BOUNDS FOR CAPACITANCE BETWEEN CIRCULAR AND SQUARE COAXIAL CONDUCTORS

A THESIS

Presented to

the Faculty of the Graduate Division

by

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

June 1961

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AND SQUARE COAXIAL CONDUCTORS

Approved:

Date Approved: May 11, 1961

ACKNOWLEDGEMENT

I wish to thank my thesis advisor, Dr. M. B. Sledd, for suggesting this topic and for his guidance during the course of this study.

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SUMMARY

The capacitance per unit length between a circular cylindrical conductor and a coaxial square conductor interior to it is given by

$$C = \frac{1}{4\pi(V_c - V_s)^2} \iint_R (\Phi_x^2 + \Phi_y^2)\, dx\, dy ,$$

where R is the annular region between the conductors and Φ is a harmonic function ($\Phi_{xx} + \Phi_{yy} = 0$) satisfying the prescribed boundary conditions ($\Phi = V_c$ on the circle, $\Phi = V_s$ on the square). The problem considered is to determine upper and lower bounds on the quadratic functional representing C.

To avoid wordiness, the capacitance per unit length between two cylindrical conductors of given cross sections is referred to simply in terms of the plane curves representing the cross sections.

It is first proved that the circle circumscribed about the inner square, when taken with the circle representing the outer conductor, yields an upper bound on the capacitance and, similarly, that the circle inscribed in the inner square, when taken with the outer circle, yields a lower bound. The difference between these bounds (and hence the uncertainty with which their arithmetic average approximtes C) is large if the length of the diagonal of the square is greater than one-fourth the diameter of the outer circle. In this case the method described in the following paragraph substantially reduces the difference between the computed bounds.

By extending the diagonals of the square till they intersect the outer circle, the region R is divided into four congruent subregions, one of which is mapped by $f(z) = z^4$ into a region between a circle A and a tear-shaped closed curve B interior to A. The smallest possible circle is circumscribed about B, and the largest possible circle is inscribed in B. It is then proved that these circles, when taken with A, yield the desired improved upper and lower bounds on the capacitance C/4 of the original subregion. An extension of the procedure gives even better bounds if they are desired.

This method may be used on any capacitor formed by two concentric cylindrical conductors, providing one is a circle and the other can be described as follows: it is a simple, closed, piecewise smooth curve consisting of n identical arc segments such that the terminal points of each segment have equal moduli, these moduli being greater than that of any point on the arc segment other than a terminal point. There are only two changes necessary: the original transformation must now be $f(z) = z^n$, and the region on which f(z) is defined must be one of the n congruent subregions into which the configuration is divided by radial lines drawn from the common axis through the terminal points of the n similar arc segments.

Finally it is shown that the approximate solution of the capacitance problem treated here is directly applicable to analogous problems in the fields of heat transfer, fluid flow, and neutron diffusion.

CHAPTER I

THE PROBLEM AND METHOD OF SOLUTION

Two coaxial cylindrical conductors and the annulus between them form a type of capacitor. The capacitance per unit axial length is defined to be the charge per unit length divided by the potential difference between the conductors when that charge is present. The determination of upper and lower bounds on the capacitance of such a system with a square inner conductor and circular outer conductor is the subject of this thesis.

The problem.--It seems intuitively obvious that an upper bound on the capacitance is obtained by replacing the inner conductor by its circumscribed circular cylinder. The truth of this statement is proved in the Appendix (Lemma 2). A lower bound is similarly obtained by replacing the inner conductor by its inscribed circular cylinder. The arithmetic average of these bounds is a good approximation to the capacitance when the ratio μ of the diagonal of the square to the diameter of the circle is small. As μ increases, however, the reliability of the approximation quickly deteriorates because the bounds which are averaged become progressively farther apart. Thus the crux of the problem is to reduce the difference between the bounds when μ is large ($1/4 < \mu < 1$).

The method of solution.--Let Γ_1 and Γ_2 be piecewise smooth, simple, closed curves with no point in common and with Γ_1 interior to Γ_2. These curves may be thought of as cross sections normal to the axes of two long cylindrical conductors. If end effects are neglected, the capacitance per unit length of such a configuration is given by the expression [1]

$$C = \frac{1}{4\pi(V_c - V_s)^2} \iint_R (\Phi_x^2 + \Phi_y^2)\, dx\, dy \,, \tag{1}$$

where R is the annular region between the conductors and Φ is a harmonic function ($\Phi_{xx} + \Phi_{yy} = 0$) satisfying the prescribed boundary conditions ($\Phi = V_c$ on Γ_2, $\Phi = V_s$ on Γ_1). In the special case where Γ_1 and Γ_2 are concentric circles of radii a and b ($a < b$), the expression for C reduces to [2]

$$C = \frac{1}{2 \ln(b/a)} \quad . \tag{2}$$

Now consider a capacitor where the inner conductor is square and the outer conductor is circular. As mentioned in the previous section, upper and lower bounds for the capacitance per unit length can be obtained by replacing the square first by its circumscribed circle and then by its inscribed circle. Formula (2) provides a way to calculate these bounds.

When μ is large, the area between the square and its circumscribed circle is appreciable compared to the area between the square and the outer conductor; a similar comment applies to the area between the square and its inscribed circle. This circumstance suggests the possibility of obtaining better bounds by replacing the circumscribed and inscribed circles by other suitably chosen curves which, while lying respectively outside and inside the square, approximate the square more closely than the circles do. Such curves may be obtained as explained in the following paragraphs.

Figure 1 is a representation in the z-plane of one-half of a cross section of the actual capacitor. The diagonals of the square representing the inner conductor have been extended until they intersect the circle representing the outer conductor. The extended diagonals thus divide the annular region between the conductors into four congruent subregions, one of which is labeled S. The boundaries of S are L_1, a segment of the outer circle; L_2 and L_4, segments of extended diagonals; and L_3, one side of the square.

Under the mapping $w = z^4$, where $z = x + iy$ and $w = u + iv$, S is mapped onto S' and L_1, L_2, L_3 onto L_1', L_2', L_3'. L_3' is the tear-shaped closed curve in Figure 2, and L_2' is the segment of the u-axis joining the cusp of L_3' with L_1'. The mapping and its inverse are one-to-one and analytic, provided the domain of definition in the z-plane is $S + L_1 + L_2 + L_3$ and the domain of definition in the w-plane is $S' + L_1' + L_2' + L_3'$ where L_2' is considered as a cut. The following facts should be noted:

1. the cusp of L_3' is the image of the point (a,a) in the z-plane, where a is the half-side of the square, and the distance of the cusp from the origin is $4a^4$;

2. the point of intersection of L_3' with the positive u-axis is the image of the point (a,0) in the z-plane, and the distance of this intersection from the origin is a^4.

Thus the extremities of L_3' on the u-axis are unequally distant from the origin. This geometrical fact motivated the construction of the curves γ_1' and γ_2' shown in Figure 4.

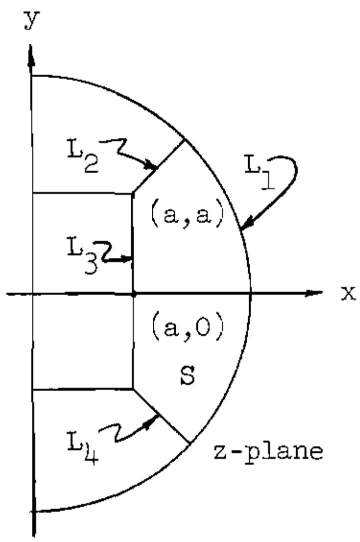


Figure 1.
Actual Configuration

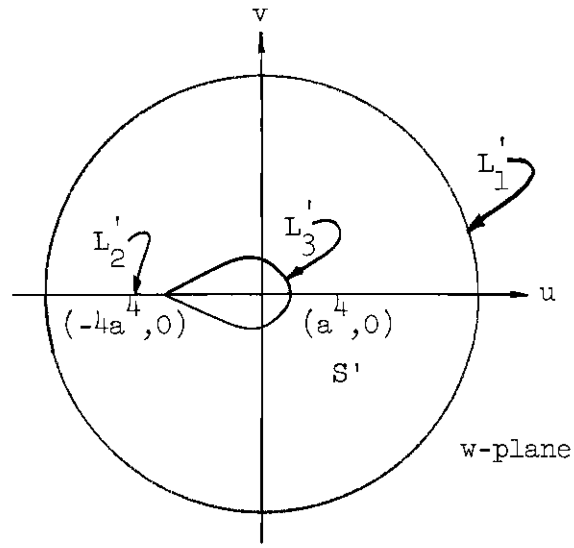


Figure 2.
Image Under z^4 of Actual Configuration

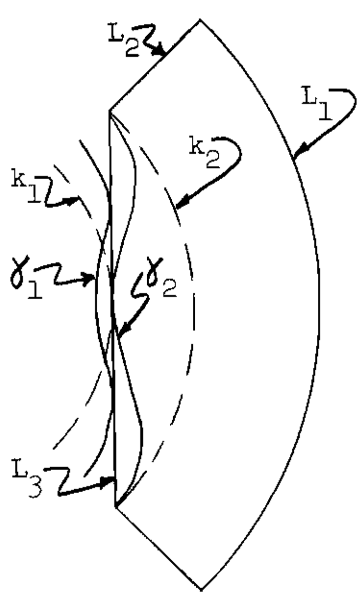


Figure 3.
Bounding Curves in
the z-plane

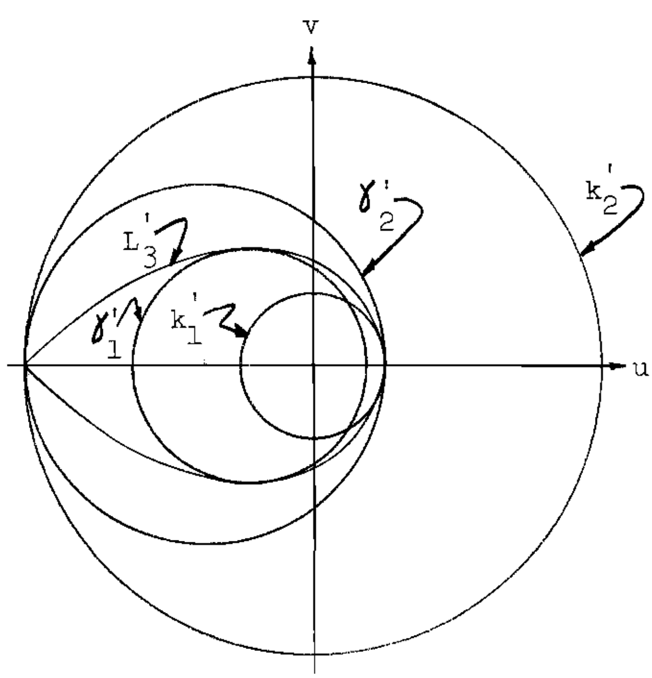


Figure 4.
Bounding Curves in the w-plane

In that figure, k_1' is the image of the circle k_1 inscribed in the square in the z-plane, and k_2' is the image of the circle k_2 circumscribed about the square in the z-plane. γ_1' is the largest circle which can be inscribed in L_3', and γ_2' is the smallest circle which can be circumscribed about it. A critical examination of the methods of constructing γ_1' and γ_2' is included in the next chapter.

Since $|w_1^{1/4}| \leq |w_2^{1/4}|$ implies $|z_1| \leq |z_2|$, where $z_1 = w_1^{1/4}$ and $z_2 = w_2^{1/4}$, the curve γ_2', lying between L_3' and k_2' in the w-plane, will have its inverse image γ_2 entirely between L_3 and k_2 in the z-plane, as shown in Figure 3. Then, by Lemma 2, the value of the integral (1) over the region bounded by γ_2, L_1, L_2, and L_4 is less than the value of this integral over the region bounded by k_2, L_1, L_2, and L_4. In Figure 3 it can be seen that γ_1 more closely approximates L_3 than k_1 does so that presumably γ_1 will yield a greater lower bound than k_1. This is later substantiated numerically.

By Lemma 4 it is known that there exists a one-to-one analytic mapping, $p = L(w)$, which takes the circles γ_2' and L_1' into concentric circles in the p-plane. γ_1' and L_1' can be similarly mapped into concentric circles in the p-plane. By applying Formula (2) to the two systems of concentric circles in the p-plane, new upper and lower bounds on the capacitance of the actual system can be found, since it is shown in the next section that under an analytic one-to-one transformation the integral (1) remains invariant. As will be seen subsequently, the new bounds are a substantial improvement over those obtained by using the inscribed and circumscribed circles.

CHAPTER II

MATHEMATICAL JUSTIFICATION OF THE METHOD

It will be shown that

1. the inverse image of γ_2' under $z = w^{1/4}$ lies exterior to L_3 in the z-plane;

2. the inverse image of γ_1' under $z = w^{1/4}$ lies interior to L_3 in the z-plane;

3. there exists a linear transformation which takes γ_2' and L_1' (γ_1' and L_1') into two concentric circles in the p-plane;

4. the value of the integral (1) remains invariant under the analytic one-to-one mappings considered.

To verify 1. it will be proved that γ_2' lies exterior to L_3' and therefore, since the mapping $w = z^{1/4}$ maintains modular inequalities, γ_2 lies exterior to L_3.

Let $z = re^{i\theta}$ and $w = se^{i\alpha}$. The equation of L_3 is $r = a \sec\theta$, where $-\pi/4 < \theta \leq \pi/4$, and therefore the equation of L_3' is, since $\alpha = 4\theta$,

$$w = r^4 e^{4i\theta} = a^4 \sec^4(\alpha/4)\left[\cos\alpha + i\sin\alpha\right].$$

Suppose, as will be verified later, that the equation of γ_2' is

$$|w + 1.5a^4| = 2.5a^4. \tag{3}$$

L_3' and γ_2' will intersect when

$$\left| a^4 \sec^4(\alpha/4)\,[\cos\alpha + i\sin\alpha] + 1.5a^4 \right| = 2.5a^4.$$

Simplification of this equation and use of the relation

$$\cos\alpha = 8\cos^4(\alpha/4) - 8\cos^2(\alpha/4) + 1$$

yield

$$[\cos^2(\alpha/4) - 1][\cos^2(\alpha/4) - 1/2][10\cos^4(\alpha/4) + 3\cos^2(\alpha/4) + 1] = 0.$$

The only real solutions of this equation are $\alpha = 0, \pm 4\pi, \pm 8\pi, \ldots$ and $\alpha = \pm\pi, +3\pi, \ldots$. Since $-\pi < \alpha \leq \pi$, L_3' and γ_2' intersect only when $\alpha = 0$ and $\alpha = \pi$. When $\alpha = \pi/2$ the moduli of L_3' and γ_2' are $8a^4/(3 + 2\sqrt{2})$ and $2a^4$. Thus, since $2 > 8/(3 + 2\sqrt{2})$, γ_2' is exterior to L_3' at $\alpha = \pi/2$. By continuity and symmetry, γ_2' is exterior to L_3' everywhere in the w-plane, and 1. is proved.

To prove 2. it is first necessary to construct γ_1'. The function which gives the perpendicular distance of a point on L_3' from the u-axis is $D(\alpha) = \left| a^4 \sec^4(\alpha/4) \sin\alpha \right|$. By maximizing $D(\alpha)$ it is found that the points on L_3' which are furthest from the u-axis have coordinates $[a^4\sec^4(\pi/6)\cos(2\pi/3), \pm a^4\sec^4(\pi/6)\sin(2\pi/3)]$. The common abcissa of these points determines the center of γ_1', and the ordinate of either point determines the radius. Hence the equation of γ_1' is

$$\left| w - a^4\sec^4(\pi/6)\cos(2\pi/3) \right| = a^4\sec^4(\pi/6)\sin(2\pi/3), \qquad (4)$$

and the same technique used to prove 1. also proves 2.

Before proving 3. it is necessary to define what is meant by symmetric points with respect to a circle whose center is on the real axis.

Definition.--Let F be a circle with center at $(g,0)$ and radius h. Let $(m,0)$ be any point on the real axis with $m \neq g$. The point $(p,0)$ is said to be the symmetric point of $(m,0)$ with respect to F if, and only if, $(m - g)(p - g) = h^2$.

Lemma 3 guarantees that there is one, and only one, pair of points which is symmetric with respect to both circles of an eccentric pair of circles, one of which is entirely inside the other. Lemma 4 exhibits the transformation which takes this eccentric pair into two concentric circles. This proves 3.

Repeated application of Lemma 5 proves 4.

CHAPTER III

SAMPLE CALCULATIONS AND RESULTS

Sample calculations.--A numerical example will be worked out to illustrate the above method. Suppose L_1 is the circle with center at (0,0) and radius one, and L_3 is the square with half-side one-third centered at (0,0). Let the region S be that shown in Figure 1. $w = z^4$ maps L_1 onto the unit circle L_1', and L_3 onto L_3', where L_3' intersects the u-axis at (-4/81,0) and (1/81,0). The equation of γ_2' is, from equation (3),

$$\left| w + 1.5/81 \right| = 2.5/81 \;;$$

and the equation of γ_1' is, from equation (4),

$$\left| w + 8/729 \right| = 8\sqrt{3}/729 \;.$$

Let the symmetric points of L_1' and γ_2' be (s,0) and (t,0). These points must satisfy the two relationships

$$(s + 1.5/81)(t + 1.5/81) = (2.5/81)^2 \;;$$

$$s\,t = 1 \;.$$

The solution of this system is s = -0.0185 and t = -53.9485. The linear transformation which takes L_1' and γ_2' into concentric circles in the p-plane is, from Lemma 4,

$$p = \frac{(w + 0.0185)(-0.0185 - 1)}{(w + 53.9485)(1 + 53.9485)} \;.$$

From the second part of Lemma 4, the ratio of the radii in the p-plane is

$$\frac{2.5}{81}\left|\frac{-53.9485}{-53.9485 + (1.5/81)}\right| = 0.0309 \; ;$$

and by equation (2) the upper bound capacitance of the L_1 - γ_2 capacitor is

$$C = \frac{1}{2\ \ln(b/a)} = 0.143 \; .$$

This result, when multiplied by four, yields the upper bound 0.572 for the capacitance per unit length of the actual capacitor. The same procedure yields, for the L_1 - γ_1 system, the lower bound C = 0.505.

The upper bound for the L_1 - k_2 system is, from equation (2),

$$C = \frac{1}{2\ \ln(3/\sqrt{2})} = 0.670 \; ,$$

and, similarly, for the L_1 - k_1 system C = 0.455.

In summary,

upper bound calculated using the circumscribed circle k_2 = 0.670;
upper bound calculated using the curve γ_2 ------------- = 0.572;
lower bound calculated using the curve γ_1 ------------- = 0.505;
lower bound calculated using the inscribed circle k_1---- = 0.455.

Results.--The following curves are depicted in Figure 5:

a. k_2-L_1 upper bound versus μ ;
b. γ_2-L_1 upper bound versus μ ;
c. γ_1-L_1 lower bound versus μ ;
d. k_1-L_1 lower bound versus μ .

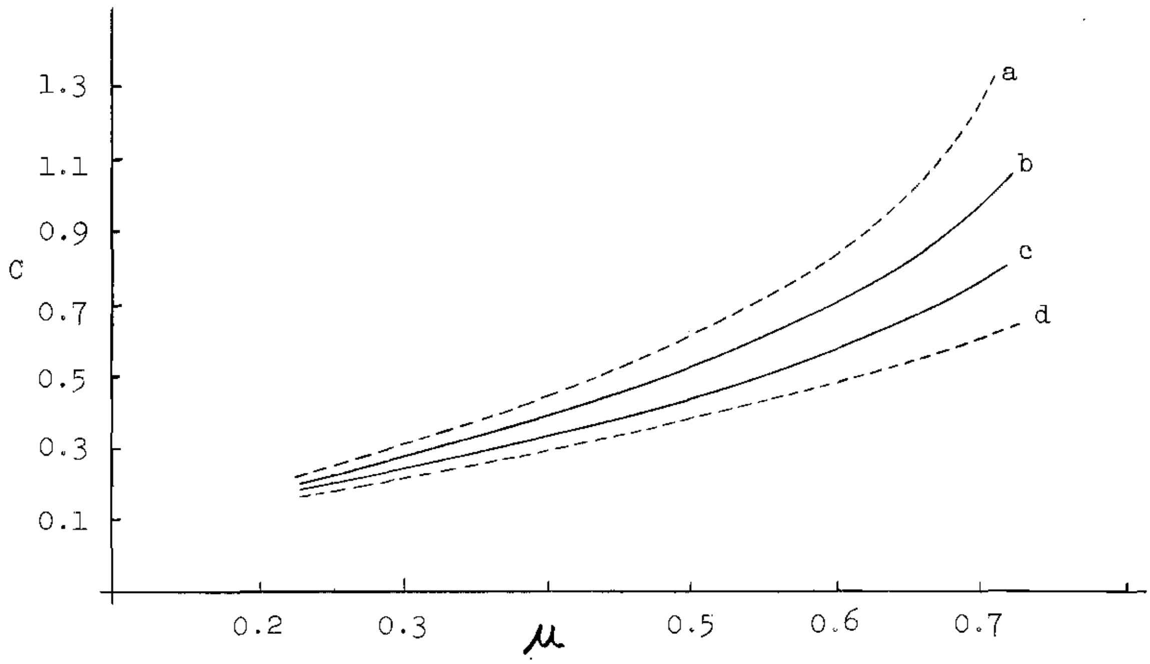


Figure 5. - Comparison of Upper and Lower Bounds

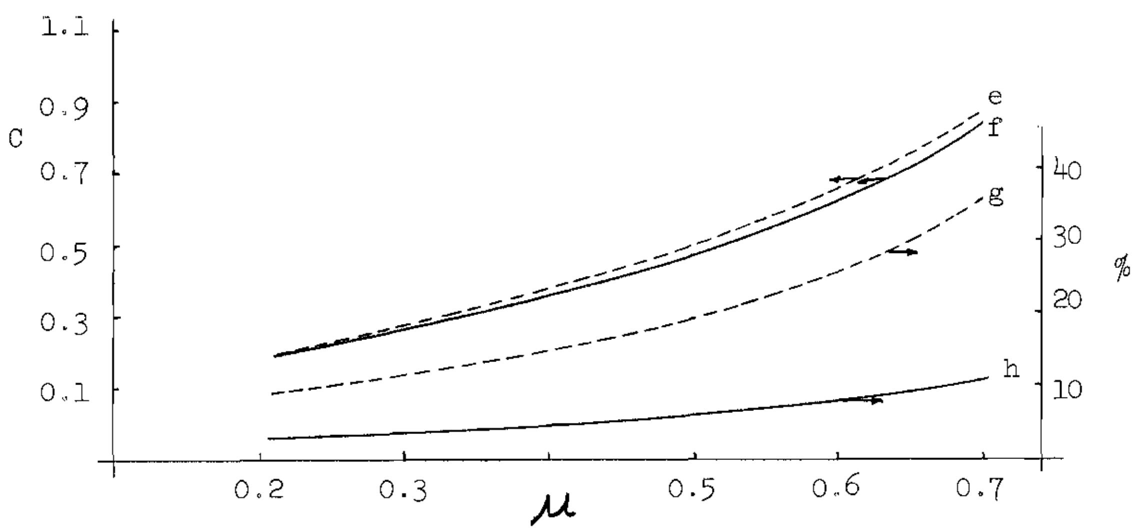


Figure 6. - Arithmetic Averages and Per Cent Error

Figure 6 contains the following curves:

e. the arithmetic average of a and d versus μ;

f. the arithmetic average of b and c versus μ;

g. the possible per cent error associated with e;

h. the possible per cent error associated with f,

where g and h are plotted using the right ordinate and e and f using the left ordinate.

Since the arithmetic averages are virtually the same, one may well ask whether all the effort which has gone into improving the bounds was worth while. The answer to this doubt lies, of course, in the fact that the difference in the arithmetic average and either bound is a measure of the uncertainty of the average as an approximation to the true result. This uncertainty has been sizably reduced by bringing the bounds closer together.

CHAPTER IV

EXTENSIONS AND ANALOGIES

Extensions to other configurations.--This method may be used on any capacitor formed by two concentric cylindrical conductors, providing one is a circle and the other can be described as follows: it is a simple, closed, piecewise smooth curve consisting of n identical arc segments such that the terminal points of each segment have equal moduli, these moduli being greater than that of any point on the arc segment other than a terminal point. There are only two changes necessary: the original transformation must now be $f(z) = z^n$, and the region on which $f(z)$ is defined must be one of the n congruent subregions into which the configuration is divided by radial lines drawn from the common axis through the terminal points of the n similar arc segments.

Improvement of results by successive application of the method.--Although the new bounds differ from their arithmetic average by less than twelve per cent for most reasonable configurations, there may be times when closer estimates are required. Such estimates may be obtained by reapplication of the method to the images of L_1 and L_3 in the p-plane, since this system satisfies all the conditions stated in the previous section. By first applying the mapping $g(p) = p^2$, $0 \leq \arg p < \pi$, and then a suitable linear transformation, the desired better bound can be found.

Analogies.--The function Φ in equation (1) has been considered only as representing electrostatic potential. As will be shown, it can also be considered as temperature, neutron flux, and fluid velocity potential. To make clear the analogies, a remark is necessary concerning Green's Theorem, which in two dimensions can be written [3],

$$\int_{\Gamma} \frac{\partial \Phi}{\partial n}\, ds - \iint_S \Phi \nabla^2 \Phi \, dx\, dy = \iint_S (\Phi_x^2 + \Phi_y^2)\, dx\, dy ,$$

where S is a doubly connected region, $\Gamma = \Gamma_1 + \Gamma_2$ is its boundary, and Φ has the necessary differentiability and integrability properties. If $\nabla^2 \Phi = 0$ in S, $\Phi = \Phi_1$ on Γ_1, and $\Phi = \Phi_2$ on Γ_2, then

$$\Phi_1 \int_{\Gamma_1} \frac{\partial \Phi}{\partial n}\, ds + \Phi_2 \int_{\Gamma_2} \frac{\partial \Phi}{\partial n}\, ds = \iint_S (\Phi_x^2 + \Phi_y^2)\, dx\, dy = I(\Phi). \tag{5}$$

Now suppose the closed region $S + \Gamma$ satisfies the conditions stated in the first section of this chapter. By different interpretations of the symbols involved, the various physical problems mentioned in the first paragraph of this section will be shown to be mathematically equivalent.

If the electrostatic problem is considered and the medium M enclosed between Γ_1 and Γ_2 has permittivity one [4], then

$$\int_{\Gamma_1} \frac{\partial \Phi}{\partial n}\, ds = -4\pi Q_1 \text{ and } \int_{\Gamma_2} \frac{\partial \Phi}{\partial n}\, ds = -4\pi Q_2 ,$$

where Q_1 and Q_2 are the charges on Γ_1 and Γ_2 per unit axial length. For S + Γ , $Q_2 = -Q_1$ [5] so that equation (5) becomes

$$\frac{Q_1}{(\Phi_2 - \Phi_1)} = \frac{I(\Phi)}{4\pi(\Phi_2 - \Phi_1)^2} .$$

The quantity $Q_1/(\Phi_2 - \Phi_1)$ is the capacity of the system, and may be thought of as the charge per unit length on Γ_1 per unit of potential difference between Γ_1 and Γ_2. This last relation is identical with equation (1).

In the thermal problem suppose that M has thermal conductivity k, and let Φ represent the temperature function T. If Q is the quantity of heat transferred across S per unit length per unit time in steady state, then [6]

$$T_1 \int_{\Gamma_1} \frac{\partial T}{\partial n}\, ds + T_2 \int_{\Gamma_2} \frac{\partial T}{\partial n}\, ds = \frac{Q}{k}(T_2 - T_1) ;$$

and from equation (5)

$$\frac{Q}{(T_2 - T_1)} = \frac{k\, I(T)}{(T_2 - T_1)^2} .$$

Now the quantity $Q/(T_2 - T_1)$ may be thought of as the quantity of heat transferred per unit length per degree of temperature difference.

In the case of neutron diffusion, consider a medium M which neither absorbs nor scatters neutrons but has diffusion coefficient D. If Φ represents the neutron flux function, and if the steady state neutron current through S is J, then [7]

$$\Phi_1 \int_{\Gamma_1} \frac{\partial \phi}{\partial n}\, ds + \Phi_2 \int_{\Gamma_2} \frac{\partial \phi}{\partial n}\, ds = \frac{J}{D}(\Phi_2 - \Phi_1) \quad ;$$

and from equation (5)

$$\frac{J}{(\Phi_2 - \Phi_1)} = \frac{D\, I(\Phi)}{(\Phi_2 - \Phi_1)^2} \quad .$$

Here the quantity $J/(\Phi_2 - \Phi_1)$ is the neutron current per unit length per unit difference in the neutron flux function.

Finally, for the fluid flow analogy, let the conductors be thought of as a source and a sink, and suppose that the flow between them is at right angles to the common axis. The fluid is incompressible and has density ρ . If W is the steady state weight flow per unit length across S, then [8]

$$V_1 \int_{\Gamma_1} \frac{\partial V}{\partial n}\, ds + V_2 \int_{\Gamma_2} \frac{\partial V}{\partial n}\, ds = \frac{W}{\rho}(V_2 - V_1) \ ;$$

and from equation (5)

$$\frac{W}{(V_2 - V_1)} = \frac{\rho\, I(V)}{(V_2 - V_1)^2} \ ,$$

where V is the velocity potential function. The quotient $W/(V_2 - V_1)$ is the weight transfer per unit length per unit difference in the velocity potential.

APPENDIX

Lemma 1.--Suppose $\Phi(x,y)$ is harmonic in a simply connected region R and continuous on Γ , the positively oriented, closed, piecewise smooth boundary of R. Then [3]

$$I_R(\Phi) = \iint_R (\Phi_x^2 + \Phi_y^2)\, dx\, dy = \int_\Gamma \Phi \frac{\partial \Phi}{\partial n}\, ds \ .$$

Corollary.--Suppose R is the doubly connected region between the two simple, closed, piecewise smooth curves C_1 and C_2. If $\Phi(x,y)$ is harmonic in R and continuous on C_1 and C_2, then [9]

$$\iint_R (\Phi_x^2 + \Phi_y^2)\, dx\, dy = \int_{C_1+C_2} \Phi \frac{\partial \Phi}{\partial n}\, ds \ .$$

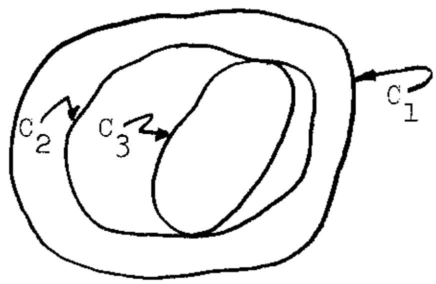
Lemma 2.--Let C_1, C_2, and C_3 be simple, closed, piecewise smooth curves, where C_2 is contained in C_1, C_3 is contained in C_2, and C_1 and C_2 have no points in common. Call R_{12} the region between C_1 and C_2, and R_{13} the region between C_1 and C_3. Let $\Phi(x,y)$ and $\psi(x,y)$ be harmonic functions, defined on R_{13} and R_{12} respectively. Also suppose $\Phi = \psi = V > 0$ on C_1, $\Phi = 0$ on C_3, and $\psi = 0$ on C_2. Then $I_{R_{12}}(\psi) \geqq I_{R_{13}}(\Phi)$.

Proof: From the corollary to Lemma 1,

$$I_{R_{12}}(\psi) = V \int_{C_1} \frac{\partial \psi}{\partial n}\, ds \text{ and } I_{R_{13}}(\Phi) = V \int_{C_1} \frac{\partial \Phi}{\partial n}\, ds \ .$$

Therefore this lemma will be proved if it can be shown that

$$\int_{C_1} \left(\frac{\partial \psi}{\partial n} - \frac{\partial \Phi}{\partial n}\right) ds \geq 0 .$$



Let $F(x,y) = \Phi(x,y) - \psi(x,y)$ in $R_{12} + C_1 + C_2$. F is harmonic; F = 0 on C_1; and $F = \Phi$ on C_2. Since Φ is harmonic in R_{13}, Φ attains its minimum on the boundary of R_{13} [10] . Therefore $\Phi \geq 0$ on C_2 since C_2 is contained in R_{13}. Then $F \geq 0$ on C_2 and, by [10], $F \geq 0$ in R_{12}.

On C_1

$$\frac{\partial F}{\partial n} = \lim_{\Delta n \to 0} \frac{F(x,y) - F(x',y')}{\Delta n} ,$$

where (x,y) lies on C_1, (x', y') lies on the inward normal constructed to C_1 at (x,y), and Δn is the distance from (x,y) to (x',y'). Since F = 0 on C_1 and $F \geq 0$ in R_{12},

$$\frac{\partial F}{\partial n} = \lim_{\Delta n \to 0} \frac{(\text{negative quantity})}{(\text{positive quantity})} \leq 0 .$$

But $\frac{\partial F}{\partial n} = \frac{\partial \Phi}{\partial n} - \frac{\partial \psi}{\partial n}$ so $\frac{\partial \psi}{\partial n} - \frac{\partial \Phi}{\partial n} \geq 0$ and $\int_{C_1} \left(\frac{\partial \psi}{\partial n} - \frac{\partial \Phi}{\partial n}\right) ds \geq 0$.

Lemma 3.--Let C_1 be the circle $|z| = 1$ and C_2 the circle $|z - a| = R$, where a is real, $0 < |a| < 1$, and $|R \pm a| < 1$ (i.e. C_2 lies entirely inside C_1). Then there exists one and only one pair or points $[(s,0),(t,0)]$ such that $(t,0)$ is the symmetric point of $(s,0)$ with respect to both C_1 and C_2. For definiteness, let $(s,0)$ be the point inside C_1.

Proof: From the definition of symmetric points, all such points with respect to C_1 must satisfy $s_1 t_1 = 1$. Similarly, for C_2, $(s_2 - a)(t_2 - a) = R^2$ must be satisfied. Any set which is symmetric with respect to C_1 and C_2 would satisfy both the above relations. Such a set, under the restriction that $(s,0)$ is inside C_1, is $[(s,0)(t,0)]$, where

$$s = \frac{1}{2a} \left\{ -(R^2 - a^2 - 1) + \sqrt{[(R - a)^2 - 1][(R + a)^2 - 1]} \right\}$$

$$t = \frac{1}{2a} \left\{ -(R^2 - a^2 - 1) - \sqrt{[(R - a)^2 - 1][(R + a)^2 - 1]} \right\} .$$

The quantity under the radical is positive, since $|R \pm a| < 1$, and therefore s and t are real. If $[(s',0), (t',0)]$ was another such set, s' and t' would also have to satisfy the first two relations simultaneously, which would yield $s' = s$ and $t' = t$, provided the point inside C_1 is called $(s',0)$.

Lemma 4.--Let C_1 and C_2 be the circles of Lemma 3. Then there exists a linear transformation $w = L(z)$ which maps C_1 and C_2 onto concentric circles in the w-plane in such a way that the image of C_2 is entirely inside the image of C_1. If $(t,0)$ is the symmetric point of the C_1C_2 system lying outside of C_1, then the ratio of the radii of the images of C_2 and C_1 in the w-plane is $R\,|t/(t - a)|$.

Proof: Such a transformation will be exhibited. It is known [11] that a linear transformation from the z-plane to the w-plane is uniquely determined if any three distinct points in the z-plane are chosen to be mapped into three distinct points in the w-plane. Choose the domain points to be (s,0) and (t,0) of Lemma 3, and (1,0), and the corresponding image points to be (0,0), ∞ , and (1,0). $w = L(z)$ can be written down:

$$w = \frac{(z - s)(1 - t)}{(z - t)(1 - s)} . \tag{6}$$

Any point on C_1 can be represented by $z = e^{i\theta}$. Since $s = 1/t$,

$$|w| = \left|\frac{s - z}{sz-1}\right| = \left|\frac{s - e^{i\theta}}{se^{i\theta} - 1}\right| = \left|\frac{(s-\cos\theta) - i\sin\theta}{(s\cos\theta - 1) + is(\sin\theta)}\right| = 1 ;$$

so C_1 maps onto the unit circle in the w-plane.

With $z = a + Re^{i\theta}$ and $s = a + R^2/(t - a)$, it follows from equation (6) that

$$|w| = \left|\frac{[Re^{i\theta} - R^2/(t-a)][1-t]}{[a+Re^{i\theta} - t][1-a-R^2/(t-a)]}\right| = \left|\frac{R[e^{i\theta}(t-a)-R][1-t]}{[a+Re^{i\theta} - t][(1-a)(t-a)-R^2]}\right|$$

$$= \left|\frac{1-t}{(1-a)(t-a)-(s-a)(t-a)}\right| R = R\left|\frac{1-t}{(t-a)(1-s)}\right|$$

$$= \left|\frac{t(1-t)}{(t-1)(t-a)}\right| R = \left|\frac{t}{t-a}\right| R = \left|\frac{1}{1-as}\right| R .$$

Thus, C_2 maps onto the circle $|w| = R|t/(t-a)|$. Since C_2 is entirely within C_1, $|1-a| > R$. Also $|s| < 1$, so $|1-as| > R$, or $R/|1-as| < R/R = 1$. Therefore the image of C_2 lies entirely inside the image of C_1.

Lemma 5.--Let $\Phi(x,y)$ be harmonic in the simply connected region R. Let $x = x(u,v)$ and $y = y(u,v)$ map R in a one-to-one fashion onto the region S in the u-v plane. Also suppose $u_x = v_y$ and $v_x = -u_y$. Then

$$\iint_R \left[\Phi_x^2(x,y) + \Phi_y^2(x,y)\right] dx\, dy =$$

$$\iint_S \left[\Phi_u^2(x(u,v),y(u,v)) + \Phi_v^2(x(u,v),y(u,v))\right] dudv.$$

Proof: The lemma will be proved by simply carrying out the indicated transformation of variables.

$$\Phi_x^2 + \Phi_y^2 = (\Phi_u u_x + \Phi_v v_x)^2 + (\Phi_u u_y + \Phi_v v_y)^2$$

$$= (\Phi_u u_x + \Phi_v v_x)^2 + (-\Phi_u v_x + \Phi_v u_x)^2$$

$$= (\Phi_u^2 + \Phi_v^2)(u_x^2 + v_x^2) .$$

From [12], the Jacobian of the transformation can be written

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}}$$

$$= \frac{1}{u_x v_y - u_y v_x} = \frac{1}{u_x^2 + v_x^2} .$$

So $\Phi_x^2 + \Phi_y^2 = (\Phi_u^2 + \Phi_v^2)$ J, and the lemma is proved.

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