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7/25/68

EXAMPLES OF TOPOLOGICAL SPACES

A THESIS

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INTRODUCTION

The purpose of this paper is to supply a rich selection of examples of topological spaces for use as counterexamples. In order to accomplish this objective, 25 topological properties are examined here for each of the 17 topological spaces included in this paper.

In Chapter 0, we are including a list of definitions of the topological properties that we are examining, and a list of non-standard notation that we are using. The reader should take note that in defining these properties, we have used the definitions with the least requirements, e.g., in many texts, a regular space is always a Hausdorff space, but in this paper, this is not the case. Also, in Chapter 0, there is a list of theorems which relate to general topological spaces. The reader should take special note of the relationships between the topological properties as depicted in Figure 0-1 and 0-2. A tabulation of the topological spaces and their properties, Table 0-1, is located at the end of Chapter 0.

In general, Chapters I-IV are arranged in the order of increasing difficulty. In particular, Chapter II is devoted to examples in metric spaces and Chapter IV is devoted to examples involving "the order topology" on linearly ordered sets. Chapter IV also contains some specialized definitions and theorems.

For each example, we first define the topological space. After each definition, there is a list of approximately ten items each of which describes a property of the space and each is followed by a short proof. Note that all of the topological properties which pertain to each topological space are tabulated in Table 0-1. However, in the list of items for each example, we have included proofs of the strongest properties which the space has or counterexamples to the weakest properties which the space does not have. All of those properties which are not listed as items may easily be verified by applying Figure 0-1, Figure 0-2, or Theorem 0-6.

CHAPTER 0

PRELIMINARIES

In the following definitions and theorems, $(X,\mathcal{T})$ represents a topological space. We have omitted those definitions which are universal.

Definitions

1. Let $x \in X$. Then N is a *neighborhood* of x if $x \in N \subset X$ and there exists a $U \in \mathcal{T}$ such that $x \in U \subset N$.

2. $(X,\mathcal{T})$ is a *first countable space* if for each $x \in X$, there exists a countable base for the collection of neighborhoods of x.

3. $(X,\mathcal{T})$ is a *second countable space* if $\mathcal{T}$ has a countable base.

4. $(X,\mathcal{T})$ is a *Lindelöf space* if each open cover of X has a countable subcover.

5. $(X,\mathcal{T})$ is a *separable space* if there exist a countable dense subset of X.

6. $(X,\mathcal{T})$ is a *compact space* if each open cover of X has a finite subcover.

7. $(X,\mathcal{T})$ is a *locally compact space* if each $x \in X$ has a compact neighborhood.

8. $(X,\mathcal{T})$ is a *sequentially compact space* if each sequence in X has a convergent subsequence.

9. $(X,\mathcal{T})$ is a *countably compact space* if each countable open cover of X has a finite subcover.

10. $(X,\mathcal{T})$ is *B.W. compact* if each infinite subset of X has a limit point in X.

11. If $\mathcal{C}$ is a cover of X, $\mathcal{D}$ is a *refinement* of $\mathcal{C}$ in case $\mathcal{D}$ is a cover of X and, for each $D \in \mathcal{D}$, there exists $C \in \mathcal{C}$ such that $D \subset C$. Note that an open refinement is a refinement consisting of open sets.

12. A family, Λ, of subsets of X is *locally-finite* if each $x \in X$ has a neighborhood which intersects at most a finite number of elements of Λ.

13. A family, Λ, of subsets of X is *point-finite* if each $x \in X$ is contained in at most finitely many elements of Λ.

14. $(X,\mathcal{T})$ is *paracompact* if each open cover of X has a locally-finite, open refinement.

15. $(X,\mathcal{T})$ is *countably paracompact* if each countable open cover of X has a locally-finite, open refinement.

16. $(X,\mathcal{T})$ is *metacompact* if each open cover of X has a point-finite, open refinement.

17. $(X,\mathcal{T})$ is *countably metacompact* if each countable open cover of X has a point-finite, open refinement.

18. $(X,\mathcal{T})$ is a *regular space* if whenever $x \in X$ and K is a closed subset of X with $x \notin K$, there exist disjoint open sets U and V such that $x \in U$ and $K \subset V$.

19. $(X,\mathcal{T})$ is a *normal space* if whenever H and K are disjoint closed subsets of X, there exist disjoint open sets U and V such that $H \subset U$ and $K \subset V$.

20. $(X,\mathcal{T})$ is a *completely normal space* if whenever H and K are separated subsets of X (i.e. $H \cap \bar{K} = \bar{H} \cap K = \phi$, where $\bar{H}$ is the closure of H), there exist disjoint open sets U and V such that $H \subset U$ and $K \subset V$.

21. A subset H of X is called a *G_δ set* if H is the intersection of a countable collection of open sets.

22. $(X,\mathcal{T})$ is a *perfectly normal space* if $(X,\mathcal{T})$ is a normal space and each closed subset of X is a G_δ set.

23. $(X,\mathcal{T})$ is a *locally-connected space* if, whenever $U \in \mathcal{T}$ and $x \in U$, the component of U which contains x is open. (This is equivalent to the usual definition.)

24. $(X,\mathcal{T})$ is a *totally disconnected space* if each component of X is a one point set.

25. Let $S \subset X$. Then, the *relative topology* on S, written $\mathcal{T}/S$, is given by $\mathcal{T}/S = \{U \cap S : U \in \mathcal{T}\}$.

<u>Notation</u>

1. Let $x \in X$. Then, $\mathcal{N}(x) = \{N : N$ is a neighborhood for $x\}$.
2. $P = \{1, 2, \ldots, n, \ldots\}$, i.e., the set of positive integers.
3. R is the set of all real numbers.
4. Q is the set of rational numbers in R.
5. If $S \subset X$, then $\mathcal{T}/S$ is the relative topology on S.

6. If $A \subset X$, then the closure of A is written $\bar{A}$.

Theorems (References are given for theorems which are not generally included in a first year course in Topology.)

1. Suppose S is an open subset of X. If $(X,\mathcal{T})$ is locally compact and regular, then $(S,\mathcal{T}/S)$ is also.

2. (a) If $(X,\mathcal{T})$ is paracompact and Hausdorff, then $(X,\mathcal{T})$ is regular ([12], page 68).

   (b) If $(X,\mathcal{T})$ is paracompact and regular, then $(X,\mathcal{T})$ is normal ([12], page 68).

   (c) If $(X,\mathcal{T})$ is Lindelöf and regular, then $(X,\mathcal{T})$ is paracompact ([7], page 211 or [10], page 74).

3. The following are equivalent:

   (a) $(X,\mathcal{T})$ is completely normal.

   (b) Each subspace of $(X,\mathcal{T})$ is a normal space.

   (c) Each open subspace of $(X,\mathcal{T})$ is a normal space.

4. If $(X,\mathcal{T})$ is a T_1 space, then $(X,\mathcal{T})$ is compact if it is B.W. compact and metacompact.

5. (Urysohn's Metrization Theorem.) If $(X,\mathcal{T})$ is T_1, regular and second countable, then $(X,\mathcal{T})$ is also a metric space.

6. Every metric space is:

   (a) First countable.

   (b) Second countable if and only if it is separable.

   (c) Perfectly normal.

   (d) Paracompact ([17], page 222).

7. If $(X,\mathcal{T})$ is Hausdorff and locally compact, then $(X,\mathcal{T})$ is regular.

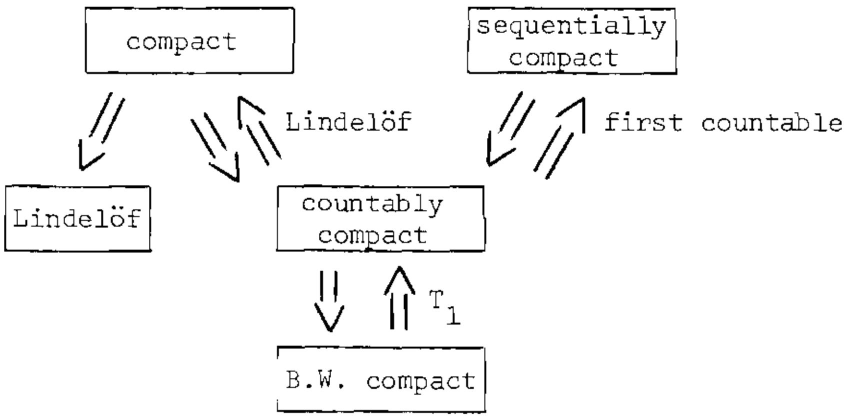
8. Figures 0-1 and 0-2.

Table 0-1 uses the following notation:

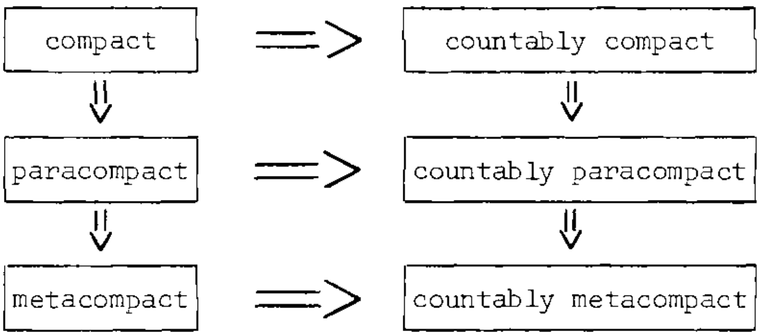
(i) "T" means the topology has that property,

(ii) "F" means the topology does not have that property, and

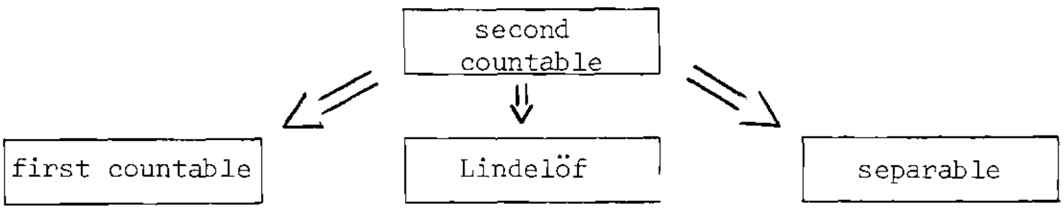
(iii) "O" means that it is not proven here whether or not the space has that property.



(a)



(b)



(c)

Figure 0-1

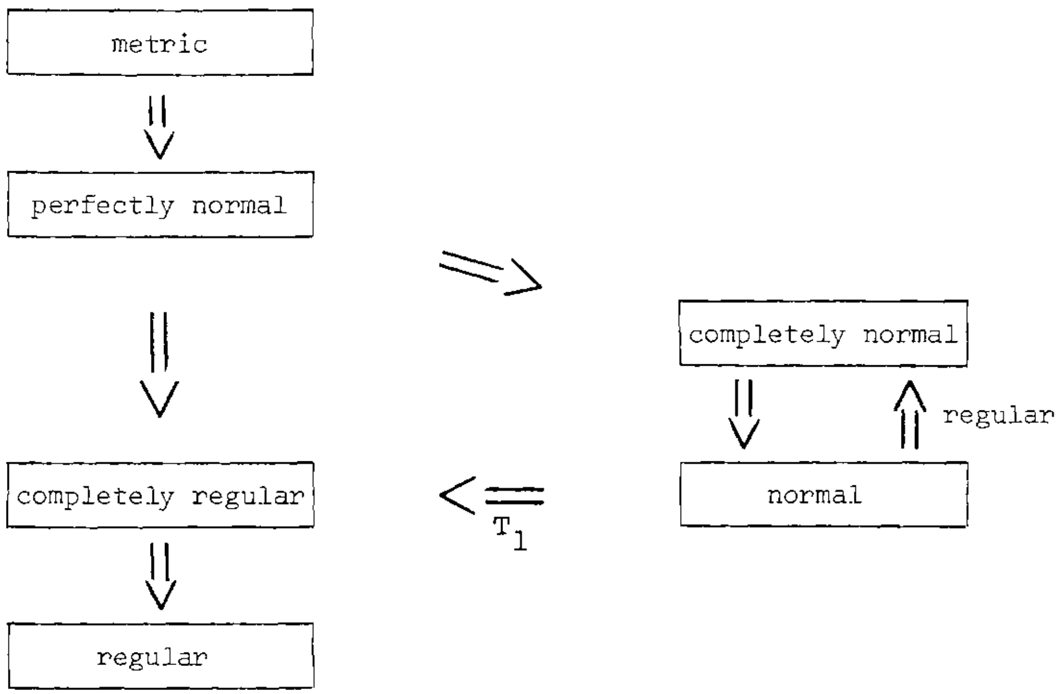

metric
perfectly normal
completely normal
regular
completely regular
T_1
normal
regular

Figure 0-2

Table 0-1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
T_o	T	F	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T_1	F	F	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T_2	F	F	F	F	T	T	T	T	T	T	T	T	T	T	T	T	T
First Countable	T	T	F	F	T	T	T	T	T	F	T	T	T	T	T	F	T
Second Countable	F	T	F	F	F	T	F	F	F	F	T	F	F	F	F	F	F
Lindelof	T	T	T	T	T	T	F	F	T	T	T	F	F	F	T	T	F
Separable	T	T	T	F	T	F	F	F	F	T	T	T	T	T	F	F	F
Locally Compact	T	T	T	F	F	0	F	F	T	F	F	F	F	F	T	T	T
Compact	F	T	T	F	F	F	F	F	T	F	F	F	F	F	T	T	F
Sequentially Compact	F	T	T	F	F	F	F	F	T	F	F	F	F	F	T	T	T
Countably Compact	F	T	T	F	F	F	F	F	T	F	F	F	F	F	T	T	T
B.W. Compact	T	T	T	F	F	F	F	F	T	F	F	F	F	F	T	T	T
Paracompact	F	T	T	F	T	T	T	T	T	T	F	F	F	F	T	T	F
Countably Paracompact	F	T	T	F	T	T	T	T	T	T	F	F	0	0	T	T	0
Metacompact	F	T	T	F	T	T	T	T	T	T	0	F	F	F	T	T	F
Countably Metacompact	F	T	T	F	T	T	T	T	T	T	0	0	0	0	T	T	0
Regular	F	F	F	F	T	T	T	T	T	T	F	F	F	T	T	T	T
Completely Regular	F	F	F	F	T	T	T	T	T	T	F	F	F	T	T	T	T
Normal	T	T	F	F	T	T	T	T	T	T	F	F	F	F	T	T	T
Completely Normal	T	T	F	F	T	T	T	T	T	T	F	F	F	F	T	T	T
Perfectly Normal	F	F	F	F	T	T	T	T	F	T	F	F	F	F	F	F	0
Metric	F	F	F	F	F	T	T	T	F	F	F	F	F	F	F	F	F
Connected	T	T	T	T	F	F	T	T	F	F	T	T	T	T	T	F	F
Locally Connected	T	T	T	T	F	F	T	T	F	F	F	F	T	T	T	F	F
Totally Disconnected	F	F	F	F	T	T	F	F	F	T	F	F	F	F	F	T	T

"T" - the topology has that property.

"F" - the topology does not have that property.

"0" - it is not proven here whether or not the space has that property.

CHAPTER I

INTRODUCTORY EXAMPLES

Example 1

The following topology is found on page 245, of [5]. It is a topology on the real line R.

Let $\mathcal{B} = \{[a,\infty) : a \in R\}$, the collection of all semi-infinite intervals which contain their left-hand end points. Note that if $A \in \mathcal{B}$ and $B \in \mathcal{B}$, then $A \cap B \in \mathcal{B}$. Then, since $R = \cup\{[a,\infty) : a \in R\}$, it follows that $\mathcal{B}$ is a base for a topology $\mathcal{T}$ for R. Notice that
$\mathcal{T} = \mathcal{B} \cup \{R,\phi\} \cup \{(a,\infty) : a \in R\}$.

1. T_o (i.e., $(R,\mathcal{T})$ is a T_o space).

Let $x,y \in R$ with $x \neq y$. We may assume $x < y$. Then, $y \in [y,\infty) \in \mathcal{T}$, but $x \notin [y,\infty)$.

2. Not T_1 (i.e., $(R,\mathcal{T})$ is not a T_1 space).

Note that any two nonempty open sets have a nonempty intersection.

3. First countable.

For each $x \in R$, $\mathcal{B}(x) = \{[x,\infty)\}$ is a countable base for the neighborhood system of x.

4. Not second countable.

Suppose that $\mathcal{D}$ is a countable subcollection of $\mathcal{T}$. Then $\mathcal{D} \subset \mathcal{T}$.

Let $A = \{a : [a,\infty) \in \mathcal{D}\}$ and $B = \{b : (b,\infty) \in \mathcal{D}\}$. Then, $A \cup B$ is countable. Hence, there exists $z \in R - (A \cup B)$. Note that $[z,\infty) \in \mathcal{T}$.

Suppose that $D \in \mathcal{D}$, $z \in D$, and $D \subset [z,\infty)$. Then, D must be of the form $[x,\infty)$ or (x,∞). Now, $z \le x$ since $D \subset [z,\infty)$, and $x \le z$ since $z \in D$. Hence, $D = [z,\infty)$ which is a contradiction since $z \notin A \cup B$. Therefore, D cannot be a base for $\mathcal{T}$.

5. Lindelöf.

Let $\mathcal{C}$ be an arbitrary open cover of R. Let $x \in R$. Then there exists $n \in P$ such that $-n \le x$ and, since $\mathcal{C}$ is a cover of R, there exists $C_n \in \mathcal{C}$ such that $-n \in C_n$. Note that $\{C_n : n \in P\}$ is a countable subcover of $\mathcal{C}$.

6. *Proposition (a):* If $x \in R$, then $\overline{\{x\}} = (-\infty,x]$.

Proof. If $y \in R$ and $x < y$, then $[y,\infty) \cap \{x\} = \phi$. Hence $y \notin \overline{\{x\}}$. If $z \in R$ and $z \le x$, each open set containing z contains $[z,\infty)$. Note that $\{x\} \subset [z,\infty)$. Hence, $z \in \overline{\{x\}}$.

Proposition (b). Suppose $A \ne \phi$ and $A \subset R$. Then, A is compact if and only if g.l.b.$A \in A$.

Proof. Suppose that A is compact, but g.l.b.$A = \alpha \notin A$. Then, either $\alpha = -\infty$ or $\alpha \in R$. If $\alpha = -\infty$, then $\mathcal{C} = \{[-n,\infty) : n \in P\}$ is clearly an open cover of A and $\mathcal{C}$ has no finite subcover. Since this contradicts A being compact, we find that $\alpha \in R$. But then $\mathcal{D} = \{[\alpha + \frac{1}{n}, \infty) : n \in P\}$ is an open cover of A and $\mathcal{D}$ has no finite subcover. But this is a contradiction; therefore g.l.b.$A = \alpha \in A$.

Suppose that $\alpha \in A$. Let $\mathcal{C}$ be an open cover of A. Then, there exists $C \in \mathcal{C}$ such that $\alpha \in C$. Clearly, $\{C\}$ is a finite subcover of $\mathcal{C}$. Hence, A is compact.

Proposition (c). Suppose $A,B \subset R$ and $A \cap \bar{B} = \bar{A} \cap B = \phi$. Then either $A = \phi$ or $B = \phi$.

Proof. If $A \neq \phi$ and $B \neq \phi$, then there exist $a,b \in R$ such that $a \in A$ and $b \in B$. Since $a \neq b$, we may assume, without loss of generality, that $a < b$. But by the proof of Proposition (a), $a \in \overline{\{b\}}$. Since $\overline{\{b\}} \subset \bar{B}$, this is a contradiction.

7. B.W. Compact.

Note that, by (6(a)) (page 10), finite sets have limit points in R.

8. Not countably metacompact.

Let $\mathcal{C} = \{[-n,\infty) : n \in P\}$. Then, $\mathcal{C}$ is a countable open cover of R.

Suppose that $\mathcal{C}$ has a point-finite open refinement, say $\mathcal{F}$. Let $x \in R$. Then x is contained in at most finitely many elements of $\mathcal{F}$, say $F_1,\ldots,F_m$ for some positive integer m. Note that $R \notin \mathcal{F}$. Hence, for each n such that $1 \leq n \leq m$, there exists $a_n \in R$ such that $F_n = [a_n,\infty)$ or $F_n = (a_n,\infty)$. Also, there exists $a \in R$ such that $a < a_n$ for $1 \leq n \leq m$. Now, there exists $F \in \mathcal{F}$ such that $a \in F$. But then $x \in F$, also. This is impossible since $F \neq F_n$ for $1 \leq n \leq m$ and $F_1,\ldots,F_m$ are the only elements of $\mathcal{F}$ which contain x. Therefore, $(X,\mathcal{T})$ is not countably metacompact.

9. Locally compact.

Let $x \in R$. Then by (6(b)) (page 10), $[x,\infty)$ is a compact neighborhood of x.

10. Not regular.

If A and B are disjoint open sets, then $A \cap \bar{B} = \bar{A} \cap B = \phi$ and, by (6(c)) (page 11) either $A = \phi$ or $B = \phi$. Therefore, it is sufficient to exhibit a nonempty closed set K and a point a such that $a \notin K$. Let $K = (-\infty, 0]$ and $a = 1$.

11. Completely normal.

By (6(c)) (page 11), there are no subsets of R, say A and B, such that $A \neq \phi$, $B \neq \phi$, and $A \cap \bar{B} = \bar{A} \cap B = \phi$.

12. Not perfectly normal and not metric.

Every perfectly normal space is regular (see Figure 0-2).

13. Separable.

P is a countable dense subset of R.

14. Connected and locally connected.

By (6(c)) of page 11, no subset of R can have a separation.

<u>Example 2</u>

The following topology is found in [17], page 105.

Let Z be the set of all integers in R. Let M be the set of even integers. Then, $\mathcal{T} = \{\phi, M, Z\}$ is clearly a topology for Z. It is obvious that $(Z, \mathcal{T})$ is second countable and compact, but not T_o.

1. *Proposition (a):* If $x \in Z$, then $\overline{\{x\}} = Z$ or $\overline{\{x\}} = Z - M$.

Proof. Clearly, $\overline{\{x\}}$ is Z or Z-M depending on whether x is even or odd, respectively.

Proposition (b): If $A,B \subset Z$ and $A \cap \bar{B} = \bar{A} \cap B = \phi$, then $A = \phi$ or $B = \phi$.

Proof. This is clear by (a) above.

2. Not regular.

Note that $0 \notin Z - M$ and $Z - M$ is a closed set. By (1(b)) (page 13), there are no disjoint open sets U and V such that $0 \in U$ and $Z - M \subset V$.

3. Completely normal.

As in Example 1, there are no separated sets to check.

4. Connected and locally connected.

By (1(b)) (page 13), no subset of Z can have a separation.

Example 3

The following topology is usually called the "co-finite topology" and can be found in [13].

Consider R, the set of real numbers, with $\mathcal{T}_F = \{U : U \subset R$ and either $(R-U)$ is a finite set or $U = \phi\}$.

1. $\mathcal{T}_F$ is a topology for R.

   (i) $\phi \in \mathcal{T}_F$ by definition. $R \in \mathcal{T}_F$ since $(R-R)$ is finite.

   (ii) Suppose A and B are in $\mathcal{T}_F$. Then, if $A = \phi$ or $B = \phi$, $A \cap B = \phi \in \mathcal{T}_F$. Otherwise, $R - A$ and $R - B$ are finite sets. Hence, $R - (A \cap B) = (R-A) \cup (R-B)$ is finite, and thus $A \cap B \in \mathcal{T}_F$.

   (iii) Suppose $A_\alpha \in \mathcal{T}_F$ for each $\alpha \in \Lambda$. If $A_\alpha = \phi$ for each $\alpha \in \Lambda$, then $\cup \{A_\alpha : \alpha \in \Lambda\} = \phi \in \mathcal{T}_F$. Otherwise, there exists $\beta \in \Lambda$ such that $(R-A)$ is finite. But then $R - \cup \{A_\alpha : \alpha \in \Lambda\} =$

$\cap \{R - A_\alpha : \alpha \in \Lambda\} \subset R - A_\beta$. Therefore, $R - \cup \{A_\alpha : \alpha \in \Lambda\}$ is finite and $\cup \{A_\alpha : \alpha \in \Lambda\} \in \mathcal{T}_F$.

2. T_1.

Let $x, y \in R$ with $x \neq y$. Then $x \notin R - \{x\}$ and $y \in R - \{x\}$. $R - \{x\} \in \mathcal{T}_F$ since $R - (R - \{x\}) = \{x\}$.

3. *Proposition (a):* If A is a closed subset of R, then A is finite or A = R.

Proof. $R - A \in \mathcal{T}_F$.

Proposition (b): R contains no disjoint, nonempty open sets.

Proof. If $A \neq \phi$ and $A \in \mathcal{T}_F$, R - A is finite. If $B \subset R$ and $A \cap B = \phi$, then B is finite and thus $B \notin \mathcal{T}_F$ unless $B = \phi$.

Proposition (c): Let A and B be nonempty subsets of R. Then, $A \cap \bar{B} = \bar{A} \cap B = \phi$ if and only if $(A \cup B)$ is finite and $A \cap B = \phi$.

Proof. Suppose $A \cap \bar{B} = \bar{A} \cap B = \phi$. Clearly, $\bar{A} \neq R$ and $\bar{B} \neq R$. By (a) above, $\bar{A}$ and $\bar{B}$ are finite. But then $A \cup B$ is finite.

Suppose $A \cup B$ is finite and $A \cap B = \phi$. Then A and B are each finite and therefore closed. Hence, $\bar{A} \cap B = A \cap B = \phi$ and $A \cap \bar{B} = A \cap B = \phi$.

Proposition (d): Let K be a nonempty subset of R. Then K is connected if and only if K has exactly one element or K is infinite.

Proof. Suppose K has more than one element. By (c) above, K has a separation if and only if K is finite.

Proposition (e): Each subset of R is compact.

Proof. Let $A \subset R$. If $A = \phi$, there is nothing to prove. Suppose $a \in A$ and $\mathcal{C}$ is an open cover of A. Then, there exists $C \in \mathcal{C}$ such that

$a \in C$. Since C is open, R - C is finite. Obviously, there is a finite subcover of $\mathcal{C}$.

4. Not T_2.

Note that R contains more than one point, but, by (3(b)) (page 14), R does not contain two disjoint, nonempty open sets.

5. Not regular.

Note that $2 \notin \{1\}$ and $\{1\}$ is a closed set. Apply (3(b)) on page 14.

6. Not normal.

Since $(R,\mathcal{T})$ is T_1 and not regular, it is not normal. (See Figure 0-2).

7. Connected.

Since R is infinite, R is connected. See (3(d)) on page 14.

8. Locally connected.

Since each nonempty open subset of R is infinite, each such subset is connected. Therefore, components of open sets are open (Theorem 0-6).

9. Separable.

Let D be any countably infinite subset of R. By (3(a)) (page 14), D is dense in R since $\bar{D} = R$.

10. Not first countable.

Let $x \in R$. Suppose that $\mathcal{B}(x) = \{B_n : n \in P\}$ is a countable subset of $\mathcal{T}_F$ such that $x \in B_n$ for each $n \in P$. It suffices to show that $\mathcal{B}(x)$ is not a base for $\mathcal{N}(x)$.

$B_n \neq \phi$ for each $n \in P$. So by definition of $\mathcal{T}_F$, $R - B_n$ is finite for each $n \in P$. Therefore, $\cup \{R - B_n : n \in P\}$ is countable. There exists $a \in R - \cup \{R - B_n : n \in P\}$ such that $a \neq x$. Then, $a \in B_n$ for each $n \in P$.

Clearly, $R - \{a\} \in \mathcal{T}_F$, $x \in R - \{a\}$, and B_n is not a subset of $R - \{a\}$ for any $n \in P$. Therefore, $\mathcal{B}(x)$ is not a base for $\mathcal{N}(x)$.

11. Compact.

See (3(e)) on page 14.

12. Sequentially compact.

Let $\{a_n : n \in P\}$ be a sequence in R. If there exists a $b \in \{a_n : n \in P\}$ such that $b = a_i$ for infinitely many $i \in P$, then there exists a subsequence $\{a_{n(k)} : k \in P\}$ such that $a_{n(k)} = b$ for each $k \in P$. Otherwise, for any fixed $N \in P$, $a_N = a_i$ for only finitely many $i \in P$. Let $x \in R$. Then, each open set which contains x contains all but a finite subcollection of $\{a_n : n \in P\}$. Therefore, $\{a_n : n \in P\}$ converges to x.

<u>Example 4</u>

The following topology is usually called the "co-countable topology" and may be found in [13].

Recall that R is the set of all real numbers. Let $\mathcal{T}_C = \{U : U \subset R \text{ and either } U = \phi \text{ or } (R-U) \text{ is a countable set}\}$. Note that, if $\mathcal{T}_C$ is a topology for R, then $\mathcal{T}_F \subset \mathcal{T}_C$.

1. $\mathcal{T}_C$ is a topology for R.

   (i) Clearly, $\phi, R \in \mathcal{T}_C$.

(ii) Suppose A, B $\in \mathcal{T}_C$. If A = ϕ or B = ϕ, A $\cap$ B = $\phi \in \mathcal{T}_C$. Otherwise, (R-A) and (R-B) are countable sets. Hence, R - (A$\cap$B) = (R-A) $\cup$ (R-B) is countable and A $\cap$ B $\in \mathcal{T}_C$.

(iii) Suppose $A_\alpha \in \mathcal{T}_C$ for each $\alpha \in \Lambda$. If $A_\alpha = \phi$ for each $\alpha \in \Lambda$, then $\cup \{A_\alpha : \alpha \in \Lambda\} = \phi \in \mathcal{T}_C$. Otherwise, there exists $\beta \in \Lambda$ such that $(R-A_\beta)$ is countable. Thus $R - \cup \{A_\alpha : \alpha \in \Lambda\} = \cap \{R - A_\alpha : \alpha \in \Lambda\} \subset (R-A_\beta)$. Therefore, $\cup \{A_\alpha : \alpha \in \Lambda\} \in \mathcal{T}_C$.

2. T_1.

Since $\mathcal{T}_F \subset \mathcal{T}_C$ and $(R,\mathcal{T}_F)$ is T_1, $(R,\mathcal{T}_C)$ is T_1.

3. Not T_2.

Clearly, there are no disjoint, nonempty, open sets in $(R,\mathcal{T}_C)$.

4. Not first countable.

In the proof of (10) of Example 3, substitute $\mathcal{T}_C$ and countable for $\mathcal{T}_F$ and finite, respectively.

5. Not separable.

Suppose D is a countable subset of R. Then R - D is a nonempty open set with D $\cap$ (R-D) = ϕ.

6. *Proposition (a):* If A is a closed subset of R, then A is countable or A = R.

Proof. If A $\neq$ R, then, since R - A is open, R - (R-A) is countable. Note A = R - (R-A).

Proposition (b): R contains no disjoint, nonempty open sets.

Proof. If $A \neq \phi$ and $A \in \mathcal{T}_C$, $R - A$ is countable. If $B \subset R$ and $B \cap A = \phi$, then B is countable and thus $B \notin \mathcal{T}_C$ unless $B = \phi$.

Proposition (c): Let A and B be nonempty subsets of R. Then, $A \cap \bar{B} = \bar{A} \cap B = \phi$ if and only if $(A \cup B)$ is countable and $A \cap B = \phi$.

Proof. Suppose $A \cap \bar{B} = \bar{A} \cap B = \phi$. Clearly, $\bar{A} \neq R$ and $\bar{B} \neq R$. By (a) above, $\bar{A}$ and $\bar{B}$ are countable. But then $A \cup B$ is countable.

Suppose $A \cup B$ is countable and $A \cap B = \phi$. Then, A and B are each countable and therefore closed. Hence, $\bar{A} \cap B = A \cap \bar{B} = A \cap B = \phi$.

Proposition (d): Let K be a nonempty subset of R. Then K is connected if and only if K has only one element or K is uncountably infinite.

Proof. Suppose K contains more than one element. By (c) above, K has a separation if and only if K is countable.

Proposition (e): No infinite subset of R is compact.

Proof. Let A_1 be an infinite subset of R. Let A be a countably infinite subset of A_1. Let $D(x) = \{x\} \cup (R-A)$ for each $x \in A$. Then $\mathcal{C} = \{D(x) : x \in A\}$ is an open cover of A_1 and $\mathcal{C}$ has no finite subcover.

7. Not regular.

Note (6(c)), page 18, and the fact that $1 \notin \{2\}$ and $\{2\}$ is a closed subset of R.

8. Lindelöf.

Let $\mathcal{C}$ be an open cover of R. Then there exists $C \in \mathcal{C}$ such that $1 \in C$. Recall that $R - C$ is countable. Therefore, $\mathcal{C}$ has a countable subcover.

9. Not locally compact.

Note that all (nonempty) neighborhoods are uncountably infinite sets and hence not compact by (6(e)), page 18.

10. Not B.W. compact.

P, the set of positive integers, is an infinite subset of R. Let $x \in R$. Then $\{x\} \cup (R-P)$ is a neighborhood of x which does not intersect $P - \{x\}$.

11. Connected.

See (6(d)), page 18.

12. Locally connected.

Open sets are connected (see (6(d)), page 18).

13. Not countably metacompact.

Let D be a countably infinite subset of R. For each $x \in D$, let $B(x) = \{x\} \cup (R-D)$. Clearly, $\mathcal{C} = \{B(x) : x \in D\}$ is a countable open cover of R. Let $\mathcal{F}$ be an open refinement of $\mathcal{C}$. It suffices to show that $\mathcal{F}$ is not a point-finite system.

Since $\mathcal{F}$ is an open cover of R and $(R,\mathcal{T}_C)$ is a Lindelöf space, there exists a countable subcover of $\mathcal{F}$, say $\mathcal{F}_o$. Without loss of generality, we may assume that $F \neq \phi$ for each $F \in \mathcal{F}_o$. It will be sufficient to show that $\mathcal{F}_o$ is an infinite subcollection of $\mathcal{F}$ and that there exists $w \in R$ such that $w \in F$ for each $F \in \mathcal{F}_o$.

Let $x,y \in D$ with $x \neq y$. Recall that $y \notin B(x) = \{x\} \cup (R-D)$. There exists $F \in \mathcal{F}_o$ such that $x \in F$. The only element of $\mathcal{C}$ which can contain F is B(x). Since $y \notin B(x)$, $y \notin F$. Therefore for each $x \in D$, there is at least one $F \in F_\circ$ such that $x \in F$. Thus, $F_\circ$ is infinite.

Since $\phi \notin F_\circ$ and $F_\circ \subset T_C$, $F \in F_\circ$ implies that $(R-F)$ is countable. Hence, there exists $w \in R - \cup \{R - F : F \in F_\circ\}$ since $F_\circ$ is also countable. Note that $w \in F$ for each $F \in F_\circ$ as desired.

Therefore, C is a countable open cover such that each open refinement of C is not a point-finite system; i.e., (R,T_C) is not countably metacompact.

Example 5

The following topology is found in [10], page 23, where it is referred to as the "lower-limit topology."

Let $B = \{[a,b) : -\infty < a < b < \infty\}$. (Recall that $[a,b) = \{x \in R : a \le x < b\}$.)

1. B is a base for a topology L on R.

   (i) Clearly, $R = \cup \{[-n,n) : n \in P\} \subset \cup B$.

   (ii) Let $[a_1,b_1)$, $[a_2,b_2) \in B$ and let $x \in [a_1,b_1) \cap [a_2,b_2)$. Then $a_1 \le x < b$ and $a_2 \le x < b$. Let $p = \max\{a_2,b_1\}$ and $q = \min\{b_1,b_2\}$. Obviously, $x \in [p,q) \in B$ and $[p,q) \subset [a_1,b_1) \cap [a_2,b_2)$.

2. First countable.

Let $x \in R$. Then, clearly, $B(x) = \{[x, x + \frac{1}{n}) : n \in P\}$ is a countable base for $N(x)$.

3. *Proposition (a):* $E \subset L$ (recall E is Euclidean topology).

Proof. It suffices to show that base elements of E, i.e. open intervals, are elements of L. Let $(a,b) \in E$ (assume $a < b$). Then, there exists $N \in P$ such that $a + \frac{1}{N} < b$. Note $(a,b) = \cup \{[a + \frac{1}{n}, b) : n \in P, n \geq N\} \in \mathcal{L}$.

Proposition (b): Let $K \subset R$. Let K' (wrt $\mathcal{L}$) be the set of limit points of K with respect to $\mathcal{L}$. Then $x \in K'$ (wrt $\mathcal{L}$) if and only if $[x, x + \frac{1}{n}) \cap (K - \{x\}) \neq \phi$ for each $n \in P$.

Proof. Recall $x \in K'$ (wrt $\mathcal{L}$) if and only if each neighborhood of x intersects $K - \{x\}$, and then note (2), page 20.

Proposition (c): Let $K \subset R$. Then K' (wrt $\mathcal{L}$) $\subset K'$ (wrt $\mathcal{E}$).

Proof. Note that $x \in K'$ (wrt $\mathcal{E}$) if $(x - \frac{1}{n}, x + \frac{1}{n}) \cap K = \phi$ for each $n \in P$.

Proposition (d): If $U \in \mathcal{B}$, then $R - U \in \mathcal{L}$.

Proof. If $U \in \mathcal{B}$, then $U = [a,b)$ for some $a,b \in R$ such that $a < b$. Let $A = \cup \{[a-n,a) : n \in P\}$ and $B = \cup \{[b,b+n) : n \in P\}$. Then $A, B, A \cup B \in \mathcal{L}$, and $R - U = R - [a,b) = (-\infty,a) \cup [b,\infty) = A \cup B \in \mathcal{L}$.

4. T_2.

Since $\mathcal{E} \subset \mathcal{L}$ and $(R,\mathcal{E})$ is T_2, $(R,\mathcal{L})$ is T_2.

5. Separable.

Consider Q, the set of rationals in R. If $[a,b) \in \mathcal{L}$, then $b > a$ and $(a,b) \in \mathcal{E}$. Then $\phi \neq Q \cap (a,b) \subset Q \cap [a,b)$. Hence, Q is dense in R.

6. Not second countable.

Let $\mathcal{D}$ be a base for $\mathcal{L}$. It suffices to show that $\mathcal{D}$ is not countable.

Let $x \in R$. Then there exists $D(x) \in \mathcal{D}$ such that $x \in D(x) \subset [x,x+1)$ since $\mathcal{D}$ is a base for $\mathcal{L}$. Let $\mathcal{D}_\circ = \{D(x) : x \in R\}$. Let $x,y \in R$

with $x \neq y$. If $x < y$, $x \notin D(y)$ and $x > y$, then $y \notin D(x)$. In either case, $D(x) \neq D(y)$ for each $x,y \in R$ with $x \neq y$. Thus $\mathcal{D}_o$ and hence $\mathcal{D}$ is uncountable since R is.

7. Not metric.

$(R,\mathcal{L})$ is separable but not second countable (see Theorem 0-6(b)).

8. Lindelöf.

It suffices to consider open coverings of R with base elements of $\mathcal{L}$.

Let $\mathcal{C}$ be an open cover of R such that $\mathcal{C} \subset \mathcal{B}$. Then elements of $\mathcal{C}$ have the form $[a,b)$ where $-\infty < a < b < \infty$. Let $\mathcal{C}_o = \{(a,b) : [a,b) \in \mathcal{C}\}$ and let $T = \cup\, \mathcal{C}_o$.

$(R,\mathcal{E})$ is a Lindelöf space. It follows that $(T, \mathcal{E}/T)$ is a Lindelöf space. Then there exists a countable subcover of $\mathcal{C}_o$, say $\mathcal{F}_o$.

We shall now show that $R - T$ is countable. Let $x \in R - T$. Then there exists $t(x) \in R$ such that $[x, t(x)) \in \mathcal{C}$ since $\mathcal{C}$ covers R. Note that $\{(x, t(x)) : x \in R - T\}$ is a collection of nonempty, disjoint intervals each of which is open in $(R,\mathcal{E})$. This is a contradiction unless $R - T$ is countable since Q is countable and there is at least one element of Q in $(x, t(x))$ for each $x \in R - T$.

It should now be clear that $\mathcal{F} = \{[a,b) : (a,b) \in \mathcal{F}_o\} \cup \{[x, t(x)) : x \in R - T\}$ is a countable subcover of $\mathcal{C}$.

9. Normal.

Let K and L be disjoint, nonempty closed subsets of $(R,\mathcal{L})$. Then for each $x \in K$, there exists $\varepsilon(x) > 0$ such that $[x,x + \varepsilon(x)) \cap L = \phi$

since x is not a limit point of L. Also, for each $y \in L$, there exists $\varepsilon(y) > 0$ such that $[y, y + \varepsilon(y)) \cap K = \phi$. Let $U = \cup \{[x, x + \varepsilon(x)) : x \in K\}$ and let $V = \cup \{[y, y + \varepsilon(y)) : y \in L\}$. Note that $K \subset U \in \mathcal{L}$ and $L \subset V \in \mathcal{L}$. We shall show next that $U \cap V = \phi$.

Let $x \in K$ and $y \in L$. Suppose $z \in [x, x + \varepsilon(x)) \cap [y, y + \varepsilon(y))$. Then it must be that $y \in [x, x + \varepsilon(x))$ or that $x \in [y, y + \varepsilon(y))$. If either case were true, it would contradict the way in which $\varepsilon(x)$ and $\varepsilon(y)$ were chosen. Hence, if $x \in K$, then $[x, x + \varepsilon(x)) \cap [y, y + \varepsilon(y)) = \phi$ for each $y \in L$. Therefore, if $x \in K$, $[x, x + \varepsilon(x)) \cap V = \phi$. It follows that $U \cap V = \phi$ as required.

10. Paracompact.

$(R, \mathcal{L})$ is Lindelöf and regular (see Theorem 0-2(c)).

11. Perfectly normal.

It will suffice, by (9) on page 22, to prove that each closed set is the intersection of a countable collection of open sets.

Let K be a nonempty closed subset of $(R, \mathcal{L})$. Let L be the closure of K with respect to the Euclidean topology on R; $L = K \cup [K' \text{ (wrt } \mathcal{E})]$. Since $(R, \mathcal{E})$ is a perfectly normal space (see Theorem 0-6(c)), there exists a countable collection of Euclidean open sets, $\mathcal{C}$, such that $\cap\, \mathcal{C} = L$. Note that $K \subset L$ and, since $\mathcal{E} \subset \mathcal{L}$, $C \in \mathcal{L}$ for each $C \in \mathcal{C}$. In the next paragraph, we shall exhibit a countable collection of open sets whose intersection contains K but does not contain a point of $L - K = [K' \text{ (wrt } \mathcal{E})] - K$.

Let $x \in L - K$. Then, since $x \notin K$ and $x \notin K' \text{ (wrt } \mathcal{L})$ (see 3(c)), by (3(b)) on page 21, there exists $t(x) \in R$ such that $t(x) > x$ and

$[x,t(x)) \cap K = \phi$. Let $x,y \in L - K$ with $x \neq y$. Assume that $(x,t(x)) \cap (y,t(y)) \neq \phi$. Then, either $x \in (y,t(y))$ or $y \in (x,t(x))$. Suppose $x \in (y,t(y))$. Then $(y,t(y))$ is a Euclidean open set which contains a limit point of K with respect to the Euclidean topology. Therefore, $(y,t(y))$ also contains a point of K. Since this contradicts the way in which $t(y)$ was chosen, it must be that $x \in (y,t(y))$. Note that this is also a contradiction, hence $(x,t(x)) \cap (y,t(y)) = \phi$ for each $x,y \in L - K$ with $x \neq y$. Then $\{(x,t(x)) : x \in L - K\}$ is a collection of nonempty disjoint Euclidean open subsets of R. Recall that this is a contradiction unless $L - K$ is countable. Hence, $L - K$ is countable.

For each $x \in L - K$, let $D(x) = (-\infty,x) \cup (x,\infty)$. Note that $K \subset D(x)$ and $D(x) \in \mathcal{L}$ for each $x \in L - K$. It follows from the preceding discussion that $K = \cap \{A : A \in \mathcal{C}$ or $A = D(x)$ for some $x \in L - K\}$ and that $\{A : A \in \mathcal{C}$ or $A = D(x)$ for some $x \in L - K\}$ is a countable collection of open sets with respect to topology $\mathcal{L}$.

12. Not locally compact.

Let $x \in R$. Suppose N is a compact neighborhood of x. There exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset N$. By (3(d)) on page 21, $R - B \in \mathcal{L}$. Therefore, B is closed and, since $B \subset N$, B is compact. There exists $a,b \in R$ with $a < b$ such that $B = [a,b)$. Note that $\mathcal{C} = \{[a,b - \frac{1}{n}) : n \in P$ and $a < b - \frac{1}{n}\}$ is an open cover of B and that $\mathcal{C}$ has no finite subcover. Therefore, x has no compact neighborhoods.

13. Not B.W. compact.

Clearly, P is an infinite subset of R with no limit points in R.

14. Totally disconnected.

Suppose that $K \subset R$ and that K contains more than one point. Let $a, b \in K$ with $a \neq b$. Let $U = (-\infty, b)$ and $V = [b, \infty)$. Then $U, V \in \mathcal{L}$ and $U \cap V = \phi$. It is clear that $\{U \cap K, V \cap K\}$ is a separation for K. Hence, if A is a connected subset of R, then A contains at most one point.

15. Not locally connected.

It is sufficient to note that components of open sets are one point sets and that one point sets are not open.

CHAPTER II

METRIC SPACES

Example 6

The following topological space is due to Niemytzki according to [17], page 41.

Let Z be the set of all integers in R. For each $x \in Z$ and each $p \in P$, define $N_x^p = \{y : y = x + rp, r \in Z\}$ and let $\mathcal{B} = \{N_x^p : x \in Z, p \in P\}$.

1. $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on Z.

(i) Note that $N_0^1 = \{y : y = r, r \in Z\} = Z$. Thus $\cup\, \mathcal{B} = Z$.

(ii) Suppose $N_x^p, N_y^q \in \mathcal{B}$ and $z \in N_x^p \cap N_y^q$. Then, for some $r_1, r_2 \in Z$, $z = r_1p + x = r_2q + y$. Let $t = pq$. Note $z \in N_z^t \in \mathcal{B}$ since $t \in P$ and $z = (0)t + Z$. Next, we shall prove that $N_z^t \subset N_x^p \cap N_y^q$.

Let $u \in N_z^t$. Then, for some $r \in Z$, $u = rt + z$. Then $u = rpq + z = rpq + r_1p + x = (rq + r_1)p + x \in N_x^p$ since $rq + r_1 \in Z$. Also, $u = rpq + r_2q + y = (rp + r_2)q + y \in N_y^q$. Hence, $N_z^t \subset N_x^p \cap N_y^q$.

2. T_2.

Let $x,y \in Z$ with $x \neq y$. Note that if $t = |x| + |y| + 1$, then t does not divide $(y-x)$ since $t > |y-x|$. Clearly, $x \in N_x^t \in \mathcal{B}$ and $y \in N_y^t \in \mathcal{B}$. Furthermore, we shall prove that $N_x^t \cap N_y^t = \phi$.

Suppose $h \in N_x^t \cap N_y^t$. Then, for some $r,s \in Z$, $h = rt + x = st + y$. It follows that $(r - s)t = y - x$. Now, $r \neq s$ since $y \neq x$. This implies that t divides $(y - x)$ and contradicts the way in which t was chosen. Hence, $N_x^t \cap N_y^t = \phi$.

3. Second countable.

It is evident that $\mathcal{B}$ is countable.

4. *Proposition (a):* If $N_x^p \in \mathcal{B}$, there exists $a \in Z$ such that $0 \leq a < p$ and $N_x^p = N_a^p$.

Proof. If $0 \leq x < p$, let $a = x$. Otherwise, $x < 0$ or $x \geq p$.

Suppose $x < 0$. Then for some unique $n \in P$, $0 \leq np + x < p$. Let $a = np + x$. Then, if $w \in N_a^p$, for some $r \in Z$, $w = rp + a$ and thus $w = rp + np + x = (r + n)p + x \in N_x^p$ since $r + n \in Z$. Also, if $w \in N_x^p$, for some $s \in Z$, $w = sp + x = sp + (a - np) = (s - n)p + a \in N_a^p$ since $s - n \in Z$. Therefore, $N_x^p = N_a^p$ and $0 \leq a < p$.

Suppose $x \geq p$. Then, for some unique $m \in P$, $0 \leq -mp + x < p$. Let $a = - mp + x$. Then, if $w \in N_a^p$, for some $r \in Z$, $w = rp + a = rp + (-mp + x) = (r - m)p + x \in N_x^p$ since $(r - m) \in Z$. Also, if $w \in N_x^p$, for some $s \in Z$, $w = sp + x = sp + (a + mp) = (s + m)p + a \in N_a^p$ since $s + m \in Z$. Therefore, $N_x^p = N_a^p$ and $0 \leq a < p$.

Proposition (b): If $N_x^p \in \mathcal{B}$, N_x^p is a closed set.

Proof. By (a) above, we may assume that $0 \leq x < p$.

Suppose $p = 1$. Then $N_x^p = N_x^1 = \{y : y = r + x, r \in Z\} = Z$. Hence, N_x^p is a closed set.

Otherwise, $p \in P$ and $p > 1$. Now, $N_0^p, N_1^p, \ldots, N_{p-1}^p$ are all open sets. Recall that, if $z \in Z$, then there exist unique integers r and t such that $z = rp + t$ and $0 \leq t < p$. Therefore, if $z \in Z$, z is in exactly one of $N_0^p, N_1^p, \ldots, N_{p-1}^p$. It follows that $Z - N_x^p = \cup \{N_y^p : 0 \leq y < p - 1 \text{ and } y \neq x\}$ and that $Z - N_x^p$ is open. Thus, N_x^p is closed.

5. Regular.

Let $x \in Z$ and let K be a closed subset of Z with $x \notin K$. Then, $Z - K \in \mathcal{N}(x)$ and there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset Z - K$. Let $U = B$ and $V = Z - B$. Then, $x \in U$, $K \subset V$, $U \cap V = \phi$, $U \in \mathcal{T}$ and, by (4(b)) on page 27, $V \in \mathcal{T}$.

6. Metric.

$(Z,\mathcal{T})$ is T_1, second countable, and regular (Theorem 0-5).

7. Totally disconnected.

Let A be a subset of Z consisting of more than one point. Let $a,b \in A$ with $a \neq b$. Let $d = 2|b - a|$. Then $a \in N_a^d$ and N_a^d is both open and closed. Suppose $b \in N_a^d$. Then there exists $r \in Z$ such that $b = rd + a$. Then $b - a = rd = 2r|b - a|$ and since $b \neq a$, $1 = 2|r|$. But this is a contradiction since $r \in Z$. Hence, N_a^d and $Z - N_a^d$ are disjoint open sets such that $a \in N_a^d$ and $b \in Z - N_a^d$. Obviously, $\{A \cap N_a^d, A \cap [Z - N_a^d]\}$ is a separation for A. Therefore, components of Z are singleton sets; i.e. sets containing exactly one point.

8. Not locally connected.

Note that components of open sets are singleton sets which are not open (Theorem 0-7).

9. Not compact.

Recall that each integer which is not 1 or -1 may be written as the product of an integer and a prime number and that there are infinitely many prime numbers (recall that 1 is not a prime number and that prime number are necessarily positive integers).

Suppose that $(Z,\mathcal{T})$ were a compact space. Let $\mathcal{C} = \{N_1^7, N_{-1}^7\} \cup \{N_0^p : p$ is a prime number$\}$. Clearly, $\mathcal{C}$ is a collection of open sets in $(Z,\mathcal{T})$. Since $1, -1 \in \cup\, \mathcal{C}$ and $N_0^p = \{z : z = np, n \in Z\}$, $\mathcal{C}$ is an open cover of Z. Therefore, $\mathcal{C}$ has a finite subcover $\mathcal{F}$ which includes N_1^7 and N_{-1}^7, say $\mathcal{F} = \{N_1^7, N_{-1}^7\} \cup \{N_0^p : p \in A\}$, where A is a finite subset of the set of the prime numbers. Note that $7 \in A$ since 7 is not an integer multiple of 3, 5, or any integer larger than 7. Let t be that positive integer which is the product of those elements of A which are numbers larger than 2; i.e., $t > 0$ and the set of integers which divide t is $\{1,-1,t,-t\} \cup \{x : a \in A - \{2\}$ and either $x = a$ or $x = -a\}$. Now, let $s = t + 2$. Note $s \in Z$.

Since 7 divides t, $s = 7n + 2$ for some integer n. Hence, $s \notin N_{-1}^7 \cup N_1^7$. Now, since $\mathcal{F}$ is a cover of Z, $s \in N_0^p$ for some $p \in A$. But, $s = t + 2$ and t is not divisible by 2. Therefore, s is not divisible by 2. Hence, $p \neq 2$. However, if p is an element of $A - \{2\}$, then p divides t but, since p does not divide 2, p does not divide s. Hence, s is an element of $Z - \cup\, \mathcal{F}$. Therefore, $\mathcal{F}$ is not a cover of Z. Hence, $\mathcal{C}$ has no finite subcover and, therefore, $(Z,\mathcal{T})$ is not a compact space.

Example 7

The following metric topology on R^2 may be found in [15], page 48.

Let d be the Euclidean metric on R^2. Let $N(z;\rho) = \{w \in R^2 : d(w,z) < \rho\}$. Then, $e : R^2 \to R$ is defined by: if p, q $\in R^2$, then (i) $e(p,q) = d(p,q)$ if p, q, and (0,0) are collinear or (ii) $e(p,q) = d\big(p,(0,0)\big) + d\big(q,(0,0)\big)$ if p, q, and (0,0) are not collinear. Let T_e be the topology generated by e.

Let $p \in R^2$ and $\varepsilon > 0$. Then, $U = \{q : q \in R^2 \text{ and } e(p,q) < \varepsilon\}$ will be one of the following sets of points: (i) if p = (0,0), then U = $N(q;\varepsilon)$, (ii) if $p \neq (0,0)$ and $d\big(p,(0,0)\big) \geq \varepsilon$, then U is an open interval centered at p on the line which passes through p and (0,0), or (iii) if $p \neq (0,0)$ and $d\big(p,(0,0)\big) \leq \varepsilon$, then U is an open interval centered at p on the line which passes through p and (0,0) plus $N\Big((0,0);\ \varepsilon - d\big(p,(0,0)\big)\Big)$. (See Figure 7-1.)

1. e is a metric for R^2.

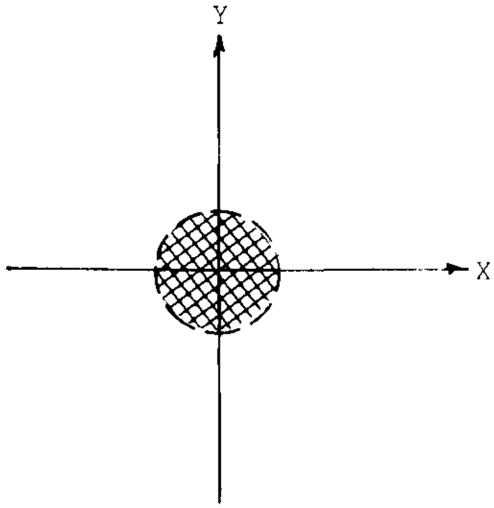
   (a) $e(p,q) \geq 0$ for each $p,q \in R^2$ since $d(p,q) \geq 0$, $d\big(p,(0,0)\big) \geq 0$, and $d\big(q,(0,0)\big) \geq 0$.

   (b) Clearly, $e(p,q) = 0$ if and only if $p = q$.

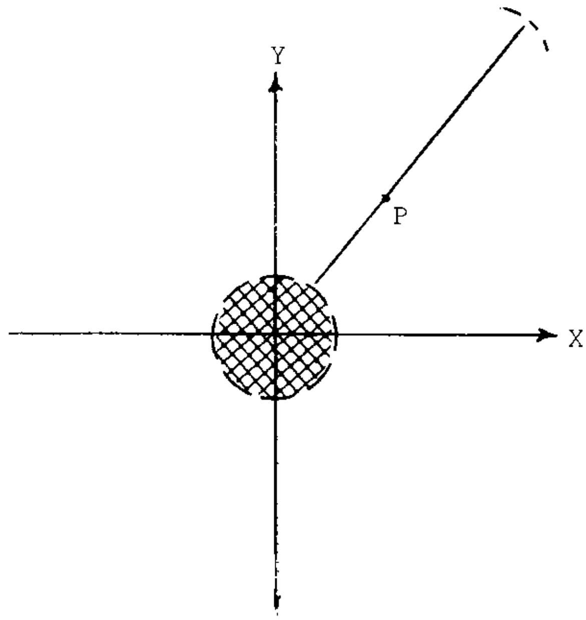
   (c) Clearly, $e(p,q) = e(q,p)$ for each $p,q \in R^2$.

   (d) Let x, y, z $\in R^2$. (i) Suppose x, y, z, and (0,0) are collinear. Then, $e(x,y) = d(x,y) \leq d(x,z) + d(z,y) = e(x,z) + e(z,y)$.

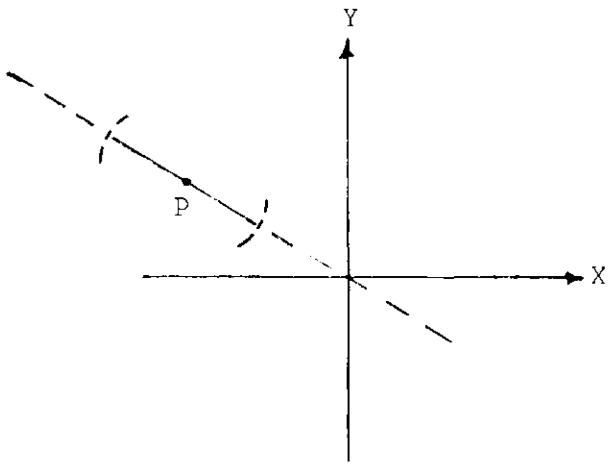
   (ii) Suppose only x, y, and (0,0) are collinear. Then, $e(x,y) = d(x,y) \leq d\big(x,(0,0)\big) + d\big(y,(0,0)\big) \leq d\big(x,(0,0)\big) +$



(a) $\{q \epsilon R^2 : e\big((0,0),q\big) < \varepsilon\}$



(b) $\{q \epsilon R^2 : e(p,q) < \varepsilon\}$



(c) $\{q \epsilon R^2 : e(p,q) < \varepsilon\}$

Figure 7-1

$d(z,(0,0)) + d(y,(0,0)) + d(z,(0,0)) = e(x,z) + e(z,y)$.

(iii) Suppose only x, z, and (0,0) are collinear. Then, $e(x,y) = d(x,(0,0)) + d(y,(0,0)) \leq d(x,z) + d(z,(0,0)) + d(y,(0,0)) = e(x,z) + e(z,y)$. (iv) Suppose only y, z, and (0,0) are collinear. This is a corollary of case (iii).

(v) Suppose no three of the points are collinear. Then, $e(x,y) = d(x,(0,0)) + d(y,(0,0)) \leq d(x,(0,0)) + d(z,(0,0)) + d(y,(0,0)) + d(z,(0,0)) = e(x,z) + e(z,y)$.

2. Not separable.

Let A be a countable subset of R^2. Then there exists a line through the origin which does not intersect $A - \{(0,0)\}$. Let p be on this line such that $p \neq (0,0)$. Then $\{z : z \in R^2 \text{ and } e(p,z) < e(p,(0,0))\}$ is open, nonempty, and does not intersect A.

3. Not Lindelöf.

Let $N^*(x;\rho) = \{z : e(x,z) < \rho\}$ for each $x \in R^2$ and $\rho > 0$. Let $\mathcal{C} = \left\{N^*((0,0);\ 1)\right\} \cup \left\{N^*\left(p;\ e(p,(0,0))\right) : p \in R^2 - \{(0,0)\}\right\}$. Then $\mathcal{C}$ is an open cover of R^2. It is evident that any countable subcollection of $\mathcal{C}$ cannot cover more than countably many lines through the origin in addition to the interior of the unit disk. Since there are uncountably many lines thru the origin, $(R^2,\mathcal{T}_e)$ is not Lindelöf.

4. Connected.

Let L be a line through the origin. Then the subspace (L,e) has the Euclidean topology and hence is connected. It follows that L is a connected set in the space (R^2,e). Note that R^2 is the union of all lines which pass through the origin. Clearly, R^2 is connected.

5. Locally connected.

Recall that $\{N^*(z;\rho) : z \in R, \rho > 0\}$ is a base for T_e. It suffices to note that each of these base elements is a connected set, for then components of open sets are open.

6. Not locally compact.

It may be noted that the origin is the only point of R^2 which does not have a compact neighborhood.

Suppose, however, that N is a compact neighborhood of the origin. There exists $\rho > 0$ such that $N^*\big((0,0) \; ; 2\rho\big) \subset N$. Let $K = \{z \in R^2 : e\big(z,(0,0)\big) \le 2\rho\}$. The reader may easily verify that the closure of $N^*\big((0,0) \; ; 2\rho\big)$ is K. Then, $K \subset N$ since N is closed and K is compact since N is compact. Let $C = \left\{N^*\big((0,0); \rho\big)\right\} \cup \left\{N^*\Big[z; e\big(z,(0,0)\big)\Big] : z \in R^2 - \{(0,0)\}\right\}$. Then, C is an open cover of R^2 and hence C is an open cover of K. Clearly, C does not contain a countable subcover. Therefore, the origin has no compact neighborhood.

7. Not B.W. compact.

Note that $A = \{(n,0) : n \in P\}$ is an infinite subset of R^2. Note that for each $z \in R^2$, $N^*(z;1)$ is an open set which contains z and does not intersect A in more than one point. Since (R,T_e) is Hausdorff, this implies that A does not have any limit points in R^2.

<u>Example 8</u>

The following metric topology on R^2 may be found in [7], page 173. This topology has the same topological properties as Example 7, although neither topology contains the other as a subcollection.

Let $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$ be points of R^2. Then $\rho : R^2 \to R$ is defined by: if $x_1 = x_2$, then $\rho(z_1,z_2) = |y_1 - y_2|$ and $\rho(z_1,z_2) = |y_1| + |y_2| + |x_1 - x_2|$ if $x_1 \neq x_2$. Base elements of the topology generated by ρ are depicted in Figure 8-1.

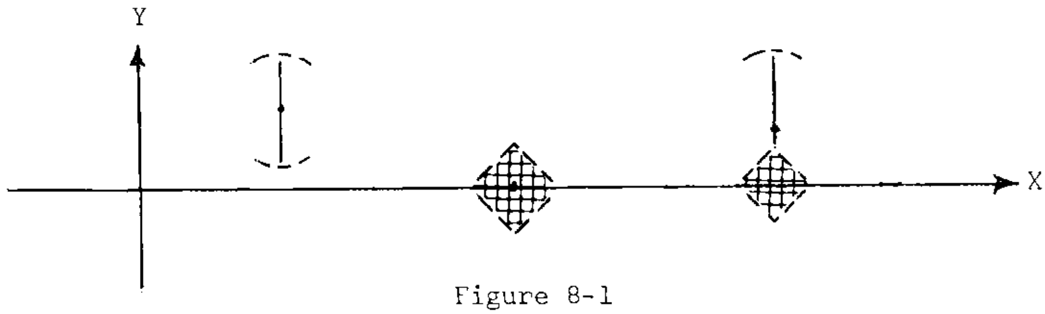


Figure 8-1

1. ρ is a metric for R^2.

(a) Clearly, $\rho(z_1,z_2) \geq 0$ for each z_1, $z_2 \in R^2$.

(b) $\rho\big((x,y), (x,y)\big) = |y - y| = 0$ for each $(x,y) \in R^2$.

(c) $\rho\big((x_1,y_1), (x_2,y_2)\big) = 0$ implies either $x_1 = x_2$ and $|y_1 - y_2| = 0$ or $|y_1| + |y_2| + |x_1 - x_2| = 0$. But in either case, $(x_1,y_1) = (x_2,y_2)$.

(d) Obviously, $\rho(z_1,z_2) = \rho(z_2,z_1)$ for each z_1, $z_2 \in R^2$.

(e) Let $z_1 = (x_1,y_1)$, $z_2 = (x_2,y_2)$, and $z_3 = (x_3,y_3)$ be points of R^2. (i) Suppose $x_1 = x_2 = x_3$. Then, $\rho(z_1,z_2) = |y_1 - y_2| \leq |y_1 - y_3| + |y_3 - y_2| = \rho(z_1,z_3) + \rho(z_3,z_2)$. (ii) Suppose

$x_1 = x_2 \neq x_3$. Then, $\rho(z_1,z_2) = |y_1 - y_2| \leq |y_1| + |y_2| \leq (|y_1| + |y_3| + |x_3 - x_1|) + (|y_2| + |y_3| + |x_3 - x_2|) = \rho(z_1,z_3) + \rho(z_3,z_2)$. (iii) Suppose $x_1 \neq x_2 = x_3$. Then, $\rho(z_1,z_2) = |y_1| + |y_2| + |x_1 - x_2| = |y_1| + |y_2| + |x_1 - x_3| \leq (|y_1| + |y_3| + |x_1 - x_3|) + |y_2 - y_3| = \rho(z_1,z_3) + \rho(z_3,z_2)$. (iv) Suppose $x_2 \neq x_1 = x_3$. This is a corollary to (iii). (v) Suppose x_1, x_2, x_3 are distinct numbers. Then, $\rho(z_1,z_2) = |y_1| + |y_2| + |x_1 - x_2| \leq |y_1| + |y_2| + |x_1 - x_3| + |x_3 - x_2| + 2|y_3| = (|y_1| + |y_3| + |x_1 - x_3|) + (|y_2| + |y_3| + |x_3 - x_2|) = \rho(z_1,z_2) + \rho(z_2,z_3)$.

2. Not separable.

Let D be any countable subset of R^2. Since there are uncountably many vertical lines in R^2, there exists $a \in R$ such that the line $x = a$ contains no point of D. Note that $\{(x,y) : x = a, y > 0\}$ is open, non-empty, and does not intersect D. Hence, D is not dense in R^2.

3. Not Lindelöf.

Let $N^*(x;\varepsilon) = \{z : z \in R^2 \text{ and } \rho(x,z) < \varepsilon\}$ for each $x \in R^2$ and each $\varepsilon > 0$. For each $a \in R$, let $U_a = \{(a,y) : y \neq 0\}$ and note U_a is open since $U_a = \cup \{N^*((a,y) ; |y|) : y \in R \text{ and } y \neq 0\}$.

Let $C = \{N^*((a,0); 1) : a \in R\} \cup \{U_a : a \in R\}$. Then, C is an open cover of R^2. Furthermore, no countable subcollection of C can cover more than $\cup \{N^*((a,0); 1) : a \in R\}$ and countably many vertical line segments more.

4. Not locally compact.

No point on the X-axis has a compact neighborhood.

Let $a \in R$ and suppose that N is a compact neighborhood of $(a,0)$. Then there exists $\varepsilon > 0$ such that $N^*\big((a,0);\ 2\varepsilon\big) \subset N$. Let $K = \{z \in R^2 : \rho\big((a,0),\ z\big) \leq 2\varepsilon\}$. The reader may easily verify that the closure of $N^*\big((a,0);\ 2\varepsilon\big)$ is K. Then, $K \subset N$ since N is closed and K is compact since N is compact. Let $\mathcal{F} = \{N^*\big((a,0);\ \varepsilon\big)\} \cup \{U_a : a \in R\}$. Then, $\mathcal{F}$ is an open cover of K and $\mathcal{F}$ does not have a finite subcover. Therefore, $(a,0)$ has no compact neighborhood.

5. Connected.

The topology on the X-axis as a subspace is Euclidean and hence the X-axis is connected as a subset of R^2. The topology on any vertical line as a subspace is Euclidean and hence any vertical line is connected as a subset of R^2. Since each vertical line has a point in common with the X-axis, and R^2 is the union of all such lines, R^2 is connected.

6. Locally connected.

It is easily seen that for each $x \in R^2$ and each $\varepsilon > 0$, $N^*(x;\varepsilon)$ is connected. Hence, components of open sets are open.

7. Not B.W. compact.

Note that $A = \{(n,0) : n \in P\}$ is an infinite subset of R^2. Since, for each $z \in R^2$, $N^*(z;1)$ is an open set which contains z and does not intersect A in more than one point, A does not have any limit points in R^2.

CHAPTER III

NON-METRIC SPACES

Example 9

The following topology may be found in [7] on page 107 where credit is given to an example considered in a paper by P. Alexandroff and P. S. Urysohn [2].

Consider the following subset of R^2 : $X = C_1 \cup C_2$ where $C_i = \{(x,y) : 0 \leq x \leq 1, y = i\}$ for $i = 1,2$. The topology on X is to be generated by a collection of neighborhoods. If $z \in C_2$, let $\mathcal{B}(z) = \left\{\{z\}\right\}$. If $z \in C_1$ and $z = (x,1)$, let $\mathcal{B}(z) = \{U_k(z) : k \in P\}$ where $U_k(z) = \{z\} \cup \{(x',y') : (x',y') \in X \text{ and } 0 < |x - x'| < \frac{1}{k}\}$. Let $\mathcal{B} = \cup \{\mathcal{B}(z) : z \in X\}$.

1. $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on X.

   (a) Clearly, $X = \cup \{B : B \in \mathcal{B}\}$.

   (b) Let $B_1,B_2 \in \mathcal{B}$ and suppose $z \in B_1 \cap B_2$. If $z \in C_2$, then $\{z\} \in \mathcal{B}$ and $z \in \{z\} \subset B_1 \cap B_2$. Otherwise, $z \in C_1$ and there exist $a,b \in [0,1]$ and $m,n \in P$ such that $B_1 = U_m\big((a,2)\big)$ and $B_2 = U_n\big((b,1)\big)$. Let $z = (x,1)$. Then $0 < |a - x| < \frac{1}{m}$ and $0 < |b - x| < \frac{1}{n}$. There exists $N \in P$ such that $0 < |x - y| < \frac{1}{N}$ implies that $0 < |a - y| < \frac{1}{m}$ and $0 < |b - y| < \frac{1}{n}$. It follows that $z \in U_N(z) \in \mathcal{B}$ and $U_N(z) \subset U_m\big((a,1)\big) \cap U_n\big((b,1)\big)$.

2. T_2.

This is obvious.

3. First countable.

Indeed, if $z \in X$, then $\mathcal{B}(z)$ is a countable base for $\mathcal{N}(z)$.

4. Not second countable.

Any base for $\mathcal{T}$ must contain the following uncountable collection of open sets, $\{\{z\} : z \in C_2\}$.

5. Not separable.

Let D be a countable subset of X. There exists $z \in C_2$ such that $z \notin D$. But $\{z\} \neq \phi$, $\{z\} \in \mathcal{T}$, and $\{z\} \cap D = \phi$. Hence, D is not dense in X.

6. Compact.

Let $\mathcal{F}$ be an open cover of X. It suffices to assume that $\mathcal{F} \subset \mathcal{B}$. (This proof may be found in [7], page 107.)

Note that $\mathcal{D} = \{F \cap C_1 : F \in \mathcal{F}\}$ is an open cover of C_1 with respect to the Euclidean topology on C_1. Then $\mathcal{D}$ has a finite subcover, say $F_1 \cap C_1, \ldots, F_n \cap C_1$ where F_i is not a one point set for $1 \leq i \leq n$, since C_1 with the Euclidean topology is compact. Then, there exist $k(i) \in P$ and $x_i \in [0,1]$ for $1 \leq i \leq n$ such that $F_i = U_{k(i)}\big((x_i,1)\big)$ for $1 \leq i \leq n$. Also, there exist $G_i \in \mathcal{F}$ such that $(x_i,2) \in G_i$ for $1 \leq i \leq n$. Thus, $\{F_1, \ldots, F_n, G_1, \ldots, G_n\}$ is a finite subcover of $\mathcal{F}$.

7. *Proposition:* Every open subspace of $(X,\mathcal{T})$ is a paracompact space.

Proof. Let A be an open subset of $(X,\mathcal{T})$. Recall that $\mathcal{T}/A$ is the relative topology on A and that $\mathcal{B}/A = \{B \cap A : B \in \mathcal{B}\}$ is a base for

$\mathcal{T}/A$. Note that $\mathcal{T}/A \subset \mathcal{T}$ and $\mathcal{B}/A \subset \mathcal{B}$ since A is open.

Let $\mathcal{F}$ be an open (wrt $\mathcal{T}/A$) cover of A in the space $(A, \mathcal{T}/A)$. Let $\mathcal{D} = \{F \cap C_1 : F \in \mathcal{F}\}$. Then, in the space $(A \cap C_1, \mathcal{T}/A \cap C_1)$, $\mathcal{D}$ is an open cover of $A \cap C_1$. Note that $(A \cap C_1, \mathcal{T}/A \cap C_1)$ is a subspace of the metric space $(C_1, \mathcal{T}/C_1)$ and hence is metric. By Theorem 0-6(d), $(A \cap C_1, \mathcal{T}/A \cap C_1)$ is paracompact. Therefore, there exists a locally-finite open (wrt $\mathcal{T}/A \cap C_1$) refinement $\mathcal{R}$ of $\mathcal{D}$.

Note that each element R of $\mathcal{R}$ is open in $(A \cap C_1, \mathcal{T}/A \cap C_1)$ and that $\cup \mathcal{R} = A \cap C_1$. For each $R \in \mathcal{R}$, define $R' = \{(x,y) : (x,1) \in R$ and $(x,y) \in A\}$. Clearly, for each $R \in \mathcal{R}$, $R' \in \mathcal{T}/A$. For each $R \in \mathcal{R}$, there exists an $F \in \mathcal{F}$ such that $R \subset F \cap C_1$. For each $R \in \mathcal{R}$, define $R^* = R' \cap F$. Then, $R^* \in \mathcal{T}/A$ since $F \in \mathcal{T}/A$.

Next, let $B = A - \cup \{R^* : R \in \mathcal{R}\}$. Note that $B \subset C_2$. I claim that $\mathcal{R}^* = \{\{z\} : z \in B\} \cup \{R^* : R \in \mathcal{R}\}$ is a locally-finite open (wrt $\mathcal{T}/A$) refinement of $\mathcal{F}$.

First, note that $A = \cup \mathcal{R}^*$ and therefore $\mathcal{R}^*$ is clearly a refinement of $\mathcal{F}$. It is also clear that $\mathcal{R}^*$ is an open (wrt $\mathcal{T}/A$) refinement of $\mathcal{F}$.

Let $z \in A$. If $z \in B$, then $\{z\}$ is an open (wrt $\mathcal{T}/A$) set which intersects only finitely many elements of $\mathcal{R}^*$. If $z \in A - B$ and $z = (x,y)$, then there exists $U \in \mathcal{T}/A \cap C_1$ such that $(x,1) \in U$ and that U intersects only finitely many elements of $\mathcal{R}$. Let $U^* = \{(x,y) : (x,1) \in U$ and $(x,y) \in A\}$. Then, $U^* \in \mathcal{T}/A$, $z \in U^*$, and clearly U^* intersects only finitely many elements of $\mathcal{R}^*$. Therefore, $\mathcal{R}^*$ is a locally-finite open (wrt $\mathcal{T}/A$) refinement of $\mathcal{F}$. Hence, $(A, \mathcal{T}/A)$ is paracompact.

8. Completely normal.

Since each open subspace is T_2 and paracompact, each subspace is normal by Theorem 0-2(a) and (b). See also Theorem 0-3.

9. Not perfectly normal.

In particular, C_1 is not the intersection of a countable collection of open sets. Suppose U is an open set which contains C_1. By the proof of (b) on page 38, X - U is finite. For otherwise, $\{V : V = U$ or $V = \{z\}$ for some $z \in C_2\}$ is an open cover of X with no finite subcover. Since X - U is finite, if $C_1 \subset A_n$ and $A_n \in \mathcal{T}$ for each $n \in P$, $X - \cap \{A_n : n \in P\} = \cup \{X - A_n : n \in P\}$ is at most countable. It follows that $C_1 \subsetneq \cap \{A_n : n \in P\}$.

10. Not connected.

Note that $\{(\frac{1}{2},2)\}$ and $\left(X - (\frac{1}{2},2)\right)$ are disjoint open sets whose union is X.

11. Not totally disconnected.

Note that C_1 is connected in the metric space $(C_1, \mathcal{T}/C_1)$. Hence, C_1 is connected in $(X,\mathcal{T})$.

12. Not locally connected.

Since we have shown that C_1 is connected and not open, it suffices to prove that C_1 is a component of X.

Suppose D is a connected subset of X and $C_1 \subsetneq D$ with $(x,2) \in D$. Note that $\{(x,2)\}$ and $X - \{(x,2)\}$ are disjoint open sets. Clearly, $\left\{\{(x,2)\}, D \cap \left(X - \{(x,2)\}\right)\right\}$ would then be a separation for D.

Example 10

The following topology first appeared in [1]. It also may be found in [7], page 109.

Let S be the following subset of R^2, S = {(m,n) : m,n are integers}. A base $\mathcal{B}$ for the topology $\mathcal{S}$ on S is to be defined here. If (m,n) $\in$ S - {(0,0)}, {(m,n)} $\in \mathcal{B}$. If B is a subset of S with (0,0) $\in$ B, then B $\in \mathcal{B}$ if and only if B is S minus at most (i) the points on a finite number of lines parallel to the axis y = 0 and (ii) a finite number of points on each of the remaining lines parallel to the axis y = 0. Note that (ii) allows the removal of a finite number of lines parallel to the axis x = 0, although (ii) certainly allows more than removing these lines.

1. $\mathcal{B}$ is a base for a topology $\mathcal{S}$ on S.

    (i) Note that S $\in \mathcal{B}$.

    (ii) Let B_1, $B_2 \in \mathcal{B}$. It is clear that $B_1 \cap B_2$ is an element of $\mathcal{B}$.

2. T_2.

Let x, y $\in$ S with x $\neq$ y. Without loss of generality, assume y $\neq$ (0,0). Then, {y} and S - {y} are the required disjoint open sets.

3. Separable.

Any countable space is separable.

4. Lindelöf.

Any countable space is Lindelöf.

5. *Proposition (a):* Let $K \subset S$. Then K is compact only if K is finite.

Proof. Suppose K is an infinite subset of S. Either $(0,0) \notin K$ or $(0,0) \in K$. Suppose $(0,0) \notin K$. Then, $\mathcal{C} = \langle\{x\} : x \in K\rangle$ is an open cover of K and $\mathcal{C}$ has no finite subcover. Suppose $(0,0) \in K$. Then either K contains an infinite set A of points on a line $y = k$ for some integer k or else K contains an infinite set B consisting of at least one point from an infinite collection of lines parallel to the axis $y = 0$. If there is such a set A and if k corresponds to A as in the definition of A above, then $U = \{(0,0)\} \cup [S - \{(m,k) : m \text{ is an integer}\}]$ is open and $\mathcal{F} = \{U\} \cup \langle\{x\} : x \in A - \{(0,0)\}\rangle$ is an open cover of K such that $\mathcal{F}$ has no finite subcover. Otherwise, there is a set B as defined previously. Now, $V = \{(0,0)\} \cup [S-B]$ is open and $\mathcal{H} = \{V\} \cup \langle\{x\} : x \in B - \{(0,0)\}\rangle$ is an open cover of K such that $\mathcal{H}$ has no finite subcover.

Proposition (b): Any subset of S is paracompact.

Proof. Let $A \subset S$ and $\mathcal{C}$ be an open cover of A in the space $(A, T/A)$. If $(0,0) \in A$, there exists $U \in \mathcal{C}$ such that $(0,0) \in U$. Otherwise, define $U = \phi$. We claim that $\mathcal{R} = \{U\} \cup \langle\{x\} : x \in A - U\rangle$ is a locally-finite open refinement of $\mathcal{C}$. If $y \in A$, then $y \in U$ or $y \in A - U$. If $y \in U$, then U is an open set which contains y and intersects exactly one element of $\mathcal{R}$. If $y \in A - U$, then $\{y\}$ is an open set which intersects only finitely many elements of $\mathcal{R}$. It is clear that $\mathcal{R}$ is a collection of open sets and that $\mathcal{R}$ is a refinement of $\mathcal{C}$.

Proposition (c): Each closed subset of S is a G_δ set.

Proof. Notice that if $B \in \mathcal{B}$, then $S - B \in \mathcal{S}$. Let K be a closed subset of S. If $(0,0) \in K$, define $A_o = \phi$. Otherwise, $(0,0) \notin K$ and there exists $A_o \in \mathcal{B}$ such that $(0,0) \in A_o \subset S - K$. In either case, $S - A_o \in \mathcal{S}$. Note that $S - (K \cup \{(0,0)\})$ is countable and write $S - (K \cup \{(0,0)\}) = \{x_n : n \in P\}$. Let $A_n = \{x_n\}$ for each $n \in P$. Then $S - A_n \in \mathcal{S}$ for each $n \in P$ since $A_n \in \mathcal{B}$ for each $n \in P$. It is now apparent that K is the intersection of a countable collection of open sets; in fact $K = \cap \{S - A_n : n = 0 \text{ or } n \in P\}$.

6. Paracompact.

See (5(b)), page 42.

7. Normal.

Each paracompact, Hausdorff space is normal (see Theorem 0-2(a), (b)).

8. Perfectly normal.

This follows from (5(c)) on page 43, and (7) on page 43.

9. Totally disconnected.

Let A be subset of S which consists of more than one point. Let $x \in A$ with $x \neq (0,0)$. Note that $\{x\}$ and $(S - \{x\})$ are disjoint open sets. Then, if A consists of more than one point, $\{\{x\}, S - \{x\}\}$ is a separation for A.

10. Not locally connected.

$\{(0,0)\}$ is a component of S and $\{(0,0)\}$ is not an open set.

11. Not B.W. compact.

Let $A = \{(n,1) : n \text{ is an integer}\}$. Then A is an infinite subset of S and A is a closed set. Clearly, no point of A is a limit point of A since $x \in A$ implies $\{x\} \in \mathcal{S}$.

12. Not first countable.

Suppose that $\{B_n : n \in P\}$ is a collection of neighborhoods of (0,0). It suffices to prove that $\{B_n : n \in P\}$ is not a base for $\mathcal{N}(0,0)$. We will now construct a neighborhood U of the origin for which $U - B_n \neq \phi$ for each $n \in P$; i.e., B_n is not a subset of U for each $n \in P$.

Without loss of generality, we may assume $B_n \in \mathcal{B}$ for each $n \in P$. Since B_1 is S minus at most finitely many lines parallel to y = 0 and finitely many points on each of the remaining lines parallel to y = 0, there exists $(m_1,n_1) \in B_1 - \{(0,0)\}$. Suppose that $(m_1,n_1),\ldots,(m_k,n_k)$ have been chosen so that $(m_j,n_j) \in B_j - \{(0,0)\}$ for $1 \le j \le k$ and $n_i \neq n_j$ for $i \neq j$. Then since B_{k+1} is S minus at most finitely many lines parallel to y = 0 and finitely many points on each of the remaining lines, $(B_{k+1} - \{(m,n_i) : (m,n_i) \in S, 1 \le i \le k, i \in P\})$ is an open set which contains the origin. Therefore, there exists $(m_{k+1}, n_{k+1}) \in B_{k+1} - \{(m,n_i) : (m,n_i) \in S, 1 \le i \le k, i \in P\}$. The above process inductively defines a sequence of points $\{(m_k,n_k) : k \in P\}$, none of which is (0,0) and no two of which belong to the same line y = p for any integer p. It follows that $U = S - \{(m_k,n_k) : k \in P\}$ is the desired neighborhood of (0,0); i.e., B_n is not a subset of U for each $n \in P$.

13. Not metric.

Every metric space is first countable.

14. Not locally compact.

Let U be any closed neighborhood of the origin. It suffices to show that U is not compact.

Note that U is obviously an open set since U contains an open neighborhood of the origin and, if $z \in U - \{(0,0)\}$, then $\{z\}$ is open. One can easily see from the definition of $\mathcal{S}$ that there exists an infinite set A with $(0,0) \notin A$ such that A contains exactly one point on those lines $y = m$ parallel to the axis $y = 0$ such that U contains a point (n,m). Clearly, U - A is open and, if $\mathcal{C} = \{U - A\} \cup \{\{z\} : z \in A\}$, then $\mathcal{C}$ is an open cover of U with no finite subcover.

Example 11

The following topology is due to R. H. Bing [3]. This topological space is an example of a connected countable Hausdorff space.

Let Z be the set of points in the plane which are on or above the X-axis and for which both co-ordinates are rational.

Define $p : Z \to R$ and $q : Z \to R$ by $p\big((x,y)\big) = x - \frac{1}{\sqrt{3}}y$ and $q\big((x,y)\big) = x + \frac{1}{\sqrt{3}}y$ for each $(x,y) \in Z$. Note that (i) $p\big((x,y)\big)$ and $q\big((x,y)\big)$ are rational if and only if $y = 0$ and (ii) if $z \in Z$, then z, $\big(p(z),0\big)$, and $\big(q(z),0\big)$ are the vertices of an equilateral triangle.

Let $\mathcal{B} = \{N(z;\varepsilon) : z \in Z,\ \varepsilon > 0\}$ where $N(z;\varepsilon) = \{z\} \cup \{(r,0) : r \in Q$ and either $|r - p(z)| < \varepsilon$ or $|r - q(z)| < \varepsilon\}$ for each $z \in Z$ and $\varepsilon > 0$.

1. $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on Z.

(i) Clearly, $Z = \cup\, \mathcal{B}$.

(ii) Suppose $N(z_1;\varepsilon_1)$, $N(z_2;\varepsilon_2) \in \mathcal{B}$ and $z \in N(z_1;\varepsilon_1) \cap N(z_2;\varepsilon_2)$. If $z_1 = z_2$, then $N(z_1;\varepsilon_1) \cap N(z_2;\varepsilon_2) = N(z_1;\varepsilon)$ where $\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$. If $z_1 \neq z_2$, then $N(z_1;\varepsilon_1) \cap N(z_2;\varepsilon_2)$ consists of open intervals of rationals along the X-axis, i.e., unions of elements of $\mathcal{B}$.

2. *Proposition (a):* If z_1, $z_2 \in Z$ and $z_1 \neq z_2$, then $p(z_1) \neq p(z_2)$ and $q(z_1) \neq q(z_2)$.

Proof. Let $z_1 = (x_1,y_1)$ and $z_2 = (x_2,y_2)$. Then, $p(z_1) - p(z_2) = (x_1-x_2) - \frac{1}{\sqrt{3}}(y_1-y_2)$, and the quantity on the right of this equation is rational if and only if $y_1 = y_2$. Therefore, $p(z_1) = p(z_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$. Similarly, $q(z_1) = q(z_2)$ if and only if $z_1 = z_2$.

Proposition (b): A subset of Z is not compact if it contains infinitely many points (x,y) where $y > 0$.

Proof. Let K be a subset of Z and suppose $\{(x,y) \in K : y > 0\}$ is infinite. Let $z \in Z$ and note that $N(z;1)$ contains at most one point (x,y) such that $y > 0$. It follows that $\mathcal{C} = \{N(z;1) : z \in K\}$ is an open cover of K and that $\mathcal{C}$ has no finite subcover.

Proposition (c): Let $b > a$ and let $U = \{(r,0) : r \in (a,b)\}$. If $K = \{(x,y) \in Z :$ either $\sqrt{3}\,(x-b) \leq y \leq \sqrt{3}\,(x-a)$ or $-\sqrt{3}\,(x-a) \leq y \leq -\sqrt{3}\,(x-b)\}$, then $K = \bar{U}$.

Proof. Let $(x,y) \in K$. Suppose $\sqrt{3}\,(x-b) \leq y \leq \sqrt{3}\,(x-a)$. It follows that $a \leq x - \frac{1}{\sqrt{3}}y \leq b$ and $a \leq p\big((x,y)\big) \leq b$. Clearly, $N\big((x,y);\varepsilon\big) \cap U \neq \phi$ for each $\varepsilon > 0$. Therefore, $(x,y) \in \bar{U}$. Suppose $-\sqrt{3}\,(x-a) \leq y \leq -\sqrt{3}\,(x-b)$. Then, $a \leq x + \frac{1}{\sqrt{3}}y \leq b$ and thus

$a \le q\big((x,y)\big) \le b$. Again, $N\big((x,y);\varepsilon\big) \cap U \ne \phi$ for each $\varepsilon > 0$. Thus $(x,y) \in \bar{U}$. Hence, $K \subset \bar{U}$.

Let $(x,y) \in S - K$. Then, (i) $y < \sqrt{3}\,(x-b)$, or (ii) $y < -\sqrt{3}\,(x-a)$, or (iii) $-\sqrt{3}\,(x-b) < y$ and $\sqrt{3}\,(x-a) < y$ (see Figure 11-1).

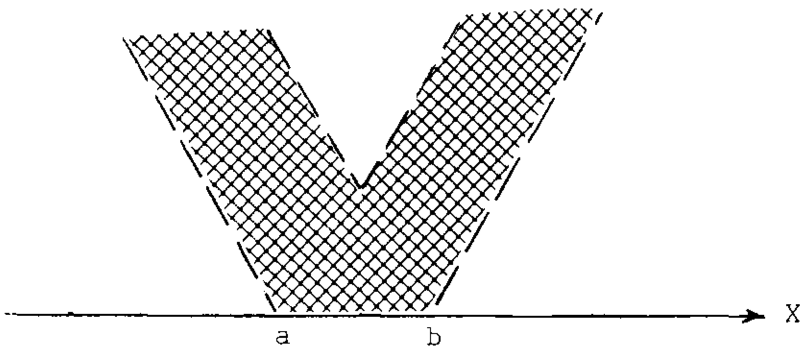


Figure 11-1

(It should be noted that for each $(a,b) \in Z$, $p\big((a,b)\big) \le q\big((a,b)\big)$.) In case (i), $b < x - \frac{1}{\sqrt{3}}\,y = p\big((x,y)\big) \le q\big((x,y)\big)$, so let $\varepsilon_1 = p\big((x,y)\big) - b$. In case (ii), $p\big((x,y)\big) \le q\big((x,y)\big) = x + \frac{1}{\sqrt{3}}\,y < a$, so let $\varepsilon_2 = a - q\big((x,y)\big)$. In case (iii), $b < q\big((x,y)\big)$ and $p\big((x,y)\big) < a$, so let $\varepsilon_3 = \min\{q\big((x,y)\big) - b,\ a - p\big((x,y)\big)\}$. Then, $N\big((x,y);\varepsilon_i\big) \cap U = \phi$ for $i = 1,2,3$. Thus, $(x,y) \in S - \bar{U}$. Therefore, $S - K \subset S - \bar{U}$. Since $K \subset \bar{U}$, $K = \bar{U}$.

3. T_2.

Let $z_1, z_2 \in Z$ with $z_1 \ne z_2$. Let $\varepsilon = \frac{1}{2}\min\{|p(z_1) - p(z_2)|,\ |q(z_1) - q(z_2)|,\ |p(z_1) - q(z_2)|,\ |q(z_1) - p(z_2)|\}$. Then, $N(z_1;\varepsilon)$ and

$N(z_2;\varepsilon)$ are the required disjoint open sets.

4. Second countable.

It is clear that $\{N(z;\frac{1}{n}) : z \in Z, n \in P\}$ is a countable base for $\mathcal{T}$.

5. Not regular.

Let $(r,0) \in Z$ and let $U = \{(x,0) : (x,0) \in Z\}$. Note that $(r,0) \in U$ and $U \in \mathcal{T}$. It suffices to show that, if $(r,0) \in V$ and $V \in \mathcal{T}$, then $\bar{V} - U \neq \phi$. However, [2(c)] on page 46 shows that, if $(r,0) \in V$ and $V \in \mathcal{T}$, then $\bar{V}$ contains points above the X-axis and thus $\bar{V} - U \neq \phi$.

6. Not locally compact.

Every locally compact, Hausdorff space is regular. (See Theorem 0-7.)

7. Not countably paracompact.

Since $(Z,\mathcal{T})$ is Hausdorff but not regular, $(Z,\mathcal{T})$ is not paracompact [see Theorem 0-2(a)]. Since $(Z,\mathcal{T})$ is Lindelöf by (4) of page 48, $(Z,\mathcal{T})$ is not countably paracompact.

8. Connected.

Suppose $\{A,B\}$ is a separation for Z. Then, A and B are each non-empty, open, and closed subsets of Z. Without loss of generality, assume that A contains a point $a = (x,0)$ for some $x \in R$. Suppose $(x,0) \in A$ for each $x \in R$. It would then follow from [2(c)], page 46, that $A = R$ since A is closed. This contradicts $\{A,B\}$ is a separation for Z. Therefore, B contains a point $b = (y,0)$ for some $y \in R$. Note that A and B are closed neighborhoods of a and b, respectively. From $(2(c))$ of page 46, Figure 11-1, and Figure 11-2, it is geometrically obvious that $A \cap B \neq \phi$. Therefore, Z has no separation; i.e., Z is connected.

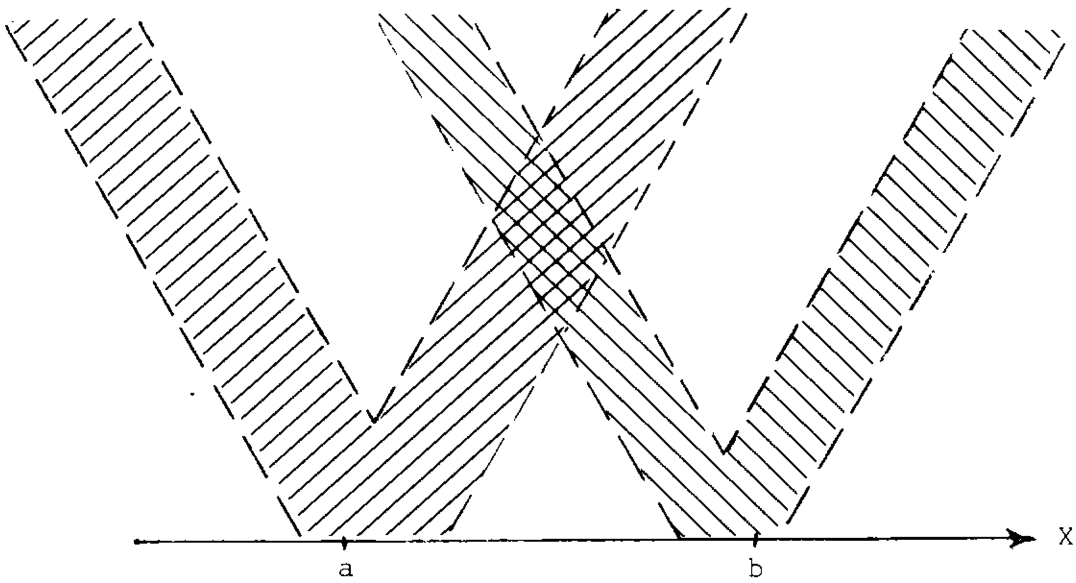


Figure 11-2

9. Not locally connected.

Note that singleton sets are not open. It is sufficient to show that $\{(0,1)\}$ is a component of $N((0,1);1)$.

Let K be a subset of $N((0,1);1)$ which contains $(0,1)$. Suppose $(r,0) \in K$ for some $(r,0) \in Z$. Let $\varepsilon = \min\{|r - p((0,1))|, |r - q((0,1))|\}$. Then $\varepsilon > 0$ since $p((0,1))$ and $q((0,1))$ are irrational whereas r is rational. Let $\delta \in Q$ be such that $0 < \delta < \varepsilon$. It is now almost obvious that $\{N((0,1);\delta), N((0,1);1) - N((0,1);\delta)\}$ is a separation for $N\big((0,1);1\big)$. It follows that $\{K \cap N\big((0,1);\delta\big)$, $K \cap [N\big((0,1);1\big) - N\big((0,1);\delta\big)]\}$ is a separation for K. Thus $\{(0,1)\}$ is a component of $N\big((0,1);1\big)$.

<u>Example 12</u>

The following topology may be found in [15], page 88.

Recall that R is the set of real numbers, Q is the set of rational numbers, and R- Q is the set of irrational numbers. Let $\mathcal{I}$ be the set of intervals in $\mathcal{E}$, i.e., $\mathcal{I} = \{(a,b) : a, b \in R \text{ and } a < b\}$. Then, the topology $\mathcal{T}$ on R is given by: $\mathcal{T} = \{G : G \subset R$ and if $p \in G$, there exists $I \in \mathcal{I}$ such that $p \in I$ and $I \cap Q \subset G\}$. Note that G is open if and only if each point of G is contained in an open (wrt $\mathcal{E}$) interval of rational points which is itself contained in G.

1. $\mathcal{T}$ is a topology for R.

   (i) Clearly, $\phi \in \mathcal{T}$ and $R \in \mathcal{T}$.

   (ii) Suppose $G_1, G_2 \in \mathcal{T}$ and $p \in G_1 \cap G_2$. Then there exists $I_1, I_2 \in \mathcal{I}$ such that $p \in I_1 \cap I_2$, $I_1 \cap Q \subset G_1$ and $I_2 \cap Q \subset G_2$. But then $p \in G_1 \cap G_2$, $I_1 \cap I_2 \in \mathcal{I}$, $(I_1 \cap I_2) \cap Q \subset G_1 \cap G_2$ and hence $G_1 \cap G_2 \in \mathcal{T}$.

   (iii) Suppose $G_\alpha \in \mathcal{T}$ for each $\alpha \in \Lambda$. Then let $G = \cup\{G_\alpha : \alpha \in \Lambda\}$. If $p \in G$, then $p \in G_\alpha$ for some $\alpha \in \Lambda$. There exists $I \in \mathcal{I}$ such that $p \in I$ and $I \cap Q \subset G_\alpha$. But then $I \cap Q \subset G$ and hence $G \in \mathcal{T}$.

2. *Proposition (a):* $\mathcal{E} \subset \mathcal{T}$.

Proof. This assertion is obvious. For if $U \in \mathcal{E}$, then U is the union of a collection of Euclidean open intervals where intersection with Q is certainly contained in U. Hence, $U \in \mathcal{T}$ and therefore $\mathcal{E} \subset \mathcal{T}$.

Proposition (b): Let $G \in \mathcal{T}$. Then there exist $U \in \mathcal{E}$ and $K \subset R - Q$ such that $K \subset U$ and $G = U - K$.

Proof. For each $g \in G$, there exists $I(g) \in \mathcal{I}$ such that $g \in I(g)$ and $I(g) \cap Q \subset G$. Let $U = \cup \{I(g) : g \in G\}$. Then $G \subset U$ and $U \cap Q \subset G$. Let $K = U - G$. Then since $U \cap Q \subset G$, $K \subset U - Q \subset R - Q$. Also, $G = U - K$.

Proposition (c): In the notation of (b) above, $\bar{G}$(wrt $\mathcal{T}$) = $\bar{U}$ (wrt $\mathcal{E}$). (The reader should recall that $\bar{G}$ (wrt $\mathcal{T}$) is the closure of G with respect to topology $\mathcal{T}$.)

Proof. We shall first show that $\bar{G}$ (wrt $\mathcal{T}$) $\subset \bar{U}$ (wrt $\mathcal{E}$). Since $\mathcal{E} \subset \mathcal{T}$, it follows that $\bar{G}$ (wrt $\mathcal{T}$) $\subset \bar{G}$ (wrt $\mathcal{E}$). But $\bar{G}$ (wrt $\mathcal{E}$) = $\overline{(U-K)}$(wrt $\mathcal{E}$) and it is easily seen, since $U \in \mathcal{E}$, that $\overline{(U-K)}$(wrt $\mathcal{E}$) = $\bar{U}$ (wrt $\mathcal{E}$). Thus, $\bar{G}$ (wrt $\mathcal{T}$) $\subset \bar{U}$ (wrt $\mathcal{E}$).

Now, we shall show that $\bar{U}$ (wrt $\mathcal{E}$) $\subset \bar{G}$ (wrt $\mathcal{T}$). Since $G = U - K$ and $K \subset R - Q$, $U \subset \bar{G}$ (wrt $\mathcal{T}$) and $\bar{U}$ (wrt $\mathcal{T}$) $\subset \bar{G}$ (wrt $\mathcal{T}$). Therefore, it will suffice to show that $\bar{U}$ (wrt $\mathcal{E}$) $\subset \bar{U}$ (wrt $\mathcal{T}$). Let $x \in \bar{U}$ (wrt $\mathcal{E}$). Recall that $U \in \mathcal{E}$, and hence there exists a sequence of rationals $\{r_n : n \in P\}$ in $U - \{x\}$ which converges (wrt $\mathcal{E}$) to x. Now, let $x \in V$ and $V \in \mathcal{T}$. There exists $I \in \mathcal{I}$ such that $x \in I$ and $I \cap Q \subset V$. But then since $I \in \mathcal{E}$, there exists $N \in P$ such that $r_n \in I$ for each $n \in P$ with $n \geq N$. It follows since $I \cap Q \subset V$ that $r_n \in V$ for each $n \in P$ with $n \geq N$. Therefore, $V \cap (U - \{x\}) \neq \phi$. Hence, $x \in \bar{U}$ (wrt $\mathcal{T}$) and $\bar{U}$ (wrt $\mathcal{E}$) $\subset \bar{U}$ (wrt $\mathcal{T}$) as required.

Note that we have shown $\bar{G}$ (wrt $\mathcal{T}$) $\subset \bar{U}$ (wrt $\mathcal{E}$) and $\bar{U}$ (wrt $\mathcal{E}$) $\subset \bar{G}$ (wrt $\mathcal{T}$). Thus, $\bar{G}$ (wrt $\mathcal{T}$) = $\bar{U}$ (wrt $\mathcal{E}$).

3. T_2.

Since $\mathcal{E} \subset \mathcal{T}$ and $(R,\mathcal{E})$ is Hausdorff, $(R,\mathcal{T})$ is Hausdorff.

4. Not Lindelöf.

Note that $Q \cup \{x\} \in \mathcal{T}$ for each $x \in R$. Then, $\mathcal{C} = \{Q \cup \{x\} : x \in R-Q\}$ is an open cover of R and $\mathcal{C}$ has no countable subcover. In fact, $\mathcal{C}$ has no proper subcover.

5. First countable.

Let $x \in R$. Let $I_n = (x - \frac{1}{n}, x + \frac{1}{n})$ for each $n \in P$. Note that $I_n \in \mathcal{I}$ for each $n \in P$. Hence, $U_n = \{x\} \cup (I_n \cap Q)$ is open (wrt $\mathcal{T}$) for each $n \in P$. It is evident that $\{U_n : n \in P\}$ is a countable base for $\mathcal{N}(x)$.

6. Not B.W. compact.

Recall that $(R,\mathcal{E})$ is not B.W. compact.

7. Separable.

Obviously, Q is a countable dense subset of R.

8. Not locally compact.

Let $x \in R$ and let $\{U_n : n \in P\}$ be the countable base for $\mathcal{N}(x)$ constructed in (5), page 52. Since $(R,\mathcal{T})$ is Hausdorff, it suffices to show that $\bar{U}_n$ is not compact for each $n \in P$. By [2(c)], page 51, $\bar{U}_n = [x - \frac{1}{n}, x + \frac{1}{n}]$ for each $n \in P$. Let $n \in P$. Then $\mathcal{C} = \{Q \cup \{x\} : x \in R\}$ is an open cover of $\bar{U}_n$ and $\mathcal{C}$ has no finite subcover. Thus $\bar{U}_n$ is not compact for each $n \in P$.

9. Not regular.

Let $I = (\pi-1, \pi+1) \in \mathcal{I}$. Let $U = \{\pi\} \cup (Q \cap I)$. Suppose V is an open set such that $\pi \in V$ and $V \subset U$. It suffices to show that $\bar{V} - U \neq \phi$.

There exists $I \in \mathcal{I}$ such that $\pi \in I$ and $I \cap Q \subset V$. Now, $I = (a,b)$ for some $a,b \in R$ with $a < b$. Also, by [2(c)] on page 51, $\bar{I}$ (wrt E) $\subset \bar{V}$. Note $\bar{I}$ (wrt E) $= [a,b]$. Since the only irrational number contained in U is π, we see that $\phi \neq [a,b] - U \subset \bar{V} - U$ as desired.

10. Connected.

Suppose $\{A,B\}$ is a separation for R. Then, A and B are each non-empty, closed and open subsets of $(R,\mathcal{T})$. Then, by [2(b)] on page 51, there exist $U \in E$ and $K \subset R - Q$ such that $K \subset U$ and $A = U - K$. By [2(c)], page 51, $\bar{U}$ (wrt E) $= \bar{A}$ (wrt $\mathcal{T}$). But $\bar{A}$ (wrt $\mathcal{T}$) $= A$ since A is a closed set in $(R,\mathcal{T})$. Since $\bar{U}$ (wrt E) $= A = U - K$, it follows that $\bar{U}$ (wrt E) $= U$; i.e., U is an open and closed set in (R,E). Therefore, $U = \phi$ or $U = R$. Now, $U \neq \phi$ since $A \neq \phi$ and $A = U - K$. Then $U = R$. But this implies that $B \subset R - Q$ which means that $B \notin \mathcal{T}$ which is also a contradiction. Hence, R has no separation in the space $(R,\mathcal{T})$; i.e., $(R,\mathcal{T})$ is connected.

11. Not locally connected.

Note that singleton sets are not open. Since Q is open, it suffices to show that components of Q are singleton sets.

Let K be a subset of Q and suppose $x,y \in K$ with $x \neq y$. There exists $t \in R - Q$ such that $x < t < y$ or $y < t < x$. It is clear that $\{K \cap (-\infty,t), K \cap (t,\infty)\}$ is a separation for K. Hence, components of Q are singleton sets.

12. Not countably paracompact.

Let $z \in R - Q$ and let $\{z_n : n \in P\}$ be a subcollection of $(R-Q)$ such that (i) $z_m = z_n$ only if $m = n$ and (ii) $\lim_{n\to\infty} d(z,z_n) = 0$, where d is the Euclidean metric. Furthermore, let $K = (R-Q) - \{z_n : n \in P\}$.

Now, $\mathcal{C} = \{K \cup \{z_n\} \cup Q : n \in P\}$ is a countable open cover of R. Let $\mathcal{F}$ be an arbitrary open refinement of $\mathcal{C}$. It suffices to show that $\mathcal{F}$ is not a locally-finite system. Toward this end note that, if $F \in \mathcal{F}$, then F contains no more than one point of $\{z_n : n \in P\}$. Hence, for each $n \in P$, there exists $F_n \in \mathcal{F}$ such that $z_n \in F_n$ and it must be that $F_n \neq F_m$ if $m \neq n$.

Let V be any open set containing z. There exists $(a,b) \in \mathcal{I}$ such that $z \in (a,b)$ and $(a,b) \cap Q \subset V$. Since $\lim_{n\to\infty} d(z,z_n) = 0$, there exists $N \in P$ such that $n \geq N$ implies that $z_n \in (a,b)$. Note that $n \geq N$ implies that $F_n \cap (a,b)$ contains a rational since F_n and (a,b) are open sets which contain z_n. It follows that if $n \geq N$, $F_n \cap V \subset F_n \cap [(a,b) \cap Q] \neq \phi$. Since V was an arbitrary open set containing z, we have shown that each open set containing z intersects infinitely many distinct $F \in \mathcal{F}$. Thus $\mathcal{F}$ is not a locally-finite system. Therefore, $\mathcal{C}$ has no locally-finite open refinements.

13. Not metacompact.

Let $\mathcal{C} = \{Q \cup \{t\} : t \in R - Q\}$. Then $\mathcal{C}$ is an open cover of R. Let $\mathcal{F}$ be an open refinement of $\mathcal{C}$. It suffices to show that $\mathcal{F}$ is not a point-finite system. Note that, for each $t \in R - Q$, there exists $F(t) \in \mathcal{F}$ such that $t \in F(t)$ and, by construction of $\mathcal{C}$, $F(s) \neq F(t)$ if if $s,t \in R - Q$ and $s \neq t$.

Now, for each $t \in R - Q$, there exists $\delta(t) > 0$ such that $(t - \delta(t), t + \delta(t)) \cap Q \subset F(t)$. For each $t \in R - Q$, let $V(t) = \{t\} \cup [Q \cap (t - \delta(t), t + \delta(t))]$. It follows that $V(t)$ is not a subset of $F(s)$ if $s,t \in R - Q$ and $s \neq t$. Hence, $V(s) \neq V(t)$ if $s,t \in R - Q$ and $s \neq t$. It will be sufficient to show that there exists $r \in R$ such that $r \in V(t)$ and hence $r \in F(t)$ for an infinite collection of points t of $R - Q$, for this would mean that $\mathcal{F}$ is not a point-finite system.

For each $n \in P$, let $A_n = \{t : t \in R - Q \text{ and } \delta(t) > \frac{1}{n}\}$. Since $\delta(t) > 0$ for each $t \in R - Q$, $R - Q = \cup \{A_n : n \in P\}$. Hence, there exists $N \in P$ such that A_N is uncountable. Then, there exists $p \in R$ such that p is a limit point (wrt $\mathcal{E}$) of A_N. Therefore, there exists an infinite set B with $B \subset A_N$ such that $t \in B$ implies that $d(p,t) < \frac{1}{2N}$. There is also a rational point $r \in R$ such that $d(p,r) < \frac{1}{2N}$. Now, for each $t \in B$, $d(t,r) \leq d(t,p) + d(p,r) < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N}$ and, since $B \subset A_N$ and r is rational, $r \in V(t) = \{t\} \cup [Q \cap (t - \delta(t), t + \delta(t))]$. Thus, $r \in V(t)$ for each $t \in B$, an infinite collection. Also, $r \in F(t)$ for each $t \in B$. Hence, $\mathcal{F}$ is not a point-finite system since $\{F(t) : t \in B\}$ is an infinite collection of elements of $\mathcal{F}$ and r is a point of R such that $r \in F(t)$ for each $t \in B$. Thus, $\mathcal{C}$ has no point-finite open refinement.

Example 13

The following topology may be found in [15], page 89.

Let H be the X-axis and let U be the set of points in the plane above the X-axis. Let $Z = U \cup H$. The topology, $\mathcal{T}_1$, for Z will be defined in terms of a base $\mathcal{B}_1$ for $\mathcal{T}_1$.

Let $\mathcal{E}$ be the Euclidean topology on Z and, for each $t \in Z$ and each $\varepsilon > 0$, let $N(t;\varepsilon)$ be the set of points in Z whose Euclidean distance from t is less than ε.

Let $(x,y) \in U$. If $0 < \varepsilon < y$, define $N^*\big((x,y);\varepsilon\big) = N\big((x,y);\varepsilon\big)$. Let $(x,0) \in H$. For each $\varepsilon > 0$, define $N^*\big((x,0);\varepsilon\big) = \{(x,0)\} \cup [N\big((x,0);\varepsilon\big) \cap U]$. (See Figure 13-1.) Let $\mathcal{B}_1 = \{N^*\big((x,y);\varepsilon\big) : (x,y) \in Z$ and (i) $y = 0$ and $\varepsilon > 0$ or (ii) $\varepsilon > 0$ and $\varepsilon < y\}$.

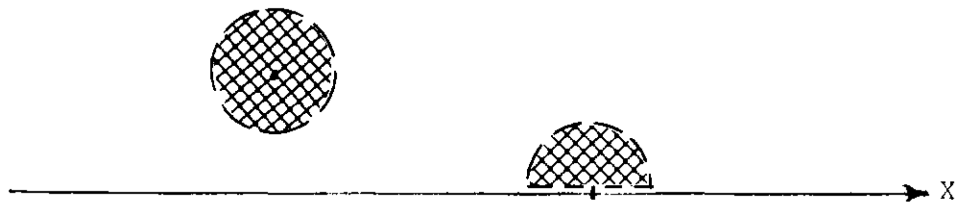


Figure 13-1

1. *Proposition:* If $V \in \mathcal{E}$, then V is the union of a subcollection of elements of $\mathcal{B}_1$.

Proof. It suffices to show that $N\big((a,b);r\big)$, where $(a,b) \in Z$ and $r > 0$, is the union of a subcollection of elements of $\mathcal{B}_1$.

(i) Suppose $r < b$. Then, $N\big((a,b);r\big) = N^*\big((a,b);r\big) \in \mathcal{B}_1$.

(ii) Suppose $b = 0$. Let $N = N\big((a,b);r\big)$. Then, $N = \cup \{N^*\big((x,0);\varepsilon\big)$: (i) $x = a$ and $\varepsilon = r$ or (ii) $(x,0) \in N$, $\varepsilon > 0$, and $N\big((x,0);\varepsilon\big) \subset N\}$. Hence, N is the union of a subcollection of elements of $\mathcal{B}_1$.

(iii) Suppose $0 < b \leq r$. Let $N = N\big((a,b);r\big)$. Then, $N = \cup \Big\{N^*\big((x,y);\varepsilon\big)$: (i) $(x,y) \in N$, $0 < y < \varepsilon$, and $N\big((x,y);\varepsilon\big) \subset N$ or (ii) $(x,y) \in N$, $y = 0$, $\varepsilon > 0$, and $N\big((x,y);\varepsilon\big) \subset N\Big\}$. Hence, N is the union of a subcollection of elements of $\mathcal{B}_1$.

Notice that, if $\mathcal{B}_1$ generates a topology $\mathcal{T}_1$, then $\mathcal{E} \subset \mathcal{T}_1$.

2. $\mathcal{B}_1$ is a base for a topology $\mathcal{T}_1$ on Z.

(i) Clearly, $Z = \cup\, \mathcal{B}_1$.

(ii) Let $N^*(t_1;r_1)$, $N^*(t_2;r_2) \in \mathcal{B}_1$. If $t_1 = t_2$, then let $r = \min\,\{r_1,r_2\}$ and $N^*(t_1;r_1) \cap N^*(t_2;r_2) = N^*(t_1;r) \in \mathcal{B}_1$. If $t_1 \neq t_2$, then it is easily seen that $N^*(t_1;r_1) \cap N^*(t_2;r_2)$ is open (wrt $\mathcal{E}$) and, by (1) of page 56, is the union of a subcollection of elements of $\mathcal{B}_1$.

3. T_2.

Since $\mathcal{E} \subset \mathcal{T}_1$ and $(R,\mathcal{E})$ is Hausdorff, $(R,\mathcal{T}_1)$ is Hausdorff.

4. First countable.

Let $(x,y) \in Z$. If $y = 0$, then $\{N^*\big((x,0);\frac{1}{n}\big) : n \in P\}$ is a countable base for $\mathcal{N}\big((x,0)\big)$. If $y > 0$, then $\{N^*\big((x,y);\frac{1}{n}\big) : n \in P$ and $\frac{1}{n} < y\}$ is a countable base for $\mathcal{N}\big((x,y)\big)$.

5. Separable.

It is clear that each element of $\mathcal{B}_1$ contains a non-empty set which is open in the Euclidean topology. Hence, the set of points in Z with both coordinates rational is a countable dense subset of Z.

6. Not Lindelöf.

Let $\mathcal{C} = \{N^*(z;1)\colon z \in Z\}$. Then, $\mathcal{C}$ is an open cover of Z. Note that for each $z \in Z$, $N^*(z;1)$ contains at most one point of the X-axis. Hence, $\mathcal{C}$ has no countable subcover.

7. Not regular.

Let $N = N^*((0,0);1)$. Then $(0,0) \in N$ and $N \in \mathcal{T}_1$. Suppose U is an open neighborhood of (0,0) and $U \subset N$. It suffices to show that $\bar{U}$ is not contained in N. There exists $\varepsilon > 0$ such that $N^*((0,0);\varepsilon) \subset U$. It is clear that $\overline{N^*((0,0);\varepsilon)} = \overline{N((0,0);\varepsilon)}$ and that $(\frac{1}{2}\varepsilon,0) \in \overline{N((0,0);\varepsilon)} \subset \bar{U}$, but $(\frac{1}{2}\varepsilon,0) \notin N$. Hence, $(Z,\mathcal{T}_1)$ is not regular.

8. Not locally compact.

Every locally compact, Hausdorff space is regular. (See Theorem 0-7.)

9. Not compact.

This follows from (8), page 58.

10. Connected.

Note that $\mathcal{T}_1/U$ is the Euclidean topology on U, the upper half plane. Since U is connected in $(U, \mathcal{T}_1/U)$, U is connected in $(Z,\mathcal{T}_1)$. Since $Z = \bar{U}$, Z is connected.

11. Locally connected.

It suffices to prove that elements of $\mathcal{B}_1$ are connected. Let $B \in \mathcal{B}_1$. Then, B contains at most one point on the X-axis. Let $D = B - H$. Clearly, $\mathcal{T}_1/D$ is metric and D is connected in $(D, \mathcal{T}_1/D)$. Now, since $D \subset B \subset \bar{D}$, B is connected.

12. Not metacompact.

Let $\mathcal{C} = \{U\} \cup \{N^*(z;1) : z \in H\}$. Then, $\mathcal{C}$ is an open cover of Z. We claim that, if $\mathcal{F}$ is an open refinement of $\mathcal{C}$, then $\mathcal{F}$ is not a point-finite system.

Suppose $\mathcal{F}$ is an open refinement of $\mathcal{C}$. Then for each $z \in H$, there exists $F(z) \in \mathcal{F}$ such that $z \in F(z)$. Furthermore, since $F(z)$ is open, $N^*(z;\frac{1}{n}) \subset F(z)$ for some $n \in P$. Let $z,w \in H$ with $z \neq w$. It follows that $z \notin F(w)$ since $F(w) \subset N^*(w;1)$.

For each $n \in P$, let $A_n = \{z : z \in H \text{ and } N^*(z;\frac{1}{n}) \subset F(z)\}$. Note that $\{A_n : n \in P\}$ is an increasing sequence such that $H = \cup \{A_n : n \in P\}$. Thus, there exists $N \in P$ such that A_N is uncountably infinite. Consider H with the Euclidean topology. A_N has a limit point, say $t = (x,0)$, in H with respect to the Euclidean topology on H. Then, there exists $B \subset A_N$ such that B is infinite and, if $s \in B$, then $d(s,t) < \frac{1}{2N}$ where d is the Euclidean metric on H.

Now, consider the point $z \in Z$ where $z = (x,\frac{1}{2N})$. Note that, if $s \in B$, then $d(s,z) \leq d(s,t) + d(t,z) < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N}$, and, since $B \subset A_N$, $N^*(s;\frac{1}{N}) \subset F(s)$ which implies that $z \in F(s)$. Thus, $z \in F(s)$ for each $s \in B$. Since $F(s) \neq F(r)$ for $r,s \in H$ with $r \neq s$ and since B is infinite, it follows that $\{F(s) : s \in B\}$ is an infinite subcollection of $\mathcal{F}$ with the property that $z \in F(s)$ for each $s \in B$. Therefore, $\mathcal{F}$ is not a point-finite system and $\mathcal{C}$ has no point-finite open refinement.

13. Not B.W. compact.

Note that H is an infinite, closed subset of $(Z,\mathcal{T}_1)$ and that $(H, \mathcal{T}_1/H)$ is the discrete topology.

Example 14

The following topology on the upper half plane may be found in [15], page 90, and [17], page 109. Wolfgang J. Thron gives credit for this example to "Niemytzki."

As in Example 13, let H be the X-axis, let U be the set of points $(x,y) \in R^2$ with $y > 0$, and let $Z = U \cup H$. Furthermore, $\mathcal{E}$ is the Euclidean topology on Z, d is the Euclidean metric on Z, and $N(z;\varepsilon) = \{w \in Z : d(z,w) < \varepsilon\}$ for each $z \in Z$ and each $\varepsilon > 0$.

For $(a,b) \in U$ and $0 < \varepsilon < b$, define $N^*\big((a,b);\varepsilon\big) = N\big((a,b);\varepsilon\big)$. For $(a,b) \in U$ and $b = \varepsilon$, define $N^*\big((a,b);\varepsilon\big) = \{(a,0)\} \cup N\big((a,b);\varepsilon\big)$. Then, let $\mathcal{B}_2 = \{N^*\big((a,b);\varepsilon\big) : (a,b) \in U$ and $0 < \varepsilon \le b$. The topology, $\mathcal{T}_2$, on Z is to have $\mathcal{B}_2$ as base. (See Figure 14-1 for pictures of typical base elements.)

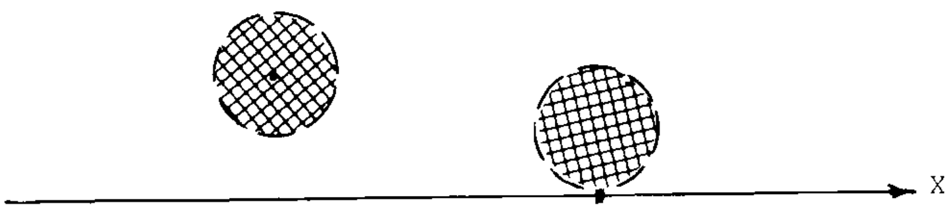


Figure 14-1

Recall that $\mathcal{B}_1$ is the base for the topology of Example 13. The reader should note that if $B \in \mathcal{B}_1$, then B is the union of a subcollection of elements of $\mathcal{B}_2$. Hence, $\mathcal{T}_1 \subset \mathcal{T}_2$ and, since $\mathcal{E} \subset \mathcal{T}_1$, $\mathcal{E} \subset \mathcal{T}_2$.

1. $\mathcal{B}_2$ is a base for a topology $\mathcal{T}_2$ on Z.

   (i) Clearly, $Z = \cup\, \mathcal{B}_2$.

   (ii) The intersection of any two elements of $\mathcal{B}_2$ is easily seen to be the union of a subcollection of elements of $\mathcal{B}_2$.

2. T_2 and not Lindelöf.

Since $(Z,\mathcal{T}_1)$ is T_2 and not Lindelöf and since $\mathcal{T}_1 \subset \mathcal{T}_2$, $(Z,\mathcal{T}_2)$ is T_2 and not Lindelöf.

3. First countable.

Suppose $(a,0) \in H$. Then, $\{N^*\left((a,\frac{1}{n});\ \frac{1}{n}\right) : n \in P\}$ is a countable base for $\mathcal{N}\left((a,0)\right)$. Suppose $(a,b) \in U$. Then, $\{N^*\left((a,b);\ \frac{1}{n}\right) : \frac{1}{n} < b$ and $n \in P\}$ is a countable base for $\mathcal{N}\left((a,b)\right)$.

4. Separable.

It is clear that each element of $\mathcal{B}_2$ contains a non-empty set which is open in the Euclidean topology. Hence, the set of rational points in Z is a countable dense subset of Z.

5. Completely regular.

Let $t \in Z$ and let K be a closed set with $t \notin K$. Then, either (i) $t \in H$ or (ii) $t \in U$.

(i) Suppose $t = (a,0) \in H$. Then, there exists $\varepsilon > 0$ such that $N^*\left((a,\varepsilon);\ \varepsilon\right) \subset Z - K$.

For each $r \in R$ with $r > 0$, let $C_r = \{z : z \in U$ and $d\left(z,\ (a,r)\right) = r\}$ (note $t \in C_r$ for each $r > 0$). Note that, if $z \in U$, then $z \in C_r$ for exactly one $r > 0$ (note that: $(x_0,y_0) \in C_r$ if and only if $(x_0,y_0) \in U$ and the center of the circle C_r is located at

$(a, \frac{y_\circ}{2} + \frac{1}{2y_\circ}[x_\circ - a]^2))$.

Define $f : (Z, \mathcal{T}_2) \to ([0,1], \mathcal{E}^1)$ (where $\mathcal{E}^1$ is Euclidean topology on $[0,1]$) by: if $z \in Z$, then

$$f(z) = \begin{cases} 0, & \text{if } z = t \\ 1, & \text{if } z \in H - \{t\} \\ \frac{1}{\varepsilon} r, & \text{if } z \in C_r \text{ and } r \le \varepsilon \\ 1, & \text{if } z \in C_r \text{ and } r > \varepsilon, \end{cases}$$

where ε is the one I have already chosen so that $N^*((a,\varepsilon); \varepsilon) \subset Z - K$.

Clearly, f is a function and $f(t) = 0$. It is not hard to see that $f[K] \subset \{1\}$. To verify that f is continuous, it suffices to show that if $0 < m < 1$, then $f^{-1}\left[[0,m)\right]$ and $f^{-1}[(m,1]]$ are open (wrt $\mathcal{T}_2$), since $f^{-1}[\cup \{A_\alpha : \alpha \in \Lambda\}] = \cup \{f^{-1}[A_\alpha] : \alpha \in \Lambda\}$ and $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$.

Let $A = \{z \in Z : z = t \text{ or } z \in C_r \text{ with } r < m\varepsilon\}$. Note, $A \in \mathcal{T}_2$ since $A = \cup \{N^*((a,r); r) : 0 < r < m\varepsilon\}$. Let $z \in A$. Then if $z = t$, $f(z) = 0$. Also, if $z \in C_r$ with $r < m\varepsilon$, then $f(z) = \frac{1}{\varepsilon} r < \frac{1}{\varepsilon}(m\varepsilon) = m$. Hence, $A \subset f^{-1}\left[[0,m)\right]$. Suppose $w \in Z - A$. Then, if $w \in C_r$ with $r \ge m\varepsilon$, $f(w) = \frac{1}{\varepsilon}(r) \ge \frac{1}{\varepsilon}(m\varepsilon) = m$ if $r \le \varepsilon$ or $f(z) = 1$ if $r > \varepsilon$. Also, if $w \in H - \{t\}$, then $f(w) = 1$. Hence, $Z - A \subset Z - f^{-1}\left[[0,m)\right]$ and therefore, $f^{-1}\left[[0,m)\right] = A \in \mathcal{T}_2$.

Let $B = \{z \in Z : z \in H - \{t\} \text{ or } z \in C_r \text{ with } r > m\varepsilon\}$. Note that $Z - \overline{N((a,m\varepsilon);m\varepsilon)}$ (where closure is taken with respect to $\mathcal{E}$) is an element of $\mathcal{E}$ and hence is an element of $\mathcal{T}_2$. Then, $B \in \mathcal{T}_2$ since

$B = Z - \overline{N((a,m\varepsilon);m\varepsilon)}$ (where closure is taken with respect to $\mathcal{E}$). Let $z \in B$. If $z \in H - \{t\}$, then $f(z) = 1$. Also, if $z \in C_r$ with $r > m\varepsilon$, then $f(z) = \frac{1}{\varepsilon} r > \frac{1}{\varepsilon}(m\varepsilon) = m$ if $r \leq \varepsilon$ or $f(z) = 1$, if $r > \varepsilon$. Hence, $B \subset f^{-1}\left[(m,1]\right]$. Suppose $w \in Z - B$. If $w = t$, then $f(w) = 0$. Also, if $w \in C_r$ with $r \leq m\varepsilon$, then $f(w) = \frac{1}{\varepsilon} r \leq \frac{1}{\varepsilon}(m\varepsilon) = m$. Thus $Z - B \subset Z - f^{-1}\left[(m,1]\right]$ and therefore, $f^{-1}\left[(m,1]\right] = B \in \mathcal{T}_2$.

(ii) Suppose $t = (a,b) \in U$. Recall that K is a closed set with $t \notin K$. There exists $\delta > 0$ such that $\delta < b$ and $N^*((a,b);\delta) \subset Z - K$.

Define $g : (Z,\mathcal{T}_2) \to ([0,1],\mathcal{E}^1)$ by: if $z \in Z$, then

$$g(z) = \begin{cases} \frac{1}{\delta} \cdot d(z,t), & \text{if } d(z,t) \leq \delta \\ 1, & \text{if } d(z,t) > \delta. \end{cases}$$

Clearly, g is a function, $g(t) = 0$, and $g[K] \subset \{1\}$. If $h : (Z,\mathcal{E}) \to ([0,1],\mathcal{E}^1)$ is defined by $h(z) = g(z)$ for each $z \in Z$, then h is easily seen to be continuous. But then g is continuous since $\mathcal{E} \subset \mathcal{T}_2$.

6. Connected.

Note that $\mathcal{T}_2/U$ is the Euclidean topology on the upper half plane. Since U is connected in $(U, \mathcal{T}_2/U)$, U is connected in $(Z,\mathcal{T}_2)$. Z is connected since $Z = \bar{U}$.

7. Locally connected.

It suffices to show that elements of $\mathcal{B}$ are connected sets. Let $B \in \mathcal{B}$. Then $(B \cap U)$ is connected in $(Z,\mathcal{T}_2)$ since $\mathcal{T}_2/(B \cap U)$ is the

Euclidean topology on $B \cap U$ and $B \cap U$ is connected in $(B \cap U, \mathcal{T}_2/B \cap U)$. It follows that B is connected in $(Z,\mathcal{T}_2)$ since $B \cap U \subset B \subset \overline{B \cap U}$.

8. Not locally compact.

Since $(Z,\mathcal{T}_2)$ is Hausdorff, it suffices to show that $\overline{N^*\big((0,2\varepsilon);2\varepsilon\big)}$ is not compact for each $\varepsilon > 0$.

Suppose that there exists $\varepsilon > 0$ such that $\bar{N}$ is compact where $N = N^*\big((0,2\varepsilon);2\varepsilon\big)$. Let $A_n = \{(x,y) \in Z : y > \frac{1}{n}\}$ for each $n \in P$. Note that $A_n \in \mathcal{E}$ for each $n \in P$ and hence $A_n \in \mathcal{T}_2$ for each $n \in P$. Let $\mathcal{C} = \{A_n : n \in P\} \cup \{N^*\big((0,\varepsilon);\varepsilon\big)\}$. It is clear that $\mathcal{C}$ is an open cover of $\bar{N}$ and that $\mathcal{C}$ has no finite subcover. Since this contradicts "$\bar{N}$ is compact," it follows that (0,0) has no compact neighborhood.

9. Not metacompact.

Let $\mathcal{C} = \{U\} \cup \{N^*\big((x,1);1\big) : (x,0) \in H\}$. Then, $\mathcal{C}$ is an open cover of Z. It suffices to prove that, whenever $\mathcal{F}$ is an open refinement of $\mathcal{C}$, $\mathcal{F}$ is not a point-finite system.

Let $\mathcal{F}$ be an open refinement of $\mathcal{C}$. For each $(x,0) \in H$, there exists $F(x) \in H$ such that $(x,0) \in F(x)$. Now, $N^*\big((x,\frac{1}{n});\frac{1}{n}\big) \subset F(x)$ for some $n \in P$ since $F(x)$ is an open set containing $(x,0)$. Let $(a,0)$, $(b,0) \in H$ with $a \neq b$. It follows that $(a,0) \notin N^*\big((b,1);1\big)$ and thus $(a,0) \notin F(b)$ since $F(b) \subset N^*\big((b,1);1\big)$.

For each $n \in P$, let $A_n = \{(x,0) : (x,0) \in H \text{ and } N^*\big((x,\frac{1}{n});\frac{1}{n}\big) \subset F(x)\}$. Note that $\{A_n : n \in P\}$ is an increasing sequence such that $H = \cup\{A_n : n \in P\}$. Hence, there exists $N \in P$ such that A_N is uncountably infinite. Consider H with the Euclidean topology. A_N has a limit point, say $t = (a,0)$, in H with respect to the Euclidean topology on H.

Then, there exists $B \subset A_N$ such that B is infinite and, if $(s,0) \in B$, then $d\big((s,0), (a,0)\big) < \frac{1}{N}$ where d is the Euclidean metric on H.

Now, consider the point $z \in Z$ and that $z = (a,\frac{1}{N})$. If $(s,0) \in B$, then $d\big((s,\frac{1}{N}), (a,\frac{1}{N})\big) = d\big((s,0), (a,0)\big) < \frac{1}{N}$. Therefore, $z = (a,\frac{1}{N}) \in N^*\big((s,\frac{1}{N}); \frac{1}{N}\big)$ for each $(s,0) \in B$ which implies that $z \in F(s)$ for each $(s,0) \in B$ (since $B \subset A_N$ and $(s,0) \in A_N$ implies that $N^*\big((s,\frac{1}{N}); \frac{1}{N}\big) \subset F(s)$). Since $F(s) \neq F(r)$ for each $(r,0)$, $(s,0) \in H$ with $r \neq s$ and since B is infinite, it follows that $\{F(s) : (s,0) \in B\}$ is an infinite subcollection of $\mathcal{F}$ with the property that $z \in F(s)$ for each $(s,0) \in B$. Therefore, $\mathcal{F}$ is not a point-finite system and $\mathcal{C}$ has no point-finite open refinement.

10. Not normal.

I have included here two proofs of the fact that $(Z,\mathcal{T}_2)$ is not a normal space. Of the proofs which appear in the literature, these are perhaps the least involved. The first proof utilizes Baire Category Theory and the proof was presented to me by Dr. William R. Smythe, Junior, of the Georgia Institute of Technology Mathematics faculty. The second proof makes use of an extension theorem for functions and the proof is suggested in [9], page 50, problem 3K.

For either proof, one must first observe that H is a closed subset of Z and that $\mathcal{T}_2/H$ is the discrete topology on H; i.e., $\{z\} \in \mathcal{T}_2/H$ for each $z \in H$.

(a) One of the Baire Category Theorems states: If (X,d) is a complete metric space and $\{D_i : i \in P\}$ is a countable collection of open subsets of X, each dense in X, then $\cap \{D_i : i \in P\}$ is dense in X (see [16], page 97). We will use the following alternate version of this theorem.

Suppose $\{E_n : n \in P\}$ is a countable collection of closed subsets of X and $\cup \{E_n : n \in P\} = X$. Then, there exists $N \in P$ such that E_N contains a nonempty open subset of (X,d). For, otherwise, $\{X - E_n : n \in P\}$ is a countable collection of open dense subsets of X whose intersection is empty; hence not dense in X.

Let $\mathcal{E}$ be the Euclidean topology on H with d as the Euclidean metric on H. Recall that (H,d) is a complete metric space.

Let $Q_\circ = \{(x,0) \in H : x \in Q\}$ where Q is the set of all rational numbers. Then $Q_\circ$ and $H - Q_\circ$ are disjoint closed subsets of H with respect to $\mathcal{T}_2/H$ and hence are closed subsets of Z (wrt $\mathcal{T}_2$).

It suffices to show that whenever V and W are open (wrt $\mathcal{T}_2$) subsets of Z such that $Q_\circ \subset V$ and $H - Q_\circ \subset W$, then $V \cap W \neq \phi$.

Since $Q_\circ$ is countable, let $Q_\circ = \{(r_n,0) : n \in P\}$. Note that $\{(r_n,0)\}$ is closed with respect to topology $\mathcal{E}$ on H for each $n \in P$. For each $n \in P$, let $F_{-n} = \{(r_n,0)\}$.

Now, since W is an open (wrt $\mathcal{T}_2$) set containing $H - Q_\circ$, for each $(t,0) \in H - Q_\circ$, there exists $\varepsilon > 0$ such that $N^*\big((t,\varepsilon);\varepsilon\big] \subset W$. Hence, if $E_n = \{(x,0) \in H - Q_\circ : N^*\big((x,\frac{1}{n}); \frac{1}{n}\big] \subset W\}$ for each $n \in P$, then $H - Q_\circ = \cup \{E_n : n \in P\}$. For each $n \in P$, let $F_n = cl(E_n)$ (wrt $\mathcal{E}$); i.e., F_n is the closure of E_n with respect to topology $\mathcal{E}$ on H.

It follows that $\{F_n : n \in P \text{ or } (-n) \in P\}$ is a countable collection of closed sets in the space $(H,\mathcal{E})$ and that $H = \cup \{F_n : n \in P \text{ or } (-n) \in P\}$. Thus, by my alternate Baire Category Theorem, there exists

$N \in P$ such that F_N contains a nonempty open (wrt E) set L, since the interior of F_{-n} is empty for each $n \in P$.

Let $(r,0) \in Q_o \cap L$. Since $Q_o \subset V$, there exists $\delta > 0$ such that $\delta < \frac{1}{N}$ and $N^*((r,\delta);\delta) \subset V$. Since $(r,\delta) \in V$, it suffices to show that $(r,\delta) \in W$. Now, since $H - Q_o$ is also dense in H (wrt E), there exists $(t,0) \in (H-Q_o) \cap L$ such that $0 < r - t < \delta$. But then, $d((t,\frac{1}{N}), (r,\delta)) \le d((t,\frac{1}{N}), (t,\delta)) + d((t,\delta), (r,\delta)) = (\frac{1}{N} - \delta) + (r-t) < \frac{1}{N} - \delta + \delta = \frac{1}{N}$. Hence, $(r,\delta) \in N^*((t,\frac{1}{N}); \frac{1}{N})$ and $N^*((t,\frac{1}{N}); \frac{1}{N}) \subset W$ (by choice of $(t,0) \in L$ and since $L \subset F_N$). Thus, $(r,\delta) \in V \cap W$ and $V \cap W \neq \phi$. Therefore, $(Z,\mathcal{T}_2)$ is not normal.

(b) We state without proof the following theorem from [7], page 68: if $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ are topological spaces with $(Y,\mathcal{S})$ a Hausdorff space, D is a dense subset of X, and f and g are continuous functions mapping $(X,\mathcal{T})$ into $(Y,\mathcal{S})$ such that $f(x) = g(x)$ for each $x \in D$, then $f(x) = g(x)$ for each $x \in X$.

Let $Q^2 = \{(x,y) \in Z : x,y \in Q \text{ and } y > 0\}$. Note that Q^2 is a dense subset of Z. Let $\mathcal{F}_1 = \{f | f : (Z,\mathcal{T}_2) \to ([0,1],\mathcal{E}) \text{ is continuous}\}$, where $\mathcal{E}$ is the Euclidean topology on $[0,1]$. Let $\mathcal{F}_2 = \{f | f : (Q^2,\mathcal{T}_2/Q^2) \to ([0,1],\mathcal{E}) \text{ is continuous}\}$. Define $F : \mathcal{F}_1 \to \mathcal{F}_2$ by $F(f)$ is the restriction of f to Q^2 for each $f \in \mathcal{F}_1$. Then, F is a function from $\mathcal{F}_1$ to $\mathcal{F}_2$. By the extension theorem which we have stated, F is also one-to-one. Hence, $|\mathcal{F}_1| \le |\mathcal{F}_2|$, where if A is a set, then $|A|$ is the cardinality of A. Let $\mathcal{F} = \{f : (Q^2,\mathcal{T}_2/Q^2) \to ([0,1],\mathcal{E})\}$. Clearly, $|\mathcal{F}_2| \le |\mathcal{F}|$ and hence, $|\mathcal{F}_1| \le |\mathcal{F}|$.

Recall that each subset of H is a closed set in $(Z,\mathcal{T}_2)$. If $(Z,\mathcal{T}_2)$ were a normal space, then corresponding to each $A \in \mathcal{P}(H)$ (where $\mathcal{P}(H)$ is the set of all subsets of H), there would exist $f_A \in \mathcal{F}_1$ such that $f_A[A] \subset \{0\}$ and $f_A[H-A] \subset \{1\}$. Let $A,B \in \mathcal{P}(H)$ with $A \neq B$. If $f_A, f_B \in \mathcal{F}_1$ were chosen as above, then necessarily $f_A \neq f_B$. Therefore, $|\mathcal{P}(H)| \leq |\mathcal{F}_1|$. Note that $|R| \lneq |\mathcal{P}(H)|$, where R is the set of real numbers, and thus $|R| \lneq |\mathcal{F}_1|$ and therefore, $|R| \lneq |\mathcal{F}|$ if $(Z,\mathcal{T}_2)$ is a normal space. Hence, to prove that $(Z,\mathcal{T}_2)$ is not a normal space, it suffices to prove that $|\mathcal{F}| \leq |R|$.

Identify with each $f \in \mathcal{F}$ a unique matrix $A(f)$ of the form $(a_{m,1})_{m \in P}$ in the following way. Since Q^2 is countable, write $Q^2 = \{r_n : n \in P\}$. Then for each $f \in \mathcal{F}$, let $A(f) = (a_{m,1})_{m \in P}$ be given by $a_{m,1} = f(r_m)$ for each $m \in P$ and for each $m \in P$, $a_{m,1}$ is to be written as a binary number with the following conventions: (i) the number "1" will be written ".111..."; (ii) if $n \in P$ and $f(r_n) \neq 1$, then $f(r_n)$ will be written as $.a_1 \, a_2 \, a_3 \, \ldots$ where $a_n \in \{0,1\}$ for each $n \in P$ and $a_n = 0$ for infinitely many indices $n \in P$ (note that this is always possible). The reader should note that each $f \in \mathcal{F}$ is assigned a unique matrix $A(f)$.

Let $f \in \mathcal{F}$. Then for each $m \in P$, the entry $f(r_m)$ in $A(f)$ is $.\,a_{m,1}a_{m,2}a_{m,3}\ldots$ where $a_{m,n} \in \{0,1\}$ for each $n \in P$ (see Figure 14-2). Define $\Phi : \mathcal{F} \to [0,1]$ by

$$\Phi(f) = .\, a_{1,1}a_{1,2}a_{2,1}a_{1,3}a_{2,2}a_{3,1}a_{1,4}a_{2,3}a_{3,2}a_{4,1}\ldots$$

where $f \in \mathcal{F}$ and f is written as above.

$$A(f) = \begin{bmatrix} .\ a_{1,1}a_{1,2}a_{1,3}a_{1,4}\cdots \\ .\ a_{2,1}a_{2,2}a_{2,3}a_{2,4}\cdots \\ .\ a_{3,1}a_{3,2}a_{3,3}a_{3,4}\cdots \\ .\ a_{4,1}a_{4,2}a_{4,3}a_{4,4}\cdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \end{bmatrix}$$

Figure 14-2

Since each $f \in F$ is associated with a unique matrix representation $A(f)$, it is clear from the definition of Φ that Φ is a function with domain F. Also, it is clear that the range of Φ is included in $[0,1]$. It is to be noted here that Φ actually maps elements of F into binary representations of numbers in $[0,1]$. Note that each number in $[0,1]$ has at most two binary representations. Hence, to prove that $|F| \leq |R|$, it suffices to prove that Φ is one-to-one.

Let $.\ a_1a_2a_3\ldots$ be a number in $[0,1]$ written in one of its binary representations. If $a_n = 1$ for each $n \in P$ and $f \in F$ such that $f(r_m) = .111\ldots$for each $m \in P$, then it is clear that $\Phi^{-1}[.a_1a_2a_3\ldots] = \{f\}$. If there exists $N \in P$ such that $a_N = 0$ and $a_n = 0$ for only finitely many $n \in P$, then $.a_1a_2a_3\ldots$ is not a binary representation that is mapped onto by Φ. To verify this statement, suppose, instead, that $g \in F$ and $\Phi(g) = .a_1a_2a_3\ldots$. Then there would exist on $M \in P$ such that $g(r_M) \neq .111\ldots$ and therefore $g(r_M)$ in its binary representation, according to the convention stated previously, must have infinitely many zeros to the right of the binary point. Clearly, this would make $.a_1a_2a_3\ldots$ have $a_n = 0$ for infinitely many $n \in P$ which is a contradiction. Finally, if $a_n = 0$ for infinitely many $n \in P$, then it is possible that Φ maps onto $\Phi.a_1a_2a_3\ldots$ but if $.a_1a_2a_3\ldots$ has another binary representation, then the other representation is not mapped onto by Φ, by the previous statements. Hence, each number in $[0,1]$ has at most one binary representation which is mapped onto by Φ. Therefore, if Φ does map onto $.a_1a_2a_3\ldots$ and $g \in F$ such that $A(g)$ is given in Figure 14-3, then it is clear that $\Phi^{-1}[.a_1a_2a_3\ldots] = \{g\}$. Thus, we have shown that each number in $[0,1]$ has at most one pre-image in F by the mapping Φ. Hence, Φ is one-to-one. Therefore, $|F| \leq |R|$ as required.

$$A(g) = \begin{bmatrix} .\ a_1a_2a_4a_7\ldots \\ .\ a_3a_5a_8a_{12}\ldots \\ .\ a_6a_9a_{13}a_{18}\ldots \\ .\ a_{10}a_{14}a_{19}a_{25}\ldots \\ \vdots\ \ \vdots\ \ \vdots\ \ \vdots\ \ \vdots\vdots\vdots \end{bmatrix}$$

Figure 14-3

In summary, we have shown that there are more pairs (K,L) of closed, disjoint subsets of Z than there are continuous functions from (Z,T_2) to $([0,1],E)$ which map K into $\{0\}$ and L into $\{1\}$. Hence, (Z,T_2) is not a normal space.

11. Not B.W. compact.

Note that H is an infinite, closed subset of $(Z, \mathcal{T}_2)$ and that $(H, \mathcal{T}_2/H)$ is the discrete topology.

CHAPTER IV

ORDER TOPOLOGIES

The definitions which follow are included here to prevent misunderstanding. In each definition, it is to be understood that S is a non-empty set and that "$<$" is a relation in $S \times S$. Also, if $(a,b) \in <$, then we shall write $a < b$.

Definitions

1. $(S,<)$ is a *partially ordered set* if "$<$" is transitive, reflexive, and antisymmetric.

2. $(S,<)$ is a *linearly ordered set* if $(S,<)$ is a partially ordered set and whenever $a,b \in S$, then $a < b$ or $b < a$.

3. $(S,<)$ is a *well-ordered set* if for each nonempty subset A of S, A contains a first element; i.e., there exists an $a \in A$ such that $a < x$ for each $x \in A$.

4. Let $(S,<)$ be a linearly ordered set. For each $a \in S$, let $R(a) = \{x \in S : a < x \text{ and } a \neq x\}$ and $L(a) = \{x \in S : x < a \text{ and } x \neq a\}$. Let $\mathcal{S} = \{R(a) : a \in S\} \cup \{L(a) : a \in S\}$. Then, the *order topology* $\mathcal{T}$ on $(S,<)$ is defined to be that topology on S which has $\mathcal{S}$ as a subbase. The base $\mathcal{B}$ for $\mathcal{T}$ which is generated by $\mathcal{S}$ is given by $\mathcal{B} = \{\phi\} \cup \{R(a) : a \in S\} \cup \{L(b) : b \in \mathcal{S}\} \cup \{R(a) \cap L(b) : a,b \in S\}$.

The theorem which follows and is proved in part is very useful in this chapter.

<u>Theorem</u>. Let S be an infinite set and let (S,<) be a linearly ordered set. Let $\mathcal{T}$ be the order topology on (S,<). Then:

(a) (S,$\mathcal{T}$) is Hausdorff.

(b) If (S,<) is a well-ordered set, then (S,$\mathcal{T}$) is a totally disconnected space.

(c) If S is compact, then (S,$\mathcal{T}$) is a completely normal space.

(d) S is compact if and only if for each set A contained in S, l.u.b. A exists as an element of S (recall l.u.b. A = a if and only if (i) $x < a$ for each $x \in A$ and (ii) $a < y$ for each y such that $x < y$ for each $x \in A$).

Proof.

(a) Let $a,b \in S$ with $a \neq b$. Then, $a < b$ or $b < a$. Without loss of generality, assume $a < b$. If there exists $c \in S$ such that $a < c < b$, $a \neq c$ and $b \neq c$, then $a \in L(c) \in \mathcal{T}$, $b \in R(c) \in \mathcal{T}$ and $R(c) \cap L(c) = \phi$. If there is no such element c, then $a \in L(b) \in \mathcal{T}$, $b \in R(a) \in \mathcal{T}$ and $L(b) \cap R(a) = \phi$.

(b) Let (S,<) be a well-ordered set. Suppose that A is a subset of S such that there exist $a,b \in A$ with $a \neq b$. It suffices to prove that A is not a connected set. Without loss of generality, we may assume that $a < b$.

Since (S,<) is a well-ordered set and $R(a) \neq \phi$, R(a) contains a first element, say c. Since $c \in R(a)$, $a < c$ and $c \neq a$. Now, R(a) and L(c) are nonempty, open, disjoint subsets of S such that $S = R(a) \cup L(c)$. Hence, R(a) and L(c) are also closed sets in (S,$\mathcal{T}$). Clearly, $\{R(a) \cap A, L(c) \cap A\}$ is a separation for A. Hence, all components of S are singleton sets.

(c) This conclusion of the theorem is stated in [17], page 108, and also in [7], page 160, where a hint for the proof is given. The proof of this part is omitted here.

(d) (i) Let S be compact and suppose there exists $A \subset S$ such that l.u.b. A does not exist as an element of S. Let $H = \{x \in S : x$ is an upper bound for $A\}$. Let $\mathcal{C} = \{R(x) : x \in H\} \cup \{L(a) : a \in A\}$ and note that $\mathcal{C}$ is a collection of open sets. Let $y \in S$. Then, either there exists an $a \in A$ such that $y \in L(a)$ or $y \in H$. If $y \in L(a)$ for some $a \in A$, then $y \in \cup\, \mathcal{C}$. Otherwise, since $y \neq$ l.u.b. A, there exists an $x \in H$ such that $x < y$ and $x \neq y$. But then $y \in R(x)$ and thus $y \in \cup\, \mathcal{C}$. Hence, $\mathcal{C}$ is an open cover of S.

Therefore, there exists a finite subcover $\mathcal{F}$ of $\mathcal{C}$. Clearly, $\mathcal{F} = \{R(x) : x \in H_o\} \cup \{L(a) : a \in A_o\}$ where H_o and A_o are each finite subsets of H and A, respectively. Notice that we have not shown that A has an upper bound.

Suppose there exists an upper bound for A in S. Then, since H_o is finite, let x_o be its first element, i.e., $x_o \in H_o$ and $x_o < x$ for each $x \in H_o$. Either $x_o \in A$ or $x_o \notin A$. If $x_o \in A$, then, since x_o is an upper bound for A, $x_o =$ l.u.b. A which is a contradiction. Hence, $x_o \notin A$. But since $x_o \in H$, $x_o \notin L(a)$ for each $a \in A_o$. Since $\mathcal{F}$ is a cover of S, $x_o \in R(x)$ for some $x \in H_o$. But this is impossible since $x_o < x$ for each $x \in H_o$. Hence, A cannot have an upper bound in S and thus $\mathcal{F} = \{L(a) : a \in A_o\}$.

Since A_o is finite, there exists an $a_o \in A_o$ such that $a < a_o$ for each $a \in A_o$. Note that this implies that $a_o \notin L(a)$ for each $a \in A_o$ and hence F is not a cover of S which contradicts the way in which F was chosen. Hence, it follows that each subset of S has a least upper bound in S.

(ii) Let S have the property that each subset of A has a least upper bound in S.

To prove that S is compact, we refer to a subbase theorem by Alexander, page 139 of [14], which states: "If $\mathcal{S}$ is a subbase for the topology of a space X such that every cover of X by members of $\mathcal{S}$ has a finite subcover, then X is compact." A proof of this theorem appears on that page also.

Let C be a cover of S by elements of $\mathcal{S}$. Then, $C = \{L(x) : x \in X\} \cup \{R(y) : y \in Y\}$ where X and Y are subsets of S.

Let x_o = l.u.b. X. Then $x < x_o$ for each $x \in X$ and hence $L(x) \subset L(x_o)$ for each $x \in X$. Note that $x_o \notin L(x_o)$. Hence, $x_o \in R(y_1)$ for some $y_1 \in Y$. Since $y_1 < x_o$ and $y_1 \neq x_o$, there exists $x_1 \in X$ such that $y_1 < x_1$ and $y_1 \neq x_1$, by definition of x_o = l.u.b. X.

Note that $y_1 < x_1$, $y_1 \neq x_1$, $x_1 \in X$, and $y_1 \in Y$ imply that $\{R(y_1), L(x_1)\}$ is a finite subcover of C. Hence, S is compact.

Example 15

The following topology may be found in [7], page 161.

The space X is $[0,1] \times [0,1]$; i.e., $X = \{(x,y) : 0 \le x \le 1$ and $0 \le y \le 1\}$. For $(x_1,y_1) \in X$ and $(x_2,y_2) \in X$, "<<" is defined by "$(x_1,y_1) << (x_2,y_2)$" if and only if either (i) $x_1 < x_2$ or (ii) $x_1 = x_2$ and $y_1 < y_2$. It is easily seen that "<<" is a linear ordering on X. Hence, let T be the order topology on X induced by "<<".

1. Compact.

By the theorem from the introduction to this chapter, it suffices to show that if $A \subset X$, then l.u.b. A exists as an element of X.

Hence, let $A \subset X$. Clearly, if $A = \phi$, then l.u.b. $A = (0,0)$. Suppose then that $A \neq \phi$.

Let $a =$ l.u.b. $\{x : (x,y) \in A$ for some $y \in [0,1]\}$, where the least upper bound is taken with respect to the usual order on $[0,1]$. Clearly, $0 \leq a \leq 1$. If $(a,y) \notin A$ for each $y \in [0,1]$, then let $b = 0$. Otherwise, let $b =$ l.u.b. $\{y : (a,y) \in A\}$ where the least upper bound is taken with respect to the usual order on $[0,1]$. We claim that l.u.b. $A = (a,b)$.

Suppose (a,b) is not an upper bound for A. Then, there exists $(x_o,y_o) \in A$ such that $(a,b) << (x_o,y_o)$ and $(a,b) \neq (x_o,y_o)$. Then, either $a < x_o$ or $a = x_o$ and $b < y_o$. If $a < x_o$, then $a \neq$ l.u.b. $\{x : (x,y) \in A$ for some $y \in [0,1]\}$ which is a contradiction. Hence, $a = x_o$ and $b < y_o$. But then $b \neq$ l.u.b. $\{y : (a,y) = (x_o,y) \in A\}$ which is a contradiction. Hence, (a,b) is an upper bound for A.

Suppose that (x_1,y_1) is an upper bound for A and that $(x_1,y_1) << (a,b)$ and $(x_1,y_1) \neq (a,b)$. Then, either $x_1 < a$ or $x_1 = a$ and $y_1 < b$. In the case that $x_1 < a$, it follows, by definition of a, that there exists $(x,y) \in A$ such that $x_1 < x \leq a$. But then $(x,y) \in A$ and $(x_1,y_1) << (x,y)$ with $(x_1,y_1) \neq (x,y)$ which contradicts the choice of (x_1,y_1). In the case that $x_1 = a$ and $y_1 < b$, it follows, by the definition of b, that there exists $y \in [0,1]$ such that $(x_1,y_1) << (a,y) = (x_1,y)$. But then $(x_1,y) \in A$ and $(x_1,y_1) << (x_1,y)$ with $(x_1,y_1) \neq (x_1,y)$

which contradicts the choice of (x_1,y_1). Hence, if (x_1,y_1) is an upper bound for A, then $(a,b) << (x_1,y_1)$. Hence, $(a,b) = \text{l.u.b. } A$.

2. Hausdorff and completely normal.

These properties follow from the theorem in the introduction to this chapter.

3. Not separable.

Let D be any countable subset of X. It suffices to show that D is not dense in X.

For each $a \in [0,1]$, define $X_a = \{(a,y) : 0 < y < 1\}$. Note that $X_a = R\big((a,0)\big) \cap L\big((a,1)\big) \in \mathcal{T}$ for each $a \in [0,1]$. Also, if $a,b \in [0,1]$ with $a \neq b$, then $X_a \cap X_b = \phi$. Hence, $\{X_a : a \in [0,1]\}$ is an uncountable collection of disjoint, nonempty, open subsets of X. Therefore, there exists $c \in [0,1]$ such that $D \cap X_c = \phi$. Thus, D is not dense in X.

4. First countable.

We state without proof that $\{L\big((0,\frac{1}{n})\big) : n \in P\}$ and $\{R\big((1,1-\frac{1}{n})\big) : n \in P\}$ are countable bases for $N\big((0,0)\big)$ and $N\big((1,1)\big)$, respectively.

Let $(a,b) \in X$ with $a \in (0,1)$. Then, $b \in (0,1)$ or $b \in \{0,1\}$. Now, there exists $N_a \in P$ such that $n \geq N_a$ implies that $a - \frac{1}{n} \in (0,1)$. Also, if $b \in (0,1)$, then there exists $N_b \in P$ such that $n \geq N_b$ implies that $b - \frac{1}{n} \in (0,1)$ and $b + \frac{1}{n} \in (0,1)$.

We claim that $\{R\big((a-\frac{1}{n}, 1)\big) \cap L\big((a,\frac{1}{n})\big) : n \in P, n \geq N_a\}$, $\{R\big((a,b-\frac{1}{n})\big) \cap L\big((a,b+\frac{1}{n})\big) : n \in P, n \geq N_b\}$, and $\{R\big((a,1-\frac{1}{n})\big) \cap L\big((a+\frac{1}{n}, 0)\big) : n \in P, n \geq N_a\}$ are countable bases for $N\big((a,0)\big)$, $N\big((a,b)\big)$ where $b \in (0,1)$, and $N\big((a,1)\big)$, respectively. We shall prove the claim for $N((a,0))$ and note that the proofs for $N((a,b))$ where $b \in (0,1)$ and $N((a,1))$ are not more complicated.

Let $(a,0) \in X$ with $a \in (0,1)$. Suppose that $(a,0) \in U$ and $U \in \mathcal{T}$. Then, there exists (s_1,s_2), $(t_1,t_2) \in X$ such that $(a,0) \in R((s_1,s_2)) \cap L((t_1,t_2)) \subset U$, since $\mathcal{B}$ is a base for $\mathcal{T}$ (see Definition 4 in the introduction to this chapter). It follows that $(s_1,s_2) \ll (a,0) \ll (t_1,t_2)$. Then, $s_1 < a$ and either $a < t_1$ or $a = t_1$ with $t_2 > 0$. In either case, $s_1 < a$ and there exists $t_3 \in (0,1)$ such that $(a,t_3) \ll (t_1,t_2)$.

Now, $s_1 < a$ implies there exists $N_1 \in P$ such that $N_1 \geq N_a$ and, if $n \geq N_1$, then $s_1 < a - \frac{1}{n}$. Also, since $t_3 \in (0,1)$, there exists $N_2 \in P$ such that $N_2 \geq N_1$ and, if $n \geq N_2$, then $\frac{1}{n} < t_3$. Hence, $(s_1,s_2) \ll (a - \frac{1}{N_2}, 1) \ll (a,0) \ll (a, \frac{1}{N_2}) \ll (a,t_3) \ll (t_1,t_2)$ and therefore $(a,0) \in R((a - \frac{1}{N_2}, 1)) \cap L((a, \frac{1}{N_2})) \subset U$ as required to verify that $\{R((a - \frac{1}{n}, 1)) \cap L((a, \frac{1}{n})) : n \in P, n \geq N_a\}$ is a countable base for $N((a,0))$.

5. *Proposition:* If $S \in \mathcal{S}$, then S is connected.

Proof. If $S \in \mathcal{S}$, then there exists $t \in X$ such that either $S = R(t)$ or $S = L(t)$. We shall show that, if $S = L(t)$, then S is connected and then note that the proof is similar in the case that $S = R(t)$.

Suppose $\{A,B\}$ is a separation for $L(t)$. Then, since $A \cap \bar{B} = \bar{A} \cap B = \phi$, A and B contain, respectively, their limit points which are also in $L(t)$ (since $A \cup B = L(t)$). We may assume (that $S \neq \phi$ and) that $(0,0) \in A$. Let $\alpha = \text{l.u.b. } A$ and note that $\alpha \leq t$. Suppose $\alpha < t$. Then, since α is clearly a limit point of A with $\alpha < t$, it follows that $\alpha \in A$. But then α would be a limit point of B and this would imply that $\alpha \in B$

which is a contradiction. Therefore, it must be that $\alpha = t$. Let β = l.u.b. $\{p : p \in A$ and $p \ll b$ for each $b \in B\}$. Note that $\beta \in A$. But, since β is also a limit point of B, $\beta \in B$ which is a contradiction. Hence, L(t) cannot have a separation {A,B}. Therefore, S = L(t) is connected.

6. Connected.

Note that $L\big((1,1)\big)$ and $R\big((0,0)\big)$ are each connected sets, $L\big((1,1)\big) \cap R\big((0,0)\big) \neq \phi$ and $X = L\big((1,1)\big) \cup R\big((0,0)\big)$. Hence, X is connected.

7. Locally connected.

Recall, from Definition 4 in the introduction to this chapter, the definition of the base $\mathcal{B}$ for $\mathcal{T}$. It is clear by (5), page 78, that $B \in \mathcal{B}$ implies that B is connected. It follows that components of open sets are open.

8. (The following propositions are useful in proving that $(X,\mathcal{T})$ is not a perfectly normal space.)

Proposition (a): Suppose $U \in \mathcal{T}$ and $(x,1) \in U$ for some $x \in [0,1]$. Then, there exists $(u,v) \subset R$ (i.e., an interval on the real line) such that $u \leq x < v$ and $\phi \neq \{(x,y) : (x,y) \in (u,v)$ and $y \in [0,1]\} \subset U$.

Proof. There exist (s_1,s_2), $(t_1,t_2) \in X$ such that $(x,1) \in R\big((s_1,s_2)\big) \cap L\big((t_1,t_2)\big) \subset U$. Note that $s_1 \leq x < t_1$. Suppose that $(a,b) \in X$ and $s_1 < a < t_1$. Then, $(a,b) \in R\big((s_1,s_2)\big) \cap L\big((t_1,t_2)\big)$. Hence, if $u = s_1$ and $v = t_1$, then $u \leq x < v$ and $\phi \neq \{(x,y) : x \in (u,v)$ and $y \in [0,1]\} \subset U$.

Proposition (b): Suppose $U \in \mathcal{T}$ and $(x,0) \in U$ for some $x \in [0,1]$. Then, there exists $(u,v) \subset R$ such that $u < x \leq v$ and $\phi \neq \{(x,y) : x \in (u,v) \text{ and } y \in [0,1]\} \subset U$.

Proof. There exist (s_1,s_2), $(t_1,t_2) \in X$ such that $(x,0) \in R((s_1,s_2)) \cap L((t_1,t_2)) \subset U$. Note that $s_1 < x \leq t_1$. Suppose that $(a,b) \in X$ and $s_1 < a < t_1$. Then, $(a,b) \in R((s_1,s_2)) \cap L((t_1,t_2))$. Hence, if $u = s_1$ and $v = t_1$, then $u < x \leq v$ and $\phi \neq \{(x,y) : x \in (u,v) \text{ and } y \in [0,1]\} \subset U$.

Proposition (c): Let $K = \{(x,y) : x \in [0,1] \text{ and } y \in \{0,1\}\}$. Let $U \in \mathcal{T}$ such that $K \subset U$. If $B = \{x : (x,y) \in U \text{ for each } y \in [0,1]\}$, then B is an open dense subset of $([0,1], \mathcal{E})$ where $\mathcal{E}$ is the Euclidean topology on $[0,1]$.

Proof. Let $x \in [0,1]$. Now, $(x,0) \in K$ and $(x,1) \in K$. Hence, $(x,0) \in U$ and $(x,1) \in U$. By Propositions (a) and (b) above, there exist $u,v \in R$ such that $u < x < v$, $(u,x) \subset B$, and $(x,v) \subset B$. If $x \in B$, then $x \in (u,v) \subset B$ and, therefore, B is open in $([0,1], \mathcal{E})$. If $x \notin B$, then it is clear that each open set in $([0,1], \mathcal{E})$ containing x intersects B and, therefore, B is dense in $([0,1], \mathcal{E})$.

9. Not perfectly normal.

Let $x \in [0,1]$. Note that $\phi \neq R((x,0)) \cap L((x,1)) \in \mathcal{T}$. Hence, if $V = \cup\{R((x,0)) \cap L((x,1)) : x \in [0,1]\}$, then $V \in \mathcal{T}$. Note that $X - V = K$ where K is defined in (8(c)) above. Hence, K is closed.

Let $\{U_n : n \in P\}$ be a countable collection of open subsets of $(X,\mathcal{T})$ such that $K \subset U_n$ for each $n \in P$. For each $n \in P$, let $B_n = \{x : (x,y) \in U_n \text{ for each } y \in [0,1]\}$. Then, by (8(c)) above, B_n is an open dense subset of $([0,1], \mathcal{E})$ for each $n \in P$. By the Baire Category Theorem on page 65, $\cap \{B_n : n \in P\}$ is dense in $([0,1], \mathcal{E})$. In particular, there exists $b \in \cap \{B_n : n \in P\}$. It follows that $\{(b,y) : y \in [0,1]\} \subset U_n$ for each $n \in P$. Hence, $\phi \neq \{(b,y) : y \in [0,1]\} \subset \cap \{U_n : n \in P\}$. Therefore, $K \subsetneq \cap \{U_n : n \in P\}$. Hence, K is a closed subset of $(X,\mathcal{T})$ such that K is not the intersection of a countable collection of open subsets of $(X,\mathcal{T})$.

10. Not metric.

Every metric space is perfectly normal.

<u>Example 16</u>

The following topology may be found in [10], page 17. The space of Example 17 is a subspace of the space of this example.

The construction of this space depends on the Well-Ordering Theorem which is also known as Zermelo's Theorem (see [10], page 17 or [11], page 52-53). The Well-Ordering Theorem states that if X is a set, then there exists a well-ordering for X.

By the Well-Ordering Theorem, there exists a well-ordering "$<_o$" for the set $R - P$, where R is the set of real numbers. Now, let $\alpha = R$ and extend "$<_o$" to another ordering "$<$" by agreeing that: (a) $x < y$ for each $x,y \in R - P$ where $x <_o y$, (b) $n < n + 1$ for each $n \in P$, (c) $n < x$ whenever $n \in P$ and $x \in R - P$, and (d) $x < \alpha$ whenever $x \in R$. The reader should have no difficulty in verifying that "$<$" is a well-ordering for $R \cup \{\alpha\}$.

Note that α has uncountably many predecessors (with respect to the ordering "<"). Hence, there exists an $\Omega \in R \cup \{\alpha\}$ such that Ω is the first element of $R \cup \{\alpha\}$ which has uncountably many predecessors.

If $\mathcal{Q}$ is the set of predecessors of Ω in $R \cup \{\alpha\}$, then the space $\mathcal{Q}^*$ for this example is given by $\mathcal{Q}^* = \mathcal{Q} \cup \{\Omega\}$. Let $\mathcal{W}^*$ be the order topology on $\mathcal{Q}^*$.

Let A be a nonempty subset of $\mathcal{Q}^*$. Since $(\mathcal{Q}^*,<)$ is a well-ordered set, A has a first element, say a. We shall use the notation a = f.e.o. A, to be read "a is the first element of A."

1. Compact.

By the theorem in the introduction to this chapter, it suffices to prove that if $A \subset \mathcal{Q}^*$, then l.u.b. A exists as an element of $\mathcal{Q}^*$. Let $A \subset \mathcal{Q}^*$. Clearly, if $\Omega \in A$, then l.u.b. $A = \Omega$. Otherwise, Ω is an upper bound for A and l.u.b. A = f.e.o. $\{x \in \mathcal{Q}^* : a < x$ for each $x \in A\}$. Hence, l.u.b. A exists as an element of $\mathcal{Q}^*$.

2. T_2, completely normal, and totally disconnected.

These properties follow from the theorem in the introduction to this chapter.

3. *Proposition:* Let A be a countable subset of $\mathcal{Q}$. Then, l.u.b. $A \in \mathcal{Q}$.

Proof. Note that, by definition of Ω, each element of $\mathcal{Q}$ has at most countably many predecessors. Let $A_1 = \{x \in \mathcal{Q} : x < a$ for some $a \in A\}$. Then, A_1 is countable. Note that l.u.b. A_1 = l.u.b. A and that $a < \Omega$ for each $a \in A_1$. By definition, Ω has uncountably many predecessors. Therefore, if t = f.e.o. $\{y \in \mathcal{Q}^* : a < y$ or $a = y$, for each $a \in A_1\}$, then $t < \Omega$, i.e., $t \in \mathcal{Q}$. Because "<" is a well-ordering, $a < t$ for each $a \in A_1$.

If $t \neq$ l.u.b. A, there exists $s \in \mathcal{Q}$ such that $s < t$ and $a \leq s$ for each $a \in A$. But then for each $a \in A_1$, $a \leq s$. This contradicts the definition of t as the first element of a certain set. Hence, $t =$ l.u.b. A and $t \in \mathcal{Q}$.

4. Not first countable.

In particular, there is no countable base for $\mathcal{N}(\Omega)$.

Suppose that $\{B_n : n \in P\}$ is a countable subcollection of $\mathcal{N}(\Omega)$. Recall, from the introduction to this chapter, the base $\mathcal{B}^*$ for $\mathcal{W}^*$. Note that there exists $\{b_n : n \in P\} \subset \mathcal{Q}$ such that $R(b_n) \subset B_n$ for each $n \in P$. By (3), page 82, if $b =$ l.u.b. $\{b_n : n \in P\}$, then $b \in \mathcal{Q}$. It follows that there exists $d \in \mathcal{Q}$ such that $b < d$. Note that $R(d) \in \mathcal{N}(\Omega)$ but that B_n is not a subset of $R(d)$ for each $n \in P$. Hence, $\{B_n : n \in P\}$ is not a base for $\mathcal{N}(\Omega)$. Therefore, $\mathcal{N}(\Omega)$ has no countable base.

5. Not separable.

Let D be a countable subset of $\mathcal{Q}^*$. It suffices to show that D is not dense in $\mathcal{Q}^*$.

Note that $D - \{\Omega\}$ is a countable subset of $\mathcal{Q}$. Hence, if $b =$ l.u.b. $(D - \{\Omega\})$, then $b \in \mathcal{Q}$. Clearly, $\phi \neq R(b) \cap L(\Omega)$ and $D \cap [R(b) \cap L(\Omega)] = \phi$.

6. Not perfectly normal.

Since $(\mathcal{Q}^*, \mathcal{W}^*)$ is Hausdorff, $\{\Omega\}$ is a closed set. In (4), page 83, suppose $\{B_n : n \in P\}$ is also a subset of $\mathcal{B}$. Then, $\{\Omega\} \subsetneq \cap \{B_n : n \in P\}$. It follows that $\{\Omega\}$ is not the intersection of a countable collection of open subsets of $\mathcal{Q}^*$.

7. Not metric.

All metric spaces are perfectly normal.

8. Not locally connected.

Since $(\mathcal{Q}^*,\mathcal{W}^*)$ is totally disconnected, it suffices to show there exists a singleton set which is not open. Clearly, $\{\Omega\}$ is not open.

9. Sequentially compact.

This follows from the proof of (3) on page 82.

Example 17

The space for this topology is $(\mathcal{Q},\mathcal{W})$ where $\mathcal{Q}$ is described in the previous example and $\mathcal{W} = \mathcal{W}^*/\mathcal{Q}$. It is trivial to show that the order topology generated by "<" on $\mathcal{Q}$ is $\mathcal{W}^*/\mathcal{Q}$.

1. T_2, completely normal, and totally disconnected.

These properties are inherited from $(\mathcal{Q}^*,\mathcal{W}^*)$.

2. Not Lindelöf.

Let $\mathcal{C} = \{L(x) : x \in \mathcal{Q}\}$. Then, $\mathcal{C}$ is an open cover of $\mathcal{Q}$ since l.u.b. $\mathcal{Q} = \Omega$. Note that $L(x)$ is a countable set for each $x \in \mathcal{Q}$. Suppose that $\mathcal{D}$ is a countable subcollection of $\mathcal{C}$. Then, since $\cup\,\mathcal{D}$ is a countable set, $\cup\,\mathcal{D} \subsetneq \mathcal{Q}$. Therefore, $\mathcal{C}$ has no countable subcover.

3. Not separable.

This follows from (5) of Example 16.

4. First countable.

Note that $\{L(2)\}$ is a countable base for $\mathcal{N}(1)$. Let $\{x\} \in \mathcal{Q} - \{1\}$. Since x has only countably many predecessors in $\mathcal{Q}$, let $\{x_n : n \in T\}$ for some $T \subset P$ be a list of the predecessors of x. Let y = f.e.o. $\{z \in \mathcal{Q}: x < z\}$. It is easily seen that $\{R(x_n) \cap L(y) : n \in P\}$ is a countable base for $N(x)$.

5. B.W. compact.

Let A_o be an infinite subset of $\mathcal{Q}$. Let A be a countably infinite subset of A_o. Then l.u.b. $A \in \mathcal{Q}$.

Let b = f.e.o. $\{q \in \mathcal{Q}$: there exists an infinite subset A(q) of A such that $a < q$ or $a = q$, for each $a \in A(q)\}$. Suppose b is not a limit point of A. Then, there exist $c,d \in \mathcal{Q}$ such that $b \in R(c) \cap L(d)$ and $[R(c) \cap L(d)] \cap [A - \{b\}] = \phi$. But then, A(b) is an infinite collection of predecessors of c in A. This contradicts the definition of b. Thus, b is a limit point of A and hence b is a limit point of A_o.

6. Locally compact.

Recall $(\mathcal{Q}^*, \mathcal{W}^*)$ is locally compact and regular and that $\mathcal{Q}$ is an open subset of $\mathcal{Q}^*$. By Theorem 0-1, $(\mathcal{Q}, \mathcal{W})$ is locally compact.

7. Not metacompact.

Any T_1-space which is B.W. compact and metacompact is compact (see Theorem 0-4).

8. Not metric.

All metric spaces are paracompact (see Theorem 0-6(d)).

9. Not locally connected.

Since $(\mathcal{Q}, \mathcal{W})$ is totally disconnected, it suffices to prove that not all singleton sets are open. If all singleton sets were open, then $(\mathcal{Q}, \mathcal{W})$ would clearly have to be metacompact. Hence, there exists $a \in \mathcal{Q}$ such that $\{a\}$ is not open.

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