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PERIODIC SOLUTIONS OF A
NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION

A THESIS

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CHAPTER I

INTRODUCTION

Consider the differential equation

$$\ddot{x} + x = \rho f(t, x, \dot{x}, \rho) \quad , \left(\cdot = \frac{d}{dt} \right) \tag{1.1}$$

together with the initial conditions

$$x(0) = a + p(\rho) \quad , p(0) = 0 \tag{1.2}$$

$$\dot{x}(0) = b + q(\rho) \quad , q(0) = 0 \quad ,$$

and the generating equation obtained by letting $\rho = 0$

$$\ddot{x} + x = 0 \tag{1.3}$$

together with the initial conditions

$$x(0) = a \tag{1.4}$$

$$\dot{x}(0) = b$$

where f is a given real function which is periodic in t with period 2π. It is desired to construct a solution of (1.1) satisfying (1.2) which is periodic in t with period 2π and which for $\rho = 0$ reduces to the solution

$$x = a \cos t + b \sin t$$

of (1.3) satisfying (1.4). Hereafter this will be referred to as the "basic problem" for (1.1)-(1.2).

In this study the problem is treated by first introducing a phase shift in (1.1)-(1.2) in order that the second initial condition in (1.2) may be taken as zero. This is done by defining

$$h(\rho) = \text{Arc tan}\,\frac{b + q(\rho)}{a + p(\rho)} = \frac{\pi}{2} - \text{Arc tan}\,\frac{a + p(\rho)}{b + q(\rho)}$$

where p and q are taken to be analytic functions of ρ at $\rho = 0$ and

$$p(\rho) = o(|\rho|^r) \quad , r \geq 1$$

and

$$q(\rho) = o(|\rho|^s) \quad , s \geq 1.$$

Without loss of generality it may be assumed that either a or b is different from zero. For if $a = b = 0$ and $r \geq s$, $\frac{p}{q}$ may be put in the form

$$\frac{p}{q} = \frac{a_1\rho^{r-s} + \bar{p}(\rho)}{b_1 + \bar{q}(\rho)}$$

where $\bar{p}(0) = \bar{q}(0) = 0$ and $b_1 \neq 0$. Similarly for $r < s$ and q/p. Hence, $h(\rho)$ will be analytic at $\rho = 0$. Define $g(\rho) = -h(\rho)$. Then (1.1)-(1.2) take the form

$$\ddot{x} + x = \rho f(t + g(\rho), x, \dot{x}, \rho) \tag{1.5}$$

$$x(0) = A_o + \lambda(\rho) \tag{1.6}$$
$$\dot{x}(0) = 0$$

and (1.3)-(1.4) become

$$\ddot{x} + x = 0 \tag{1.7}$$

$$x(0) = A_0 \tag{1.8}$$
$$\dot{x}(0) = 0$$

The basic problem will be to construct a solution, $x(t,\rho)$ of (1.5) satisfying (1.6) which is periodic in t with period 2π and which for $\rho = 0$ reduces to the solution

$$x(t,0) = A_0 \cos t$$

of (1.7) satisfying (1.8).

The quantity A_0 and the functions g and λ are to be determined as functions of ρ such that the basic problem will have a solution. It will be seen that there can be a solution to the basic problem only for certain pairs (A_0,g_0), called admissible, where $g_0 = g(0)$. An admissible pair (A_0,g_0) then determines a pair $(\lambda'(0),g'(0))$ and this in turn determines the pair $(\lambda''(0),g''(0))$, and so on, where the primes denote differentiation with respect to ρ.

The basic problem connected with the corresponding autonomous system

$$\ddot{x} + x = \rho f(x,\dot{x},\rho) \tag{1.9}$$

$$x(0) = a + p(\rho)\ ,\ p(0) = 0 \tag{1.10}$$
$$\dot{x}(0) = b + q(\rho)\ ,\ q(0) = 0$$

is considered by Malkin [1], Stoker [2], Andronow and Chaikin [3], Coddington and Levinson[4] and others. Here a simple change of the independent variable always enables one to take the second initial condition in (1.10) to be zero without changing the differential equation (1.9). Once a solution is found, the inverse change of variable yields a solution of (1.9)-(1.10). Proskuryakov [5] solves the basic problem for (1.9)-(1.10) with the second initial condition zero and with f analytic in its three arguments. In this work a similar method of solution is used to solve the basic problem for the system (1.5)-(1.6).

After independent work was begun on this study, Proskuryakov [6] published the solution of the basic problem for (1.1)-(1.2). He constructs a solution under the assumption that f is periodic in t with period 2π, continuous in all its arguments, and analytic in $(x, \dot{x}, \rho)$ for each fixed t. The method of solution requires that f be expanded in powers of ρ. By treating the differential equation (1.1)-(1.2) as it stands (i.e., without first reducing the second initial condition to zero) he was able to avoid all differentiation of f with respect to t. However, in the differential equation (1.5)-(1.6) treated in this study the phase shift in the first argument of f makes it necessary to differentiate f with respect to t and hence a hypothesis stronger than continuity in t is essential. In fact f is assumed to be analytic in all its arguments. The method of solution also requires g and λ to be expanded in powers of ρ and hence these functions are assumed to be analytic functions of ρ.

The thesis is divided into three parts. Chapter Two is devoted to an exposition of basic concepts and theorems from the theory of differential equations which are needed to accomplish the main task. Most of these are known in some sense but are included for the sake of completeness. Chapter Three contains an exposition of the main theoretical problem described above, together with an example. The appendix contains some theorems from the theory of functions of several complex variables pertinent to the development of chapters two and three.

CHAPTER II

AUXILIARY CONCEPTS AND THEOREMS

In this chapter various basic concepts and theorems are presented to support the work done in Chapter Three. The existence and uniqueness theorem (Theorem 1) and the theorem on periodic solutions (Theorem 3) are basic in the theory of differential equations. The definition of multiple roots and the theorem on interchange of order of differentiation (Theorem 2) are more specialized to handle this particular problem.

<u>Definition of Multiple Roots for a System of Equations</u>.-- For the equation $h(x) = 0$, the definition of multiple roots is quite simple. Suppose h is sufficiently smooth so that there is no question about differentiability. The equation is said to have an n - fold root at $x = x_o$ if $h(x_o) = h'(x_o) = \dots = h^{(n-1)}(x_o) = 0$, but $h^{(n)}(x_o) \neq 0$.

In the case of a system

$$S: \quad \begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0 \end{aligned}$$

one considers the two curves in the (x, y) plane determined by $f(x, y) = 0$ and $g(x, y) = 0$. Let (x_o, y_o) be a point at which $f(x_o, y_o) = g(x_o, y_o) = 0$. In a neighborhood of (x_o, y_o) $f(x, y) = 0$ defines y as a function of x, $y = y_1(x)$, and $g(x, y) = 0$ defines y as a function of x, $y = y_2(x)$. The system S is said to have an n - fold root at (x_o, y_o) if

$$D^k y_1(x_o) = D^k y_2(x_o) \text{ for } k = 1,2,\ldots,n-1$$

and

$$D^n y_1(x_o) \neq D^n y_2(x_o) \qquad , \text{ where}$$

D^k is the operator $\frac{d^k}{dx^k}$.

Note that the definition for the system S reduces to the simpler case above when $f(x, y) = y - h(x)$ and $g(x, y) = y$.

To illustrate the concept of multiple roots the case of double roots will be considered. Suppose S has a double root at (x_o, y_o) and further suppose that f_y and g_y are different from zero. By the definition of double root it must be true that

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{g_x}{g_y} \tag{2.1}$$

at (x_o, y_o) and

$$-(f_y)^{-2}\left[f_y\left(f_{xx} + f_{xy}\frac{dy}{dx}\right) - f_x\left(f_{xy} + f_{yy}\frac{dy}{dx}\right)\right] \tag{2.2}$$
$$\neq -(g_y)^{-2}\left[g_y\left(g_{xx} + g_{xy}\frac{dy}{dx}\right) - g_x\left(g_{xy} + g_{yy}\frac{dy}{dx}\right)\right].$$

Note that (2.1) is equivalent to saying that the Jacobian

$$\frac{\partial(f, g)}{\partial(x_o, y_o)} = 0.$$

Using (2.1) in (2.2) one sees that for the case of double roots,

$$f_y[g_y^2 f_{xx} - 2g_x g_y f_{xy} + g_x^2 f_{yy}] - \qquad (2.3)$$

$$g_y[f_y^2 g_{xx} - 2f_x f_y g_{xy} + f_x^2 g_{yy}] \neq 0.$$

The following general existence and uniqueness theorem for complex systems is needed:

Theorem 1.-- Let

1. I_ρ be the region $|\rho - \rho_o| < c$, $c > o$, where ρ is the vector $\rho = (\rho_1, \ldots, \rho_n)$, ρ_i, $i = 1,\ldots,n$ complex;
2. I_g be the interval $|g - g_o| < c'$, $c' > o$, g real;
3. D be a domain of (t, w) space, t real and w the vector $w = (w_1,\ldots,w_n)$, w_i complex $i = 1,\ldots, n$;
4. D_ρ be the set of points (t, w, ρ) such that $(t, w) \varepsilon D$ and $\rho \varepsilon I_\rho$ and;
5. $D_{g\rho}$ be the set of points $(t + g, w, \rho)$ such that $(t, w, \rho)\ \varepsilon D_\rho$, $g \varepsilon I_g$.

Further, let $f(s,w,\rho)$ be continuous in (s,w,ρ) on $D_{g\rho}$ and for each fixed s let f be analytic in (w,ρ), where $f = (f_1,\ldots,f_n)$. For $\rho = \rho_o$ let $p(t,g_o,\rho_o)$ be a solution of

$$w' = f(t + g_o, w, \rho_o)$$

on some interval I: $a \leq t \leq b$ [i.e. $(t,p(t,g_o,\rho_o))\ \varepsilon D$ for $t \varepsilon I$]

satisfying $p(\tau) = w_o$ where $\tau \varepsilon I$.

Then there exists a $\delta > o$ such that for any $(\bar{w}, \rho, g) \varepsilon U_{g\rho}$,

$U_{g\rho}$: $|\bar{w} - w_o| + |g - g_o| + |\rho - \rho_o| < \delta$,

there exists a unique solution

$$h = h(t,g,\bar{w},\rho) \text{ of}$$
$$w' = f(t + g,w,\rho)$$

for $t \varepsilon I$ with $h(\tau,g,\bar{w},\rho) = \bar{w}$. Moreover, h is continuous in $(t,g,\bar{w},\rho)$ and for each fixed t and g, h is analytic in $(\bar{w},\rho)$.

<u>Proof</u>.-- Choose a δ_1 such that the closed $(t + g,w,\rho)$ region

$$R_{g\rho} = \left\{ (t + g,w,\rho) : \ a \leq t \leq b, \ |w - w_o| + |\rho - \rho_o| + |g - g_o| \leq \delta_1 \right\}$$

is in $D_{g\rho}$. Let

$$R^* = \left\{ (t,g,w,\rho) : \ a \leq t \leq b, \ |w - w_o| + |\rho - \rho_o| + |g - g_o| < \delta_1 \right\}$$

Define the successive approximations h_i by

$$h_o(t,\tau,g,\bar{w},\rho) = p(t,g_o,\rho_o) + \bar{w} - w_o \tag{2.4}$$

$$h_{n+1}(t,\tau,g,\bar{w},\rho) = \bar{w} + \int_\tau^t f(s + g,h_n(s,\tau,g,\bar{w},\rho),\rho)\, ds \tag{2.5}$$

Since p is a solution of a differential equation on [a,b], p is a continuous function of t on [a,b] and from (2.4) it is seen that h_o is continuous in $(t,g,\bar{w},\rho)$ for t on [a,b] and any choice of the other variables. In particular then, h_o is continuous in $(t,g,\bar{w},\rho)$ on R^*. Similarly for each fixed $t \varepsilon [a,b]$, h_o is analytic in $(\bar{w},\rho)$ and hence this is true at least on R^*.

As an induction hypothesis assume that h_n is continuous for $(t,g,\bar{w},\rho) \varepsilon R^*$ and that for each fixed t and g, h_n is analytic in $(\bar{w},\rho)$. By Theorem 3 of the appendix h_{n+1} is a continuous function of $(t,g,\bar{w},\rho)$ in R^*, and by Theorem 5 of the appendix for each fixed t and g, h_{n+1} is analytic in $(\bar{w},\rho)$. Hence all the iterants possess the desired property. The next step is to show the uniform convergence of the series

$$\sum_{n=1}^{\infty} [h_n - h_{n-1}] \text{ on } R^* \quad .$$

From (2.4),

$$|h_o(t,\tau,g,w,\rho) - p(t,g_o,\rho_o)| = |\bar{w} - w_o| \qquad (2.6)$$

and

$$h_1(t,\tau,g,w,\rho) - h_o(t,\tau,g,\bar{w},\rho) =$$

$$\int_\tau^t f(s + g,\ h_o(s,\tau,g,\bar{w},\rho),\ \rho)\ ds - [p - w_o].$$

But $p - w_o = \int_\tau^t p'(s)\ ds = \int_\tau^t f(s + g_o,\ p(s),\ \rho_o)\ ds.$

Thus,

$$|h_1(t,\tau,g,\bar{w},\rho) - h_o(t,\tau,g,\bar{w},\rho)| = \qquad (2.7)$$

$$\left|\int_\tau^t f(s + g, h_o(s),\rho) - f(s + g_o,p(s),\ \rho_o)\ ds\right|.$$

Since f is continuous on the compact set $R_{g\rho}$, it is uniformly continuous so that given $\varepsilon > o$, there exists a δ_ε such that

$$|f(s + g, h_o(s), \rho) - f(s + g_o, p(s), \rho_o)| < \varepsilon \tag{2.8}$$

for $a \leq s \leq b$ and

$$|\bar{w} - w_o| + |g - g_o| + |\rho - \rho_o| < \min(\delta_1, \delta_\varepsilon). \tag{2.9}$$

Using (2.8) in (2.7), one obtains

$$|h_1(t, \tau, g, \bar{w}, \rho) - h_o(t, \tau, g, \bar{w}, \rho)| < \varepsilon\ |t - \tau|$$

provided (2.9) holds. Now since f is analytic in w, f_w is continuous on the compact set $R_{g\rho}$ and hence is bounded; $|f_w| \leq K$. Also since $R_{g\rho}$ is convex, the mean value theorem may be applied to show that f satisfies a Lipschitz condition in w on $R_{g\rho}$ uniformly with respect to its other arguments. Note that since f and w are vectors, f_w is a matrix $A = (a_{ij})$, and the norm is defined by the relation

$$|A| = \sum_{i,j=1}^{n} |a_{ij}|.$$

As an induction hypothesis assume that for some n

$$|h_n - h_{n-1}| \leq \varepsilon K^{n-1} \left|\frac{t - \tau}{n!}\right|^n$$

where K is the Lipschitz constant for f.

Then

$$|h_{n+1} - h_n| \leq \int_\tau^t |f(s + g, h_n, \rho) - f(s + g, h_{n-1}, \rho)|\ ds$$

$$\leq \int_\tau^t K|h_n - h_{n-1}|\ ds \leq \int_\tau^t \varepsilon K^n \left|\frac{s - \tau}{n!}\right|^n ds$$

$$= \varepsilon K^n \frac{|t - \tau|^{n+1}}{(n+1)!} \leq \varepsilon K^n \frac{(b - a)^{n+1}}{(n+1)!}.$$

Thus for all n

$$|h_n - h_{n-1}| \leq \varepsilon K^{n-1} \frac{|t - \tau|^n}{n!} .$$

Thus,

$$|h_n - p| \leq |h_n - h_{n-1}| + |h_{n-1} - h_{n-2}| + \ldots$$

$$+ |h_1 - h_o| + |h_o - p| \leq$$

$$\frac{\varepsilon}{K}\left\{\frac{[K(b - a)]^n}{n!} + \frac{[K(b - a)]^{n-1}}{(n-1)!} + \ldots + K(b - a)\right\} + \delta_\varepsilon$$

$$\leq \frac{\varepsilon}{K} (e^{K(b - a)} - 1) + \delta_\varepsilon$$

Now let ε be given such that $\frac{\varepsilon}{K} (e^{K(b - a)} - 1) < \frac{\delta_1}{2}$. Then choose $\delta_\varepsilon < \frac{\delta_1}{2}$. Finally let the δ in the statement of the theorem be equal to δ_ε. Hence $|h_n - p| < \delta_1$ for all n and the point $(t, h_n(t, \tau, g, \bar{w}, \rho))$ remains in the region $\left\{a \leq t \leq b, |w - p| < \delta_1\right\}$ for all $(\tau, \bar{w}, \rho) \in D_\rho$.

The series

$$h_o + \sum_{n=1}^{\infty} [h_n - h_{n-1}] = \lim_{n\to\infty} h_n$$

is majorized by the series for $e^{K(b - a)}$ and hence converges uniformly to a limit function

$$\lim_{n\to\infty} h_n = h(t,\tau,g,\bar{w},\rho).$$

Since the convergence is uniform, h is continuous in $(t,g,\bar{w},\rho)$ on R^* and by Theorem 6 of the Appendix, h is analytic in $(\bar{w},\rho)$ for each fixed t and g.

To see that h is a solution let $f_n = f(s + g,h_n,\rho)$. Then for any r and for n sufficiently large,

$$\begin{aligned} |f_n - f_{n+r}| &\leq |f(s + g,h_n,\rho) - f(s + g,h_{n+r},\rho)| \\ &\leq K\,|h_n - h_{n+r}| < \varepsilon \end{aligned}$$

for all $(t,g,\bar{w},\rho) \varepsilon R^*$ since $h_n \to h$ uniformly in R^*. Thus,

$$\lim_{n\to\infty} \int_\tau^t f_n ds = \int_\tau^t \lim_{n\to\infty} f_n ds = \int_\tau^t f(s + g,\lim_{n\to\infty} h_n,\rho)ds = \int_\tau^t f(s + g,h,\rho)ds.$$

Thus, in (2.5) the limit may be taken on both sides to obtain

$$h(t,\tau,g,\bar{w},\rho) = \bar{w} + \int_\tau^t f(s + g,h(s),\rho)\, ds$$

which is equivalent to

$$h' = f(t + g,h,\rho).$$

Uniqueness of h is obtained as in Coddington and Levinson [7].

Corollary.-- In the previous theorem if f is assumed to be analytic in all its arguments then the solution

$$h = h(t,g,\bar{w},\rho)$$

will be analytic in all its arguments.

Proof.-- This follows from the uniform convergence of the successive approximations and Theorem 6 of the Appendix.

Theorem 2.— Consider the differential equation

$$\ddot{x} + x = \rho f(t + g(\rho), x, \dot{x}, \rho) \tag{2.10}$$

$$x(t = o) = A_o + \lambda$$

$$\dot{x}(t = o) = o$$

where $x = x(t, A_o, g, \lambda, \rho)$ and $f(s, x, \dot{x}, \rho)$ is analytic in $(s, x, \dot{x}, \rho)$. Suppose that the solution x can be written as

$$x(t, A_o, g, \lambda, \rho) = (A_o + \lambda) \cos t + \tag{2.11}$$

$$+ \sum_{n=1}^{\infty} \left\{ C_n(t, A_o, g) + \frac{\partial C_n}{\partial \lambda} \lambda + \frac{1}{2} \frac{\partial^2 C_n}{\partial \lambda^2} \lambda^2 + \dots \right\} \rho^n$$

with its first derivative given by

$$\dot{x}(t, A_o, g, \lambda, \rho) = -(A_o + \lambda) \sin t + \tag{2.12}$$

$$+ \sum_{n=1}^{\infty} \left\{ \dot{C}_n(t, A_o, g) + \frac{\partial \dot{C}_n}{\partial \lambda} \lambda + \frac{1}{2} \frac{\partial^2 \dot{C}_n}{\partial \lambda^2} \lambda^2 + \dots \right\} \rho^n$$

Then in (2.11) and (2.12), differentiation with respect to λ can be replaced by differentiation with respect to A_o at $\rho = \lambda = o$, i.e.,

$$\frac{\partial^{m+n} x(t, A_o, g, o, o)}{\partial \lambda^m \, \partial \rho^n} = \frac{\partial^{m+n} x(t, A_o, g, o, o)}{\partial A_o^m \, \partial \rho^n} \tag{2.13}$$

$m, n \geq o$ and similarly for $\dot{x}$.

Proof.-- Let

$$\rho f(t + g(\rho),\ x(t,A_o,g,\lambda,\rho),\ \dot{x}(t,A_o,g,\lambda,\rho),\ \rho)$$

$$= \rho\bar{f}(t,A_o,g,\lambda,\rho).$$

Now expand $\rho\bar{f}$ about $\rho = \lambda = 0$

$$\rho\bar{f} = \sum_{m,n=o}^{\infty} \frac{1}{m!\ n!} \frac{\partial^{m+n}\bar{f}(t,A_o,g,o,o)}{\partial\lambda^m\ \partial\rho^n} \lambda^m\rho^{n+1} \tag{2.14}$$

Substitute (2.14) and (2.11) into (2.10) and equate coefficients of $\lambda^m\rho^{n+1}$ to get

$$\frac{\partial^{m+n+1}\ \ddot{x}\ (t,A_o,g,o,o)}{\partial\lambda^m\partial\rho^{n+1}} + \frac{\partial^{m+n+1}\ x(t,A_o,g,o,o)}{\partial\lambda^m\partial\rho^{n+1}} \tag{2.15}$$

$$= (n+1)\ \frac{\partial^{m+n}\ \bar{f}(t,A_o,g,o,o)}{\partial\lambda^m\partial\rho^n}$$

Using the fact that $x(o,A_o,g,\lambda,\rho) = A_o + \lambda$ and $\dot{x}(o,A_o,g,\lambda,\rho) = o$

the following initial conditions for (2.15) must be satisfied,

$$\frac{\partial^{m+n+1}\ x(o,A_o,g,o,o)}{\partial\lambda^m\partial\rho^{n+1}} = \frac{\partial^{m+n+1}\ \dot{x}(o,A_o,g,o,o)}{\partial\lambda^m\partial\rho^{n+1}} = o \tag{2.16}$$

By variation of parameters, the solution of (2.15) and (2.16) is

$$\frac{\partial^{m+n+1}\ x(t,A_o,g,o,o)}{\partial\lambda^m\partial\rho^{n+1}} = \tag{2.17}$$

$$(n+1)\int_{o}^{t} \frac{\partial^{m+n} \bar{f}(s,A_o,g,o,o)}{\partial\lambda^m \partial\rho^n} \sin(t-s)\, ds \quad .$$

Now to start an induction argument, notice from (2.11) and (2.12) that (2.13) is true for $n = o$ and all m. As an induction step assume that (2.13) is true for $n = 1, 2, \ldots, K$ and all m. Then

$$\frac{\partial^{m+k} \bar{f}(t,A_o,g,o,o)}{\partial\lambda^m \partial\rho^k} = \tag{2.18}$$

$$G_{mk}\left(\frac{\partial x}{\partial\rho}, \frac{\partial x}{\partial A_o}, \ldots \frac{\partial^{m+k} x}{\partial A_o^{\,m} \partial\rho^k}, \frac{\partial \dot{x}}{\partial\rho}, \frac{\partial \dot{x}}{\partial A_o}, \ldots \right.$$

$$\left. \ldots \frac{\partial^{m+k} \dot{x}}{\partial A_o^{\,m} \partial\rho^k}, g, g' \ldots \right) \Bigg|_{\rho = \lambda = o} =$$

$$\frac{\partial^{m+k} \bar{f}(t,A_o,g,o,o)}{\partial A_o^{\,m} \partial\rho^k}$$

Upon substituting (2.18) under the integral sign in (2.17) one gets

$$\frac{\partial^{m+k+1} x(t,A_o,g,o,o)}{\partial\lambda^m \partial\rho^{k+1}} = \frac{\partial^{m+k+1} x(t,A_o,g,o,o)}{\partial A_o^{\,m} \partial\rho^{k+1}}$$

Since the procedure was independent of m, it has been shown that (2.13) now holds for $n = k + 1$ and all m. This completes the induction.

The following criterion for periodic solutions of nonlinear systems is needed.

Theorem 3.— Consider the following system of differential equations and initial conditions:

$$x' = f(t,x,\rho) \qquad (2.19)$$
$$x(o) = x_o$$

where x and f are real vectors and ρ is a parameter. Suppose that f is periodic in t with period 2π and that f and f_x are continuous in (t,x,ρ) for $-\infty < t < \infty$, $|x| \leq a$, $|\rho| \leq \rho_o$ where a and ρ_o are constants. Let $x(t,x_o,\rho)$, or simply $x(t)$, be a solution of (2.19) defined on $-\infty < t < \infty$. Then a necessary and sufficient condition that $x(t)$ be periodic with period 2π is that

$$x(o) = x(2\pi). \qquad (2.20)$$

Proof.— First assume that $x(t)$ is 2π periodic, i.e., $x(t + 2\pi) = x(t)$ for all t. Then in particular for $t = o$, (2.20) holds.

Now assume that (2.20) holds. Define $\overline{x}(t) = x(t + 2\pi)$. The solution of (2.19) satisfies

$$x(t) = x_o + \int_\tau^t f(s,x(s),\rho)\, ds$$

and therefore

$$\overline{x}(t) = x_o + \int_\tau^{t+2\pi} f(s,x(s),\rho)\, ds.$$

The new function $\overline{x}(t)$ satisfies the differential equation in (2.19) since

$$\overline{x}'(t) = f(t + 2\pi, x(t + 2\pi),\rho)$$
$$= f(t,\overline{x}(t),\rho)$$

and furthermore

$$\bar{x}(o) = x_o + \int_o^{2\pi} f(s,x(s),\rho)ds.$$

However, (2.20) implies that

$$\int_o^{2\pi} f(s,x(s),\rho)ds = o$$

so that

$$\bar{x}(o) = x_o.$$

Thus $\bar{x}(t)$ and $x(t)$ satisfy the same differential equation and initial conditions and by uniqueness $\bar{x}(t) = x(t)$, or

$$x(t + 2\pi) = x(t).$$

The last relation states that $x(t)$ is a 2π periodic solution.

CHAPTER III

CONSTRUCTION OF PERIODIC SOLUTIONS

In this chapter a procedure for the construction of periodic solutions is presented and the results are summarized near the end of the chapter in the form of two theorems. An example is worked at the end to illustrate the construction procedure.

Consider the differential equation

$$\ddot{x} + x = \rho F(t + g(\rho), x, \dot{x}, \rho) \tag{3.1}$$

with initial conditions

$$x(o) = A_o + \lambda(\rho) \tag{3.2}$$

$$\dot{x}(o) = o \quad .$$

Suppose the following hypotheses are satisfied:

(H_1) F is periodic in t with period 2π

(H_2) $F(s,x,\dot{x},\rho)$ is analytic in $(s,x,\dot{x},\rho)$

(H_3) $g(\rho)$ and $\lambda(\rho)$ are analytic functions of ρ in some neighborhood of $\rho = o$, $g(o) = g_o$, and $\lambda(o) = o$.

The generating equation $(\rho = o)$ is

$$\ddot{x} + x = o \tag{3.3}$$

with initial conditions

$$x(o) = A_o \tag{3.4}$$

$$\dot{x}(o) = o$$

which has the solution

$$x(t,A_o) = A_o \cos t. \tag{3.5}$$

It is required to find a periodic solution of (3.1) which for $\rho = o$ reduces to the solution (3.5) of (3.3). It is known by Theorem 1 of Chapter I that the solution $x(t,A_o,g,\lambda,\rho)$ of (3.1) exists for $o \leq t \leq 2\pi$ and is analytic in λ and ρ in some neighborhood of $\lambda = \rho = o$ and thus may be expanded in powers of λ and ρ.

$$x(t,A_o,g,\lambda,\rho) = x_o + B_1\lambda = \tag{3.6}$$

$$C_1\rho + D_1\lambda\rho + B_2\lambda^2 + \ldots,$$

where the coefficients are appropriate derivatives of x evaluated at $\lambda = \rho = o$ and are functions of t,A_o,g_o and various other derivatives of g. Since at $\lambda = \rho = o$, $x = A_o \cos t$, it must be the case that $x_o = A_o \cos t$ in expression (3.6).

Expand F in terms of ρ and λ and substitute (3.6) into (3.1). Notice that since every term on the right of (3.1) has at least one factor of ρ, the following set of relations must hold:

$$\ddot{B}_n(t) + B_n(t) = o,\ n = 1,2,\ldots \quad .$$

Since at $t = o$, (3.6) must reduce to $A_o + \lambda$, the following set of initial conditions must hold:

$$B_1(o) = 1 \qquad \dot{B}_1(o) = o$$

$$B_n(o) = o \qquad \dot{B}_n(o) = o \quad , n = 2,3,\ldots \quad .$$

Hence $B_1(t) = \cos t$ and $B_n(t) = o$ for $n = 2,3,\ldots$. Now apply Theorem 7 of the Appendix to rewrite the solution (3.6) as

$$x(t,A_o,g,\lambda,\rho) = (A_o + \lambda)\cos t +$$

$$\sum_{n=1}^{\infty}\left\{c_n(t,A_o,g) + \frac{\partial c_n}{\partial\lambda}\lambda + \frac{1}{2}\frac{\partial^2 c_n}{\partial\lambda^2}\lambda^2 + \ldots\right\}\rho^n,$$

where all the coefficients are evaluated at $\lambda = \rho = o$ and

$$c_n = \frac{1}{n!}\frac{\partial^n x}{\partial\rho^n} .$$

Applying Theorem 2 of Chapter II, one readily sees that the solution may finally be written as

$$x(t,A_o,g,\lambda,\rho) = (A_o + \lambda)\cos t + \tag{3.7}$$
$$\sum_{n=1}^{\infty}\left\{c_n(t,A_o,g) + \frac{\partial c_n}{\partial A_o}\lambda + \frac{1}{2}\frac{\partial^2 c_n}{\partial A_o^2}\lambda^2 + \ldots\right\}\rho^n.$$

Note that

$$\dot{x}(t,A_o,g,\lambda,\rho) = -(A_o + \lambda)\sin t +$$

$$\sum_{n=1}^{\infty}\left\{\dot{c}_n + \frac{\partial\dot{c}_n}{\partial A_o}\lambda + \frac{1}{2}\frac{\partial^2\dot{c}_n}{\partial A_o^2}\lambda^2 + \ldots\right\}\rho^n.$$

From (3.7) one sees that it is necessary only to calculate the c_n, $n = 1,2,\ldots$; the remaining terms may be obtained by differentiation with respect to A_o.

Now expand F in terms of ρ.

$$\rho F = \sum_{n=0}^{\infty} \left(\frac{d^n F}{d\rho^n}\right)_o \frac{1}{n!} \rho^{n+1} \tag{3.8}$$

where the subscript (o) indicates that the term in parentheses is to be evaluated at $\rho = \lambda = o$, i.e.,

$$(F)_o = F(t + g_o, A_o \cos t_1, - A_o \sin t, o).$$

Note that

$$\left(\frac{dF}{d\rho}\right)_o = \left(\frac{\partial F}{\partial t}\right)_o g'(o) + \left(\frac{\partial F}{\partial x}\right)_o c_1 + \left(\frac{\partial F}{\partial \dot{x}}\right)_o \dot{c}_1 + \left(\frac{\partial F}{\partial \rho}\right)_o .$$

The coefficient of ρ^n in (3.8) is

$$H_n(t) = \frac{1}{(n-1)!} \left(\frac{d^{n-1} F}{d\rho^{n-1}}\right)_o . \tag{3.9}$$

Substitute (3.8) and (3.7) into (3.1) and equate to zero the coefficients of ρ^n. The result is

$$[\ddot{c}_n + \frac{\partial \ddot{c}_n}{\partial A_o} \lambda + \ldots] + [c_n + \frac{\partial c_n}{\partial A_o} \lambda + \ldots] = H_n(t),$$

but since this must hold for all λ in a neighborhood of $\lambda = o$, it holds also for $\lambda = o$ with the result,

$$\ddot{c}_n(t) + c_n(t) = H_n(t) . \tag{3.10}$$

Since at $t = o$, $x(o, A_o, g, \lambda, \rho) = A_o + \lambda$ one sees from (3.7) that

$$c_n(o) = \dot{c}_n(o) = o, \quad n = 1, 2, \ldots \tag{3.11}$$

It should be realized that c_n is actually a function of t, A_o and $\frac{d^i g(o)}{d\rho^i}$, $i = 1, \ldots, n-1$. However for the present it is not necessary to show this dependence explicitly. It will be shown later just how the parameters $\frac{d^i g(o)}{d\rho^i}$ can be handled.

The solution of (3.10) and (3.11) is, for $n = 1, 2, \ldots$,

$$c_n(t) = \int_o^t H_n(s) \sin (t - s)\, ds \tag{3.12}$$

$$\dot{c}_n(t) = \int_o^t H_n(s) \cos (t - s)\, ds \,.$$

A few of the H_n will now be calculated.

$$H_1(t) = F(t + g_o, A_o \cos t, - A_o \sin t, o) \tag{3.13}$$

$$H_2(t) = (F_t)_o\, g'(o) + (F_x)_o\, c_1(t) + (F_{\dot{x}})_o\, \dot{c}_1(t) + (F_\rho)_o \tag{3.14}$$

$$H_3(t) = \frac{1}{2} (F_{tt})_o [g'(o)]^2 + (F_{tx})_o\, c_1\, g'(o) + (F_{t\dot{x}})_o\, \dot{c}_1\, g'(o) + (F_{t\rho})_o \cdot$$

$$g'(o) + \frac{1}{2} (F_t)_o\, g''(o) + \frac{1}{2} (F_{xx})_o\, c_1^2 + (F_{x\dot{x}})_o\, c_1 \dot{c}_1 + (F_{x\rho})\, c_1 + (F_x)_o \cdot$$

$$c_2 + \frac{1}{2}(F_{\dot{x}\dot{x}})_o \dot{c}_1^{\,2} + (F_{\dot{x}\rho})_o \dot{c}_1 + (F_{\dot{x}})_o \dot{c}_2 + \frac{1}{2}(F_{\rho\rho})_o . \tag{3.15}$$

$$H_4(t) = \frac{1}{2}(F_t)_o g'''(o) + \dots$$

$$H_5(t) = \frac{1}{2}(F_t)_o g^{(4)}(o) + \dots$$

The next step is to expand λ and g in powers of ρ and to determine the coefficients in their expansions.

$$\lambda(\rho) = \sum_{n=1}^{\infty} A_n \rho^n \tag{3.16}$$

$$g(\rho) = g_o + \sum_{n=1}^{\infty} G_n \rho^n = \sum_{n=o}^{\infty} \frac{1}{n!}\left(\frac{d^n g}{d\rho^n}\right)_o \rho^n \tag{3.17}$$

Since periodic solutions of (3.1) are desired one may now impose some periodicity conditions on x and $\dot{x}$. By Theorem 3 of Chapter II, it is enough to require that $x(2\pi) = x(o)$ and $\dot{x}(2\pi) = \dot{x}(o)$. Expressing the fact that $x(2\pi) = x(o)$, one obtains

$$\sum_{n=1}^{\infty}\left\{c_n(2\pi) + \frac{\partial c_n}{\partial A_o}\lambda + \frac{1}{2}\frac{\partial^2 c_n}{\partial A_o^2}\lambda^2 + \dots\right\}\rho^n = o \tag{3.18}$$

Substitute (3.16) into (3.18) and equate to zero the coefficients of like powers of ρ to obtain the following:

$$c_1(2\pi) = o$$

$$c_2(2\pi) + A_1\, c_{1A_o} = o$$

$$c_3(2\pi) + A_2 c_{1A_o} + \frac{1}{2} A_1^2\, c_{1A_oA_o} + A_1 c_{2A_o} = o \qquad (3.19)$$

$$c_4(2\pi) + A_3 c_{1A_o} + A_1 A_2 c_{1A_oA_o} + \frac{1}{6} A_1^3\, c_{1A_oA_oA_o} + A_1 c_{3A_o} + A_2 c_{2A_o} +$$

$$+ \frac{1}{2} A_1^2\, c_{2A_oA_o} = o$$

$$\vdots$$

where $c_{1A_oA_o} = \dfrac{\partial^2 c_1}{\partial A_o{}^2}$ etc.

Now expressing the fact that $\dot{x}(2\pi) = \dot{x}(o)$, one obtains in the same way

$$\sum_{n=1}^{\infty} \left\{ \dot{c}_n(2\pi) + \frac{\partial \dot{c}_n}{\partial A_o} \lambda + \frac{1}{2} \frac{\partial^2 \dot{c}_n}{\partial A_o{}^2} \lambda^2 + \ldots \right\} \rho^n = o \quad (3.20)$$

and

$$\dot{c}_1(2\pi) = o$$

$$\dot{c}_2(2\pi) + A_1 \dot{c}_{1A_o} = o$$

$$\dot{c}_3(2\pi) + A_o \dot{c}_{1A_o} + \frac{1}{2} A_1^2\, \dot{c}_{1A_oA_o} + A_1 \dot{c}_{2A_o} = o$$

$$(3.21)$$

$$\dot{c}_4(2\pi) + A_3 \dot{c}_{1A_o} + A_1 A_2 \dot{c}_{1A_oA_o} + \frac{1}{6} A_1^3\, \dot{c}_{1A_oA_oA_o}$$

$$+ A_1 \dot{c}_{3A_o} + A_2 \dot{c}_{2A_o} + \frac{1}{2} A_1^2 \dot{c}_{2A_oA_o} = o$$

$\vdots$

where $\dot{c}_{1A_oA_o} = \dfrac{\partial^2 \dot{c}_1}{\partial A_o^2}$, etc.

Equations (3.19) and (3.21) thus form a set of necessary and sufficient conditions for the existence of a periodic solution of (3.1).

Some relations will now be derived showing explicitly the dependence of c_n on $\dfrac{d^i g(o)}{d\rho^i}$, $i = 1,2,\ldots, n - 1$. Let D_g be an operator which takes the total derivative with respect to ρ of a function holding g fixed, i.e.,

$$D_g F = \frac{\partial F}{\partial x} \frac{\partial x}{\partial \rho} + \frac{\partial F}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial \rho} + \frac{\partial F}{\partial \rho} ,$$

where

$$\left(\frac{\partial^n x}{\partial \rho^n}\right)_o = n!\, E_n, \quad \left(\frac{\partial^n \dot{x}}{\partial \rho^n}\right)_o = n!\, \dot{E}_n,$$

$$E_n = \int_o^t H_n^* (s) \sin (t - s)\, ds,$$

$$\dot{E}_n = \int_o^t H_n^* (s) \cos (t - s)\, ds,$$

and

$$H_n^* = \frac{1}{(n - 1)!} \left(D_g^{\,n - 1} F\right)_o .$$

Note that the E_n are functions of t, A_o, and g_o. The c_n and E_n now satisfy the following relationships:

$$c_1(t) = E_1(t)$$

$$c_2(t) = g'(o)\, E_{1g_o} + E_2(t)$$

$$c_3 = \frac{1}{2}\, [g'(o)]^2\, E_{1g_og_o} + g'(o)\, E_{2g_o} + $$

$$+ \frac{1}{2}\, g''(o)\, E_{1g_o} + E_3$$

$$(3.22)$$

$$c_4 = \frac{1}{6}\, [g'(o)]^3\, E_{1g_og_og_o} + \frac{1}{2}\, [g'(o)]^2\, E_{2g_og_o} +$$

$$+ \frac{1}{2}\, g''(o)\, E_{2g_o} + g'(o)\, E_{3g_o} +$$

$$+ \frac{1}{2}\, g'(o)\, g''(o)\, E_{1g_og_o} + \frac{1}{6}\, g'''(o)\, E_{1g_o} + E_4$$

$$\vdots$$

and similarly for the $\dot{c}_n$. Use of the relations (3.22) to rewrite conditions (3.19) and (3.21) yields the following:

$$E_1(2\pi,\ g_o,\ A_o) = o$$

$$(3.23)$$

$$\dot{E}_1(2\pi,\ g_o,\ A_o) = o\ ,$$

$$E_2(2\pi,\ g_o,\ A_o) + G_1E_{1g_o} + A_1E_{1A_o} = o$$

$$(3.24)$$

$$\dot{E}_2(2\pi, g_o, A_o) + G_1\dot{E}_{1g_o} + A_1\dot{E}_{1A_o} = o,$$

$$E_3 + G_2E_{1g_o} + \frac{1}{2} G_1^2 E_{1g_og_o} + G_1E_{2g_o} + \tag{3.25}$$

$$+ A_2E_{1A_o} + \frac{1}{2} A_1^2 E_{1A_oA_o} + A_1E_{2A_o} + A_1G_1E_{1A_og_o} = o$$

$$\dot{E}_3 + G_2\dot{E}_{1g_o} + \frac{1}{2} G_1^2 \dot{E}_{1g_og_o} + G_1\dot{E}_{2g_o} +$$

$$+ A_2\dot{E}_{1A_o} + \frac{1}{2} A_1^2 E_{1A_oA_o} + A_1\dot{E}_{2A_o} + A_1G_1\dot{E}_{1A_og_o} = o,$$

$$E_4 + A_3E_{1A_o} + G_3E_{1g_o} + \frac{1}{6} G_1^3 E_{1g_og_og_o} + \tag{3.26}$$

$$+ \frac{1}{2} G_1^2 E_{2g_og_o} + G_2E_{2g_o} + G_1E_{3g_o} + G_1G_2E_{1g_og_o} +$$

$$+ A_1A_2E_{1A_oA_o} + \frac{1}{6} A_1^3 E_{1A_oA_oA_o} + (A_2G_1 + A_1G_2) E_{1g_oA_o} +$$

$$+ A_2E_{2A_o} + \frac{1}{2} A_1^2 G_1E_{1A_oA_og_o} + \frac{1}{2} A_1^2 E_{2A_oA_o} +$$

$$+ \frac{1}{2} A_1G_1^2 E_{1A_og_og_o} + A_1G_1E_{2A_og_o} + A_1E_{3A_o} = o$$

$$\dot{E}_4 + A_3\dot{E}_{1A_o} + G_3\dot{E}_{1g_o} + \frac{1}{6} G_1^3 \dot{E}_{1g_og_og_o} +$$

$$+ \frac{1}{2} G_1^2 \dot{E}_{2g_o g_o} + G_2 \dot{E}_{2g_o} + G_1 \dot{E}_{3g_o} + G_1 G_2 \dot{E}_{1g_o g_o} +$$

$$+ A_1 A_2 \dot{E}_{1A_o A_o} + \frac{1}{6} A_1^3 \dot{E}_{1A_o A_o A_o} + (A_2 G_1 + A_1 G_2) \dot{E}_{1g_o A_o} +$$

$$+ A_2 \dot{E}_{2A_o} + \frac{1}{2} A_1^2 G_1 \dot{E}_{1A_o A_o g_o} + \frac{1}{2} A_1^2 \dot{E}_{2A_o A_o} +$$

$$+ \frac{1}{2} A_1 G_1^2 \dot{E}_{1A_o g_o g_o} + A_1 G_1 \dot{E}_{2A_o g_o} + A_1 \dot{E}_{3A_o} = 0$$

$$\vdots$$

Equations (3.23) form a system for the determination of g_o and A_o. If (3.23) has simple roots (see definition in Chapter II) then the Jacobian

$$J = \frac{\partial(E_1, \dot{E}_1)}{\partial(A_o, g_o)} \neq 0$$

and the system (3.24) may be solved for a unique pair A_1, G_1. Substitute these values of A_1 and G_1 into (3.25) and note that there results a linear system for the determination of A_2 and G_2. The determinant of coefficients is again J and (3.25) may be solved for a unique pair A_3 and G_3. Now looking at formulas (3.9) and (3.15) and also noting that

$$\frac{\partial c_1(t)}{\partial g_o} = \int_0^t D_1 F \sin (t - s)\, ds$$

where $D_1F(t,x,\dot{x},\rho) = \dfrac{\partial F}{\partial t}$, one sees that $c_n(2\pi)$ will contain terms in $G_{n-2}, G_{n-3}, \ldots, G_2, G_1$ and a term $\dfrac{\partial c_1}{\partial g_o} G_{n-1}$ which, due to (3.22) is equal to $\dfrac{\partial E_1}{\partial g_o} G_{n-1}$. Moreover, it is seen that $c_n(2\pi)$ will contain no other terms in G_{n-1}. Now consider (3.16) and (3.18) and notice that upon collecting the coefficients of ρ^n one obtains

$$c_n(2\pi) + \frac{\partial c_1}{\partial A_o} A_{n-1} + \text{ other terms in } A_{n-2}, \ldots, A_2, A_1$$

and furthermore $\dfrac{\partial c_1}{\partial A_o} A_{n-1}$ is the only term in A_{n-1}. Again due to (3.22) $\dfrac{\partial c_1}{\partial A_o} = \dfrac{\partial E_1}{\partial A_o}$. The same comments apply to $\dot{c}_n$ and the series (3.20).

Hence, just as (3.24) and (3.25) give recursive linear systems (with determinant J) for the determination of A_1, G_1 and A_2, G_2, the system obtained by equating to zero the coefficients of ρ^n in (3.18) and (3.20) gives a linear system (with determinant J) for the determination of A_{n-1} and G_{n-1}. Thus the coefficients A_n and G_n may be uniquely determined in a recursive manner.

It should be noted that the Jacobian J above is the same as the crucial Jacobian in the work of Coddington and Levinson [8] for the case of a second order equation, and that throughout their work this Jacobian does not vanish.

If the system (3.23) has double roots then the Jacobian J vanishes for these roots and the procedure just described fails. The vanishing of the Jacobian J gives rise to some supplementary conditions which must be satisfied.

It is well known that the equation

$$\begin{bmatrix} E_{1A_o} & E_{1g_o} \\ \dot{E}_{1A_o} & \dot{E}_{1g_o} \end{bmatrix} \begin{bmatrix} A_1 \\ G_1 \end{bmatrix} = - \begin{bmatrix} E_2 \\ \dot{E}_2 \end{bmatrix},$$

which is just equation (3.24) in matrix form, will have a solution if and only if the rank of

$$\begin{bmatrix} E_{1A_o} & E_{1g_o} \\ \dot{E}_{1A_o} & \dot{E}_{1g_o} \end{bmatrix} \tag{3.27}$$

is equal to the rank of

$$\begin{bmatrix} E_{1A_o} & E_{1g_o} & E_2 \\ \dot{E}_{1A_o} & \dot{E}_{1g_o} & \dot{E}_2 \end{bmatrix} . \tag{3.28}$$

But, since the determinant of (3.27) is just the zero Jacobian J, the matrix (3.28) must have rank < 2. Thus, every 2 X 2 sub-matrix of (3.28) must have determinant zero.

Hence,

$$\frac{E_{1A_o}}{\dot{E}_{1A_o}} = \frac{E_{1g_o}}{\dot{E}_{1g_o}} = \frac{E_2}{\dot{E}_2} \tag{3.29}$$

must be satisfied.

A procedure will now be described for solving (3.25) for A_1. Since $J = o$, we may eliminate A_2, G_2 and G_1 from the relations (3.25) in the following manner. Multiply the first equation in (3.25) by $(\dot{E}_{1g_o})^2$ and the second by $(E_{1g_o})^2$. Then from (3.24) substitute for G_1 and finally multiply the first equation by E_{1g_o}, the second by $\dot{E}_{1g_o}$, and subtract the second from the first. The result is the following quadratic expression for the determination of A_1:

$$P_o A_1^2 + P_1 A_1 + P_2 = o \tag{3.30}$$

where

$$P_o = E_{1g_o} \left[\dot{E}^2_{1g_o} E_{1A_oA_o} - 2\dot{E}_{1g_o} \dot{E}_{1A_o} E_{1g_oA_o} + \dot{E}^2_{1A_o} E_{1g_og_o} \right] -$$

$$\dot{E}_{1g_o} \left[E^2_{1g_o} \dot{E}_{1A_oA_o} - 2E_{1g_o} E_{1A_o} \dot{E}_{1g_oA_o} + E^2_{1A_o} \dot{E}_{1g_og_o} \right],$$

$$P_1 = E_{1g_o} \left[E_{2A_o} \dot{E}^2_{1g_o} - E_{2g_o} \dot{E}_{1g_o} \dot{E}_{1A_o} \right.$$

$$\left. - \dot{E}_2 E_{1A_og_o} \dot{E}_{1g_o} - E_2 E_{1A_og_o} \dot{E}_{1A_o} \right] -$$

$$\dot{E}_{1g_o}[\dot{E}_{2A_o}E^2_{1g_o} - \dot{E}_{2g_o}E_{1g_o}E_{1A_o} -$$

$$- E_2\dot{E}_{1A_og_o}E_{1g_o} - \dot{E}_2\dot{E}_{1A_og_o}E_{1A_o}],$$

and

$$P_2 = E_{1g_o}[E_3\dot{E}^2_{1g_o} - \dot{E}_2E_{2g_o}\dot{E}_{1g_o} + \frac{1}{2}\dot{E}^2_2E_{1g_og_o}] -$$

$$\dot{E}_{1g_o}[\dot{E}_3E_{1g_o}^2 - E_2\dot{E}_{2g_o}E_{1g_o} + \frac{1}{2}E^2_2\dot{E}_{1g_og_o}].$$

If it happens that both E_{1g_o} and $\dot{E}_{1g_o}$ are zero, then an analogous procedure using E_{1A_o} and $\dot{E}_{1A_o}$ yields a quadratic in B_1.

Having determined A_1 from (3.30), one may obtain G_1 from one of the relations (3.24).

Since it was assumed that (g_o, A_o) is a double root of (3.23), equation (2.3) holds and thus $P_o \neq o$. Hence, (3.30) determines two values for A_1. In the case that the roots of (3.30) are complex conjugate, no real solution exists. Even in the case of real roots they might be distinct giving rise to two separate determinations of λ and g as functions of ρ and hence two separate solutions. Note that this does not contradict the uniqueness theorem since there is one and only one solution for each set of initial conditions,

$$x(o) = A_o + \lambda, \; \dot{x}(o) = o.$$

There is however no guarantee of finding a unique solution to the basic problem as stated in the introduction, i.e., there may be several periodic solutions of (3.1) which for $\rho = o$ reduce to the solution (3.5) of the generating equation (3.3). Moreover, since (3.23) may have several distinct real roots, there may be several available solutions of the generating system.

Having obtained A_1 and G_1, one may calculate the quantities A_2 and G_2 in the following manner. Multiply the first equation of (3.26) by $\dot{E}_2$ and the second by E_2 and subtract the second from the first. This eliminates A_3 and G_3 and leaves one equation in the two unknowns, A_2 and G_2. Another equation in A_2 and G_2 is obtained by adding the two equations in (3.25). The system is now linear in A_2, G_2 with determinant

$$\Delta = (E_2 + \dot{E}_2)\Delta_1$$

where

$$\Delta_1 =$$

$$E_{2A_o} \dot{E}_{1g_o} - E_{1g_o} \dot{E}_{2A_o} - \dot{E}_{1A_o} E_{2g_o} + E_{1A_o} \dot{E}_{2g_o} +$$

$$A_1[\dot{E}_{1g_o} E_{1A_oA_o} - E_{1g_o} \dot{E}_{1A_oA_o} - \dot{E}_{1A_o} E_{1A_og_o} + E_{1A_o} \dot{E}_{1A_og_o}] +$$

$$G_1[\dot{E}_{1g_o} E_{1A_og_o} - E_{1g_o} \dot{E}_{1A_og_o} - \dot{E}_{1A_o} E_{1g_og_o} + E_{1A_o} \dot{E}_{1g_og_o}] .$$

Jacobian with three zero elements.— Suppose here, without loss of generality, that the Jacobian

$$J = \frac{\partial(E_1,\ \dot{E}_1)}{\partial(A_o,\ g_o)}$$

vanishes at (A_o, g_o) with $E_{1A_o} \neq o$ and $\dot{E}_{1A_o} = E_{1g_o} = \dot{E}_{1g_o} = o$. The first equation of (3.24) may be solved for A_1 with the supplementary condition that $\dot{E}_2 = o$. With this value of A_1, the second equation of (3.25) may be solved for G_1. The first equation of (3.25) may then be solved for A_2 and the second equation of (3.26) for G_2 and so on. If, however, all the coefficients of G_1 in the second equation of (3.25) are zero another supplementary condition, $\dot{E}_3(2\pi) = o$, arises and the second equation of (3.26) must be used to determine A_1.

Jacobian with four zero elements.—Suppose that the Jacobian J vanishes at $(A_o,\ g_o)$ with four zero elements. Then from (3.24) it is seen that the two supplementary conditions

$$E_2(2\pi,\ g_o, A_o) = o$$

$$\dot{E}_2(2\pi,\ g_o, A_o) = o$$

must be satisfied. Equations (3.25) now give a system of two non-linear equations in the two unknowns A_1 and G_1. As another supplementary condition, assume that these nonlinear equations can be solved for a pair (A_1, G_1). Substituting these values of A_1 and G_1 into equations (3.26) one obtains a linear system for the determination of A_2 and G_2 and so on. If, however, all the coefficients of A_1 and G_1 in equations (3.25) are zero then the supplementary conditions

$$E_3(2\pi,\ g_o,\ A_o) = o \qquad\qquad \dot{E}_3(2\pi,\ g_o,\ A_o) = o$$

must be satisfied and A_1, G_1 may be determined from equations (3.26).

Jacobian with a row of zeros.-- Suppose the Jacobian J vanishes at $(A_o,\ g_o)$ with $E_{1A_o} = E_{1g_o} = o$ but $\dot{E}_{1A_o} \neq o$ and $\dot{E}_{1g_o} \neq o$. This gives rise to the supplementary condition

$$E_2(2\pi,\ g_o,\ A_o) = o$$

which must be satisfied. The second equation in (3.24) and the first in (3.25) now provide a non-linear system for the determination of A_1 and G_1. Substituting these values of A_1 and G_1 into the second equation of (3.25) and the first equation of (3.26), one obtains a linear system for A_2 and G_2 and so on. If, however, in the first equation of (3.25) the coefficients of A_1 and G_1 are all zero then the supplementary condition

$$E_3(2\pi,\ g_o,\ A_o) = o$$

must be satisfied and A_1 and G_1 may be determined from the first equation of (3.26) and the second equation of (3.24).

Construction of solution.-- Having determined the coefficients A_n and G_n one may now construct the solution of (3.1). Collecting coefficients of like powers of ρ in (3.7) one obtains

$$x(t,\rho) = x_o(t) + x_1(t)\,\rho + x_2(t)\,\rho^2 + \dots$$

where

$$x_o(t) = A_o \cos t$$

$$x_1(t) = A_1 \cos t + E_1(t)$$

$$x_2(t) = A_2 \cos t + E_2(t) + G_1 E_{1g_o}(t) + A_1 E_{1A_o}(t)$$

$$x_3(t) = A_3 \cos t + E_3(t) + G_2 E_{1g_o}(t) + \frac{1}{2} G_1^2 E_{1g_o g_o}(t) +$$

$$+ G_1 E_{2g_o}(t) + A_2 E_{1A_o}(t) + \frac{1}{2} A_1^2 E_{1A_o A_o} + A_1 E_{2A_o}(t) + A_1 G_1 E_{1A_o g_o}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .$$

The principal results of this investigation may be summarized in the following two theorems.

Theorem 1.-- Consider the differential equation (3.1) with initial conditions (3.2) and suppose that H_1, H_2 and H_3 stated after formula (3.2) are satisfied. Suppose that the Jacobian J evaluated at (A_o, g_o) is different from zero. Then it is possible to determine the coefficients A_n and G_n recursively and to construct a unique solution of (3.1) satisfying (3.2) which for $\rho = o$ reduces to the solution of

(3.3) which satisfies (3.4).

Remark.-- The solution is unique in the sense that given a root (A_o, g_o) of (3.23) there is one and only one solution of (3.1) which for $\rho = o$ reduces to (3.5). This corresponds exactly to the case discussed in Coddington and Levinson [9] where it is assumed that the Jacobian does not vanish.

Theorem 2.-- Consider the differential equation (3.1) with the initial conditions (3.2) and suppose that H_1, H_2, and H_3 stated after formula (3.2) are satisfied. Suppose that the Jacobian J evaluated at (A_o, g_o) vanishes in one of the following ways:

1. (A_o, g_o) is a double root of (3.23);
2. J has a row of zeros, the other row consists of non-zero elements;
3. J has three zero elements, the other element is non-zero;
4. J has four zero elements.

Also assume that all of the supplementary conditions imposed by the vanishing of J are satisfied. Then it is possible to determine recursively the coefficients A_n and G_n and to construct a solution (not necessarily unique) of (3.1) which satisfies (3.2) and which for $\rho = o$ reduces to the solution of (3.3) which satisfies (3.4).

Remark.-- The vanishing of J with two zeros in one column or by simple cancellation of its terms is treated in Case 1.

Example.--

Consider the following equation:

$$\ddot{x} + x = \rho(ax + dx^2 + bx^3 + c \cos t)$$

$$x(o) = \bar{A}_o + p(\rho)$$

$$\dot{x}(o) = \bar{B}_o + q(\rho)$$

where ρ is a small parameter. By introducing the phase shift g, one may assume that $\dot{x}(o) = o$. Thus the equation under consideration is

$$\ddot{x} + x = \rho(ax + dx^2 + bx^3 + c \cos [t + g(\rho)])$$

$$x(o) = A_o + \lambda(\rho) \tag{3.31}$$

$$\dot{x}(o) = o \; .$$

Suppose

$$a = 4 \qquad b = -4 \tag{3.32}$$

$$c = -\frac{16}{9} \qquad d = \sqrt{\frac{3}{10}} \; .$$

After carrying out the necessary integrations, one sees that

$$E_1(2\pi) = \pi c \sin g_o \tag{3.33}$$

$$\dot{E}_1(2\pi) = \pi(aA_o + \frac{3}{4}bA_o^3 + c \cos g_o) \; .$$

Now substitute (3.32) into (3.33) and equate both expressions of (3.33) to zero so that relations (3.23) may be satisfied. The result is

$$- \frac{16}{9} \pi \sin g_o = o$$

and

$$\pi (4A_o - 3A_o^3 - \frac{16}{9} \cos g_o) = o$$

with roots

$$g_o = o \tag{3.34}$$

$$A_o = 2/3,\ 2/3,\ - 4/3 \ .$$

Now suppose that it is desired to construct a solution of (3.31) which for $\rho = o$ reduces to

$$x(t,\ o) = 2/3 \cos t,$$

corresponding to the double root 2/3.

The four elements in the Jacobian J are

$$E_{1g_o} = \pi c \cos g_o \tag{3.35}$$

$$E_{1A_o} = o$$

$$\dot{E}_{1g_o} = - \pi c \sin g_o$$

$$\dot{E}_{1A_o} = \pi (a + \frac{9}{4} b A_o^2) \ .$$

For the values (3.32), $g_o = o$, and $A_o = 2/3$ the first expression in (3.35) is different from zero and the last two expressions vanish. Hence this is the case discussed in chapter two in which the Jacobian J vanishes with three zero elements and the new condition $\dot{E}_2(2\pi) = o$ must be satisfied. After a little calculation, one finds

$$\dot{E}_2(2\pi) =$$

$$\pi A_o^3 \left\{ 5/6 A_o d^2 - \left(\frac{a}{3} + \frac{3}{4} bA_o^2\right) d + \left(\frac{ab}{32} + \frac{3}{64} b^2 A_o^2\right) \right\} \qquad (3.36)$$

which indeed vanishes for the values (3.32), $g_o = o$, and $A_o = 2/3$.

The first equation of (3.24) reduces to

$$E_2(2\pi) + G_1 E_{1g_o} = o ,$$

and upon calculating E_2 it is seen that $E_2(2\pi) \equiv o$. Hence $G_1 = o$ and the second equation of (3.25) reduces to

$$\dot{E}_3(2\pi) + \frac{1}{2} A_1^2 \dot{E}_{1A_oA_o} + A_1 \dot{E}_{2A_o} = o \qquad (3.37)$$

from which A_1 may be determined. For the example under consideration,

$$\mathring{E}_3(2\pi) = \pi A_o^3 (K_1 A_o^4 + K_2 A_o^3 + K_3 A_o^2 + K_4 A_o + K_5)$$

where

$$K_1 = \frac{219}{(64)^2} b^3 + \frac{19}{24} bd^2 - \frac{113}{256} db^2$$

$$K_2 = \frac{763}{384} bd^2 - \frac{5}{18} d^3$$

$$K_3 = \frac{1}{16} bd^2 - \frac{5}{9} d^3 + \frac{465}{6144} ab^2 - \frac{255}{576} abd$$

$$K_4 = \frac{201}{144} ad^2$$

$$K_5 = \frac{3}{412} a^2 b - \frac{1}{9} a^2 d ,$$

$$\dot{E}_{1A_oA_o} = \frac{9}{2} \pi b A_o ,$$

and

$$\dot{E}_{2A_o} = \pi A_o^3 \left(\frac{5}{6} d^2 - \frac{3}{2} bdA_o + \frac{3}{32} b^2 A_o\right) .$$

It is easy to show that (3.37) will have real roots if the values (3.32), $g_o = o$, and $A_o = 2/3$ are used.

The calculated value for E_1 is $E_1(t) = \frac{1}{8} bA_o^3 \cos t \sin^2 t +$

$$\frac{dA_o^3}{3} (2 - \cos t - \cos^2 t).$$

It is now possible to write down the first two terms in the expanded solution.

$$x(t,\rho) = A_o \cos t +$$

$$+ \rho[A_1 \cos t + \frac{1}{8} bA_o^3 \cos t \sin^2 t +$$

$$+ \frac{dA_o^3}{3} (2 - \cos t - \cos^2 t)] + \dots .$$

Higher order terms may be found recursively.

The first equation in (3.25) reduces to

$$E_3(2\pi) + G_2 E_{1g_o} = o$$

and upon carrying out the necessary calculations one sees that $E_3(2\pi) = o$. Hence $G_2 = o$ and the expansion of g in powers of ρ contains no powers of ρ less than three.

APPENDIX

In this appendix some theorems from the theory of functions of several complex variables are presented. Most of the theorems are well known for the case of functions of one complex variable and are proved in standard textbooks such as Ahlfors[10] and Titchmarsh [11]. For the case of functions of several complex variables those proofs which could not be found in the literature are presented here for the sake of completeness. The principle reference is Bochner and Martin [12].

<u>Definition 1</u>.-- By the norm $|w|$ of a complex n component vector w is meant the following:

$$|w| = \sum_{i=1}^{n} |w_i|, \quad |w_i| = [(\mathrm{Re}\ w_i)^2 + (\mathrm{Im}\ w_i)^2]^{1/2}$$

where Re w_i and Im w_i denote respectively the real and imaginary parts of w_i.

<u>Definition 2</u>.-- Let $F = (F_1, \ldots, F_n)$ be a vector function defined on a region D of the n complex dimensional w space where $w = (w_1, \ldots, w_n)$ with w_i complex. The function F is said to be analytic at a point $w_o = (w_{1o}, w_{2o}, \ldots, w_{no})$ if each F_j can be represented by an absolutely convergent power series

$$F_j(w_1\ldots,w_n) = \sum_{\substack{m_i = o \\ i = 1,\ldots,n}}^{\infty} A_{m_1 m_2 \ldots m_n} (w_1 - w_{10})^{m_1} (w_2 - w_{20})^{m_2} \ldots (w_n - w_{no})^{m_n}$$

in some neighborhood $|w - w_o| < \rho$, $\rho > o$. A function is said to be analytic in a domain D if it is analytic at each point of D. By a theorem in Bochner and Martin [13],

$$A_{m_1 m_2 \ldots m_n} = \frac{1}{m_1!\, m_2! \ldots m_n!} \frac{\partial^{m_1 + m_2 + \ldots + m_n} F_j(w_o)}{\partial w_1^{m_1} \partial w_2^{m_2} \ldots \partial w_n^{m_n}}$$

An equivalent definition of analyticity is supplied by the following theorem which is quoted without proof from Bochner and Martin [14].

Theorem 1.-- If a function $F(w_1,\ldots,w_n)$, all w_i complex, is continuous in a domain D, and if in the neighborhood of every point it is analytic in each variable, then F(w) is analytic in D.

Note .-- "F analytic in D" implies analytic in the sense of definition 2.

In all of the theorems which follow F is taken to be a scalar function. The same theorem may be stated for vector functions since each component of F may be treated separately.

Theorem 2.-- Let F be a scalar function of p complex vector arguments, $F = F(w_1,\ldots,w_p)$, $w_i = (w_{i1},\ldots,w_{in})$ $i = 1,\ldots,p$. Suppose F is defined and analytic on a region D of $(w_1,\ldots,w_p)$ space of np complex dimensions. Let G_{jk} $\begin{matrix} j = 1,\ldots,p \\ k = 1,\ldots,n \end{matrix}$ be a region in the w_{jk} plane. Let the boundary of G_{jk} be a closed curve c_{jk} which is piecewise differentiable. Define

$$c = c_{11} \times c_{12} \times \cdots \times c_{np} \quad \text{and}$$

$$G = G_{11} \times G_{12} \times \cdots \times G_{np}$$

where the symbol X denotes topological product. Suppose $c \cup G \subset D$ Then for $(w_1,\ldots,w_p) \in G$, Cauchy's integral formula takes the following form:

$$F(w_1,\ldots,w_p) = \left(\frac{1}{2\pi i}\right)^{np} \int_{c_1} \int_{c_2} \cdots \tag{1}$$

$$\cdots \int_{c_p} \frac{F(\bar{w}_1,\bar{w}_2,\ldots\bar{w}_p)\, d^*(\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_p)}{(\bar{w}_1 * w_1)(\bar{w}_2 * w_2)\ldots(\bar{w}_p * w_p)}$$

$$\int_{c_j} = \int_{c_{j1}} \int_{c_{j2}} \cdots \int_{c_{jn}}$$

$$(\bar{w}_j * w_j) = (\bar{w}_{j1} - w_{j1})(\bar{w}_{j2} - w_{j2}) \ldots (\bar{w}_{jn} - w_{jn})$$

$$j = 1,\ldots, p$$

and

$$d^{*}\ (\bar{w}_1,\ldots,\bar{w}_n) = \prod_{\substack{j=1,\ldots,p\\ k=1,\ldots,n}} d\bar{w}_{jk} \quad \text{where}$$

the symbolic product may be arranged in any order, i.e., the integration may be carried out in any order.

Proof.— Since F is analytic in D, it is analytic in each variable separately for all combinations of the other variables. By repeated application of Cauchy's integral formula for functions of one complex variable, one may write

$$F(w_1,\ldots,w_p) = \left(\frac{1}{2\pi i}\right)^{np} \int_{c_{11}} \frac{d\bar{w}_{11}}{\bar{w}_{11} - w_{11}} \int_{c_{12}} \frac{d\bar{w}_{12}}{\bar{w}_{12} - w_{12}} \cdots$$

$$\cdots \qquad \int_{c_{np}} \frac{F(\bar{w}_1,\bar{w}_2,\ldots,\bar{w}_p)}{\bar{w}_{np} - w_{np}} \quad d\bar{w}_{np} \quad .$$

Now since F is continuous on the manifold c, it is uniformly continuous and bounded on c and thus the integration may be carried out in any order.

Theorem 3.— Let D be a region of (ζ,w,z) space and let D_s be a region of ζ space where $w = (w_1,\ldots,w_n)$ and $z = (z_1,\ldots,z_n)$ and w_i, z_i, ζ are complex. Let c be a rectifiable path lying in D_s with τ_1 as one endpoint and τ as a variable point on c. Define

$$D_{\zeta S} = \left\{ (s + \zeta,\ w,\ z) : (\zeta,\ w,\ z)\ \varepsilon D,\ s \varepsilon D_S \right\}$$

and suppose $F(t,w,z)$ is continuous at each point of $D_{\zeta S}$ where F is a scalar function. Then

$$G(\tau,\zeta,w,z) = \int_{\tau_1}^{\tau} F(s + \zeta,w,z)\, ds$$

is continuous at each $(\tau,\zeta,w,z)\ \varepsilon c \times D$.

Proof.-- Let (τ_o,ζ_o,w_o,z_o) be an arbitrary point of $c \times D$ and define the set

$$V = \left\{ (s + \zeta_o,w_o,z_o) : s \varepsilon c \right\}.$$

Now V is a compact subset of the open set $D_{\zeta S}$ and hence there exists a compact subset K of $D_{\zeta S}$ with V on its interior $V \subset K \subset D_{\zeta S}$. Let d be the distance from V to the boundary of K. Now for (τ,ζ,w,z) $\varepsilon c \times D$,

$$|G(\tau_o,\zeta_o,w_o,z_o) - G(\tau,\zeta,w,z)| \leq \tag{2}$$

$$\int_c |F(s + \zeta_o,w_o,z_o) - F(s + \zeta,w,z)|\ |ds| +$$

$$\int_{\tau}^{\tau_o} |F(s + \zeta_o,w_o,z_o)|\ |ds| ,$$

where the latter integral is taken along c.

Continuity of F on K implies that F is bounded on K by M and that F is uniformly continuous on K. Hence given $\varepsilon > o$, there exists

$\delta > o$ such that

$$|\zeta - \zeta_o| + |w - w_o| + |z - z_o| < \min(\delta, d)$$

implies

$$|F(s + \zeta_o, w_o, z_o) - F(s + \zeta, w, z)| < \frac{\varepsilon}{2L} \quad \text{for all}$$

s on c where L is the length of c.

Also, since the path c is regular, there exists a $\delta_1 > o$ such that

$$\int_{\tau}^{\tau_o} |ds| < \frac{\varepsilon}{2M}$$

whenever $|\tau - \tau_o| < \delta_1$. Hence, for

$$|\tau - \tau_o| + |\zeta - \zeta_o| + |w - w_o| + |z - z_o| < \min(\delta, d, \delta_1),$$

$$|G(\tau_o, \zeta_o, w_o, z_o) - G(\tau, \zeta, w, z)| <$$

$$\int_c \frac{\varepsilon}{2L} |ds| + \int_{\tau}^{\tau_o} M|ds| < \varepsilon.$$

Since $(\tau_o, \zeta_o, w_o, z_o)$ is an arbitrary point of c X.D, G is continuous on c X.D.

Corollary. — Theorem 3 is true when D and D_s are compact sets.

This follows from the estimate (2) and the uniform continuity and boundedness of F on $D_{\zeta s}$.

Theorem 4.-- Let $F(z_1,\ldots,z_n)$ be a function of n complex variables. Let s_i be a region in z_i plane, $i = 1,\ldots,n$, and suppose that s_i has a rectifiable boundary curve c_i. Let $s = s_1 X s_2 X .. X s_n$ and $c = c_1 X c_2 X \ldots X c_n$ where X denotes topological product. Suppose that F is continuous on the manifold c. Then for all $m_i \geq 1$, $i = 1,\ldots,n$, (m_i integers)

$$G_{m_1\ldots m_n}(z_1,\ldots,z_n) = \int_{c_1}\int_{c_2}\ldots\int_{c_n} \frac{F(\bar{z}_1,\ldots,\bar{z}_n)\, d\bar{z}_1 \ldots d\bar{z}_n}{(\bar{z}_1 - z_1)^{m_1} \ldots (\bar{z}_n - z_n)^{m_n}}$$

is analytic in s and for $i = 1,2,\ldots,n$, and

$$\frac{\partial G_{m_1\ldots m_n}}{\partial z_i} = m_i G_{m_1\ldots m_{i-1}\,(m_i+1)m_{i+1}\ldots m_n} .$$

Proof.-- In view of Theorem 1 of the Appendix, it is necessary only to show that $G_{m_1\ldots m_n}$ is continuous in $(z_1,\ldots,z_n)$ and it is analytic in each variable separately for all combinations of the other variables. A proof will be given for the case $n = 2$ with $z_1 = z$, $z_2 = w$, $m_1 = r$, and $m_2 = s$. Define

$$g_{rs}(z,w;\bar{z},\bar{w}) = \frac{F(\bar{z},\bar{w})}{(\bar{z} - z)^r\, (\bar{w} - w)^s}$$

for $\bar{z}$ on c_1, $\bar{w}$ on c_2, $z \varepsilon s_1$, and $w \varepsilon s_2$. The function g_{rs} satisfies the hypotheses of Theorem 3 of the Appendix and the Corollary following that theorem. Thus,

$$\bar{g}_{rs}\ (z,\ w;\ \bar{z}) = \int_{c_2} g_{rs}\ (z,\ w;\ \bar{z},\ \bar{w})\ d\bar{w}$$

is continuous in all its variables. Applying the theorem again to $\bar{g}$, one obtains

$$G_{rs}\ (z,\ w) = \int_{c_1} \bar{g}_{rs}\ (z,\ w;\ \bar{z})\ d\bar{z}$$

as a continuous function of (z, w).

Now let $z \varepsilon s_1$ be arbitrary but fixed. Define the function

$$h_r\ (z,\ \bar{w}) = \int_{c_1} \frac{F(\bar{z},\ \bar{w})\ d\bar{z}}{(\bar{z} - z)^r}$$

For fixed z and for $\bar{w}$ on c_2, the integrand satisfies the hypotheses of the corollary to Theorem 3 of the appendix and thus $h_r(z, \bar{w})$ is continuous in $\bar{w}$ on c_2. Now h_r satisfied the hypothesis of Lemma 3 in Ahlfors [15] and thus

$$G_{rs}\ (z,\ w) = \int_{c_2} \frac{h_r\ (z,\ \bar{w})\ d\bar{w}}{(\bar{w} - w)^s} \quad ,$$

for fixed z, is an analytic function of w for w in s_2, and

$$\frac{\partial G_{rs}}{\partial w} = sG_{r\ (s+1)} \quad .$$

A similar argument applies for z by letting w be fixed. This proves the theorem.

<u>Theorem 5</u>.— Let $F(s, w_1, \ldots, w_p)$ be a scalar complex function of

$(s, w_1, \ldots, w_p)$ for $(w_1, \ldots, w_p) \varepsilon D$ and s on a regular contour c in s space where $w_i = (w_{i1}, \ldots, w_{in})$, w_{ij} complex, and D is a region of $(w_1, \ldots, w_n)$ space of np complex dimensions. Further let F be analytic in D for each s on c and continuous in $(s, w_1, \ldots, w_p)$. Then

$$G(w_1, \ldots, w_p) = \int_c F(s, w_1, \ldots, w_p)\, ds$$

is an analytic function of $(w_1, \ldots, w_p)$ in D.

Proof.— By Theorem 3 of the appendix G is continuous in all its arguments. Since F is analytic for fixed s, it is analytic in w_{11} for fixed s and all other w_{ij} fixed. By a theorem for one complex variable(Titchmarsh [16]), one obtains that G is analytic in w_{11} for any combination of the other w_{ij}. A similar argument applies to all the other w_{ij} taken one at a time. Hence G is continuous in all its variables and analytic in each variable separately for all combinations of the other variables. Now apply Theorem 1 of the appendix to obtain the conclusion.

Theorem 6.— Suppose that each member of the sequence

$$\left\{ F_i\ (z_1, \ldots, z_p) \right\}$$

is analytic in a region D, where the F_i are scalar complex functions and $z_j = (z_{j1}, \ldots, z_{jn})$, each z_{jk} complex. Further suppose that

$$\sum_{i=1}^{\infty} F_i = F(z_1, \ldots, z_p)$$

converges uniformly on every compact subset of D. Then the function F is analytic in D. Moreover,

$$\sum_{i=1}^{\infty} \frac{\partial F_i}{\partial z_{jk}} = \frac{\partial F}{\partial z_{jk}}$$

converges uniformly on every compact subset of D.

Proof.-- Actually each F_i is a function of the np complex variable z_{jk}, $j = 1,\ldots, p$; $k = 1,\ldots, n$. Since each F_i is analytic in all z_{jk}, it is continuous in all z_{jk}. By uniform convergence the limit function F is continuous in all z_{jk}. Consider F_i as a function of z_{11} with all other z_{jk} fixed. Now one may apply the theorem for the case of one complex variable (Ahlfors [17]) to obtain that F is analytic in z_{11} for all combinations of the other z_{jk} and that

$$\frac{\partial F}{\partial z_{11}} = \sum_{i=1}^{\infty} \frac{\partial F_i}{\partial z_{11}}$$

converges uniformly on every compact subset of D. A similar argument applies to the other z_{jk} taken one at a time. Analyticity of F can now be inferred from Theorem 1 of the appendix.

Theorem 7.-- Let $F(z_1, z_2)$ be a scalar function of two scalar complex variables. Suppose that F is analytic at $P_o:(z_{10}, z_{20})$ and its double series expansion converges in R, a neighborhood of P_o. Then at each point of R, F has the following alternative representation:

$$F(z_1, z_2) =$$

$$\sum_{n=o}^{\infty} \left\{ a_{on} + a_{1n}(z_1 - z_{10}) + a_{2n}\,(z_1 - z_{10})^2 + \cdots \right\} (z_2 - z_{20})^n \qquad (3)$$

where

$$a_{on}(z_1) = \frac{1}{n!} \quad \frac{\partial^n F}{\partial z_2^{\,n}} \quad (z_1, z_{20})$$

and

$$a_{kn} = \frac{1}{K!} \quad \frac{\partial^K a_{on}}{\partial z_1^{\,K}} \quad (z_{10})$$

Proof.— Since F is analytic at P_o,

$$F(z_1, z_2) = \sum_{m,n=o}^{\infty} a_{mn}(z_1 - z_{10})^m\,(z_2 - z_{20})^n \qquad (4)$$

where

$$a_{mn} = \frac{1}{m!n!} \quad \frac{\partial^{m+n} F(z_{10}, z_{20})}{\partial z_1^{\,m}\,\partial z_2^{\,n}}$$

and the series (4) converges absolutely in R. Let P: $(\bar{z}_1, \bar{z}_2)$ be an arbitrary point of R and let

$$h_{mn} = a_{mn}(\bar{z}_1 - z_{10})^m\,(\bar{z}_2 - z_{20})^n\,.$$

Then

$$\sum_{m,n=o}^{\infty} h_{mn}$$

Converges absolutely and by a theorem in Apostol [18],

$$F(\bar{z}_1, \bar{z}_2) = \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} h(m,n) =$$

$$\sum_{n=o}^{\infty} (\bar{z}_2 - z_{20})^n \left\{ \sum_{m=o}^{\infty} a_{mn} (\bar{z}_1 - z_{10})^m \right\}$$

which is just (3) evaluated at $(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. Since P was an arbitrary point of R, the theorem follows.

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