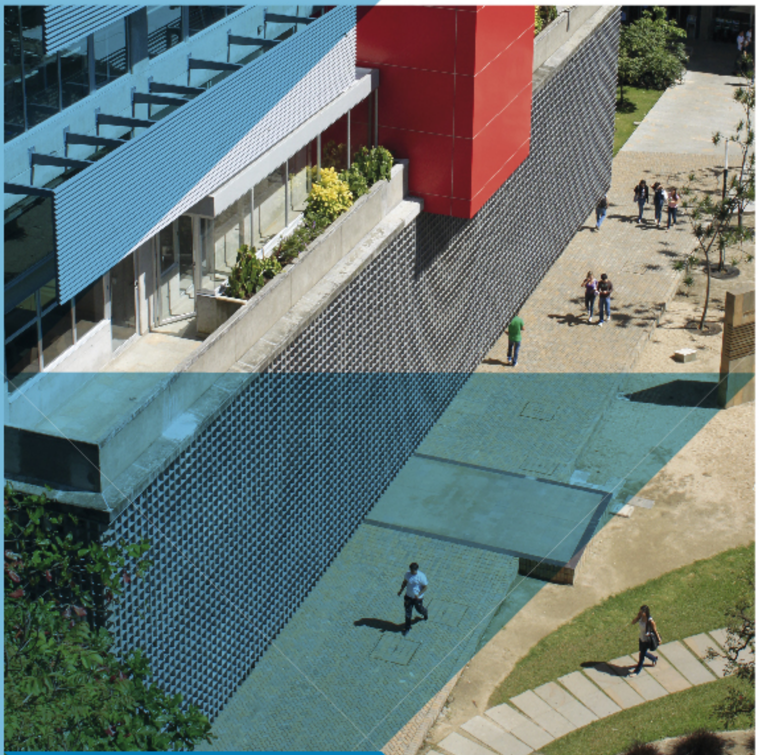

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What is the effect of sample and prior distributions on a Bayesian autoregressive linear model? An application to piped water consumption

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Abstract

In this paper we analyze the effect of four possible alternatives regarding the prior distributions in a linear model with autoregressive errors to predict piped water consumption: Normal-Gamma, Normal-Scaled Beta two, Studentized-Gamma and Student's t-Scaled Beta two. We show the effects of these prior distributions on the posterior distributions under different assumptions associated with the coefficient of variation of prior hyperparameters in a context where there is a conflict between the sample information and the elicited hyperparameters. We show that the posterior parameters are less affected by the prior hyperparameters when the Studentized-Gamma and Student's t-Scaled Beta two models are used. We show that the Normal-Gamma model obtains sensible outcomes in predictions when there is a small sample size. However, this property is lost when the experts overestimate the certainty of their knowledge. In the case that the experts greatly trust their beliefs, it is a good idea to use Student's t distribution as the prior distribution, because we obtain small posterior predictive errors. In addition, we find that the posterior predictive distributions using one of the versions of Student's t as prior are robust to the coefficient of variation of the prior parameters. Finally, it is shown that the Normal-Gamma model has a posterior distribution of the variance concentrated near zero when there is a high level of confidence in the experts' knowledge: this implies a narrow posterior predictive credibility interval, especially using small sample sizes.

JEL Classification: C11, C53
Keywords: Autoregressive model, Bayesian analysis, Forecast, Robust prior

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1 Introduction

Although the concepts of Bayesian analysis hold true for any sample size, it is interesting to examine the effects of the prior distributions on the posterior distributions given different sample sizes. In particular, it is well known that the prior distributions play a relatively greater role than do the data when the sample size is small (Greenberg, 2008). Therefore, the effect of the prior distributions on Bayesian inference is enormous when there are few data, and under this circumstance, the method that is chosen to build the prior distributions is very relevant. More specifically, we analyze the effect of four possible alternatives regarding the prior distributions in a linear model with autoregressive errors: Normal-Gamma, Normal-Scaled Beta two, Studentized-Gamma and Student's t-Scaled Beta two. We study the effect of these prior distributions on the parameters and predictive posterior distributions under different assumptions related to the coefficient of variation of the prior parameters.

The concept of probability from a Bayesian point of view is associated with the uncertainty of the occurrence of an event. In this scenario, the experts' degree of belief about an event can be tackled from either a subjective or objective perspective. The construction of prior distributions based on the subjective approach should be adopted in scenarios where it is tenable (Berger, 2006). However, this methodology is strongly influenced by the experts' perception of reality (Garthwaite et al., 2004); and unfortunately, experimental exercises have shown that human beings use heuristic strategies to make statistical statements which lead to biased affirmations (Kahneman, 2011). It does not matter which technique is used, the main objective in science is to maximize the process of learning from observation. This observation can be compiled from data or from the researcher's experience. However, what happens when there is a conflict between the sample information and the prior distributions? A possible solution is to use robust priors (Fúquene et al., 2009, 2012). In particular, we perform an elicitation procedure with experts from the main piped water company of the Metropolitan Area of Medellín (Colombia), and obtain the mean prior elasticities associated with the average household consumption of piped water from stratum four of this service.[1] As will be shown, there is a conflict between the

[1]The residential consumers of utilities in Colombia are classified by strata. This is done in order to give subsidies to poor people who are in strata one and two. On the other hand, rich people, who are in strata five

elicited parameters obtained from the experts' beliefs and the sample information. Therefore, robust prior distributions are of great help to obtain sensible outcomes under this circumstance.

The main goal in this paper is to show the effects on the parameters and predictive posterior distributions of four different combinations of prior distributions in a linear model with autoregressive errors applied to the piped water consumption in the Metropolitan Area of Medellín (Colombia). We analyze the effects on this model of different sample sizes in a context where there is a conflict between the sample information and the elicited parameters.

We show that the posterior localization parameters are less affected by the prior hyperparameters when the Studentized-Gamma and Student's t-Scaled Beta two models are used. In addition, the Normal-Gamma model generates sensible outcomes in predictions when there is a small sample size (10 observations). However, this property is lost when the experts overestimate the certainty of their knowledge of the phenomenon. In case the experts greatly trust their beliefs, it is a good idea to use Student's t distribution as the prior distribution because we obtain small posterior predictive errors. In addition, we find that the posterior predictive distributions using a Student's t as prior are robust to the coefficient of variation of the prior hyperparameters. Finally, it is shown that the Normal-Gamma model has a posterior distribution of the variance concentrated near zero when there is a high prior coefficient of variation: this implies a narrow posterior predictive credibility interval, especially with small sample sizes.

After this introduction, we outline the principal statements about our model in Section 2. Section 3 shows the principal outcomes of our analysis. And finally, we make some concluding remarks in Section 4.

2 Bayes regression with autoregressive errors

We study the average household piped water consumption of strata four in the Metropolitan Area of Medellín (Colombia). We propose a linear model with autoregressive errors due to

and six, have to pay contributions to the system in order to subsidize the poor. Finally, people who are in the medium strata (three and four) do not pay any contribution or obtain any subsidy.

having time series data which imply an inertial effect on consumption. We have quarterly data from 1985 to 2010, and estimate the model (Eqs. 1 and 2) with different sample sizes and prior distributions.

$$log\left\{cme_t\right\} = \beta_0 + \beta_1 log\left\{I_t\right\} + \beta_2\left\{n_t\right\} + \beta_3 log\left\{p_t\right\} + \mu_t \tag{1}$$

where

$$\mu_t = \phi\mu_{t-1} + \epsilon_t \tag{2}$$

$t = 1, 2, \ldots, T$ and $\epsilon \overset{i.i.d}{\sim} (0, \sigma^2_\epsilon)$.

$log\left\{cme_t\right\}$: natural logarithm of the average consumption of piped water.

$log\left\{I_t\right\}$: natural logarithm of average real income of household.

n_t: average number of people in household.

$log\left\{p_t\right\}$: natural logarithm of the real price of piped water.

μ_t: autocorrelated stochastic perturbation.

We must estimate β_1 and β_3, which are the income and price demand elasticities, and β_2, which is the semi-elasticity of piped water consumption with respect to the number of people in the household. In addition, ϕ captures the inertial effect on consumption.

Initially, we assume that the prior distributions are $\beta \sim \mathcal{N}_K(\beta_0, B_0)$, $\phi \propto \mathcal{N}(\phi_0, \sigma^2_{\phi_0}) I_{\phi\in(-1,1)}$ and $\sigma^2_\epsilon \sim \mathcal{IG}(\alpha_0/2, \delta_0/2)$ where $I_{\phi\in(-1,1)}$ denotes the indicator function of the set $(-1, 1)$ (Chib, 1993). It can be shown that the posterior distributions are $\beta|y_t, x_t, \sigma^2_\epsilon, \phi \sim \mathcal{N}_K(\bar{\beta}, \bar{B})$, $\sigma^2_\epsilon|y_t, x_t, \beta, \phi \sim \mathcal{IG}(\alpha_1/2, \delta_1/2)$ and $\phi|y_t, x_t, \beta, \sigma^2_\epsilon \propto \mathcal{N}(\bar{\phi}, \bar{\sigma}^2_\phi) I_{\phi\in(-1,1)}$ where $y_t = log\left\{cme_t\right\}$, $x_t = [1, log\left\{I_t\right\}, n_t, log\left\{p_t\right\}]'$ and

$$\bar{B} = \left[\sigma_\epsilon^{-2}\left\{\frac{x_1 x_1'}{1-\phi^2} + \sum_{t=2}^{T}(x_t - \phi x_{t-1})(x_t - \phi x_{t-1})'\right\} + B_0^{-1}\right]^{-1} \tag{3}$$

$$\bar{\beta} = \bar{B}\left[\sigma_\epsilon^{-2}\left\{\frac{y_1 x_1}{1-\phi^2} + \sum_{t=2}^{T}(x_t - \phi x_{t-1})(y_t - \phi y_{t-1})\right\} + B_0^{-1}\beta_0\right] \tag{4}$$

$$\alpha = \alpha_0 + T \tag{5}$$

$$\delta = \delta^0 + \frac{(y_1 - x_1'\beta)^2}{1-\phi^2} + \sum_{t=2}^{T} \left((y_t - \phi y_{t-1}) - (x_t - \phi x_{t-1})'\beta)^2\right) \tag{6}$$

$$\bar{\sigma}_\phi^2 = \left(\sigma_\epsilon^{-2} \sum_{t=2}^{T} (y_{t-1} - x_{t-1}'\beta)^2 + \sigma_{\phi_0}^{-2}\right)^{-1} \tag{7}$$

$$\bar{\phi} = \bar{\sigma}_\phi^2 \left(\sigma_\epsilon^{-2} \sum_{t=2}^{T} (y_t - x_t'\beta)(y_{t-1} - x_{t-1}'\beta) + \phi_0 \sigma_{\phi_0}^{-2}\right) \tag{8}$$

In addition, we use as prior distributions for β a $t_K(\beta_0, B_0, 2)$. This is a robust prior distribution (Fúquene et al., 2009). Moreover, we use as prior distribution for the variance a $\mathcal{SB}2(0.5, 0.01, 100)$, which is a "non-informative" distribution. The idea is to analyze the consequences for the posterior parameter and predictive distributions associated with different prior distributions. We do not get any analytical solution in these circumstances.

2.1 The hyperparameters of the prior distributions

Following Gelman (2006), we use a "non-informative" prior distribution in the variance parameter. It is well known that the $\mathcal{IG}(e, e)$ distribution implies an improper prior when $e \to 0$, and so in our analysis we use $\sigma_\epsilon^2 \sim \mathcal{IG}(0.001, 0.001)$ as the prior density for the variance parameter; however, we use informative distributions in the case of localization parameters, then the hyperparameters of these prior distributions must be fixed. Therefore, we employ elicitation techniques in order to assign the proper values to these hyperparameters. We elicit an expert from the most important public utility company in the Metropolitan Area of Medellín (Colombia). This person has worked in the company for 12 years, and her work is directly related to forecasts of piped water consumption in the residential sector. So, we guess that this person is an expert in this service.

Regarding the elicitation procedure, the main objective is to convert the expert's knowledge into probabilistic statements: a mean elasticity or semi-elasticity in this case. The fundamental steps in this process are (Kadane and Wolfson, 1998):

1. Establishing the general framework of the elicitation process.

2. Checking the consistency of the expert's statements.
3. Obtaining a mean of elicited parameters.

An important issue in an elicitation process is how people perceive reality, and the way that people assign statistical statements to events. In particular, people use heuristics to make statistical statements, and these heuristics can cause bias (Tversky and Kahneman, 1974, 1973). Obviously, these heuristics are based on the available information, where recent events have a more important impact than past events. Furthermore, people make estimates by starting from an initial value that is adjusted to yield a final answer. Generally, this adjustment is typically insufficient. This phenomenon is reinforced by conservatism, which means that the updating process of prior statistical statements, given new information, is lower than the statements deduced from the Bayes theorem. Moreover, Tversky and Kahneman (1971) have shown that individuals incorrectly think that the characteristics of any sample are the same as the characteristics of the population, even in the case of small samples. Finally, Fischhoff and Beyth (1975) have shown that prior knowledge of an event causes some distortions in the memory that can affect the elicitation procedure. As we can see, the elicitation procedure has a lot of shortcomings; we try to take into account all these in our elicitation process. However, it is quite difficult to accomplish this task.

Our analysis is focused on the income and price demand elasticities, and the semi-elasticity regarding the average number of people living in the household. The reason is that these parameters are more approachable by the expert's knowledge. Regarding the covariance matrix, Beach and Swenson (1966) have shown that experts have difficulty giving information about a covariance matrix. Furthermore, Keren (1991) shows that experts have a tendency to overestimate their knowledge regarding parameters, which implies narrow credibility intervals. As a consequence, we assume that there is no covariance between the parameters, and additionally, we analyze different scenarios of the variance of parameters.

On the other hand, we estimate this model using quarterly data from 1985 to 2010 where the source of this dataset is Empresas Públicas de Medellín, the most important public utility service company in Medellín (Colombia). The estimation is done for stratum four in the Metropolitan Area of Medellín.

We can observe the mean of the elicited parameters in Table 1. As we can see in this table, there is a conflict between the elicited mean and the sample information. For instance, the elicited mean of the price demand elasticity is equal to -0.10 while we obtain -0.17 using sample information. The former value means that according to the expert's information, an increment of 10% in the price implies a reduction of 9.5% in the water consumption. On the other hand, the same price's increment implies a reduction of 15.6% using the sample information. It can be a good idea to use robust prior distributions under such circumstances (Fúquene et al., 2009).

3 Results

We assign different levels to the prior variances $\sigma^2_{\beta_i}$ in all our models, so that $(\sigma_{\beta_i}/\beta_i) * 100\% = \{10\%, 30\%, 60\%, 100\%, 130\%\}$. We also perform our estimations with different sample sizes. Given that we have data from 1985q1 to 2010q3, we take the last observations to perform our estimations with $n = \{10, 100\}$. The idea is to study the impact of different prior models and sample sizes on the posterior parameter and predictive distributions.[2]

We use the Metropolis-Hastings algorithm to perform all our estimations (Metropolis et al., 1953; Hastings, 1970). Therefore, we know from the theory of Markov chains that our chains eventually converge to the stationary distribution, which is also our target distribution. Subsequently, we implement some visual and formal tests to check this assumption. In particular, we make autocorrelation graphs of the chains and carry out the Gelman and Rubin (1992) test, and find a good mixing of our chains.[3]

[2]All our estimations are performed in the R package (R Development Core Team, 2011) and JAGS (Just Another Gibbs Sampler, http://mcmc-jags.sourceforge.net/).

[3]All tests are performed in the library coda (Plummer et al., 2012) in the R package, and are available upon request.

3.1 Posterior location parameter estimates

First, we show the posterior parameter estimates on varying the variance level and sample size for four models: Normal-Gamma, Normal-Scaled Beta two, Studentized-Gamma and Student's t-Scaled Beta two.

Normal-Gamma Model

As we can see in Tables 2, 3 and 4, the coefficient of variation of the hyperparameters implies a high degree of variability between the posterior median estimates of each parameter. This phenomenon is greater with a large sample size because a large sample size can give more evidence against prior information. As a consequence, small changes in the variance level of the hyperparameters may cause large changes in estimates of the posterior coefficients. On the other hand, when there is a small sample size, the posterior coefficients' medians are anchored to the elicited parameters, and the coefficient of variation does not matter in most cases. This fact is present although we use "non-informative" prior distributions for the variance of the models. Finally, we can observe from these tables that a high level of coefficient of variation means a wider inter quantile range. In this case, the range decreases with the sample size.

Normal-Scaled Beta two Model

Regarding the posterior estimates of the localization parameters using a Normal-Scaled Beta two model, we practically observe the same pattern that we saw in the Normal-Gamma model (see Tables 5, 6 and 7). However, the evolution in the posterior parameter estimates when the prior coefficient of variation changes is more consistent in the Normal-Scaled Beta two model. More specifically, there are some abrupt changes in the Normal-Gamma model when the coefficient of variation is 130%, especially when there is a small sample. This phenomenon is not evident in the Normal-Scaled Beta two model.

Studentized-Gamma Model

If we compare the Normal-Gamma model or the Normal-Scaled Beta two model with the Studentized-Gamma model, we find that the Studentized-Gamma model converges faster to the sample information than the other two models, and additionally, the posterior median parameters using this model have less variability when the level of variance fluctuates. That is, the Studentized-Gamma model is less affected by the prior hyperparameters. This phenomenon is particularly relevant when the sample size is large and the prior variance is small compared to the prior mean. See Tables 8, 9 and 10.

Student's t-Scaled Beta two model

Regarding the Student's t-Scaled Beta two model, we find similar outcomes as with the Studentized-Gamma model. See Tables 11, 12 and 13.

The main conclusion of these exercises is that when there is a small sample size, the prior hyperparameters have a huge effect on the posterior outcomes: this effect might be a little bit mitigated when the Studentized-Gamma and Student's t-Scaled Beta two models are used when the expert's beliefs have a high degree of uncertainty associated with them.[4]

3.2 Posterior predictive distribution

Despite the fact that we have data from 1985q1 to 2010q3, we estimate our models using 2009q3 as the last observation, with different sample sizes $n = \{10, 100\}$ from this observation. Then, we evaluate the predictive capacity of our models using the data from 2009q4 to 2010q1.

Perhaps the most relevant finding from this exercise is that the posterior predictive distributions using the Studentized-Gamma model and Student'st-Scaled Beta two model as prior distributions are basically the same, that is, these posterior distributions are robust to the coefficient of variation of the prior parameters. Therefore, we just show in Table 14 the results of

[4] We can see in the Annex (5) the box plots associated with the coefficients under different models and different sample sizes (see Figs. 1, 2 and 3).

the posterior predictive distribution when $(\sigma_{\beta_i}/\beta_{0i}) * 100\% = 10\%$.

As we can see in Table 14, the posterior prediction error using the Studentized-Gamma and Student's t-Scaled Beta two models are smaller than when using the Normal-Gamma and Normal-Scaled Beta two models when the level of coefficient of variation is small, as well as the sample size. The average errors are 46.5% and 48.3% in the case of the Normal-Gamma and Normal-Scaled Beta two models, while those errors are 19.9% and 18.7% in the case of the Studentized-Gamma and Student's t-Scaled Beta two models. However, for the former models, this pattern changes when the coefficient of variation increases. Specifically, the average error decreases from 37.6% and 37.9% (when the coefficient level is 30%) to 13.4% and 12.6% when this coefficient is 60% (see Table 15).

We show in Table 15 that the average errors in the Normal-Gamma model decrease with the variance level, that is, a large coefficient of variation implies a small prediction error. In particular, a coefficient of variation equal to 130% gives an average prediction error equal to 6.32% in this case. This pattern is not clear in the case of the Normal-Scaled Beta two model, where the average error has a 'U' form, that is, low and high levels of the coefficient of variation imply high predictive errors, while medium values of the coefficient of variation give low levels of predictive errors. Furthermore, we can see from these tables that the credibility intervals of the Normal-Gamma model are the narrowest when the coefficient of variation is large. This is explained by the posterior estimation of the model's variance (see subsection 3.3).

Those outcomes are apparently not intuitive because we established in the previous subsection that with a small sample, the posterior parameter estimates from the Studentized-Gamma and Student's t-Scaled Beta two models are less affected by the hyperparameters than are those of the Normal-Gamma and Normal-Scaled Beta two models. The reason why we get better predictive result using a Normal-Gamma model with a high degree of uncertainty about experts' beliefs is due to the constant parameter. Forecasts are so sensitive to this parameter; unfortunately, this coefficient is normally omitted in structural elicitation procedures.

Table 1: Parameter estimates: Elicited and sample information.

Parameter	Elicitation	Data
$\hat{\beta}_1$	0.10	0.16
$\hat{\beta}_2$	0.30	0.43
$\hat{\beta}_3$	-0.10	-0.17

Table 2: Summary of posterior distributions for the income elasticity β_1 under different levels of prior variance $\frac{\sigma_{\beta_1}}{\beta_1} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_1}}{\beta_1} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	0.0659	0.0930	0.0997	0.1066	0.1374
	30%	-0.0179	0.0812	0.1010	0.1208	0.2238
	60%	-0.1609	0.0595	0.1005	0.1422	0.3471
	100%	-0.2389	0.0331	0.1006	0.1686	0.4614
	130%	-0.4085	0.0238	0.0949	0.1656	0.5637
$n = 100$	10%	0.0612	0.0936	0.1002	0.1070	0.1402
	30%	-0.0024	0.0910	0.1100	0.1294	0.2082
	60%	-0.0171	0.1234	0.1495	0.1752	0.3025
	100%	-0.0307	0.1365	0.1670	0.1974	0.3622
	130%	0.0194	0.1446	0.1729	0.2021	0.3683

Table 3: Summary of posterior distributions for semi-elasticity of number of people β_2 under different levels of prior variance $\frac{\sigma_{\beta_2}}{\beta_2} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_2}}{\beta_2} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	0.1941	0.2791	0.2994	0.3197	0.4143
	30%	-0.0344	0.2384	0.3006	0.3612	0.6387
	60%	-0.3979	0.1994	0.3181	0.4415	0.9692
	100%	-0.7992	0.1702	0.3689	0.5668	1.4537
	130%	-0.8933	0.1679	0.2980	0.4472	1.5784
$n = 100$	10%	0.2045	0.2843	0.3042	0.3240	0.4243
	30%	0.3611	0.4602	0.4798	0.4993	0.5843
	60%	0.3803	0.4537	0.4695	0.4861	0.5666
	100%	0.3783	0.4522	0.4699	0.4877	0.5625
	130%	0.3787	0.4479	0.4634	0.4788	0.5500

Table 4: Summary of posterior distributions for price elasticity β_3 under different levels of prior variance $\frac{\sigma_{\beta_3}}{\beta_3} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_3}}{\beta_3} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	-0.1380	-0.1067	-0.1000	-0.0933	-0.0633
	30%	-0.2132	-0.1199	-0.0996	-0.0798	0.0135
	60%	-0.3385	-0.1373	-0.0969	-0.0582	0.1441
	100%	-0.4925	-0.1625	-0.0935	-0.0269	0.2533
	130%	-0.5166	-0.1484	-0.0648	0.0174	0.4133
$n = 100$	10%	-0.1366	-0.1068	-0.1001	-0.0932	-0.0657
	30%	-0.2296	-0.1468	-0.1305	-0.1138	-0.0401
	60%	-0.2418	-0.1642	-0.1476	-0.1306	-0.0496
	100%	-0.2434	-0.1675	-0.1483	-0.1293	-0.0384
	130%	-0.2538	-0.1638	-0.1470	-0.1303	-0.0545

Table 5: Summary of posterior distributions for income elasticity β_1 under different levels of prior variance $\frac{\sigma_{\beta_1}}{\beta_1} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_1}}{\beta_1} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	0.0635	0.0931	0.1000	0.1066	0.1351
	30%	-0.0128	0.0795	0.0999	0.1194	0.2131
	60%	-0.1121	0.0604	0.0998	0.1408	0.3246
	100%	-0.3340	0.0320	0.1018	0.1674	0.4575
	130%	-0.3607	0.0152	0.0994	0.1860	0.5839
$n = 100$	10%	0.0625	0.0935	0.0999	0.1068	0.1396
	30%	0.0058	0.0908	0.1092	0.1287	0.2125
	60%	0.0067	0.1230	0.1492	0.1756	0.2833
	100%	0.0050	0.1358	0.1676	0.1974	0.3438
	130%	-0.0203	0.1396	0.1720	0.2031	0.3730

Table 6: Summary of posterior distributions for semi-elasticity of number of people β_2 under different levels of prior variance $\frac{\sigma_{\beta_2}}{\beta_2} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_2}}{\beta_2} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	0.1864	0.2803	0.3000	0.3204	0.4136
	30%	-0.0849	0.2419	0.3020	0.3622	0.6313
	60%	-0.2827	0.2055	0.3243	0.4433	0.9787
	100%	-0.6325	0.1704	0.3689	0.5708	0.4614
	130%	-1.0461	0.1664	0.4175	0.6659	1.8176
$n = 100$	10%	0.1768	0.2837	0.3041	0.3246	0.4139
	30%	0.3555	0.4604	0.4800	0.4996	0.6019
	60%	0.3708	0.4529	0.4692	0.4850	0.5693
	100%	0.3654	0.4522	0.4692	0.4869	0.5767
	130%	0.3699	0.4526	0.4696	0.4865	0.5736

Table 7: Summary of posterior distributions for price elasticity β_3 under different levels of prior variance $\frac{\sigma_{\beta_3}}{\beta_3} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_3}}{\beta_3} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	-0.1371	-0.1068	-0.1001	-0.0933	-0.0586
	30%	-0.2238	-0.1196	-0.0997	-0.0793	0.0101
	60%	-0.3085	-0.1386	-0.0978	-0.0571	0.1335
	100%	-0.4442	-0.1600	-0.0920	-0.0264	0.3101
	130%	-0.5624	-0.1727	-0.0874	0.0020	0.4302
$n = 100$	10%	-0.1465	-0.1071	-0.1003	-0.0935	-0.0631
	30%	-0.2360	-0.1462	-0.1293	-0.1124	-0.0361
	60%	-0.2443	-0.1656	-0.1482	-0.1308	-0.0552
	100%	-0.2624	-0.1679	-0.1492	-0.1304	-0.0452
	130%	-0.2596	-0.1676	-0.1491	-0.1310	-0.0358

Table 8: Summary of posterior distributions for income elasticity β_1 under different levels of prior variance $\frac{\sigma_{\beta_1}}{\beta_1} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_1}}{\beta_1} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	-0.2142	0.0917	0.0998	0.1083	0.2472
	30%	-0.4500	0.0770	0.1006	0.1248	0.6259
	60%	-0.9284	0.0535	0.1027	0.1504	1.3464
	100%	-1.4976	0.0184	0.0993	0.1795	1.7151
	130%	-2.2123	0.0008	0.1012	0.2055	2.5727
$n = 100$	10%	0.0144	0.0980	0.1060	0.1178	0.2741
	30%	0.0108	0.1076	0.1289	0.1561	0.3632
	60%	-0.0203	0.1201	0.1479	0.1787	0.3350
	100%	-0.0274	0.1279	0.1602	0.1920	0.3620
	130%	-0.0204	0.1320	0.1639	0.1963	0.3693

Table 9: Summary of posterior distributions for semi-elasticity of number of people β_2 under different levels of prior variance $\frac{\sigma_{\beta_2}}{\beta_2} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_2}}{\beta_2} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	-0.2311	0.2758	0.2999	0.3250	1.0916
	30%	-0.9980	0.2422	0.3093	0.3825	3.9549
	60%	-2.1270	0.2084	0.3435	0.4977	3.1598
	100%	-2.9453	0.1784	0.4012	0.6646	5.6110
	130%	-3.1843	0.1614	0.4375	0.7750	4.3472
$n = 100$	10%	0.3822	0.4737	0.4907	0.5056	0.5654
	30%	0.3712	0.4598	0.4761	0.4934	0.5648
	60%	0.3789	0.4568	0.4738	0.4899	0.5656
	100%	0.3893	0.4566	0.4733	0.4907	0.5810
	130%	0.3768	0.4550	0.4726	0.4908	0.5726

Table 10: Summary of posterior distributions for price elasticity β_3 under different levels of prior variance $\frac{\sigma_{\beta_3}}{\beta_3} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_3}}{\beta_3} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	-0.2422	-0.1081	-0.1002	-0.0923	0.1166
	30%	-0.7944	-0.1241	-0.0993	-0.0733	0.9372
	60%	-0.8128	-0.1416	-0.0943	-0.0429	2.8426
	100%	-1.6399	-0.1654	-0.0874	-0.0015	3.1558
	130%	-2.1081	-0.1802	-0.0806	0.0308	3.0872
$n = 100$	10%	-0.2328	-0.1360	-0.1171	-0.1055	-0.0694
	30%	-0.2537	-0.1567	-0.1389	-0.1219	-0.0457
	60%	-0.2558	-0.1628	-0.1452	-0.1275	-0.0499
	100%	-0.2425	-0.1650	-0.1469	-0.1284	-0.0310
	130%	-0.2480	-0.1666	-0.1474	-0.1274	-0.0341

Table 11: Summary of posterior distributions for income elasticity β_1 under different levels of prior variance $\frac{\sigma_{\beta_1}}{\beta_1} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_1}}{\beta_1} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	-0.1746	0.0918	0.0999	0.1079	0.3159
	30%	-0.4358	0.0747	0.1000	0.1248	0.7921
	60%	-1.3186	0.0535	0.1016	0.1506	1.4169
	100%	-1.4947	0.0149	0.0966	0.1768	1.9064
	130%	-1.7872	0.0001	0.1012	0.2001	2.1374
$n = 100$	10%	0.0428	0.0983	0.1063	0.1123	0.3598
	30%	0.0009	0.1076	0.1291	0.1562	0.3357
	60%	-0.0367	0.1207	0.1489	0.1785	0.3618
	100%	-0.0241	0.1288	0.1608	0.1926	0.3414
	130%	-0.0460	0.1317	0.1648	0.1983	0.3823

Table 12: Summary of posterior distributions for semi-elasticity number of people β_2 under different levels of prior variance $\frac{\sigma_{\beta_2}}{\beta_2} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_2}}{\beta_2} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	-0.2711	0.2771	0.3006	0.3250	1.0675
	30%	-0.7706	0.2396	0.3111	0.3890	3.0840
	60%	-1.4452	0.2059	0.3380	0.4984	4.3999
	100%	-1.5909	0.1763	0.3953	0.6618	3.3711
	130%	-2.1597	0.1844	0.4546	0.7829	5.0511
$n = 100$	10%	0.3785	0.4712	0.4892	0.5044	0.5654
	30%	0.3649	0.4598	0.4774	0.4935	0.5651
	60%	0.3814	0.4576	0.4745	0.4915	0.5742
	100%	0.3857	0.4563	0.4731	0.4911	0.5916
	130%	0.3569	0.4565	0.4743	0.4923	0.5739

Table 13: Summary of posterior distributions for price elasticity β_3 under different levels of prior variance $\frac{\sigma_{\beta_3}}{\beta_3} * 100\%$

Sample size	Variance level $\frac{\sigma_{\beta_3}}{\beta_3} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	-0.3930	-0.1080	-0.0999	-0.0913	0.2323
	30%	-1.0507	-0.1226	-0.0984	-0.0737	0.5060
	60%	-1.4826	-0.1412	-0.0942	-0.0439	2.1076
	100%	-1.6051	-0.1646	-0.0873	-0.0023	3.7133
	130%	-1.8982	-0.1818	-0.0813	0.0247	2.2815
$n = 100$	10%	-0.2476	-0.1387	-0.1180	-0.1061	-0.0655
	30%	-0.2515	-0.1565	-0.1377	-0.1205	-0.0454
	60%	-0.2567	-0.1624	-0.1440	-0.1258	-0.0324
	100%	-0.2412	-0.1648	-0.1469	-0.1272	-0.0438
	130%	-0.2613	-0.1649	-0.1460	-0.1267	-0.0459

Table 14: First level for prior variance of β_i: $((\sigma_{\beta_i}/\beta_i) * 100\% = 10\%)$ and $n = 10$

Model	Period	Real value	Predic. value	Lower HPD interval	Upper HPD interval	Error rate
Normal-Gamma	2009 Q4	14.48	8.01	3.76	14.74	48.42%
	2010 Q1	13.88	8.25	3.86	15.17	45.49%
	2010 Q2	13.78	7.99	3.89	14.86	46.46%
	2010 Q3	13.64	8.15	3.83	15.02	45.21%
Normal-Scaled Beta2	2009 Q4	14.48	7.54	3.66	13.45	50.40%
	2010 Q1	13.88	7.76	3.76	13.83	47.35%
	2010 Q2	13.78	7.52	3.65	13.43	48.38%
	2010 Q3	13.64	7.68	3.72	13.69	47.07%
Studentized-Gamma	2009 Q4	14.48	16.38	12.28	20.77	15.07%
	2010 Q1	13.88	16.86	12.68	21.39	22.11%
	2010 Q2	13.78	16.34	12.23	20.71	19.55%
	2010 Q3	13.64	16.68	12.51	21.13	22.86%
Student's t-Scaled Beta2	2009 Q4	14.48	16.25	12.76	20.39	13.90%
	2010 Q1	13.88	16.73	13.14	21.00	21.00%
	2010 Q2	13.78	16.21	12.79	20.38	18.40%
	2010 Q3	13.64	16.54	13.00	20.76	21.76%

We can see in Table 16 that the average predictive error decreases with sample size. In particular, we have average predictive errors of approximately 19% using the Normal-Gamma and Normal-Scaled Beta two models, and errors near 4% using the Studentized-Gamma and Student's t-Scaled Beta two models when the sample size is 100 and the coefficient of variation of the hyperparameters is 10%. On the other hand, if we use just 10 observations, the average predictive errors are 47% in the case of the Student's t models, but 19% using Normal models (see Table 14).

Table 15: Other levels for prior variance of β_i: $((\sigma_{\beta_i}/\beta_i) * 100\% = 30\%, 60\%, 100\%, 130\%)$ and $n = 10$.

Model	Period	Real value	Predic. value	Lower HPD interval	Upper HPD interval	Error rate
Normal-Gamma: (30%)	2009 Q4	14.48	10.62	4.57	19.82	39.32%
	2010 Q1	13.88	10.94	4.81	20.45	36.92%
	2010 Q2	13.78	10.60	4.64	19.88	37.71%
	2010 Q3	13.64	10.82	4.63	20.13	36.71%
Normal-Scaled Beta2: (30%)	2009 Q4	14.48	9.40	4.72	15.98	40.23%
	2010 Q1	13.88	9.68	4.96	16.54	36.94%
	2010 Q2	13.78	9.38	4.76	15.97	38.04%
	2010 Q3	13.64	9.57	4.81	16.26	36.64%
Normal-Gamma: (60%)	2009 Q4	14.48	15.05	10.97	18.93	10.74%
	2010 Q1	13.88	15.52	11.43	19.50	14.79%
	2010 Q2	13.78	15.02	10.99	18.88	13.05%
	2010 Q3	13.64	15.34	11.33	19.37	15.31%
Normal-Scaled Beta2: (60%)	2009 Q4	14.48	15.05	11.48	18.55	9.77%
	2010 Q1	13.88	15.53	11.77	18.97	14.09%
	2010 Q2	13.78	15.03	11.37	18.42	12.26%
	2010 Q3	13.64	15.34	11.81	18.96	14.62%
Normal-Gamma: (100%)	2009 Q4	14.48	15.95	11.85	20.10	13.36%
	2010 Q1	13.88	16.53	12.60	20.82	19.93%
	2010 Q2	13.78	15.93	12.06	20.21	17.14%
	2010 Q3	13.64	16.28	12.26	20.51	20.32%
Normal-Scaled Beta2: (100%)	2009 Q4	14.48	15.84	12.30	19.73	12.33%
	2010 Q1	13.88	16.41	13.01	20.34	18.85%
	2010 Q2	13.78	15.82	12.28	19.64	16.07%
	2010 Q3	13.64	16.18	12.85	20.21	19.28%
Normal-Gamma: (130%)	2009 Q4	14.48	14.47	12.88	16.34	4.13%
	2010 Q1	13.88	14.91	13.85	16.74	7.49%
	2010 Q2	13.78	14.43	13.00	16.31	5.42%
	2010 Q3	13.64	14.74	13.24	16.53	8.24%
Normal-Scaled Beta2: (130%)	2009 Q4	14.48	15.96	12.30	19.73	13.20%
	2010 Q1	13.88	16.60	13.01	20.34	20.11%
	2010 Q2	13.78	15.94	12.29	19.64	16.96%
	2010 Q3	13.64	16.33	12.85	20.21	20.35%

Table 16: First level for prior variance of β_i: $((\sigma_{\beta_i}/\beta_i) * 100\% = 10\%)$ and $n = 100$.

Model	Period	Real value	Predic. value	Lower HPD interval	Upper HPD interval	Error rate
Normal-Gamma	2009 Q4	14.48	11.92	8.01	16.78	21.48%
	2010 Q1	13.88	12.28	8.23	17.26	18.15%
	2010 Q2	13.78	11.90	7.99	16.75	19.17%
	2010 Q3	13.64	12.14	8.11	17.05	17.91%
Normal-Scaled Beta2	2009 Q4	14.48	11.70	7.94	16.47	22.35%
	2010 Q1	13.88	12.05	8.18	16.98	18.79%
	2010 Q2	13.78	11.68	7.93	16.45	19.91%
	2010 Q3	13.64	11.92	8.09	16.78	18.51%
Studentized-Gamma	2009 Q4	14.48	13.80	13.37	14.24	4.72%
	2010 Q1	13.88	14.49	14.11	14.90	4.39%
	2010 Q2	13.78	13.80	13.89	14.24	1.24%
	2010 Q3	13.64	14.17	13.75	14.61	3.94%
Student's t-Scaled Beta2	2009 Q4	14.48	13.81	13.39	14.28	4.62%
	2010 Q1	13.88	14.51	14.08	14.88	4.49%
	2010 Q2	13.78	13.81	13.37	14.23	1.30%
	2010 Q3	13.64	14.19	13.75	14.64	4.05%

We show in Table 17 that a small increase in the coefficient of variation implies a significant reduction in predictive error associated with the Normal-Gamma and Normal-Scaled Beta two models. For instance, the average predictive error with a variance level of 30% is 3.84% and 3.83% using those models, respectively. These errors are 19.18% and 19.89% using a coefficient of variation equal to 10%. In general, we can see from this table that the average posterior predictive errors are similar for a specific cut-off in the coefficient of variation, this cut-off is 30% in this case. The explanation for this fact is related to the structural change in the posterior estimations when there is more uncertainty in the experts' beliefs. Therefore, it does not matter if there is an increase in the sample size when the prior coefficients are so tied to a specific value. This overestimation of the trust in the experts' knowledge can cause a conflict between the sample information and the experts' beliefs, which might generate predictive errors.

Finally, the posterior predictive errors using the Normal distributions are comparable to those from using Student's t distribution when the sample size is 100 observations. Again, this phenomenon is present once a high level of the coefficient of variation for the prior parameters is achieved.

3.3 Posterior model's variance estimates

We can see in Tables 18 and 19 that the posterior variance estimates in the Normal-Gamma and Normal-Scaled Beta two models have a great level of variability between the coefficients of variation. We also observe from these tables that the posterior median estimates of the variance decrease with the coefficient of variation. There is a trade-off, where a high level of certainty in the location parameters implies a small scale parameter. There is also an abrupt change in the posterior median of the scale parameter when the coefficient of variation is 130% in the Normal-Gamma model. This is the reason why the credibility interval in this case is too narrow. Analyzing the Studentized-Gamma and Student's t-Scaled Beta two models, one observes that there is no relation between the level of prior uncertainty and the posterior scale parameter. As we can see in Tables 20 and 21, the posterior median estimates of the variance of the models are robust to the coefficients of variation of the location parameters.

Regarding the sample size, we see in these tables that a small sample size implies a large variance of the models. This is due to the fact that when there is a small sample size, the effect of a non-informative prior distribution is greater on the posterior distribution compared with the case where there is a large sample size.[5]

4 Concluding Remarks

We find in our application that the posterior predictive distributions using the Studentized-Gamma or Student's t-Scaled Beta two as priors are robust to the coefficient of variation of the hyperparameters. Moreover, if experts greatly trust in their beliefs, we obtain small posterior predictive errors using these distributions as prior.

Regarding the Normal-Gamma and Normal-Scaled Beta two models, we obtain sensible outcomes in predictions when there is a small sample size. However, this property is lost when the experts overestimate the certainty of their knowledge. Especially if a Normal-Gamma model is used. In this particular model, the posterior distribution of the variance is concentrated near zero when a high level of uncertainty about the experts' beliefs are present, which implies a narrow posterior predictive credibility interval, particularly with small sample sizes. This conclusion is in accordance with the results in the school example reported by Gelman (2006). This phenomenon is less severe in the Normal-Scaled Beta two model.

[5]Fig. 4 depicts the box plot of scale parameters under different models and sample sizes.

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5 Annex

Table 17: Other levels for prior variance of β_i: $((\sigma_{\beta_i}/\beta_i) * 100\% = 30\%, 60\%, 100\%, 130\%)$ and $n = 100$.

Model	Period	Real value	Predic. value	Lower HPD interval	Upper HPD interval	Error rate
Normal-Gamma (30%)	2009 Q4	14.48	13.86	13.27	14.49	4.36%
	2010 Q1	13.88	14.54	13.97	15.11	4.77%
	2010 Q2	13.78	13.86	13.27	14.47	1.85%
	2010 Q3	13.64	14.23	13.61	14.84	4.38%
Normal-Scaled Beta2 (30%)	2009 Q4	14.48	13.85	13.22	14.49	4.42%
	2010 Q1	13.88	14.54	13.94	15.11	4.72%
	2010 Q2	13.78	13.85	13.25	14.49	1.86%
	2010 Q3	13.64	14.22	13.60	14.86	4.32%
Normal-Gamma (60%)	2009 Q4	14.48	13.99	13.56	14.44	3.42%
	2010 Q1	13.88	14.64	14.26	15.04	5.47%
	2010 Q2	13.78	13.94	13.56	14.37	1.58%
	2010 Q3	13.64	14.38	13.91	14.82	5.47%
Normal-Scaled Beta2 (60%)	2009 Q4	14.48	13.99	13.54	14.43	3.41%
	2010 Q1	13.88	14.65	14.24	15.04	5.48%
	2010 Q2	13.78	13.95	13.53	14.36	1.59%
	2010 Q3	13.64	14.39	13.92	14.84	5.48%
Normal-Gamma (100%)	2009 Q4	14.48	14.04	13.59	14.52	3.06%
	2010 Q1	13.88	14.69	14.30	15.12	5.81%
	2010 Q2	13.78	13.99	13.55	14.42	1.77%
	2010 Q3	13.64	14.45	13.97	14.94	5.99%
Normal-Scaled Beta2 (100%)	2009 Q4	14.48	14.05	13.59	14.52	3.02%
	2010 Q1	13.88	14.70	14.28	15.10	5.84%
	2010 Q2	13.78	13.99	13.58	14.43	1.80%
	2010 Q3	13.64	14.46	13.98	14.94	6.02%
Normal-Gamma (130%)	2009 Q4	14.48	14.11	13.71	14.53	2.61%
	2010 Q1	13.88	14.74	14.38	15.11	6.20%
	2010 Q2	13.78	14.05	13.68	14.43	2.02%
	2010 Q3	13.64	14.52	14.06	14.95	6.49%
Normal-Scaled Beta2 (130%)	2009 Q4	14.48	14.06	13.62	14.54	2.93%
	2010 Q1	13.88	14.71	14.31	15.13	5.93%
	2010 Q2	13.78	14.00	13.60	14.43	1.83%
	2010 Q3	13.64	14.48	13.98	14.97	6.16%

Table 18: Summary of posterior distributions for σ^2 under different levels of prior variance $\frac{\sigma_{\beta_i}}{\beta_i} * 100\%$ for $i = 0, 1, 2, 3$. using Normal-Gamma model

Sample size	Variance level $\frac{\sigma_{\beta_i}}{\beta_i} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	0.8859	1.3122	1.4557	1.6196	2.4728
	30%	0.3494	1.0643	1.2485	1.4487	2.6848
	60%	0.1345	0.2669	0.3312	0.4241	1.9433
	100%	0.1345	0.2553	0.3085	0.3819	1.4825
	130%	0.0160	0.0346	0.0533	0.1193	1.0262
$n = 100$	10%	1.1642	1.5654	1.6543	1.7465	2.2771
	30%	0.0777	0.0988	0.1046	0.1109	0.1450
	60%	0.0439	0.0554	0.0587	0.0625	0.0851
	100%	0.0434	0.0553	0.0587	0.0625	0.0892
	130%	0.0400	0.0497	0.0522	0.0549	0.0745

Table 19: Summary of posterior distributions for σ^2 under different levels of prior variance $\frac{\sigma_{\beta_i}}{\beta_i} * 100\%$ for $i = 0, 1, 2, 3.$ using Normal-Scaled Beta two model

Sample size	Variance level $\frac{\sigma_{\beta_i}}{\beta_i} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	0.8756	1.2859	1.4221	1.5849	2.5246
	30%	0.3831	0.9822	1.1477	1.3338	2.5930
	60%	0.1135	0.2447	0.2988	0.3731	1.4208
	100%	0.1229	0.2374	0.2860	0.3487	1.2484
	130%	0.1123	0.2380	0.2848	0.3499	1.0843
$n = 100$	10%	1.1630	1.5530	1.6420	1.7340	2.2550
	30%	0.0746	0.0992	0.1048	0.1110	0.1508
	60%	0.0452	0.0551	0.0584	0.0621	0.0849
	100%	0.0447	0.0550	0.0583	0.0619	0.0876
	130%	0.0442	0.0551	0.0584	0.0623	0.0881

Table 20: Summary of posterior distributions for σ^2 under different levels of prior variance $\frac{\sigma_{\beta_i}}{\beta_i} * 100\%$ for $i = 0, 1, 2, 3.$ using Studentized-Gamma model

Sample size	Variance level $\frac{\sigma_{\beta_i}}{\beta_i} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	0.1329	0.2555	0.3098	0.3842	1.4449
	30%	0.1258	0.2567	0.3116	0.3862	1.7726
	60%	0.1197	0.2536	0.3089	0.3803	1.3317
	100%	0.1246	0.2558	0.3103	0.3825	1.7946
	130%	0.1075	0.2541	0.3098	0.3836	1.5762
$n = 100$	10%	0.0448	0.0566	0.0602	0.0640	0.0878
	30%	0.0431	0.0555	0.0591	0.0630	0.0878
	60%	0.0421	0.0557	0.0590	0.0628	0.0898
	100%	0.0422	0.0557	0.0591	0.0629	0.0859
	130%	0.0423	0.0557	0.0584	0.0629	0.0845

Table 21: Summary of posterior distributions for σ^2 under different levels of prior variance $\frac{\sigma_{\beta_i}}{\beta_i} * 100\%$ for $i = 0, 1, 2, 3$ using Student's t-Scaled Beta two model

Sample size	Variance level $\frac{\sigma_{\beta_i}}{\beta_i} * 100\%$	Min.	1st Qu.	Median	3rd Qu.	Max.
$n = 10$	10%	0.1149	0.2400	0.2897	0.3531	1.2479
	30%	0.1193	0.2394	0.2878	0.3511	1.4062
	60%	0.1031	0.2391	0.2873	0.3522	1.2467
	100%	0.1154	0.2370	0.2843	0.3467	1.0696
	130%	0.1161	0.2386	0.2858	0.3502	1.3698
$n = 100$	10%	0.0435	0.0561	0.0596	0.0634	0.0851
	30%	0.0444	0.0556	0.0589	0.0627	0.0929
	60%	0.0426	0.0554	0.0587	0.0626	0.0831
	100%	0.0428	0.0553	0.0587	0.0625	0.0848
	130%	0.0437	0.0554	0.0587	0.0624	0.0908

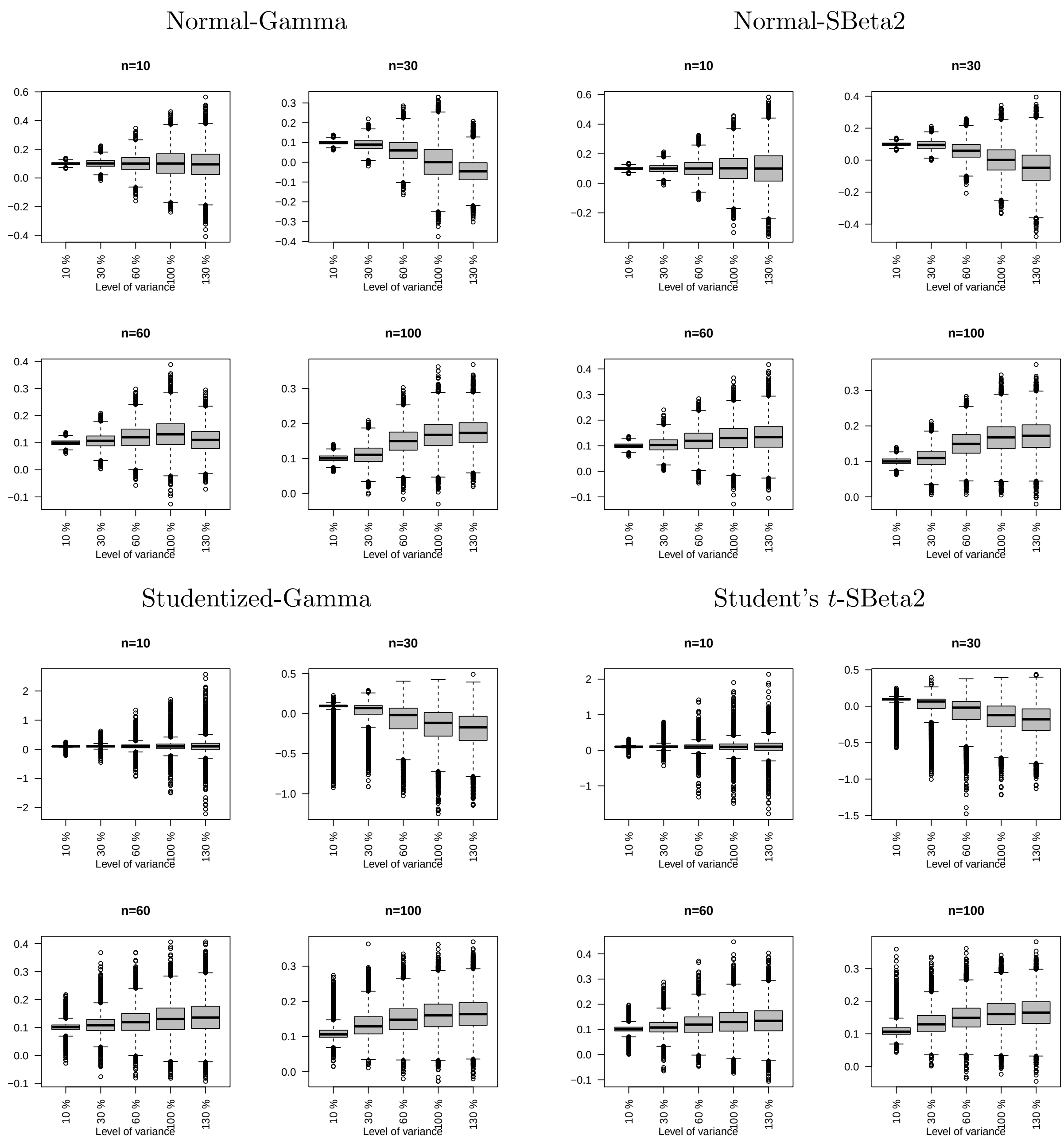

Figure 1: Posterior distributions for the income elasticity β_1 under different levels of $\sigma^2_{\beta_1}$ and $n = 10, 30, 60, 100$

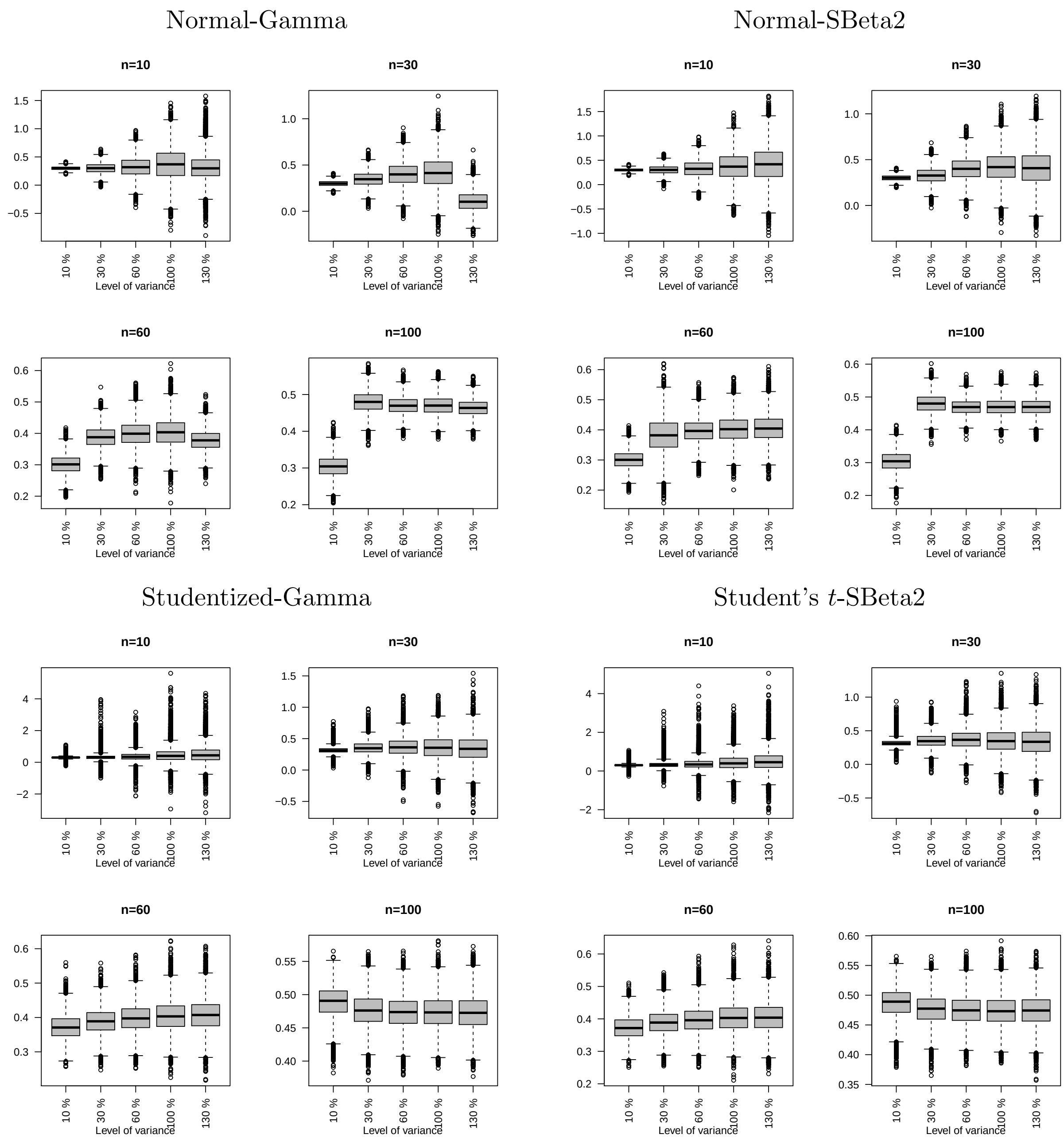

Figure 2: Posterior distributions for the semi-elasticity of number of people β_2 under different levels of $\sigma^2_{\beta_2}$ and $n = 10, 30, 60, 100$

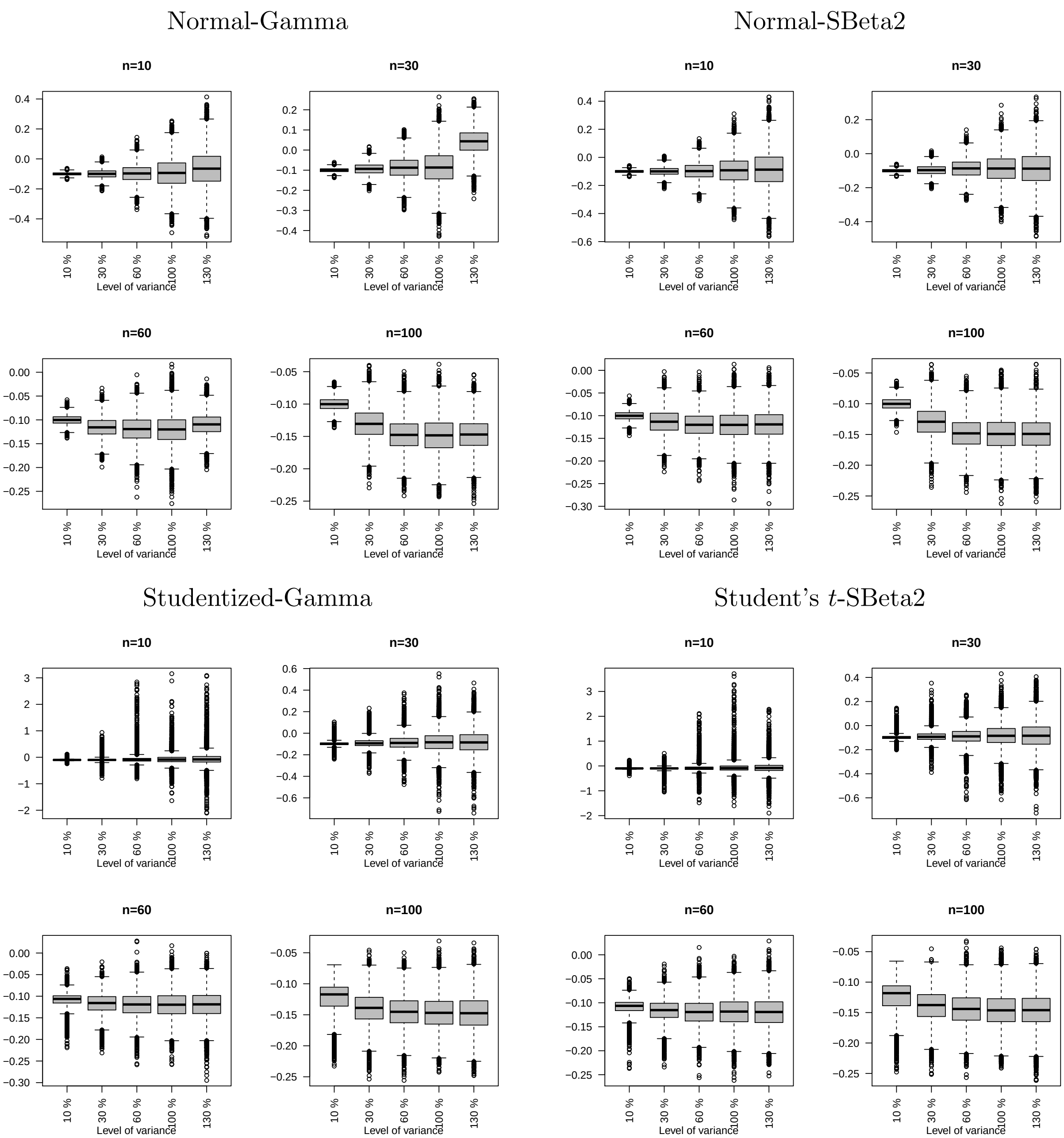

Figure 3: Posterior distributions for the price elasticity β_3 under different levels of $\sigma^2_{\beta_3}$ and $n = 10, 30, 60, 100$

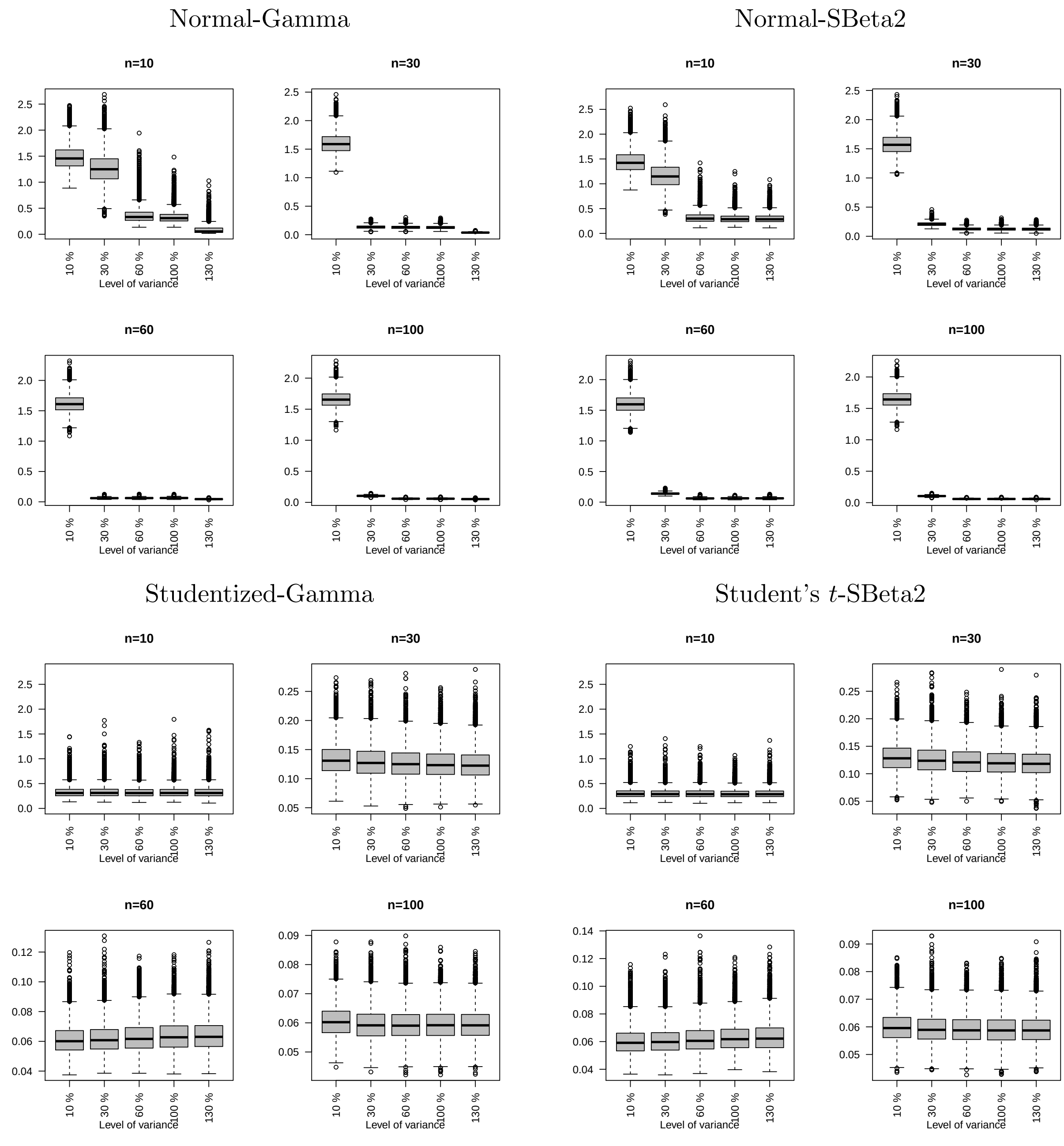

Figure 4: Posterior distributions for σ under different levels of $\sigma^2_{\beta_i}$ and $n = 10, 30, 60, 100$