Applied Mathematical Sciences, Vol. 6, 2012, no. 69, 3411 - 3423

Wavelet-Petrov-Galerkin Method for the Numerical Solution of the KdV Equation

Jairo Villegas G. and Jorge Castaño B.

Departamento de Ciencias Básicas
Universidad EAFIT
Medellín, Colombia
javille@eafit.edu.co

Julio Duarte V. and Esper Fierro Y.

Departamento de Matemáticas
Universidad Surcolombiana
Neiva, Colombia

Abstract

The development of numerical techniques for obtaining approximate solutions of partial differential equations has very much increased in the last decades. Among these techniques are the finite element methods and finite difference. Recently, wavelet methods are applied to the numerical solution of partial differential equations, pioneer works in this direction are those of Beylkin, Dahmen, Jaffard and Glowinski, among others. In this paper, we employ the Wavelet-Petrov-Galerkin method to obtain the numerical solution of the equation Korterweg-de Vries (KdV).

Mathematics Subject Classification: 35A40, 65D05, 65M12, 65M15

Keywords: KdV equation, soliton, wavelet, Wavelet-Petrov-Galerkin Method

1 Introduction

The main objective of this paper is to present a numerical solution of Korteweg-de Vries equation(KdV)

$$\frac{\partial u}{\partial t} + \mu u \frac{\partial u}{\partial x} + \epsilon \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in \mathbb{R}, \ \ t > 0, \tag{1}$$

where μ and ϵ are positive constants, using the method of Petrov-Galerkin-Wavelet. This equation appears in the study of waves in shallow water in the fluid dynamics [9, 10, 15]. Equation KdV satisfied the property that the non linear term uu_x and the dispersion u_{xxx} balance each other thereby generating wave solutions which propagate maintaining same form throughout. The term soliton was coined by Zabusky and Kruskal to describe this solitary wave, solution of the KdV equation [2, 9, 11].

Daubechies presents a method to construct wavelets with compact support and scale functions with arbitrary regularity and zero momentum [8]. However, the price for these good properties is the absence of symmetry and ample support. This disadvantage disappears in the context of biorthogonal wavelets, a concept introduced by Cohen, Daubechies and Feauveau in [6]. In this context, two non orthogonal base functions $\psi_{j,k}$ and $\psi^*_{j,k}$, also called wavelets, are constructed based on the translated scale functions ψ y ψ^*.

As opposed to Galerkin's method, where the same base functions are used as both test and admissible, in the Petrov-Galerkin method, test and trial functions are different. In the Petrov-Galerkin approximation by biorthogonal wavelets, the idea is to have one of the families of base functions as admissible and its dual as test functions. The advantage of this method is the preconditioning and discretization of the wavelets for adaptive algorithms [5, 17, 19]. Therefore, the technique wavelets provide efficient numerical methods, as alternative to the classical methods [1, 7, 12, 13, 17].

The aim of this paper is to study the precision of Petrov-Galerkin's method by using biorthogonal wavelets, in the solution of KdV equation $u_t + \mu\, uu_x + \epsilon\, u_{xxx} = 0$ with the initial condition $u(x, 0) = u_0(x)$. Instead of multi-level wavelet bases, we expand the approximate solutions in terms of scale functions $\phi_{m,k}(x)$ of only one level as a basis for admissible functions, while the dual $\phi^*_{m,k}(x)$ are the test functions. The study of convergence is realized through the Fourier analysis.

The paper is organized as follows: After some preliminary remarks in Section 1, in Section 2, we give and discuss some facts showing where the method fundamental ideas come from. In Section 3 we discuss some aspects and results concerning convergence and stability in the context of Petrov-Galerkin, related to KdV equation

2 Wavelet-Petrov-Galerkin method

The method of Petrov-Galerkin is a particular case of a more general method, known as Weighted Residue Method [10, 15]. Let us consider now the weak formulation of KdV equation (1). Let $0 \leq \alpha \leq 3$ y $\beta = 3 - \alpha$, for any test function v, $\beta-$regular, we have

$$\int_{\mathbb{R}} v\frac{\partial u}{\partial t}dx + \mu\int_{\mathbb{R}} vu\frac{\partial u}{\partial x}dx + \epsilon\int_{\mathbb{R}} v\frac{\partial^3 u}{\partial x^3}dx = 0.$$

Then the weak formulation can be expressed as

$$\Big(\frac{\partial u}{\partial t}, v\Big) + \mu\Big(u\frac{\partial u}{\partial x}, v\Big) + \epsilon(-1)^\beta\Big(\frac{\partial^\alpha u}{\partial x^\alpha}, \frac{d^\beta v}{dx^\beta}\Big) = 0, \tag{2}$$

where $(\cdot,\cdot)$ is the inner product in $L_2(\mathbb{R})$. Recall that a function g is $r-$regular, if there is a constant $M_{s,n} > 0$ with $\left|g^{(s)}(x)\right| \leq M_{s,n}(1+|x|)^{-n}$, for each $x \in \mathbb{R}$, all index s with $0 \leq s \leq r$ and all $n \in \mathbb{N}$ [16].

Applying now Petrov-Galerkin method, taking as admissible functions $\phi_{h,k}(x) = h^{-1/2}\phi(h^{-1}x - k)$, $k \in \mathbb{Z}$, where ϕ is a real valued function, $r-$ regular, $r \geq 1$ and $h > 0$ is the discretization step. The approximation spaces $V_h \subset L_2(\mathbb{R})$ are generated by $\{\phi_{h,k}(x), k \in \mathbb{Z}\}$, and the exact solution of equation KdV (1) is approximated by the expression $u_h(x,t) = \sum_k U_k(t)\phi_{h,k}(x)$. Similarly, test functions are taken in the form $\phi^*_{h,k}(x)$, defined in terms of a r^*-regular function dual ϕ^*, with $r + r^* \geq 3$.

In the weak formulation (2), we choose $\alpha \leq r$ such that $\beta \leq r^*$. If we replace u by the solution $u_h(x,t)$ and v for each test function $\phi^*_{h,l}(x)$, we obtain

$$\begin{aligned}
&\left(\frac{\partial u}{\partial t}, v\right) + \mu\left(u\frac{\partial u}{\partial x}, v\right) + \epsilon(-1)^\beta\left(\frac{\partial^\alpha u}{\partial x^\alpha}, \frac{d^\beta v}{dx^\beta}\right) = \\
&\int_{\mathbb{R}} \frac{\partial}{\partial t}\left(\sum_k \phi_{h,k}(x)U_k(t)\right)\phi^*_{h,l}(x)dx + \\
&+ \mu\int_{\mathbb{R}}\left(\sum_s U_s(t)\phi_{hs}(x)\right)\left(\frac{\partial}{\partial x}\sum_k U_k(t)\phi_{hk}(x)\right)(\phi^*_{h,l}(x))dx + \\
&+ (-1)^\beta\epsilon\int_{\mathbb{R}}\left(\frac{\partial^\alpha}{\partial x^\alpha}\sum_k U_k(t)\phi_{hk}(x)\right)\left(\frac{d^\beta}{dx^\beta}\phi^*_{h,l}(x)\right)dx \\
&= h^{-1}\sum_k\int_{\mathbb{R}}\phi(h^{-1}x-k)\phi^*(h^{-1}x-l)\frac{dU_k(t)}{dt}dx + \\
&+ \mu h^{-3/2}\sum_s\sum_k\int_{\mathbb{R}}\phi(h^{-1}x-s)\phi^*(h^{-1}x-l)\frac{d}{dx}\phi(h^{-1}x-k)U_s(t)U_k(t)dx + \\
&+ h^{-1}\epsilon(-1)^\beta\sum_k\int_{\mathbb{R}}\frac{d^\alpha}{dx^\alpha}\phi(h^{-1}x-k)\frac{d^\beta}{dx^\beta}\phi^*(h^{-1}x-l)U_k(t)dx = 0.
\end{aligned}$$

If now $U_k(t) = U_k$ and introduce the change of variable $y = h^{-1}x - k$, the above expression can now be written as

$$\sum_k a(k)\frac{dU_k}{dt} + \mu h^{-3/2}\sum_s\sum_k b(l,k)U_sU_k + h^{-3}\epsilon\sum_k c(k)U_k = 0,$$

where $a(k) = \int_{\mathbb{R}} \phi(y)\phi^*(y-k)dy$, $b(l,k) = \int_{\mathbb{R}} \frac{d\phi(y)}{dy}\phi(y-l)\phi^*(y-k)dy$, $c(k) = (-1)^\beta \int_{\mathbb{R}} \frac{d^\alpha\phi(y)}{dy^\alpha}\frac{d^\beta\phi^*(y-k)}{dy^\beta}dy$. The coefficients U_k are determined from the following system of ordinary differential equations

$$\sum_k a(l-k)\frac{d}{dt}U_k + \mu h^{-3/2}\sum_s\sum_k b(s-k,l-k)U_sU_k + h^{-3}\epsilon\sum_k c(l-k)U_k = 0. \tag{3}$$

In matrix form, this last equation is

$$\frac{d}{dt}LU + \mu U^T MU + \epsilon NU = 0 \tag{4}$$

where $U = \big(U_k\big)$, $L(l,k) = a(l-k)$, $M(l,k,s) = h^{-3/2}b(l-k,l-s)$, $N(l,k) = h^{-3}c(l-k)$. The initial conditions $U_k(0)$, $k \in \mathbb{Z}$, are the coefficients of $u_h(x,0) = R_hu_0 \in V_h$, where R_h is some initial approximation scheme to be fixed below.

With a time increment $\Delta t = t_{n+1} - t_n$ and applying the trapezoidal rule we obtain $\frac{dU}{dt} = \frac{U^{n+1}-U^n}{\Delta t}$, where $U^n = U(n\Delta t)$, $n \geq 0$, and equation (4) becomes

$$L\left[\frac{U^{n+1}-U^n}{\Delta t}\right] + \mu U^T MU + \epsilon NU = 0.$$

Now setting $G(U) = \mu U^T MU + \epsilon NU$ we have

$$L\big(U^{n+1} - U^n\big) + \frac{G(U^{n+1}) + G(U^n)}{2}\Delta t = 0, \tag{5}$$

and this equation is finally solved by using Newton's iteration method.

3 Convergence and stability results

So far we have only required that functions ϕ and ϕ^* enjoy some regularity. However, to obtain good approximation results, additional conditions to be discussed in this section are necessary.

Let us begin solving linearized KdV equation by method of the Fourier transform [13]. That is applying Fourier transform to equation

$$u_t + \mu\, u_x + \epsilon\, u_{xxx} = 0 \tag{6}$$

with the same initial condition $u(x,0) = u_0(x)$, we obtain the differential equation $\hat{u}_t + i(\mu\omega - \epsilon\,\omega^3)\hat{u} = 0$, where $\hat{u} = \hat{u}(\omega,t)$ is the Fourier transform, whose solution is then $\hat{u}(\omega,t) = \hat{u}_0(\omega)e^{-it(\mu\omega-\epsilon\,\omega^3)}$.

Now, $u(x,t)$ is obtained by using inverse Fourier transform, that is,

$$u(x,t) = \mathcal{F}^{-1}\Big(e^{-it(\mu\omega-\epsilon\,\omega^3)}\hat{u}_0(\omega)\Big).$$

Defining the bounded linear operator $E(t)$ on $L_2(\mathbb{R})$ by

$$E(t)v := \mathcal{F}^{-1}\big(e^{-it(\mu\omega-\epsilon\,\omega^3)}\hat{v}(\omega)\big)$$

the solution u can be written as $u(x,t) = E(t)u_0(x)$.

On the other hand, the weak formulation of linearized KdV equation (6) is

$$\int_{\mathbb{R}} vu_t dx + \mu \int_{\mathbb{R}} vu_x dx + \epsilon \int_{\mathbb{R}} vu_{xxx} dx = 0, \tag{7}$$

where $v \in C_0^\infty(\mathbb{R})$ is a $\beta-$regular test function, $0 \le \alpha \le 3$ and $\beta = 3-\alpha$. The equation (7) can be expressed as $\left(\frac{\partial u}{\partial t}, v\right) + \mu\left(\frac{\partial u}{\partial x}, v\right) + \epsilon(-1)^\beta\left(\frac{\partial^\alpha u}{dx^\alpha}, \frac{d^\beta v}{dx^\beta}\right) = 0$, where $(\cdot,\cdot)$ is the inner product on $L_2(\mathbb{R})$.

As in the non linear case, considering spaces $V_h \subset L_2(\mathbb{R})$ with the approximate solution $u_h(x,t) = \sum_k U_k(t)\phi_{hk}(x)$ and replacing u by u_h and v by $\phi^*_{hl}(x) = h^{-1/2}\phi^*(h^{-1}x - l)$ we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}} vu_t dx + \mu \int_{\mathbb{R}} vu_x dx + \epsilon(-1)^\beta \int_{\mathbb{R}} \left(\frac{\partial^\alpha u}{dx^\alpha}\frac{\partial^\beta v}{\partial x^\beta}\right) dx = \\
&= h^{-1}\sum_k \int_{\mathbb{R}} \phi^*(h^{-1}x - l)\phi(h^{-1}x-k)\frac{dU_k(t)}{dt}dx + \\
&+ \mu h^{-1}\sum_k \int_{\mathbb{R}} \phi^*(h^{-1}x-l)\frac{d}{dx}\phi(h^{-1}x-k)U_k(t)dx \\
&+ \epsilon(-1)^\beta h^{-1}\sum_k \int_{\mathbb{R}} \frac{d^\alpha}{dx^\alpha}\phi(h^{-1}x-k)\frac{d^\beta}{dx^\beta}\phi^*(h^{-1}x-l)U_k(t)dx = 0
\end{aligned}$$

and with the change of variable $y = h^{-1}x - k$ and $U_k(t) = U_k$ one obtains

$$\sum_k a(l-k)\frac{dU_k}{dt} + h^{-1}\mu\sum_k d(l-k)U_k + \epsilon(-1)^\beta h^{-3}\sum_k c(l-k)U_k = 0$$

which finally can be written as

$$\sum_k \Big[a(l-k)\frac{dU_k}{dt} + h^{-1}\big[\mu d(l-k) + h^{-2}\epsilon c(l-k)\big]U_k\Big] = 0, \tag{8}$$

where $d(k) = \int_{\mathbb{R}} \frac{d\phi(x)}{dx}\phi^*(x-k)dx$. In analogous manner, we get the system equivalent to (5)

$$\sum_k a(l-k)\big[U_k^{n+1} - U_k^n\big] + h^{-1}\Delta t\sum_k\big[\mu d(l-k) + h^{-2}\epsilon c(l-k)\big]\frac{U_k^{n+1}+U_k^n}{2} = 0. \tag{9}$$

Equations (8) and (9) are in the form of discrete convolution. Hence, using again discrete Fourier transform $\tilde{a}(\omega)=\sum_{k\in\mathbb{Z}} a(k)e^{-ik\omega}=\sum_{k\in\mathbb{Z}} a_k e^{-ik\omega}$, where $a=(\dots,a_{-1},a_0,a_1,\dots)\in\ell^2(\mathbb{Z})$, we have respectively

$$\tilde{a}(\omega)\frac{d}{dt}\widetilde{U}(\omega,t)+h^{-1}[\mu\tilde{d}(\omega)+h^{-2}\epsilon\tilde{c}(\omega)]\widetilde{U}(\omega,t)=0$$

$$\tilde{a}(\omega)[\widetilde{U}^{n+1}(\omega)-\widetilde{U}^{n}(\omega)]+h^{-1}\Delta t[\mu\tilde{d}(\omega)+h^{-2}\epsilon\tilde{c}(\omega)]\frac{\widetilde{U}^{n+1}(\omega)+\widetilde{U}^{n}(\omega)}{2}=0,$$

the first of these equations can be written as

$$\frac{d}{dt}\widetilde{U}(\omega,t)+h^{-1}\Big[\frac{\mu\tilde{d}(\omega)+h^{-2}\epsilon\tilde{c}(\omega)}{\tilde{a}(\omega)}\Big]\widetilde{U}(\omega,t)=0$$

or in shorter form $\frac{d}{dt}\widetilde{U}(\omega,t)+\frac{W_h(\omega)}{h}\widetilde{U}(\omega,t)=0$ where

$$W_h(\omega)=\frac{\mu\tilde{d}(\omega)+h^{-2}\epsilon\tilde{c}(\omega)}{\tilde{a}(\omega)} \tag{10}$$

the solution of differential equation is $\widetilde{U}(\omega,t)=ce^{-\left(\frac{W_h(\omega)t}{h}\right)}$ and the initial condition for $t=0$ is $\widetilde{U}(\omega,0)=c$ hence $\widetilde{U}(\omega,t)=\widetilde{U}(\omega,0)e^{-\left(\frac{W_h(\omega)t}{h}\right)}$. Now, as to the second equation, we have

$$\widetilde{U}^{n+1}(\omega)-\widetilde{U}^{n}(\omega)+h^{-1}\Delta t\Big[\frac{\mu\tilde{d}(\omega)+h^{-2}\epsilon\tilde{c}(\omega)}{\tilde{a}(\omega)}\Big]\frac{\widetilde{U}^{n+1}(\omega)+\widetilde{U}^{n}(\omega)}{2}=0$$

and grouping terms it follows that

$$\widetilde{U}^{n+1}(\omega)\left[1+\Delta t\frac{W_h(\omega)}{2h}\right]-\widetilde{U}^{n}(\omega)\left[1-\Delta t\frac{W_h(\omega)}{2h}\right]=0$$

therefore, the solution of the difference equation with the given initial value is

$$\widetilde{U}^{n}(\omega)=\left[\frac{1-\left(\frac{\Delta t}{2h}\right)W_h(\omega)}{1+\left(\frac{\Delta t}{2h}\right)W_h(\omega)}\right]^n\widetilde{U}(\omega,0)$$

or more concisely $\widetilde{U}^{n}(\omega)=[A_h(\omega)]^n\widetilde{U}(\omega,0)$, where

$$A_h(\omega)=\left[\frac{1-\left(\frac{\Delta t}{2h}\right)W_h(\omega)}{1+\left(\frac{\Delta t}{2h}\right)W_h(\omega)}\right].$$

The Fourier transform of the solution $u_h(x,t) = \sum_k U_k(t)\phi_{hk}(x)$ is given by

$$\hat{u}_h(\omega,t) = \sum_k U_k(t)h^{-1/2}\int_{\mathbb{R}} \phi(h^{-1}x-k)e^{-ix\omega}dx, \qquad k \in \mathbb{Z}$$

and with the change of variable $y = h^{-1}x - k$ it becomes

$$\begin{aligned}\hat{u}_h(\omega,t) &= \sum_k U_k(t)h^{1/2}\int_{\mathbb{R}} e^{-i(y+k)h\omega}\phi(y)dy = h^{1/2}\widetilde{U}(h\omega,t)\hat{\phi}(h\omega)\\ &= \widetilde{U}(h\omega,0)e^{-\left(\frac{W_h(h\omega)t}{h}\right)}h^{1/2}\hat{\phi}(h\omega)\end{aligned}$$

since $u_h(x,0) = R_h u_0(x)$, $\hat{u}_h(\omega,0) = \mathcal{F}\big(R_h u_0\big)(\omega) = \widetilde{U}(h\omega,0)h^{1/2}\hat{\phi}(h\omega)$ and hence, $\hat{u}_h(\omega,t) = e^{-\left(\frac{W_h(h\omega)t}{h}\right)}\mathcal{F}\big(R_h u_0\big)(\omega)$. By taking inverse Fourier transform, we obtain $u_h(x,t) = \mathcal{F}^{-1}\Big[e^{-\left(\frac{W_h(h\omega)t}{h}\right)}\mathcal{F}\big(R_h u_0\big)(\omega)\Big]$. Now, if the operator $F_h(t)$ is defined by $F_h(t)v = \mathcal{F}^{-1}\Big[e^{-\left(\frac{W_h(h\omega)t}{h}\right)}\hat{v}\Big]$ the solution $u_h(x,t)$ can be expressed as $u_h(x,t) = F_h(t)\big[R_h u_0(x)\big]$.

Observe that $F_h(t)v$ can be written in terms of discrete convolution, that is, $F_h(t)v(x) = \sum_k f_k\Big(\frac{t}{h}\Big)v(x-kh)$, where $f_k(t/h)$ are the Fourier coefficients of the exponential $e^{-W_h(\omega)\,t/h}$. In the same way, the discrete solution $u_h^n(x) = \sum_k U_k^n\phi_{hk}(x)$ has Fourier transform $\hat{u}_h^n(\omega) = \sum_k h^{-1/2}U_k^n\int_{\mathbb{R}}\phi(h^{-1}x - k)e^{-ix\omega}dx$. Again setting $y = h^{-1}x - k$ yields

$$\hat{u}_h^n(\omega) = \sum_k U_k^n e^{-i\omega kh}h^{1/2}\int_{\mathbb{R}}\phi(y)e^{-ih\omega y}dy = \widetilde{U}^n(h\omega)h^{1/2}\hat{\phi}(h\omega),$$

but $\widetilde{U}^n = \widetilde{U}(\omega,0)[A_n(\omega)]^n$, and thus, $\hat{u}_h^n(\omega) = \widetilde{U}(h\omega,0)[A_n(h\omega)]^n h^{1/2}\hat{\phi}(h\omega) = [A_n(h\omega)]^n\mathcal{F}\big(R_h u_0\big)(\omega)$ so when applying inverse Fourier transform one obtains

$$u_h^n(x) = \mathcal{F}^{-1}\Big([A_n(h\omega)]^n\mathcal{F}\big(R_h u_0\big)(\omega)\Big) = G_h^n\Big(R_h u_0(x)\Big),$$

where $G_h^n v = \mathcal{F}^{-1}\Big([A_n(h\omega)]^n\hat{v}\Big)$.

Note that the method is stable if $\mathrm{Re}\big(W_h(\omega)\big) \geq 0$ for each real ω; here $\mathrm{Re}(z)$ is the real part of the complex number z. The method is called conservative if $\mathrm{Re}\big(W_h(\omega)\big) = 0$ for each ω, or dissipative if $\mathrm{Re}\big(W_h(\omega)\big) > 0$ over some interval.

Finally we will study pointwise convergence of the approximate solutions $u_h(x,t)$ and $u_h^n(x)$ at the mesh points $x = hk$. With the propose of avoiding errors due to the approximation of the initial data, we will assume that $R_h u_0$ interpolates u_0 at such points.

We will also assume that ϕ is $r-$regular, $\hat{\phi}(0) \neq 0$ and $\hat{\phi}(\omega)$ has zeros of order $p+1$ for all points $\omega = 2k\pi$, $k \in \mathbb{Z}$ nonzero, for some integer $p \geq 0$. The set of all functions satisfying these properties is denoted by $\mathcal{H}_{r,p}$.

3.1 Convergence

Let us suppose now that $\phi \in \mathcal{H}_{r,p}$ and $\phi^* \in \mathcal{H}_{r^*,p^*}$. For $0 \leq \alpha \leq r$ and $0 \leq \beta \leq r^*$, let us define the $2\pi-$periodic function $\zeta_{\alpha,\beta}(\omega) = \sum_{k\in\mathbb{Z}} I_{\alpha,\beta}(k)e^{-ik\omega}$ where $I_{\alpha,\beta}(k) = \int_{-\infty}^{\infty} \frac{d^\alpha\phi}{dx^\alpha}(x)\frac{d^\beta\phi^*}{dx^\beta}(x-k)dx$, then $\zeta_{\alpha,\beta}(\omega)$ defines a $C^\infty-$function and $\zeta_{\alpha,\beta}(\omega) = i^{\alpha-\beta}\omega^{\alpha+\beta}\hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)} + O\big(\omega^{p+p^*+2}\big)$, when $\omega \to 0$.

In fact, to apply Parceval's relation we let $f(x) = \frac{d^\alpha\phi}{dx^\alpha}(x)$ and $g(x) = \frac{d^\beta\phi^*}{dx^\beta}(x-k)$, and so,

$$\begin{aligned} I_{\alpha,\beta}(k) &= \int_{-\infty}^{\infty} \frac{d^\alpha\phi}{dx^\alpha}(x)\frac{d^\beta\phi^*}{dx^\beta}(x-k)dx = \int_{-\infty}^{\infty} f(x)\overline{g(x-k)}dx \\ &= 2\pi\int_{\mathbb{R}} f(x)\left[\int_{\mathbb{R}} \overline{\hat{g}(\omega)}e^{-i\omega(x-k)}d\omega\right]dx = 2\pi\mathcal{F}\big[\hat{f}\overline{\hat{g}}\,\big](-k). \end{aligned}$$

Hence, $\zeta_{\alpha,\beta}(\omega) = \sum_k I_{\alpha,\beta}(k)e^{-ik\omega} = \sum_k 2\pi\mathcal{F}\big[\hat{f}\overline{\hat{g}}\,\big](-k)e^{-ik\omega}$, if we take $h(\omega) = \big(\hat{f}\overline{\hat{g}}\big)(\omega)$, using Poisson summation formula [16] we have $\zeta_{\alpha,\beta}(\omega) = \sum_k \hat{f}(\omega + 2k\pi)\overline{\hat{g}(\omega+2k\pi)}$, Lemarié in [14, Lemme 1, p 159] shows that the function $\zeta_{\alpha,\beta}(\omega) \in C^\infty$. Now, $\mathcal{F}\left[\frac{d^\alpha\phi}{dx^\alpha}\right](\omega) = i^\alpha\omega^\alpha\hat{\phi}(\omega)$ and $\mathcal{F}\left[\frac{d^\beta\phi^*}{dx^\beta}\right](\omega) = i^\beta\omega^\beta\hat{\phi}^*(\omega)$ and therefore,

$$\zeta_{\alpha,\beta}(\omega) \;=\; i^{\alpha-\beta}\sum_{k\in\mathbb{Z}}(\omega+2k\pi)^{\alpha+\beta}\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}. \tag{11}$$

This last expression can be written as

$$\zeta_{\alpha,\beta}(\omega) = i^{\alpha-\beta}\left(\omega^{\alpha+\beta}\hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)} + \sum_{k\neq 0}(\omega+2k\pi)^{\alpha+\beta}\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}\right),$$

and if now $\mathcal{R}_{\alpha,\beta}(\omega) = \sum_{k\neq 0}(\omega+2k\pi)^{\alpha+\beta}\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}$ it follows that $\zeta_{\alpha,\beta}(\omega) = i^{\alpha-\beta}\Big(\omega^{\alpha+\beta}\hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)} + \mathcal{R}_{\alpha,\beta}(\omega)\Big)$. Since $\hat{\phi}(\omega)$ and $\hat{\phi}^*(\omega)$ have zeros of order $p+1$ and p^*+1, respectively, $\mathcal{R}_{\alpha,\beta}(\omega)$ has zeros of order $p+p^*+2$, and moreover it is a $C^\infty-$function. So, $\mathcal{R}_{\alpha,\beta}(\omega) = O\big(\omega^{p+p^*+2}\big)$ if $\omega \to 0$. Consequently, $\zeta_{\alpha,\beta}(\omega) = i^{\alpha-\beta}\omega^{\alpha+\beta}\hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)} + O\big(\omega^{p+p^*+2}\big)$.

Let us assume now that $\phi \in \mathcal{H}_{r,p}$ and $\phi^* \in \mathcal{H}_{r^*,p^*}$ satisfy interpolation condition $\sum_k \phi(k)e^{-ik\omega} = \sum_k \hat{\phi}(\omega+2k\pi) \neq 0$, for all real ω, where $r \geq 1$, $r + r^* \geq 3$, as well as the stability condition $\mathrm{Re}\big(W_h(\omega)\big) \geq 0$ for each real ω. Then, for the smooth initial data u_0 and every $T > 0$, there is a constant $C > 0$, independent on h, Δt and u_0, such that $0 \leq t \leq T$ and $0 \leq n\Delta t \leq T$, and it follows that

$$\begin{aligned} \|u(\cdot,t) - u_h(\cdot,t)\|_{2,h} &= \|u(\cdot,t) - F_h(t)u_0(\cdot,t)\|_{2,h} \\ &\leq Ch^{p+p^*-1}\|u_0\|_{H^{p+p^*+2}} \end{aligned} \tag{12}$$

$$\begin{aligned} \|u(\cdot,n\Delta t)-u_h^n\|_{2,h} &= \|u(\cdot,n\Delta t)-G_h^n u_0\|_{2,h} \\ &\leq C\big(h^{p+p^*-1}+\Delta t^2\big)\|u_0\|_{H^{p+p^*+2}}. \end{aligned} \tag{13}$$

In fact, from (11) we have

$$\zeta_{0,0}(\omega)=\tilde{a}(\omega)=\sum_k \hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}=\sum_k a(k)e^{-ik\omega}$$

$$\tilde{d}(\omega)=\zeta_{1,0}(\omega)=i\sum_k(\omega+2k\pi)\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}$$

$$\tilde{c}(\omega)=(-1)^\beta\zeta_{\alpha,\beta}(\omega)=-i\sum_k(\omega+2k\pi)^{\alpha+\beta}\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}.$$

Therefore, by replacing these terms in (10) one obtains

$$W_h(\omega)=\frac{i\sum_{k\in\mathbb{Z}}\left[\mu(\omega+2k\pi)-h^{-2}\epsilon(\omega+2k\pi)^3\right]\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}}{\sum_{k\in\mathbb{Z}}\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}}. \tag{14}$$

Decomposing the sums of this last expression for $k=0$ and $k\neq 0$, we obtain

$$\begin{aligned} \hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)}\left[W_h(\omega)-i\big(\mu\omega+\epsilon h^{-2}\omega^3\big)\right]+W_h(\omega)O(\omega^{p+p^*+2}) &= \\ O(\omega^{p+p^*+2})+h^{-2}O(\omega^{p+p^*+2}) \end{aligned}$$

by using properties of asymptotic developments such as $W_h(\omega)O(\omega^{p+p^*+2})\to 0$ if $\omega\to 0$, $\hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)}=1+O(\omega^{p+p^*})$ and $O\big((h\omega)^{p+p^*+2}\big)=O\big(h^{p+p^*+2}\big)O\big(\omega^{p+p^*+2}\big)$, see for example [3], $W_h(\omega)=i\big(\mu\omega+\epsilon h^{-2}\omega^3\big)+O(\omega^{p+p^*+2})+h^{-2}O(\omega^{p+p^*+2})$ and hence

$$\begin{aligned} W_h(h\omega) &= i\big(\mu h\omega+\epsilon h^{-2}(h\omega)^3\big)+O\big((h\omega)^{p+p^*+2}\big)+h^{-2}O\big((h\omega)^{p+p^*+2}\big) \\ &= ih\omega(\mu-\epsilon\omega^2)+O\big(\omega^{p+p^*+2}\big)\Big[O\big(h^{p+p^*+2}\big)+h^{-2}O\big(h^{p+p^*+2}\big)\Big]. \end{aligned}$$

On the other hand,

$$\begin{aligned} u(x,t)-u_h(x,t) &= E(t)v(x)-F_h(t)v(x) \\ &= \mathcal{F}^{-1}\Big[\big(e^{i\omega(\mu-\epsilon\omega^2)t}-e^{-W_h(h\omega)t/h}\big)\hat{v}(\omega)\Big] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\big(e^{i\omega(\mu-\epsilon\omega^2)t}-e^{-W_h(h\omega)t/h}\big)\hat{v}(\omega)e^{i\omega x}d\omega \end{aligned}$$

but $\left|e^{i\omega(\mu-\epsilon\omega^2)t} - e^{-W_h(h\omega)t/h}\right| \leq Ce^{-tO(\omega^{p+p^*+2})O(h^{p+p^*-1})} \leq Ch^{p+p^*-1}|\omega|^{p+p^*+2}$. Finally we have

$$\begin{aligned}\|u(x,t)-F_h(x,t)\|_{2,h} &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{i\omega(\mu-\epsilon\omega^2)t} - e^{-W_h(h\omega)t/h}\right||\hat{u}_0(\omega)||e^{i\omega x}|d\omega \\ &\leq Ch^{p+p^*-1}\int_{-\infty}^{\infty}|\omega|^{p+p^*+2}|\hat{u}_0(\omega)|d\omega \\ &\leq Ch^{p+p^*-1}\int_{-\infty}^{\infty}\left(1+|\omega|^2\right)^{p+p^*+2}|\hat{u}_0(\omega)|^2 d\omega \\ &= Ch^{p+p^*-1}\|u_0\|_{H^{p+p^*+2}}.\end{aligned}$$

Inequality (13) is proved in analogous manner.

3.2 Stability Conditions

Let us begin by giving some hypothesis guaranteing stability conditions. Suppose that $\hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)}$ is real for all $\omega \in \mathbb{R}$. Then equation (14) implies that $\mathrm{Re}\big(W_h(\omega)\big) = 0$, which means that the method is stable and conservative. Suppose now that test functions are obtained from $\varphi_s(x) = \phi^*(x-s)$, translated version of ϕ^*, for some $s > 0$. Since $\widehat{\varphi}_s(\omega) = e^{-is\omega}\widehat{\phi^*}(\omega)$, $\widehat{\varphi}_s(\omega+2k\pi) = e^{-is(\omega+2k\pi)}\widehat{\phi^*}(\omega+2k\pi)$ and so $\overline{\widehat{\varphi}_s(\omega+2k\pi)} = e^{is\omega}\overline{\widehat{\phi^*}(\omega+2k\pi)}\,\xi_s$, where $\xi_s = e^{i2ks\pi}$. In this case equation (14) becomes

$$W_{h,s}(\omega) = \frac{i\sum_k[\mu(\omega+2k\pi)-\epsilon h^{-2}(\omega+2k\pi)^3]\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}\,\xi_s}{\sum_k\hat{\phi}(\omega+2k\pi)\overline{\hat{\phi}^*(\omega+2k\pi)}\,\xi_s},$$

if we let $q = 1-s$, then $\xi_q = e^{i2kq\pi} = e^{i2k(1-s)\pi} = e^{2k\pi i}e^{-i2ks\pi} = \overline{\xi_s}$, hence bearing in mind that $\hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)}$ is real, we have $\overline{W_{h,s}(\omega)} = -W_{h,q}(\omega)$ and this in turn implies $\mathrm{Re}(W_{h,q}(\omega)) = -\mathrm{Re}(W_{h,s}(\omega))$. This property is used in the following conclusions for ϕ and ϕ^* when it holds that $\hat{\phi}(\omega)\overline{\hat{\phi}^*(\omega)}$ is real

- For $s = 1/2$, $\mathrm{Re}(W_{h,q}(\omega)) = 0$ for all real ω and the method is stable and conservative.

- If for some $0 < s < 1$, it is stable and conservative, then so it is for translation parameter $1-s$.

- If for some $0 < s < 1$, it is stable and dissipative, then for the translation parameter $1-s$ it becomes unestable.

Time	**Error**
0	2,53E-18
0,1	0,04114137
0,2	0,04298284
0,3	0,05190608
0,4	0,06032352
0,5	0,07034124
0,6	0,08141202
0,7	0,09275702
0,8	0,10438704
0,9	0,11619728

Error with $h = 2^{-5}$

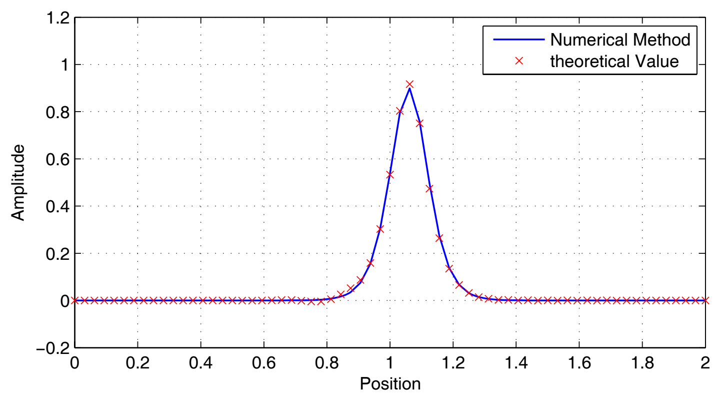

Figure 1 Comparison between the it method and the exact solution

Figure 1 shows the approximate solution and the exact solution, which was obtain from $u(x,t) = \frac{c}{2}\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right),\ c > 0.$

ACKNOWLEDGEMENTS. We wish to thank the Dirección de Investigación y Docencia de la Universidad EAFIT. Also, Andrus Giraldo for his help in the implementation of the algorithms.

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Received: March, 2012