189-1242 (1975)

References

- Eguchi, S.: *Second order efficiency of minimum coontrast estimator in a curved exponential family*. Annals of Statistics. 11, 793-803 (1983)
- Furuichi, S.: *Fundamental properties of Tsallis relative entropy*. J. Math. Phys. 45, 4868-4877 (2004)
- Gibilisco, P., Pistone, G.: *Connections on non-parametric statistical manifolds by Orlicz space geometry*. Infinite Dimensional Analysis Quantum Probability and Related Topics. 1, 325-347 (1998)
- Kadets, M.I., Kadets, V.M: *series in Banach spaces*, Birkaaauser Verlang, Besel. Conditional and undconditional convergence, Traslated for the Russian by Andrei Iacob. (1997).
- Kulback, S., Leibler, R.A.: *On Information and Sufficiency*. Annals of Mathematics and Statistics. 22, 79-86 (1951)
- Loaiza, G., Quiceno, H.R.: *A q-exponential statistical Banach manifold*. Journal of Mathematical Analysis and Applications. 398, 446-476 (2013).
- Pistone, G.: *k-exponential models from the geometrical viewpoint*. The European Physical Journal B. Springer Berlin. Online 70 29-37 (2009)
- Pistone, G., Sempi, C.: *An infinite-dimensional geometric structure on the space of all the probability measures equivalent to a given one*. The Annals of statistics. 23(5), 1543–1561 (1995).
- Tsallis, C.: *Possible generalization of Boltzmann-Gibbs statistics*. J.Stat. Phys. 52, 479-487 (1988)

THANKS