SINGULAR SELF-ADJOINT BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

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TABLE OF CONTENTS

SUMMARY

The purpose of this study is to extend the theory of singular self-adjoint boundary value problems for scalar differential equations to systems of first order differential equations.

Let the n-by-n matrices P_0 and P_1 be continuous for $a \leq t \leq b$. Moreover, let P_0' be continuous and $\det P_0(t) \neq 0$. Let x be a vector with n components and let

$$Lx = P_0 x' + P_1 x .$$

Let P^* denote the adjoint of P, that is, the transposed conjugate, and let

$$L^+ x = -(P_0^* x)' + P_1^* x .$$

If u and v are vectors with components u_j, v_j, let

$$u \cdot v = u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n .$$

It is shown that

$$Lu \cdot v - u \cdot L^+ v = (P_0 u \cdot v)' .$$

Let $L = L^+$, that is, $P_0 + P_0^* = 0$; $P_0' = P_1 - P_1^*$. Let M and N be n-by-n constant matrices and let $Bx = Mx(a) + Nx(b)$. Suppose M and N are such that for any $u, v \in c'[a,b]$ and satisfying $Bu = Bv = 0$

$$\int_a^b Lu \cdot v \, dt = \int_a^b u \cdot Lv \, dt .$$

Assuming that the eigenfunctions $\{x_j\}$ of the nonsingular self-adjoint problem

$$Lx = \ell x \qquad Bx = 0$$

form a complete orthonormal set, theory for the singular problem is developed. The singular problem refers to the problem resulting from either the finite interval [a,b] becoming infinite or the coefficients in the differential operator having a sufficiently singular behavior at a or b.

Special attention is given the real 2-by-2 case. For this case the idea of limit-point and limit-circle is developed. Completeness and expansion theorems analogous to those for singular self-adjoint second-order problems are proved. The results of the development of the real 2-by-2 case are used to prove completeness and expansion theorems for the n-by-n complex case.

In the final chapter, as an illustration of the possible applications to physical problems of the theory previously developed, the problem of heat conduction in an infinite composite solid is studied.

CHAPTER I

INTRODUCTION

The purpose of this study is to extend the theory for singular self-adjoint boundary value problems for a scalar linear differential operator to a special class of systems of linear first-order differential equations.

If u and v are vectors with components u_j, v_j, let

$$u \circ v = u_1 \bar{v}_1 + u_2 \bar{v}_2 + \cdots + v_n \bar{v}_n . \tag{1.1}$$

Let P_0 and P_1 be n-by-n matrices of scalar functions. Let the operator L be defined as

$$L\, x(t) = P_0(t)\, x'(t) + P_1(t)\, x(t) . \tag{1.2}$$

<u>Definition 1.1</u>.--A boundary value problem on a finite interval $[a,b]$ with the differential equation $Lx = \ell x$ is said to be self-adjoint if for any $u, v \in c'[a,b]$ and satisfying the boundary conditions

$$\int_a^b Lu \cdot v\, dt = \int_a^b u \cdot Lv\, dt . \tag{1.3}$$

Let P^* denote the transposed conjugate of P and let

$$L^+ x(t) = -(P_0^* x)' + P_1^* x . \tag{1.4}$$

Of special interest is the class of self-adjoint boundary value problems is which $L = L^+$, that is, $P_0 + P_0^* = 0$ and $P_0' = P_1 - P_1^*$. There is an important relation connecting L and L^+ known as the Lagrange identity.

<u>Lemma 1.1(Lagrange)</u>.--If $u(t)$ and $v(t)$ are arbitrary n-dimensional vector functions ε $c'[a,b]$, then

$$v^*Lu - (L^+v)^* u = (v^*P_0 u)' \tag{1.5}$$

<u>Proof</u>.--The left side of (1.5) is by (1.4, 1.2, 1.1)

$$v^*P_0 u' + v^*P_1 u + (P_0^{*\prime} v + P_0^* v')^* u - (P_1^* v)^* u$$

$$= v^*P_0 u' + v^*P_1 u + v^*P_0' u + v^{*\prime} P_0 u - v^*P_1 u$$

$$= v^*P_0 u' + v^*P_0' u + v^{*\prime} P_0 u = (v^*P_0 u)' .$$

Let $P_0(t)$ and $P_1(t)$ be continuous n-by-n matrices of possibly complex valued functions on the interval $a \leq t \leq b$. It will be assumed that $P_0(t)$ is nonsingular and $P_0'(t)$ is continuous on δ: $a \leq t \leq b < \infty$. Define B_δ by

$$B_\delta x = Mx(a) + Nx(b)$$

where M and N are constant matrices, Suppose that for any u,v ε $c'[a,b]$ and satisfying

$$B_\delta u = B_\delta v = 0 ,$$

it is true that

$$\int_a^b Lu \cdot v \, dt = \int_a^b u \cdot Lv \, dt .$$

Let $L = L^+$. It has been established in [5] that the eigenfunctions $\{x_j\}$ of the self-adjoint problem

$$\pi: \quad Lx = \ell x \qquad B_\delta x = 0$$

form a complete orthonormal set, with the Parseval equality

$$\int_a^b f(t) \cdot f(t) \, dt = \sum_{j=-\infty}^{\infty} \left| \int_a^b f(t) \cdot x_j(t) \, dt \right|^2$$

and the expansion formula

$$f(t) = \sum_{j=-\infty}^{\infty} \left(\int_a^b f(t) \cdot x_j(t) \, dt \right) x_j(t)$$

valid for any vector function f satisfying

$$\int_a^b f(t) \cdot f(t) \, dt < \infty .$$

By multiplying the differential equation $Lx = \ell x$ on the left by $P_0^{-1}(t)$ it is seen that the problem π has the form

$$x'(t) = R(t) \, x(t) + \ell s(t) \, x(t) \qquad B_\delta x = 0$$

where the matrices $R(t)$, $s(t)$ and $s'(t)$ are continuous and $s(t)$ is nonsingular. A more general problem of this form where $s(t)$ is just assumed to be continuous and not identically equal to the zero matrix is treated by a different method in [1] and [2].

In the following chapters the nature of the problem π will be studied as the interval $\delta \to (c,d)$ where there is a singularity at either c, d or both. The singularity may be in the form of a singular behavior of some or all of the coefficients in the operator L or perhaps c or d is infinite.

The real 2-by-2 case, where P_0 and P_1 are 2-by-2 matrices of real functions, denoted by π_{2r} will be given special attention and much of the theory for second-order scalar boundary value problems will be extended to this case. The method used to study this case was first proposed by Hermann Weyl. A short history of the method is given in [6]. The theory for the real 2-by-2 case will be developed first and the results brought out in this development will be used to treat the n-by-n complex case.

While the method used in this study is applicable only to self-adjoint problems, there are methods for treating problems which may not be self-adjoint. Such an approach is furnished by the Cauchy integral method which depends on the calculus of residues and general theorems in the theory of functions. This method is exploited extensively by Tilchmarsh in [7].* Although methods of this type are applicable to self-adjoint problems they are generally much more difficult, less direct and give less insight into the problem than the method that will be used here.

Occasionally it may be desirable to make a linear transformation of a boundary value problem either to simplify the differential operator

*This book has large bibliography listing nearly every significant publication on the subject of eigenfunction expansions up to about 1956.

or to obtain a new problem in which the operator is now such that $L = L^+$. The following theorems may be of use in making a suitable linear transformation.

<u>Theorem 1.1</u>.--Suppose P_o, P_o' and $\widetilde{P}_1$ are continuous 2-by-2 matrices of real functions. Let P_o be nonsingular. If $P_o + P_o^t = 0$ then there is a nonsingular linear transformation such that the equation

$$P_o x' + \widetilde{P}_1 x = \ell x$$

is transformed into an equation

$$P_o y' + P_1 y = \ell y$$

where $P_o + P_o^t = 0$ and $P_o' = P_1 - P_1^t$.

<u>Proof</u>.--Since $P_o + P_1^t = 0$ and P_o is real, P_o has the form

$$P_o(t) = \begin{pmatrix} 0 & P(t) \\ -P(t) & 0 \end{pmatrix} \tag{1.6}$$

and without loss of generality it may be assumed that $P(t) > 0$. Consider the effect of the transformation K where

$$K = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$$

and T is a real function of the class c'.

$$(T^{-1} P_o T) y' + (T^{-1} P_o T' + T^{-1} \widetilde{P}_1 T) y = \ell y ,$$

$$P_o y' + (T^{-1} P_o T' + \tilde{P}_1) y = \ell y \text{ .}$$

All that remains is to demonstrate there is a function T of the class c' such that

$$P_o' = (T^{-1} P_o T' + \tilde{P}_1) - (T^{-1} P_o T' + \tilde{P}_1)^t \qquad (1.7)$$

Let

$$\tilde{P}_1 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix};$$

then (1.7) is equivalent to

$$\begin{pmatrix} 0 & P' \\ -P' & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2PT'}{T} + P_{12} - P_{21} \\ \frac{-2PT'}{T} + P_{21} - P_{12} & 0 \end{pmatrix}.$$

By choosing T as a solution of the first order linear differential equation

$$P' = \frac{2PT'}{T} + P_{12} - P_{21}$$

(1.7) is satisfied and the theorem is proved.

<u>Theorem 1.2</u>.--If the boundary value problem

$$\pi: \quad Lx = \ell x, \quad B_\delta x = Mx(a) + Nx(b) = 0$$

is self-adjoint then the boundary value problem

$$\tilde{\pi}: \quad \tilde{L}y = \ell y, \quad B_\delta = MU(a)\ y(a) + NU(b)\ y(b) = 0,$$

resulting from applying the linear transformation $x = Uy$, where U is a unitary matrix of functions of class c', is self-adjoint. Moreover if L is such that $L = L^+$ then $\tilde{L} = \tilde{L}^+$.

Proof.--Let $Lx = P_0 x' + P_1 x$ and suppose the problem π is self-adjoint. It must be shown that for any functions $\tilde{R}$ and $\tilde{S}$ satisfying $\tilde{B}_\delta \tilde{R} = \tilde{B}_\delta \tilde{S} = 0$ that

$$\int_a^b (\tilde{S}^* \tilde{L}\tilde{R} - (\tilde{L}\tilde{S})^* \tilde{R})\, dt = 0 . \tag{1.8}$$

Suppose $\tilde{R}$ and $\tilde{S}$ satisfy $\tilde{B}_\delta \tilde{R} = \tilde{B}_\delta \tilde{S} = 0$ and let $R = U\tilde{R}$, $S = U\tilde{S}$, then R and S satisfy $B_\delta R = B_\delta S = 0$ and since π is assumed to be self-adjoint

$$\int_a^b (S^* LR - (LS)^* R)\, dt = 0 . \tag{1.9}$$

Applying the transformation $x = Uy$ results in

$$Ly = (U^* P_0 U) y' + (U^* P_0 U' + U^* P_1 U) y$$

$$\int_a^b \tilde{S}^* \tilde{L}\tilde{R}\, dt = \int_a^b \tilde{S}^* [(U^* P_0 U)\tilde{R}' + (U^* P_0 U')\tilde{R} + (U^* P_1 U)\tilde{R}]\, dt$$

$$\int_a^b \tilde{S}^* \tilde{L}\tilde{R}\, dt = \int_a^b (S^* U U^* P_0 U[U^{*\prime} R + U^* R'] + S^* U U^* P_0 U' U^* R + S^* U U^* P_1 U U^* R)\, dt$$

$$\int_a^b \tilde{S}^* \tilde{L}\tilde{R}\, dt = \int_a^b (S^* P_0 R' + S^* P_0 [U U^{*\prime} + U' U^*] R + S^* P_1 R)\, dt$$

$$\int_a^b \tilde{S}^* \tilde{L}\tilde{R}\, dt = \int_a^b (S^* P_0 R' + S^* P_1 R)\, dt = \int_a^b S^* LR\, dt . \tag{1.10}$$

Similar calculation yields

$$\int_a^b (\widetilde{L}\widetilde{S})^* \widetilde{R}\, dt = \int_a^b (LS)^* R\, dt \,. \tag{1.11}$$

Since (1.9, 1.10, 1.11) imply (1.8) $\widetilde{\pi}$ is therefore self-adjoint and the first part of the theorem is proved.

Suppose $L = L^+$; then

$$P_o + P_o^* = 0 \qquad P_o' = P_1 - P_1^* \,. \tag{1.12}$$

Using (1.11) it follows that

$$(U^*P_oU) + (U^*P_oU)^* = U^*P_oU + U^*P_o^*U = U^*(P_o + P_o^*)\, U = 0 \,,$$

$$(U^*P_oU)' = U^{*\prime}P_oU + U^*P_o'U + U^*P_oU',$$

$$(U^*P_oU)' = U^{*\prime}P_oU + U^*P_1U - U^*P_1^*U + U^*P_oU'. \tag{1.13}$$

$$(U^*P_oU' + U^*P_1U) - (U^*P_oU' + U^*P_1U)^*$$
$$= U^*P_oU' + U^*P_1U - U^{*\prime}P_oU - U^*P^*U \,,$$

$$(U^*P_oU' + U^*P_1U) - (U^*P_oU' + U^*P_1U)^*$$
$$= U^*P_oU' + U^*P_1U + U^{*\prime}P_oU - U^*P_1^*U \,. \tag{1.14}$$

Comparing (1.13) and (1.14) observe that

$$(U^*P_oU)' = (U^*P_oU' + U^*P_1U) - (U^*P_oU' + U^*P_1U)^* .$$

Thus $\widetilde{L} = \widetilde{L}^+$ and the second part of the theorem is proved.

Let (M:N) denote the matrix

$$\begin{pmatrix} M_{11} & \cdots & M_{1n} & N_{11} & \cdots & N_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ M_{m1} & \cdots & M_{mn} & N_{m1} & \cdots & N_{mn} \end{pmatrix} .$$

If $L = L^+$ and the rank of (M:N) is the same as that of P_o a sufficient condition for the problem π to be self-adjoint is that

$$M\, P_o^{-1}(a)\, M^* \;=\; N\, P_o^{-1}(b)\, N^* . \tag{1.15}$$

A proof of (1.15) may be found in [3]. If $L = L^+$, since it has already been shown that $\tilde{L} = \tilde{L}^+$, a sufficient condition for $\tilde{\pi}$ to be self-adjoint is that

$$MU(a)\, [U^*(a)\, P_o(a)\, U(a)]^{-1}\, (M\, U(a))^*$$
$$= N\, U(b)\, [U^*(b)\, P_o(b)\, U(b)]^{-1}\, (N\, U(b))^* . \tag{1.16}$$

Clearly (1.15) implies (1.16). If $L = L^+$, the rank of (M:N) is the same as that of P_o and π is self-adjoint, then another proof that $\tilde{\pi}$ is self-adjoint has been given.

Chapters II through V deal with the real 2-by-2 problem, π_{2r}. Since in this case P_o has the form (1.6) the Lagrange identity (1.5) yields the following important formula known as Green's formula

$$\int_a^b (LU\cdot V) - U\cdot LV)\, dt \;=\; [uv](b) - [uv](a) \tag{1.17}$$

where

$$[uv](t) \;=\; P(t)\, (u_2(t)\, \bar{v}_1(t) - u_1(t)\, \bar{v}_2(t)) .$$

The treatment of a problem in which there is singular behavior of some or all of the coefficients in the operator L at b, the end point of the finite interval $[a,b]$, is completely analogous to the treatment of a problem in which there is no singularity in the coefficients in L but the problem is considered on the semi-infinite interval $[0,\infty)$. It should be noted that all results obtained on $[0,\infty)$ are valid in the case of an interval $[a,b)$ where the coefficients in L have a singular behavior at b. Similar remarks hold concerning the case where coefficients are singular at a and b.

CHAPTER II

THE LIMIT-POINT AND LIMIT-CIRCLE CASES

Definition 2.1.--An operator is said to be of the limit-circle type at infinity if every solution of $Lx = \ell_0 x$ satisfies

$$\int_0^\infty \varphi \circ \varphi \, dt < \infty \tag{2.1}$$

for some complex number ℓ_0; otherwise L is said to be of the limit-point type at infinity.

The next theorem shows that the classification depends only on L and not on the choice of ℓ_0.

Theorem 2.1.--If every solution of $Lx = \ell_0 x$ is of class $L^2(0,\infty)$, that is, satisfies (2.1) for some complex number ℓ_0, then, for arbitrary complex ℓ, every solution of $Lx = \ell x$ is of class $L^2(0,\infty)$.

Proof.--Suppose every solution $Lx = \ell_0 x$ is of class $L^2(0,\infty)$. Let φ and ψ be two linearly independent solutions of $Lx = \ell_0 x$ and let x be any solution of $Lx = \ell x$ which may be written as

$$Lx = \ell_0 x + (\ell - \ell_0)x ,$$

or equivalently

$$P_0 x' + P_1 x = \ell_0 I x + (\ell - \ell_0) I x .$$

By the variation of constants formula [3], considering $(\ell - \ell_o)P^{-1}x$ to be the inhomogeneous term it is seen that

$$x(t) = \Phi(t)\,K + \Phi(t)\int_c^t \Phi^{-1}(s)\;(\ell - \ell_o)\;P_o^{-1}(s)\;x(s)\,ds \qquad (2.2)$$

where

$$\Phi = \begin{pmatrix} \varphi_1(t,\ell_o) & \psi_1(t,\ell_o) \\ \varphi_2(t,\ell_o) & \psi_2(t,\ell_o) \end{pmatrix}$$

and K is a constant matrix. Equation (2.2) may be written as

$$x(t) = \Phi(t)K + \Phi(t)\int_c^t \frac{\begin{pmatrix} \psi_2(s) & -\psi_1(s) \\ -\varphi_2(s) & \varphi_1(s) \end{pmatrix}\begin{pmatrix} 0 & -(\ell-\ell_o) \\ (\ell-\ell_o) & 0 \end{pmatrix}\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} ds}{[\varphi_1(s)\;\psi_2(s) - \psi_1(s)\;\varphi_2(s)]\;p(s)} \;. \qquad (2.3)$$

Notice that

$$P(s)\;[\varphi_1(s)\;\psi_2(s) - \psi_1(s)\;\varphi_2(s)] = [\varphi\;\overline{\psi}]\;(s)\;.$$

By Green's formula (1.17) it can be shown that $[\varphi\;\overline{\psi}](s)$ is a constant. Thus it may be assumed φ and ψ are such that $[\varphi\;\overline{\psi}](s) = 1$ and (2.3) becomes

$$x(t) = \Phi(t)K + \Phi(t)\int_c^t \begin{pmatrix} \psi_2(s) & -\psi_1(s) \\ -\varphi_2(s) & \varphi_1(s) \end{pmatrix}\begin{pmatrix} 0 & -(\ell - \ell_o) \\ (\ell - \ell_o) & 0 \end{pmatrix}\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} ds \;.$$

Define

$$\| x \|_c = \left(\int_c^t |x|^2 \, dt \right)^{\frac{1}{2}}$$

where

$$|x| = |x_1| + |x_2| \, .$$

Since φ and ψ are of the class $L^2(0,\infty)$ to each $M > 0$ there is a $C > 0$ such that $\| \varphi \|_c \leq M$ and $\| \psi \|_c \leq M$ for all $t \geq C$.

$$M_1(s) = \begin{pmatrix} \psi_2(s) & -\psi_1(s) \\ -\varphi_2(s) & \varphi_1(s) \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & -(\ell - \ell_0) \\ (\ell - \ell_0) & 0 \end{pmatrix},$$

$$\left| \int_c^t M_1(s) \, M_2 x(s) \, ds \right| \leq \int_c^t |M_1(s)| |M_2| |x(s)| \, ds \, ,$$

$$\left| \int_c^t M_1(s) \, M_2 x(s) \, ds \right| \leq 2 |\ell - \ell_0| \int_c^t (|\varphi(s)| + |\psi(s)|) |x(s)| \, ds. \quad (2.4)$$

The Schwarz inequality used on the right side of the inequality (2.4) yields

$$\left| \int_c^t M_1(s) \, M_2 x(s) \, ds \right| \leq 2 |\ell - \ell_0| [\| \varphi \|_c(t) + \| \psi \|_c(t)] \| x \|_c(t). \quad (2.5)$$

Using (2.5) in (2.3) gives

$$|x(s)| \leq |\Phi(s)| |K| + |\Phi(s)| \; 2 \; |\ell - \ell_0| [\| \varphi \|_c(s)$$

$$+ \| \psi \|_c(s)] \; \| x \|_c(s) \, ,$$

or

$$|x(s)| \leq |K|(|\varphi(s)| + |\psi(s)|) + 4M|\ell - \ell_o|(|\varphi(s)| + |\psi(s)|)\ \|x\|_c(s) \ . \tag{2.6}$$

From (2.6) and the definition of $\|x\|_c$ it is clear that

$$\|x\|_c \leq \left\{ \int_c^t \left(|K|[|\varphi(s)| + |\psi(s)|] + 4M|\ell - \ell_o|[|\varphi(s)| + |\psi(s)|]\ \|x\|_c(s) \right)^2 ds \right\}^{\frac{1}{2}} . \tag{2.7}$$

The Minkowski inequality applied to (2.7) gives

$$\|x\|_c \leq \left\{ \int_c^t [4M|\ell - \ell_o|(|\varphi| + |\psi|)]\|x\|_c^2\, ds \right\}^{\frac{1}{2}} + \left\{ \int_c^t [|K|(|\varphi| + |\psi|)]^2\, ds \right\}^{\frac{1}{2}} ,$$

$$\|x\|_c \leq \{4M|\ell - \ell_o|\ \|x\|_c + |K|\}\ \|\psi + \varphi\|_c ,$$

$$\|x\|_c \leq 2M\{4M|\ell - \ell_o|\ \|x\|_c + |K|\}$$

$$= 8M^2|\ell - \ell_o|\ \|x\|_c + 2M|K| \ .$$

Let C be sufficiently large so that $|\ell - \ell_o|M^2 < 1/16$; then

$$\|x\|_c \leq \frac{1}{2}\|x\|_c + 2M|K| ,$$

$$\|x\|_c \leq 4M|K| \ .$$

Since the right side of this inequality is independent of t and since

$$\int_c^t (|x_1|^2 + |x_2|^2)\, dt = \int_c^t x \cdot x\, dt \le \| x \|_c ,$$

the theorem is proved.

In the next theorem the geometric significance of the terms limit-point and limit-circle becomes apparent.

Theorem 2.2.--If $\operatorname{Im} \ell \neq 0$ and φ and ψ are linearly independent solutions of $Lx = \ell x$ satisfying

$$P^{\frac{1}{2}}(0)\ \psi_1(0,\ell) = \sin\alpha \qquad P^{\frac{1}{2}}(0)\ \psi_2(0,\ell) = \cos\alpha \tag{2.8}$$

$$P^{\frac{1}{2}}(0)\ \varphi_1(0,\ell) = -\cos\alpha \qquad P^{\frac{1}{2}}(0)\ \varphi_2(0,\ell) = \sin\alpha ,$$

then the solution $x = \varphi + m\psi$ satisfies the real boundary condition

$$x_1(b) \cos\beta - x_2(b) \sin\beta = 0, \tag{2.9}$$

if and only if m lies on a circle C_b in the complex plane whose equation is $[xx](b) = 0$. As $b \to \infty$ either $C_b \to C_\infty$ a limit-circle, or $C_b \to m_\infty$ as limit-point. All solutions of $Lx = \ell x$ are $L^2(0,\infty)$ in the former case, and if $\operatorname{Im} \ell = 0$, exactly one linearly independent solution is $L^2(0,\infty)$ in the latter case. Moreover, in the limit-circle case, a point is on the limit-circle C_∞ if and only if $[xx](\infty) = 0$.

Proof.--Assume $x = \varphi + m\psi$ and satisfies (2.9), then

$$[\varphi_1(b) + m\psi_1(b)] \cos\beta - [\varphi_2(b) + m\psi_2(b)] \sin\beta = 0. \tag{2.10}$$

Expanding (2.10) and solving for m it is seen that

$$m = -\frac{\varphi_1(b)\cos\beta - \varphi_2(b)\sin\beta}{\psi_1(b)\cos\beta - \psi_2(b)\sin\beta} = -\frac{\varphi_2(b)\cot\beta - \varphi_2(b)}{\psi_1(b)\cot\beta - \psi_2(b)} .$$

Define

$$A = \varphi_1(b), \qquad C = \psi_1(b), \qquad z = \cot\beta ;$$
$$B = -\varphi_2(b), \qquad D = -\psi_2(b),$$

then

$$m = -\frac{Az + B}{Cz + D}, \qquad z = -\frac{B + Dm}{A + Cm} .$$

Setting $\operatorname{Im} z = 0$ to obtain the image of the real axis the equation

$$(\bar{A} + \bar{C}\bar{m})(B + Dm) - (A + Cm)(\bar{B} + \bar{D}\bar{m}) = 0 \qquad (2.11)$$

is obtained, which is the equation for the circle C_b. It follows that the center of C_b is

$$\tilde{m}_b = \frac{A\bar{D} - B\bar{C}}{\bar{C}D - C\bar{D}} \qquad (2.12)$$

and the radius is

$$r_b = \frac{|AD - BC|}{|\bar{C}D - C\bar{D}|} \qquad (2.13)$$

$$[\varphi\psi](b) = P(b)[\varphi_2(b)\,\bar{\psi}_1(b) - \varphi_1(b)\,\bar{\psi}_2(b)] = P(b)\,(A\bar{D} - B\bar{C})$$

$$[\psi\psi](b) = P(b)[\psi_2(b)\,\bar{\psi}_1(b) - \psi_1(b)\,\bar{\psi}_2(b)] = P(b)\,(C\bar{D} - D\bar{C}) .$$

Green's formula gives

$$[\varphi\bar{\psi}](b) - [\varphi\bar{\psi}](0) = 0$$

$$[\varphi\bar{\psi}](b) = P(0)[\varphi_2(0)\ \psi_1(0) - \varphi_1(0)\ \psi_2(0)] = 1 = P(b)\ (AD - BC)\,,$$

so that

$$\tilde{m}_b = -\frac{[\varphi\psi](b)}{[\psi\psi](b)}\,, \qquad r_b = \frac{1}{|[\psi\psi](b)|}\ .$$

Substitution of $x = \varphi + m\psi$ in $[xx](b)$ shows that the equation of C_b (2.11) is

$$[xx](b) = 0\ . \tag{2.14}$$

Setting $|\tilde{m} - \tilde{m}_b| < r_b$, substituting for $\tilde{m}_b$ from (2.12), and r_b from (2.13), and comparing the resulting inequality with (2.11) it follows that the interior of C_b is given by

$$\frac{[xx](b)}{[\psi\psi](b)} < 0\ . \tag{2.15}$$

By Green's formula

$$2i\ \mathrm{Im}\ \ell\int_0^b (\psi\cdot\psi)\ dt = [\psi\psi](b) - [\psi\psi](0)\ . \tag{2.16}$$

The initial conditions imply $[\psi\psi](0) = 0$ and $[xx](0) = -2i\ \mathrm{Im}\ m$. Green's formula used on x gives

$$2i\ \mathrm{Im}\ \ell\int_0^b (x\cdot x)\,dt = [xx](b) - [xx](0)\ . \tag{2.17}$$

Thus (2.15) becomes

$$\int_0^b (x \cdot x)\, dt < \frac{\mathrm{Im}\, m}{\mathrm{Im}\, \ell}, \qquad \mathrm{Im}\, \ell \neq 0, \tag{2.18}$$

and the equation of C_b (2.14) becomes

$$\int_0^b (x \cdot x)\, dt = \frac{\mathrm{Im}\, m}{\mathrm{Im}\, \ell}, \qquad \mathrm{Im}\, \ell \neq 0. \tag{2.19}$$

The remainder of the proof is identical to that for the second order scalar case given in [3] and will be omitted.

Since the treatment of a problem differs considerably depending on whether the given operator L is in the limit-point or limit-circle case, it is important to know if an operator is in a particular case.

<u>Theorem 2.3</u>.--If

$$\int_c^{\infty} \frac{1}{P(t)}\, dt = \infty,$$

then L is in the limit-point case at infinity.

<u>Proof</u>.--Suppose ψ and φ are two linearly independent solutions of $Lx = \ell x$ where ℓ is real and where φ and ψ satisfy the initial conditions (2.8). φ and ψ are linearly independent. Now suppose φ and ψ are of the class $L^2(0,\infty)$.

$$\int_0^t (L\varphi \cdot \psi - \varphi \cdot L\psi)\, dt = \int_0^t [\ell(\varphi \cdot \psi) - \bar{\ell}(\varphi \cdot \psi)]\, dt = 0$$

$$\int_0^t (L\varphi \cdot \psi - \varphi \cdot L\psi)\, dt = [\varphi\psi](b) - [\varphi\psi](0) = 0.$$

From the initial conditions that φ and ψ satisfy (2.8) it is seen that $[\varphi\psi](0) = 1$; thus

$$P(t)[\varphi_2(t)\ \psi_1(t) - \varphi_1(t)\ \psi_2(t)] = 1 ,$$

$$\varphi_2(t)\ \psi_1(t) - \varphi_1(t)\ \psi_2(t) = \frac{1}{P(t)} ,$$

$$\int_c^\infty \varphi_2(t)\ \psi_1(t)\ dt - \int_c^\infty \varphi_1(t)\ \psi_2(t)\ dt = \int_c^\infty \frac{1}{P(t)}\ dt . \qquad (2.20)$$

The Schwarz inequality and the fact that φ and ψ are of the class $L^2(0,\infty)$ implies the left side of (2.20) is finite, but this contradicts the hypothesis that

$$\int_c^\infty \frac{1}{P(t)}\ dt = \infty .$$

CHAPTER III

COMPLETENESS AND EXPANSION THEOREMS IN THE LIMIT-POINT CASE

Consider the boundary value problem

$$Lx = \ell x \tag{3.1}$$

with boundary conditions

$$x_1(0) \cos \alpha - x_2(0) \sin \alpha = 0 \tag{3.2}$$

$$x_1(b) \cos \beta - x_2(b) \sin \beta = 0 .$$

Assume u, v are of the class c' and satisfy (3.2). Then by using Green's formula it will be shown that $[uv](0) = 0$ and $[uv](b) = 0$ and hence that the boundary value problem (3.1), (3.2) is self-adjoint.

$$- \sin \beta \, \overline{u}(b) \, [u_1(b) \cos \beta - u_2(b) \sin \beta] = 0 \tag{3.3}$$

$$\cos \beta \, u_2(b) \, [\overline{u}_1(b) \cos \beta - \overline{u}_2(b) \sin \beta] = 0 . \tag{3.4}$$

Adding (3.3) and (3.4) gives

$$\overline{v}_1(b) \, u_2(b) - [\overline{v}_1(b) \, u_1(b) + u_2(b) \, \overline{v}_2(b)] \sin \beta \cos \beta = 0 \tag{3.5}$$

$$- \cos \beta \, \overline{v}_2(b) \, [u_1(b) \cos \beta - u_2(b) \sin \beta] = 0 . \tag{3.6}$$

$$\sin\beta\; u_1(b)\,[\overline{v}_1(b)\cos\beta - \overline{v}_2(b)\sin\beta] = 0 . \tag{3.7}$$

Adding (3.6) and (3.7) gives

$$-\overline{v}_2(b)\,u_1(b) + [u_1(b)\,\overline{v}_1(b) + u_2(b)\,\overline{v}_2(b)]\sin\beta\cos\beta = 0 . \tag{3.8}$$

Adding (3.5) and 3.8) gives

$$\overline{v}_1(b)\,u_2(b) - \overline{v}_2(b)\,u_1(b) = 0$$

which implies $[uv] = 0$. By letting $b = 0$ in the above calculations it is seen that $[uv](0) = 0$.

Since the problem (3.1),(3.2) is self-adjoint the eigenfunctions $\{\theta_{bn}\}$ with corresponding eigenvalues $\{\lambda_n\}$ form a complete orthonormal set. For a proof see [5].

The function ψ defined on page (15) satisfies

$$P^{\frac{1}{2}}(0)\,\psi_1(0,\ell) = \sin\alpha , \quad P^{\frac{1}{2}}(0)\,\psi_2(0,\ell) = \cos\alpha ;$$

hence $\psi(0,\ell)$ satisfies

$$\psi_1(0,\ell)\cos\alpha - \psi_2(0,\ell)\sin\alpha = 0 , \tag{3.9}$$

and no solution of $Lx = \ell x$ independent of ψ can satisfy condition (3.9). To see this, suppose there exists a solution of $\widetilde{\psi}$ of $Lx = \ell x$ independent of ψ and satisfying (3.9). Then

$$\psi_1(0,\ell)\cos\alpha - \psi_2(0,\ell)\sin\alpha = 0 \tag{3.10}$$

$$\widetilde{\psi}_1(0,\ell)\cos\alpha - \widetilde{\psi}_2(0,\ell)\sin\alpha = 0 . \tag{3.11}$$

Assume $\alpha \neq 0, \pi/2$ and $\tilde{\psi}(0,\ell) \neq 0$, $\psi_1(0,\ell) \neq 0$; then there is a number $\delta \neq 0$ such that

$$\delta\psi_1(0,\ell) = \tilde{\psi}_1(0,\ell) . \tag{3.12}$$

Equations (3.10), (3.11) and (3.12) imply

$$\delta\psi_1(0,\ell) \cos\alpha = \tilde{\psi}_1(0,\ell) \cos\alpha = \delta\psi_2(0,\ell) \sin\alpha = \tilde{\psi}_2(0,\ell) \sin\alpha$$

and therefore

$$\delta\psi_2(0,\ell) = \tilde{\psi}_2(0,\ell) .$$

Thus it is seen that $\tilde{\psi}$ and $\delta\psi$ are both solutions of $Lx = \ell x$ satisfying the same initial conditions. The uniqueness theorems of differential equations imply $\tilde{\psi} = \delta\psi$. This is a contradiction since it was assumed $\tilde{\psi}$ and ψ were linearly independent. Similar arguments take care of the cases $\alpha = 0, \pi/2$.

The eigenfunctions $\{\theta_{bn}\}$ may be written as $\theta_{bn} = r_{bn}\psi(t,\lambda_{bn})$ where r_{bn} is a constant and $\{\lambda_{bn}\}$ is the sequence of eigenvalues corresponding to the complete set of eigenfunctions $\{\psi(t,\lambda_{bn}\}$.

The completeness theorem (1.1) applied to any continuous vector function f defined on $0 \leq t < \infty$ which vanishes outside $0 \leq t \leq C$ where $0 < C < b$ yields

$$\int_0^b f \cdot f \, dt = \sum_{n=1}^{\infty} |r_{bn}|^2 \; \left| \int_0^b f(t) \cdot \psi(t,\lambda_{bn}) \, dt \right|^2 \tag{3.13}$$

Let the transform of f with respect to ψ be g, that is, let

$$g(\lambda) = \int_0^\infty f(t) \cdot \psi(t,\lambda)\, dt ,$$

and let ρ_b be a monotone non-decreasing step function of λ having jumps of $|r_{bn}|^2$ at each eigenvalue λ_{bn} and otherwise constant. Assume further that $\rho_b(\lambda + 0) = \rho_b(\lambda)$ and $\rho_b(0) = 0$; then the Parseval equality (3.13) may be written as

$$\int_0^\infty f \cdot f\, dt = \int_{-\infty}^\infty |g(\lambda)|^2 d\rho_b(\lambda) . \qquad (3.14)$$

<u>Definition 3.1</u>.--Let $L^2(\rho)$ denote the set of all functions h which are measurable with respect to the Lebesque-Stieltjes measure defined by the monotone non-decreasing function ρ and such that

$$\int_{-\infty}^\infty |h(\lambda)|^2\, d\rho(\lambda) < \infty .$$

<u>Theorem 3.1</u>.--Let L be in the limit-point case at ∞. Than

(i) There exists a monotone non-decreasing function ρ on $-\infty < \lambda < \infty$ such that

$$\rho(\lambda) - \rho(\mu) = \lim_{b \to \infty} [\rho_b(\lambda) - \rho_b(\mu)] \qquad (3.15)$$

at points of continuity λ, μ of ρ.

(ii) If $f \in L^2(0,\infty)$ there exists a function $g \in L^2(\rho)$ such that

$$\lim_{a \to \infty} \int_{-\infty}^{\infty} |g(\lambda) - \int_0^a f(t) \cdot \psi(t,\lambda)\, dt|^2 \, d\rho(\lambda) = 0 , \tag{3.16}$$

and

$$\int_0^\infty (f \cdot f)\, dt = \int_{-\infty}^{\infty} |g(\lambda)|^2 \, d\rho(\lambda) . \tag{3.17}$$

(iii) The integral

$$\int_{-\infty}^{\infty} g(\lambda)\, \psi(t,\lambda)\, d\rho(\lambda) \tag{3.18}$$

converges in $L^2(0,\infty)$ to f, that is,

$$\lim_{(u,v) \to (-\infty,\infty)} \int_0^\infty |f(t) - \int_\mu^v g(\lambda)\, \psi(t,\lambda)\, d\rho(\lambda)|^2 \, dt = 0 . \tag{3.19}$$

(iv) If m_∞ is the limit-point, considered as a function of ℓ,

$$\rho(\lambda) - \rho(\mu) = \lim_{\varepsilon \to 0+} \frac{1}{\pi} \int_\mu^\lambda \operatorname{Im} m_\infty (v + i\varepsilon)\, dv \tag{3.20}$$

at points of continuity λ, μ of ρ and inversely

$$m_\infty(\ell) - m_\infty(\ell_0) = \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - \ell} - \frac{1}{\lambda - \ell_0}\right) d\rho(\lambda) + C(\ell - \ell_0) \tag{3.21}$$

where C is a nonnegative constant, and $\operatorname{Im} \ell_0 \neq 0$.

<u>Proof</u>.--Let $m_b(\ell)$ be a point on the circle C_b where $\operatorname{Im} \ell > 0$. Then the completeness theorem (3.14) applied on $[a,b]$ to the solution $x_b = \varphi + m_b\psi$ of $Lx = \ell x$ with

$$x_{b1}(b) \cos\beta - x_{b2}(b) \sin\beta = 0$$

yields

$$\int_0^b x_b \cdot x_b \, dt = \sum_{n=1}^{\infty} |r_{bn}|^2 \, \left|\int_0^b x_b(t) \cdot \psi(t,\lambda_{bn}) \, dt\right|^2. \qquad (3.22)$$

Green's formula gives

$$(\ell - \lambda_{bn}) \int_0^b x_b \cdot \psi \, dt = [x_b\psi](b) - [x_b\psi](0) \, .$$

Because of the boundary conditions satisfied by x_b and ψ, $[x_b\psi](b) = 0$ and $[x_b\psi](0) = 1$. Using this in equation (3.22) it becomes

$$\int_0^b (x_b \cdot x_b) \, dt = \sum_{n=1}^{\infty} \frac{|r_{bn}|^2}{|\ell - \lambda_{bn}|^2} \, . \qquad (3.23)$$

Taking into account the way ρ_b was defined (3.23) may be written as

$$\int_0^b (x_b \cdot x_b) \, dt = \int_{-\infty}^{\infty} \frac{d\rho_b(\lambda)}{|\lambda - \ell|^2} \, . \qquad (3.24)$$

Since m_b is on C_b

$$\int_0^b (x_b \cdot x_b) \, dt = \frac{\operatorname{Im} m_b(\ell)}{\operatorname{Im} \ell} \, , \qquad \operatorname{Im} \ell \neq 0. \qquad (2.19)$$

From (3.24) and (2.19) there results

$$\int_{-\infty}^{\infty} \frac{d\rho_b(\lambda)}{|\lambda - \ell|^2} = \frac{\operatorname{Im} m_b(\ell)}{\operatorname{Im} \ell} \, , \qquad \operatorname{Im} \ell \neq 0 \, . \qquad (3.25)$$

The proof of the Parseval equality follows from the use of the Helly selection theorem to establish the existence of a limiting function ρ

and the subsequent use of an integration theorem in essentially the same manner as in the proof of the second order scalar case. For a proof of the selection and integration theorems see [4].

To prove the expansion formula (3.19) the following is needed.

Lemma 3.1.--If f_1, f_2 ε $L^2(0,\infty)$ and g_1, g_2 are the corresponding transforms, then

$$\int_0^\infty f_1 \cdot f_2 \, dt = \int_{-\infty}^\infty g_1 \bar{g}_2 \, d\rho(\lambda) \; . \tag{3.26}$$

Proof.--Let f_{11}, f_{12} and f_{21}, f_{22} be the components of f_1 and f_2 respectively.

$$4 \, f_1 \cdot f_2 = 4 \, f_{11} \, \bar{f}_{21} + 4 \, f_{12} \, \bar{f}_{22} \; . \tag{3.27}$$

The scalar identity

$$4 \, f_a \cdot \bar{f}_b = |f_a + f_b|^2 - |f_a - f_b|^2 + i|f_a + if_b|^2 - i|f_a - if_b|^2 \tag{3.28}$$

applied to the right side of (3.27) yields

$$\begin{aligned} 4 \, f_1 \cdot f_2 = {} & |f_{11} + f_{21}|^2 - |f_{11} - f_{21}|^2 + i|f_{11} + if_{21}|^2 \\ & - i|f_{11} - if_{21}|^2 + |f_{12} + f_{22}|^2 - |f_{12} - f_{22}|^2 \\ & + i|f_{12} + if_{22}|^2 - i|f_{12} - if_{22}|^2 \; . \end{aligned} \tag{3.29}$$

Equation (3.29) may be written

$$4\, f_1 \cdot f_2 = (f_1 + f_2)\cdot(f_1 + f_2) - (f_1 - f_2)\cdot(f_1 - f_2) \tag{3.30}$$

$$+\, i(f_1 + if_2)\cdot(f_1 + if_2) - i(f_1 - if_2)\cdot(f_1 - if_2)\ .$$

Integrating both sides of (3.30) from 0 to ∞ gives

$$\int_0^\infty 4\, f_1 \cdot f_2 \, dt = \int_0^\infty (f_1 + f_2)\cdot(f_1 + f_2)\, dt \tag{3.31}$$

$$- \int_0^\infty (f_1 - f_2)\cdot(f_1 - f_2)\, dt$$

$$+ i\int_0^\infty (f_1 + if_2)\cdot(f_1 + if_2)\, dt$$

$$- i\int_0^\infty (f_1 - if_2)\cdot(f_1 - if_2)\, dt\ .$$

Let g be the transform of $C_1 f_1 + C_2 f_2$ and g_1, g_2 the transforms of f_1 and f_2 respectively.

$$g(\lambda) = \int_{-\infty}^\infty [C_1 f_1 + C_2 f_2](t)\cdot\psi(t,\lambda)\, dt$$

$$g(\lambda) = C_1 \int_{-\infty}^\infty f_1(t)\cdot\psi(t,\lambda)\, dt + C_2 \int_{-\infty}^\infty f_2(t)\cdot\psi(t,\lambda)\, dt$$

$$g(\lambda) = C_1 g_1(\lambda) + C_2 g_2(\lambda)\ . \tag{3.32}$$

Now (3.32), (3.31) and (3.17) imply

$$4\int_0^\infty f_1(t)\cdot f_2(t)\, dt = \int_{-\infty}^\infty |g_1(\lambda) + g_2(\lambda)|^2\, d\rho(\lambda)$$

$$- \int_{-\infty}^\infty |g_1(\lambda) - g_2(\lambda)|^2\, d\rho(\lambda) +$$

$$+ i \int_{-\infty}^{\infty} |g_1(\lambda) + ig_2(\lambda)|^2 \, d\rho(\lambda)$$

$$- i \int_{-\infty}^{\infty} |g_1(\lambda) - ig_2(\lambda)|^2 \, d\rho(\lambda) ,$$

$$4 \int_0^{\infty} f_1(t) \cdot f_2(t) \, dt = 4 \int_{-\infty}^{\infty} g_1 \bar{g}_2 d\rho(\lambda) .$$

With (3.26) established the remainder of the proof follows that given for the second order scalar case in [3].

CHAPTER IV

THE LIMIT-CIRCLE CASE

If L is in the limit-circle case at infinity, the circles $C_b(\ell)$ converge to a circle $C_\infty(\ell)$ as $b \to \infty$ for each ℓ, $\text{Im}\,\ell \neq 0$. A point $\hat{m}_\infty(\ell_0)$ on the circle $C_\infty(\ell)$ is the limit point of a sequence $m(\ell, b_j, \beta_j)$, $j = 1, 2, \ldots$, with $b_j \to \infty$ as $j \to \infty$.

Let m_j denote the function of ℓ given by $m_j(\ell) = m(\ell, b_j, \beta_j)$. Let ρ_j denote the step function ρ_b associated with the condition β_j at b_j.

<u>Theorem 4.1</u>.--Let $\hat{m}_\infty(\ell_0)$ be a point on $C_\infty(\ell_0)$ and (b_j, β_j) a sequence such that $m(\ell_0, b_j, \beta_j) = m_j(\ell_0)$ tends to $\hat{m}_\infty(\ell_0)$ as $j \to \infty$. Then for all ℓ

$$\lim_{j \to \infty} m_j(\ell) = \hat{m}_\infty(\ell) \tag{4.1}$$

and

$$\hat{\rho}(\lambda) - \hat{\rho}(\mu) = \lim_{j \to \infty} [\rho_j(\lambda) - \rho_j(\mu)] \tag{4.2}$$

at points of continuity λ, μ of $\hat{\rho}$. $\hat{m}_\infty$ is a meromorphic function of ℓ, real for real ℓ, and with poles and zeros that are real and simple. $\hat{\rho}$ is a step function discontinuous at the poles, $\ell = \lambda_k$, $k = 1, 2, \ldots$ of $\hat{m}_\infty$ only and with a jump at λ_k equal to minus the residue of $\hat{m}_\infty$ at λ_k. the functions ψ_k, where $\psi_k(t) = \psi(t, \lambda_k)$, form a complete orthogonal family in $L^2(0,\infty)$. If $\hat{x}_\infty$ is the function defined by $\hat{x}_\infty(\ell) = \varphi(t,\ell_o) + \hat{m}_\infty(\ell_o)\,\psi(t,\ell_o)$, then

$$[\psi_k \hat{x}_\infty](\infty) = 0 \tag{4.3}$$

for all k. On the other hand, for ψ_ℓ, where $\psi_\ell(t) = \psi(t,\ell)$,

$$[\psi_\ell x_\infty](\infty) \neq 0, \qquad \ell \neq \lambda_k, \quad k = 1, 2, \ldots .$$

<u>Proof</u>.--Let

$$x_j(t,\ell) = \varphi(t,\ell) + m_j(\ell)\,\psi(t,\ell); \tag{4.4}$$

then Green's formula yields

$$\int_o^{b_j} [Lx_j(t,\ell)\cdot \overline{x}_j(t,\ell_o) - x_j(t,\ell)\cdot L\overline{x}_j(t,\ell_o)]\,dt$$

$$= [x_j(\ell)\ \overline{x}_j(\ell_o)](b_j) - [x_j(\ell)\ \overline{x}_j(\ell_o)](0) . \tag{4.5}$$

Since $x_j(t,\ell)$ and $\overline{x}_j(t,\ell_o)$ satisfy the same boundary conditions at b_j

$$[x_j(t,\ell)\ \overline{x}_j(t,\ell_o)](b_j) = 0 .$$

Therefore (4.5) becomes

$$(\ell - \ell_o)\int_o^{b_j} x_j(t,\ell)\cdot\overline{x}_j(t,\ell_o)\,dt = -[x_j(t,\ell)\ \overline{x}_j(t,\ell_o)](0). \tag{4.6}$$

$$- [x_j(\ell)\ \overline{x}_j(\ell_o)](0) = - P(0)\Big(\{\varphi_2(0,\ell) + m_j(\ell)\ \psi_2(0,\ell)\} \times \{\varphi_1(0,\ell_o) + m_j(\ell_o)\ \psi_1(0,\ell_o)\} - \{\varphi_1(0,\ell) + m_j(\ell)\ \psi_1(0,\ell)\}\{\varphi_2(0,\ell_o) + m_j(\ell_o)\ \psi_2(0,\ell_o)\}\Big) . \tag{4.7}$$

Since ψ and φ satisfy the initial conditions (2.8), (4.7) becomes

$$- [x_j(\ell)\ \overline{x}_j(\ell_o)](0) = m_j(\ell) - m_j(\ell_o) .$$

Using this in (4.6) it is seen that

$$m_j(\ell) - m_j(\ell_o) = (\ell - \ell_o) \int_0^{b_j} x_j(t,\ell) \cdot \overline{x}_j(t,\ell_o)\ dt . \tag{4.8}$$

Putting (4.4) in (4.8) results in

$$m_j(\ell) = \frac{m_j(\ell_o) + (\ell - \ell_o) \int_0^{b_j} \varphi(t,\ell)\ x_j(t,\ell_o)\ dt}{1 - (\ell - \ell_o) \int_0^{b_j} \psi(t,\ell)\ x_j(t,\ell_o)\ dt} .$$

The remainder of the proof, except for the orthogonality of the ψ_k's which will be proved later, is essentially the same as in the proof for the second order scalar case given in [3].

Let $\hat{D}$ denote the set of all functions u such that

(i) u is differentiable on $0 \le t \le b$ for all $b < \infty$,

(ii) u and Lu ε $L^2(0,\infty)$,

(iii) $u_1(0) \cos \alpha - u_2(0) \sin \alpha = 0$,

(iv) $[u\hat{x}_\infty](\infty) = 0$.

Let

$$\hat{G}(t,\tau,\ell_o) = \begin{cases} \begin{pmatrix} \psi_1(t,\ell_o)\ \hat{x}_{\infty 1}(\tau,\ell_o) & \psi_1(t,\ell_o)\ \hat{x}_{\infty 2}(\tau,\ell_o) \\ \psi_2(t,\ell_o)\ \hat{x}_{\infty 1}(\tau,\ell_o) & \psi_2(t,\ell_o)\ \hat{x}_{\infty 2}(\tau,\ell_o) \end{pmatrix} & t \leq \tau \\ \\ \begin{pmatrix} \psi_1(\tau,\ell_o)\ \hat{x}_{\infty 1}(\tau,\ell_o) & \psi_2(\tau,\ell_o)\ \hat{x}_{\infty 1}(\tau,\ell_o) \\ \psi_1(\tau,\ell_o)\ \hat{x}_{\infty 2}(\tau,\ell_o) & \psi_2(\tau,\ell_o)\ \hat{x}_{\infty 2}(\tau,\ell_o) \end{pmatrix} & t > \tau \end{cases} \tag{4.9}$$

Let

$$G_1(t,\tau,\ell_o) = \hat{G}(t,\tau,\ell_o) \qquad t < \tau ,$$

$$G_2(t,\tau,\ell_o) = \hat{G}(t,\tau,\ell_o) \qquad t > \tau .$$

For any $f \varepsilon L^2(0,\infty)$ let

$$\hat{g}(\ell_o)\ f(t) = \int_0^\infty \hat{G}(t,\tau,\ell_o)\ f(\tau)\ d\tau . \tag{4.10}$$

The integral in (4.10) is absolutely convergent since f, ψ and $\hat{x}_\infty$ are in $L^2(0,\infty)$.

<u>Theorem 4.2</u>.--For any $f \varepsilon L^2(0,\infty)$ the function $u = \hat{g}(\ell_o)f \varepsilon \hat{D}$ and $(L - \ell_o)u = f$. Conversely, if $u \varepsilon \hat{D}$, then $f = (L - \ell_o)u \varepsilon L^2(0,\infty)$ and $u = \hat{g}(\ell_o)f$.

<u>Proof</u>.--To prove the first half of the theorem let $u = \hat{g}(\ell_o)f$; then

$$Lu = L\int_0^\infty \hat{G}(t,\tau,\ell_o)\ f(\tau)\ d\tau = L\int_t^\infty G_1(t,\tau,\ell_o)\ f(\tau)\ d\tau$$

$$+ L\int_0^\infty G_2(t,\tau,\ell_o)\ f(\tau)\ d\tau ,$$

$$Lu = \int_t^\infty LG_1(t,\tau,\ell_o)\ f(\tau)\ d\tau + \int_o^t LG_2(t,\tau,\ell_o)\ f(\tau)\ d\tau$$

$$+ P_o(t)\ [- G_1(t,t+0,\ell_o) + G_2(t,t-0,\ell_o)]\ f(t)\ ,$$

$$Lu = \ell_o\Big(\int_t^\infty G_1(t,\tau,\ell_o)\ f(\tau)\ d\tau + \int_o^t G_2(t,\tau,\ell_o)\ f(\tau)\ d\tau\Big)$$

$$+ P_o(t)[G_2(t,t-0,\ell_o) - G_1(t,t+0,\ell_o)]\ f(t)\ . \qquad (4.12)$$

From (4.9) it may be seen that

$$G_2(t,t,\ell_o) - G_1(t,t,\ell_o)$$

$$= \begin{pmatrix} 0 & -(\hat{x}_{\infty 2}(t,\ell_o)\ \psi_1(t,\ell_o) - x_{\infty 1}(t,\ell_o)\ \psi_2(t,\ell_o)) \\ \hat{x}_{\infty 2}(t,\ell_o)\ \psi_1(\tau,\ell_o) - x_{\infty 1}(t,\ell_o)\ \psi_2(t,\ell_o) & 0 \end{pmatrix}$$

$$G_2(t,t,\ell_o) - G_1(t,t,\ell_o) = \begin{pmatrix} 0 & -1/P(t)[\hat{x}\ \overline{\psi}](t) \\ 1/P(t)[\hat{x}\ \overline{\psi}](t) & 0 \end{pmatrix}$$

Green's formula gives

$$[\hat{x}\ \overline{\psi}](t) - [\hat{x}\ \overline{\psi}](0) = 0\ .$$

From the boundary condition that φ and ψ satisfy $[\hat{x}\ \overline{\psi}](0)$ may be calculated.

$$[\hat{x}\ \overline{\psi}](0) = P(0)\Big(\{\varphi_2(0) + m_\infty\psi_2(0)\}\psi_1(0) - \{\varphi_1(0) + m_\infty\psi_1(0)\}\psi_2(0)\Big)\ ,$$

$$[\hat{x}\ \overline{\psi}](0) = P(0)\{\varphi_2(0)\ \psi_1(0) - \varphi_1(0)\ \psi(0)\} = 1\ .$$

Therefore

$$G_2(t,t,\ell_o) - G_1(t,t,\ell_o) = P_o^{-1}(t) . \tag{4.13}$$

By using (4.13) in (4.12) it is clear that

$$Lu = \ell_o u + f .$$

To see that $u \varepsilon \hat{D}$ observe that (i) has already been shown. u is in $L^2(0,\infty)$ since ψ and $\hat{x}$ are in $L^2(0,\infty)$. Since $Lu = \ell_o u + f$ and $f \varepsilon L^2(0,\infty)$ then $Lu \varepsilon L^2(0,\infty)$ and u satisfies (ii).

$$u_1(0) = \int_0^\infty \psi_1(0,\ell_o)\ \hat{x}_{\infty 1}(\tau,\ell^o)\ f_1(\tau)\ d\tau - \int_0^\infty \psi_1(0,\ell_o)\ \hat{x}_{\infty 2}(\tau,\ell_o)\ f_2(\tau)\ d\tau , \tag{4.14}$$

$$u_2(0) = \int_0^\infty \psi_2(0)\ \hat{x}_{\infty 1}(\tau)\ f_1(\tau)\ d\tau - \int_0^\infty (\psi_2(0)\ \hat{x}_{\infty 2}(\tau)\ f_2(\tau)\ d\tau . \tag{4.15}$$

From (4.14) and (4.15) and the fact that ψ satisfies (iii) it is clear that u must also satisfy (iii).

$$u_1(\infty) = \int_0^\infty \psi_1(\tau,\ell_o)\ \hat{x}_{\infty 1}(\infty,\ell_o)\ f_1(\tau)\ d\tau - \int_0^\infty \psi_2(\tau,\ell_o)\ \hat{x}_{\infty 1}(\infty,\ell_o)\ f_2(\tau)\ d\tau \tag{4.16}$$

$$u_2(\infty) = \int_0^\infty \psi_1(\tau)\ \hat{x}_{\infty 2}(\infty,\ell_o)\ f_1(\tau)\ d\tau - \int_0^\infty \psi_2(\tau,\ell_o)\ \hat{x}_{\infty 2}(\infty,\ell_o)\ f_2(\tau)\ d\tau . \tag{4.17}$$

$$u(\infty) = \hat{x}_\infty(\infty)\left[\int_0^\infty \psi_1(\tau,\ell_0)\ f_1(\tau)\ d\tau - \int_0^\infty \psi_2(\tau,\ell_0)\ f_2(\tau)\ d\tau\right],$$

$$u(\infty) = C\ \hat{x}_\infty(\infty)\ . \tag{4.18}$$

From (4.18) it is clear that

$$[u\ \hat{x}_\infty](\infty) = [C\ \hat{x}_\infty\hat{x}_\infty](\infty) = C[\hat{x}_\infty\hat{x}_\infty](\infty).$$

By theorem 2.2 $[\hat{x}_\infty\hat{x}_\infty](\infty) = 0$ and consequently u satisfies (iv).

The second half of the theorem will now be proved. Let $u \varepsilon \hat{D}$ and let $f = Lu - \ell_0 u$. Then $f \varepsilon L^2(0,\infty)$ and from the first part of the theorem $\hat{g}(\ell_0)\ f$ is of the class $\hat{D}$. Thus if $\omega = u - \hat{g}(\ell_0)\ f$ then $\omega \varepsilon \hat{D}$ and

$$L\omega = Lu - L\ \hat{g}(\ell_0)\ f = Lu - \ell_0\hat{g}(\ell_0)\ f - f\ ,$$

$$L\omega = Lu - \ell_0\hat{g}(\ell_0)\ f - Lu + \ell_0 u\ ,$$

$$L\omega = \ell_0 u - \ell_0\hat{g}(\ell_0)\ f = \ell_0[u - \hat{g}(\ell_0)\ f] = \ell_0\omega\ .$$

Hence $\omega = C_1\psi + C_2\hat{x}_\infty$ for some constants C_1 and C_2. Since ω satisfies (iii)

$$[C_1\psi_1(0) + C_2\hat{x}_{\infty 1}(0)]\cos\alpha - [C_1\psi_2(0) + C_2\hat{x}_{\infty 2}(0)]\sin\alpha = 0\ ,$$

and since ψ satisfies (iii) there results

$$C_2[\hat{x}_{\infty 1}(0)\cos\alpha - \hat{x}_{\infty 2}(0)\sin\alpha] = 0\ .$$

Unless $C_2 = 0$ then $\hat{x}_\infty$ satisfies (iii). If $\hat{x}_\infty$ satisfied (iii) then since $\hat{x}_\infty = \varphi + m_\infty \psi$, ψ and φ would satisfy (iii), but this is not possible since φ and ψ are linearly independent. Thus $C_2 = 0$. Since ω satisfies (iv)

$$[\omega\hat{x}_\infty](\infty) = [C_1\psi\hat{x}_\infty](\infty) = C_1[\psi\hat{x}_\infty](\infty) = 0 .$$

From the proof of theorem 2.2 it follows that, if $\tilde{m}_\infty$ is the center of the circle $C_\infty(\ell_0)$, then

$$[\varphi\psi](\infty) + \tilde{m}_\infty [\psi\psi](\infty) = 0$$

and the reciprocal of the radius of $C_\infty(\ell_0)$ is

$$|[\psi\psi](\infty)| > 0$$

$$\hat{x}_\infty = \varphi + \tilde{m}_\infty \psi$$

$$[\psi\hat{x}_\infty](\infty) = [\psi(\varphi + \hat{m}_\infty \psi)](\infty)$$

$$[\psi\hat{x}_\infty](\infty) = [\psi\varphi](\infty) + \bar{\hat{m}}_\infty [\psi\psi](\infty)$$

$$[\psi\hat{x}_\infty](\infty) = -\bar{\tilde{m}}_\infty [\psi\psi](\infty) + \bar{\hat{m}}_\infty [\psi\psi](\infty) .$$

Thus $[\psi\hat{x}_\infty](\infty) \neq 0$ and therefore $C_1 = 0$ and the theorem is proved.

<u>Theorem 4.3</u>.--The boundary value problem

$$Lx = \ell x, \qquad \cos\alpha\, x_1(0) - \sin\alpha\, x_2(0) = 0, \qquad [x\, \hat{x}_\infty](\infty) = 0$$

is self-adjoint; that is, for all u and v of class $\hat{D}$

$$\int_0^\infty (Lu \cdot v \, dt = \int_0^\infty u \cdot (Lv) \, dt \, . \tag{4.19}$$

Proof.--

$$- \overline{v}_1(0) \, [\cos \alpha \; u_1(0) - \sin \alpha \; u_2(0)] = 0 \tag{4.20}$$

$$u(0) \, [\cos \alpha \; \overline{v}_1(0) - \sin \alpha \; \overline{v}_2(0)] = 0 \, . \tag{4.21}$$

Addition of (4.20) and (4.21) gives

$$u_2(0) \; \overline{v}_1(0) - u_1(0) \; \overline{v}_2(0) = 0$$

which implies $[uv](0) = 0$ so that Green's formula implies (4.19) is equivalent to

$$[uv](\infty) = 0 \, . \tag{4.22}$$

From theorem 4.2 there exists $f, g \;\varepsilon\; L^2(0,\infty)$ such that $u = \hat{g}(\ell_0) \, f$ and $v = \hat{g}(\ell_0) \, g$.

$$u(\infty) = \hat{x}_\infty(\infty) \left[\int_0^\infty \psi_1(\tau) \; f_1(\tau) \; d\tau - \int_0^\infty \psi_2(\tau) \; f_2(\tau) \; d\tau\right]$$

$$u(\infty) = C_1 \hat{x}_\infty(\infty) \, . \tag{4.23}$$

Similarly

$$v(\infty) = C_2 \hat{x}_\infty(\infty) \, . \tag{4.24}$$

From (4.23) and (4.24) it is clear that

$$[uv](\infty) = [C_1 \hat{x}_\infty C_2 \hat{x}_\infty](\infty) = C_1 \overline{C}_2 [\hat{x}_\infty \hat{x}_\infty](\infty) = 0 \, .$$

Thus (4.22) which is equivalent to (4.19) is established and the theorem is proved.

Since ψ_k satisfies (4.3), ψ_k is of class $\hat{D}$ and (4.19) shows the ψ_k's are orthogonal, thus completing the proof of theorem 4.1.

CHAPTER V

SINGULAR BEHAVIOR AT BOTH ENDS OF AN INTERVAL

In this chapter the coefficients in L will be assumed to be continuous and the nature of the problem, as both ends points of the interval under consideration approach infinity, will be sutdied.

Let $\varphi_1(t,\ell)$ and $\varphi_2(t,\ell)$ be solutions to $Lx = \ell x$ satisfying the initial conditions

$$\begin{aligned} P^{\frac{1}{2}}(0)\ \varphi_{11}(0,\ell) &= 0, \qquad P^{\frac{1}{2}}(0)\ \varphi_{21}(0,\ell) = 1, \\ P^{\frac{1}{2}}(0)\ \varphi_{12}(0,\ell) &= 1, \qquad P^{\frac{1}{2}}(0)\ \varphi_{22}(0,\ell) = 0\ . \end{aligned} \tag{5.1}$$

Let δ: $a \leq t \leq b$ by any finite interval containing zero, and consider the self-adjoint boundary value problem on δ:

$$Lx = \ell x$$

$$\cos\alpha\ x_1(a) - \sin\alpha\ x_2(a) = 0$$

$$\cos\beta\ x_1(b) - \sin\beta\ x_2(b) = 0 \tag{5.2}$$

where $0 < \alpha,\beta < \pi$.

From [5] it may be seen there exists a sequence of real eigenvalues $\{\lambda_{\delta n}\}$, $n = 1,2,\ldots$ and a complete orthonormal set of eigenfunctions $\{h_{\delta n}\}$. For any $f \in L^2(a,b)$ the Parseval equality holds.

$$\int_{\delta} f(t) \cdot f(t)\ dt = \sum_{n=1}^{\infty} \left| \int_{\delta} f(t) \cdot h_{\delta n}(t)\ dt \right|^2 \tag{5.3}$$

If $f_1, f_2 \in L^2(a,b)$ then by (3.26)

$$\int_{\delta} f_1(t) \cdot f_2(t)\ dt = \sum_{n=1}^{\infty} \left(\int_{\delta} f_1 \cdot h_{\delta n}\ dt\right) \overline{\left(\int_{\delta} f_2 \cdot h_{\delta n}\ dt\right)} \tag{5.4}$$

Since φ_1, φ_2 form a basis for the solutions of $Lx = \ell x$

$$h_{\delta n}(t) = r_{\delta n1}\, \varphi_1(t,\lambda_{\delta n}) + r_{\delta n2}\, \varphi_2(t,\lambda_{\delta n}) \tag{5.5}$$

where $r_{\delta n1}$ and $r_{\delta n2}$ are complex constants. Placing (5.5) in (5.3) the Parseval equality may be written

$$\int_{\delta} f \cdot f\ dt = \int_{-\infty}^{\infty} \sum_{j,k}^{2} \overline{g}_{\delta j}(\lambda)\ g_{\delta k}(\lambda)\ d\rho_{jk}(\lambda) \tag{5.6}$$

where

$$g_{\delta j} = \int_{\delta} f(t) \cdot \varphi_j(t,\lambda)\ dt, \qquad j = 1,2 \tag{5.7}$$

The matrix $\rho_{\delta} = (\rho_{\delta jk})$, called the spectral matrix associated with the problem (5.2), consists of step functions with jumps at the eigenvalues $\lambda_{\delta n}$ given by

$$\rho_{\delta jk}(\lambda_{\delta n} + 0) - \rho_{\delta jk}(\lambda_{\delta n} - 0) = \sum r_{\delta mj}\, \overline{r}_{\delta mk}, \qquad (j,k) = 1,2 \tag{5.8}$$

where the sum is taken over all m such that $\lambda_{\delta m} = \lambda_{\delta n}$ since there may be more than one $h_{\delta n}$ corresponding to $\lambda_{\delta n}$.

Let $P_\delta(\lambda + 0) = P_\delta(\lambda)$ and $P_\delta(0)$ be the zero matrix. P_δ possesses the properties:

(i) P_δ is Hermitian,

(ii) $P_\delta(\Delta) = P_\delta(\lambda) - P_\delta(\mu)$ is positive semidefinite if $\lambda > \mu$ $(\Delta = (-,\lambda])$.

(iii) The total variation of $P_{\delta jk}$ is finite on every finite λ interval.

It will now be shown that as $\delta \to (-\infty,\infty)$ there exists a limiting matrix ρ having the properties (i) through (iii).

Let $x_a = \varphi_1 + m_a\varphi_2$ be a solution of $Lx = \ell x$ $(\operatorname{Im} \ell \neq 0)$ satisfying the boundary condition

$$\cos\alpha\ x_1(a) - \sin\alpha\ x_2(a) = 0 \tag{5.9}$$

and similarly let $x_b = \varphi_1 + m_b\varphi_2$ be a solution of the same equation satisfying

$$\cos\beta\ x_1(b) - \sin\beta\ x_2(b) = 0\ . \tag{5.10}$$

Green's matrix for the problem (5.2) is given by

$$G_\delta(t,\tau,\ell) =$$

$$\begin{cases} \dfrac{1}{m_a(\ell) - m_b(\ell)} \begin{pmatrix} x_{a1}(t,\ell)\ x_{b1}(\tau,\ell) & x_{a1}(t,\ell)\ x_{b2}(\tau,\ell) \\ x_{a2}(t,\ell)\ x_{b1}(\tau,\ell) & x_{a2}(t,\ell)\ x_{b2}(\tau,\ell) \end{pmatrix} & t \le \tau \\ \\ \dfrac{1}{m_a(\ell) - m_b(\ell)} \begin{pmatrix} x_{a1}(\tau,\ell)\ x_{b1}(t,\ell) & x_{a2}(\tau,\ell)\ x_{b1}(t,\ell) \\ x_{a1}(\tau,\ell)\ x_{b2}(t,\ell) & x_{a2}(\tau,\ell)\ x_{b2}(t,\ell) \end{pmatrix} & t > \tau \end{cases} \tag{5.11}$$

$$f_1(t) = \begin{cases} \dfrac{x_a(t,\ell)\ x_{b1}(0,\ell)}{m_a(\ell) - m_b(\ell)}, & t \leq 0 \\[2ex] \dfrac{x_b(t,\ell)\ x_{a1}(0,\ell)}{m_a(\ell) - m_b(\ell)}, & t > 0 \end{cases}$$

$$f_2(t) = \begin{cases} \dfrac{x_a(t,\ell)\ x_{b2}(0,\ell)}{m_a(\ell) - m_b(\ell)}, & t \leq 0 \\[2ex] \dfrac{x_b(t,\ell)\ x_{a2}(0,\ell)}{m_a(\ell) - m_b(\ell)}, & t > 0 \end{cases}$$

Note that f_1, f_2 are the first and second columns of $G_\delta(t,0,\ell)$. The completeness relationship (5.4) applied to the functions f_j and f_k yields

$$\int_\delta f_j \cdot f_k \, dt = \sum_{n=1}^{\infty} \left(\int_\delta f_j \cdot h_{\delta n} dt\right) \left(\overline{\int_\delta f_k \cdot h_{\delta n}\, dt}\right) \tag{5.12}$$

$$x_{a_1}(0,\ell) = \varphi_{11}(0) + m_a\, \varphi_{21}(0) = m_a / P^{\frac{1}{2}}(0)$$

$$x_{a2}(0,\ell) = \varphi_{12}(0) + m_a\, \varphi_{22}(0) = 1/ P^{\frac{1}{2}}(0)$$

$$x_{b1}(0,\ell) = \varphi_{11}(0) + m_b\, \varphi_{21}(0) = m_b / P^{\frac{1}{2}}(0)$$

$$x_{b2}(0,\ell) = \varphi_{12}(0) + m_b\, \varphi_{22}(0) = 1 / P^{\frac{1}{2}}(0)$$

$$[x_b x_b](0) = P(0)\,[x_{b2}(0)\,\bar{x}_{b1}(0) - x_{b1}(0)\,\bar{x}_{b2}(0)]$$

$$[x_b x_b](0) = P(0)\left(\frac{\bar{m}_b}{P^{\frac{1}{2}}(0)\,P^{\frac{1}{2}}(0)} - \frac{m_b}{P^{\frac{1}{2}}(0)\,P^{\frac{1}{2}}(0)}\right) = -(m_b - \bar{m}_b)$$

$$[x_a x_a](0) = P(0)\,[x_{a2}(0)\,\bar{x}_{a1}(0) - x_{a1}(0)\,\bar{x}_{a2}(0)]$$

$$[x_a x_a](0) = P(0)\left(\frac{\bar{m}_a}{P(0)} - \frac{\bar{m}_b}{P(0)}\right) = -(m_a - \bar{m}_a)$$

Using the above formulas and Green's formula the integrals in (5.12) can be evaluated.

$$\int_\delta f_1 \cdot f_2\,dt = \frac{1}{P^{\frac{1}{2}}(0)[m_a(\ell) - m_b(\ell)]}\,\frac{m_b}{P^{\frac{1}{2}}(0)[m_a(\ell) - m_b(\ell)]}\int_a^0 x_a \cdot x_a\,dt$$

$$+ \frac{1}{P^{\frac{1}{2}}(0)[m_a(\ell) - m_b(\ell)]}\,\frac{m_a}{P^{\frac{1}{2}}(0)[m_a(\ell) - m_b(\ell)]}\int_0^b x_b \cdot x_b\,dt\,.$$

$$\int_\delta f_1 \cdot f_2\,dt = \frac{1}{P(0)\,|m_a(\ell) - m_b(\ell)|^2}\left(-m_b\int_0^a x_a \cdot x_a\,dt + m_a\int_0^b x_b \cdot x_b\,dt\right).$$

$$2i\ \mathrm{Im}\,\ell\int_0^a x_a \cdot x_a\,dt = [x_a x_a](a) - [x_a x_a](0)$$

$$2i\ \mathrm{Im}\,\ell\int_0^a x_a \cdot x_a\,dt = -(\bar{m}_a - m_a)$$

$$2i\ \mathrm{Im}\,\ell\int_0^a x_a \cdot x_a\,dt = 2i\ \mathrm{Im}\,m_a\,.$$

$$\int_0^a x_a \cdot x_a \, dt = \frac{\operatorname{Im} m_a}{\operatorname{Im} \ell} \tag{5.13}$$

$$\int_0^b x_b \cdot x_b \, dt = \frac{\operatorname{Im} m_b}{\operatorname{Im} \ell} \tag{5.14}$$

$$\int_\delta f_1 \cdot f_2 \, dt = \frac{1}{P(0) \ |m_a(\ell) - m_b(\ell)|^2 \ \operatorname{Im} \ell} \, m_a \operatorname{Im} m_b - m_b \operatorname{Im} m_a \tag{5.15}$$

$$\int_\delta f_1 \cdot h_{\delta n} \, dt = \frac{1}{P^{\frac{1}{2}}(0)[m_a(\ell) - m_b(\ell)]} \left(m_b \int_a^0 x_a \cdot h_{\delta n} \, dt + m_a \int_0^b x_b \cdot h_{\delta n} \, dt \right)$$

$$[x_a h_{\delta n}] (0) = P(0) \, [x_{a2}(0) \, \overline{h_{\delta n1}(0)} - x_{a1}(0) \, \overline{h_{\delta n2}(0)}]$$

$$[x_a \, h_{\delta n}](0) = P(0) \left(\frac{1}{P^{\frac{1}{2}}(0)} \frac{\overline{r}_{\delta n2}}{P^{\frac{1}{2}}(0)} - \frac{m_a}{P^{\frac{1}{2}}(0)} \frac{\overline{r}_{\delta n1}}{P^{\frac{1}{2}}(0)} \right) = \overline{r}_{\delta n1} - m_a \, \overline{r}_{\delta n1}$$

$$[x_b \, h_{\delta n}](0) = \overline{r}_{\delta n2} - m_b \, \overline{r}_{\delta n1}$$

$$(\ell - \lambda_{\delta n}) \int_a^0 x_a \cdot h_{\delta n} \, dt = [x_a \, h_{\delta n}](0) = \overline{r}_{\delta n2} - m_a \, \overline{r}_{\delta n1}$$

$$(\ell - \lambda_{\delta n}) \int_\delta f_1 \cdot h_{\delta n} \, dt$$

$$= \left(\frac{1}{P^{\frac{1}{2}}(0) \ [m_a(\ell) - m_b(\ell)]} \right) \left(m_b [x_a \, h_{\delta n}](0) - m_a [x_b \, h_{\delta n}](0) \right)$$

$$(\ell - \lambda_{\delta n}) \int_\delta f_1 \cdot h_{\delta n} \, dt = \frac{(m_a - m_b) \ \overline{r}_{\delta n2}}{P^{\frac{1}{2}}(0) \ (m_a - m_b)}$$

$$\int_\delta f_1 \cdot h_{\delta n} \, dt = - \frac{\overline{r}_{\delta n1}}{P^{\frac{1}{2}}(0) \ (\ell - \lambda_{\delta n})} \tag{5.16}$$

$$\int_\delta f_2 \cdot h_{\delta n}\, dt = -\frac{\bar{r}_{\delta n2}}{P^{\frac{1}{2}}(0)\ (\ell - \lambda_{\delta n})} \tag{5.17}$$

Using (5.15), (5.16) and (5.17) in (5.12) with $j - 1$, $k = 2$ results in

$$\frac{m_a \operatorname{Im} m_b - m_b \operatorname{Im} m_a}{P(0)\ |m_a - m_b|^2 \operatorname{Im} \ell} = \sum_{n=1}^{\infty} \frac{\bar{r}_{\delta n1}\ r_{\delta n2}}{P(0)\ |\ell - \lambda_{\delta n}|^2}$$

$$\frac{\operatorname{Im} M_{12}}{\operatorname{Im} \ell} = \int_{-\infty}^{\infty} \frac{d\rho_{12}}{|\lambda - \ell|^2}$$

where

$$M_{12} = \frac{1}{2}[m_a(\ell) + m_b(\ell)][m_a(\ell) - m_b(\ell)]^{-1} .$$

Further similar calculation shows that

$$\int_{-\infty}^{\infty} \frac{d\rho_{\delta jk}(\lambda)}{|\lambda - \ell|^2} = \frac{\operatorname{Im} M_{\delta jk}(\ell)}{\operatorname{Im} \ell} \tag{5.18}$$

where

$$M_{\delta 11} = [m_a(\ell) - m_b(\ell)]^{-1}$$

$$M_{12} = M_{21} = \frac{1}{2}[m_a(\ell) + M_b(\ell)][m_a(\ell) - m_b(\ell)]^{-1}$$

$$M_{\delta 22} = m_a(\ell)\ m_b(\ell)[m_a(\ell) - m_b(\ell)]^{-1} .$$

Using formulas (5.13) and (5.14) it may be argued exactly as in the scalar second order case that

$$\int_{-\infty}^{\infty} \frac{|d\rho_{\delta jk}(\lambda)|}{1 + \lambda^2} < K$$

where K is some constant. The Helly selection theorem may be applied since on any finite interval $\rho_{\delta jk}$ is of bounded variation and hence may be written as the difference of two non-decreasing functions. As in the proof of theorem 3.1 it follows that there exists a sequence of intervals $\delta_n = [a_n, b_n]$, $\delta_n \to (-\infty,\infty)$, and corresponding boundary conditions prescribed by α_n, β_n, such that $\rho_{\delta n jk}(\lambda)$ tends to a limit $\rho_{jk}(\lambda)$ as $n \to \infty$. The matrix $\rho = (\rho_{jk})$ possesses the properties (i) through (iii).

If L is in the limit-point case at $-\infty$ and ∞, ρ is unique since in this situation both m_a and m_b tend to points $m_{-\infty}$ and m_{∞} and we have the following theorem.

Theorem 5.1.--Let L be in the limit-point case at $-\infty$ and ∞. There exists a non-decreasing Hermitian matrix $\rho = (\rho_{jk})$ whose elements are of bounded variation on every finite λ interval, and which is essentially unique in the sense that

$$\rho_{\delta jk}(\lambda) - \rho_{\delta jk}(\mu) \to \rho_{jk}(\lambda) - \rho_{jk}(\mu), \qquad (\delta \to (-\infty,\infty))$$

at points of continuity λ, μ of ρ_{jk}. Further

$$\rho_{jk}(\lambda) - \rho_{jk}(\mu) = \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{\mu}^{\lambda} \operatorname{Im} M_{jk}(v + i\varepsilon)\, dv$$

where

$$M_{11}(\ell) = [m_{-\infty}(\ell) - m_{\infty}(\ell)]^{-1}$$

$$M_{12}(\ell) = M_{21}(\ell) = \frac{1}{2}[m_{-\infty}(\ell) + m_{\infty}(\ell)][m_{-\infty}(\ell) - m_{\infty}(\ell)]^{-1}$$

$$M_{22}(\ell) = m_{-\infty}(\ell)\, m_{\infty}(\ell)\, [m_{-\infty}(\ell) - m_{\infty}(\ell)]^{-1} .$$

Since the existence of a limiting matrix has been established a completeness and expansion theorem analogous to that of theorem 3.1 may be proved by an argument which parallels that in the proof of theorem 3.1.

The spectrum associated with a problem for which ρ is uniquely determined is the set of all nonconstancy points of ρ. The point spectrum is the set of all discontinuity points of ρ, and the continuous spectrum is the set of continuity points of ρ which are in the spectrum. Points in the point spectrum are called eigenvalues and solutions of the problem for such points are called eigenfunctions. Every eigenfunction is of class $L^2(0,\infty)$ for a boundary value problem on $(0,\infty)$ and $L^2(-\infty,\infty)$ for a boundary value problem on $(-\infty,\infty)$.

Consider a boundary value problem on $0 \leq t < \infty$ which has a spectral function ρ and let $\tilde{\lambda}$ be in the point spectrum. Define $f(t) = \psi(t,\tilde{\lambda})$ for $t < a$ and $f = 0$ for $t > a$. Let

$$\int_0^a f(t) \cdot \psi(t,\lambda)\, dt = g_a(\lambda) .$$

Then the Parseval equality may be written

$$\int_0^a f(t) \cdot f(t)\, dt = \int_{-\infty}^{\infty} |g_a(\lambda)|^2\, d\rho(\lambda)$$

$$\int_0^a f(t) \cdot f(t)\ dt \ \geq \ |g_a(\tilde{\lambda})|^2 \left[\rho(\tilde{\lambda} + 0) - \rho(\tilde{\lambda} - 0)\right]$$

$$\int_0^a (f(t) \cdot f(t)\ dt \ \geq \ \left|\int_0^a f(t) \cdot \psi(t,\tilde{\lambda})\ dt\right|^2 \tilde{r} \tag{5.19}$$

where $\tilde{r}$ is the jump at $\tilde{\lambda}$. Because of the way f was defined (5.19) becomes

$$\int_0^a \psi(t,\tilde{\lambda}) \cdot \psi(t,\tilde{\lambda})\ dt \ \geq \ \tilde{r}\left(\int_0^a \psi(t,\tilde{\lambda}) \cdot \psi(t,\tilde{\lambda})\ dt\right)^2$$

$$\int_0^a \psi(t,\tilde{\lambda}) \cdot \psi(t,\tilde{\lambda})\ dt \ < \ \frac{1}{\tilde{r}} \,. \tag{5.20}$$

Since a is arbitrary (5.20) implies $\psi(t,\tilde{\lambda})$ is of the class $L^2(0,\infty)$.

Now consider a boundary value problem on $-\infty < t < \infty$ where the limit matrix ρ has a discontinuity at $\lambda = \tilde{\lambda}$. Let the discontinuities of ρ_{11}, ρ_{12}, ρ_{21} and ρ_{22} at $\tilde{\lambda}$ be d_{11}, d_{12}, d_{21} and d_{22} respectively. From (5.5) it may be seen that a solution to the problem is

$$h_m(t) \ = \ r_{m1}\ \varphi_1(t,\tilde{\lambda}) + r_{m2}\ \varphi_2(t,\tilde{\lambda})$$

and that

$$d\rho_{jk}(\tilde{\lambda}) \ = \ \rho_{jk}(\tilde{\lambda} + 0) - \rho_{jk}(\tilde{\lambda} - 0) \ = \ \sum_{m=1}^{2} r_{mj}\ \bar{r}_{mk} \,.$$

The sum over m is necessary since corresponding to $\tilde{\lambda}$ there may be two linearly independent solutions of $Lx = \tilde{\lambda}x$ satisfying the given boundary conditions at infinity. Thus in general the matrix $\rho(\tilde{\lambda} + 0) - \rho(\tilde{\lambda} - 0)$ may have the form

$$\rho(\tilde{\lambda} + 0) - \rho(\tilde{\lambda} - 0) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \tag{5.21}$$

$$= \begin{pmatrix} r_{11}\bar{r}_{11} + r_{21}\bar{r}_{21} & r_{11}\bar{r}_{12} + r_{21}\bar{r}_{22} \\ r_{12}\bar{r}_{11} + r_{22}\bar{r}_{21} & r_{12}\bar{r}_{12} + r_{22}\bar{r}_{22} \end{pmatrix} .$$

Since $\tilde{\lambda}$ is real and the boundary conditions that h_1, φ_1, φ_2 satisfy are real it may be assumed that the r_{jk} are real and from (5.21) it is seen that $d_{12} = d_{21}$.

$$d_{12}^2 = r_{12}r_{11}r_{11}r_{12} + r_{22}r_{21}r_{11}r_{22} + r_{12}r_{11}r_{21}r_{22} + r_{22}r_{21}r_{21}r_{22}$$

$$d_{11}d_{22} = r_{11}r_{11}r_{12}r_{12} + r_{11}r_{11}r_{22}r_{22} + r_{21}r_{21}r_{12}r_{12} + r_{21}r_{21}r_{22}r_{22}$$

$$d_{11}d_{22} - d_{12}^2 = r_{11}r_{11}r_{22}r_{22} - r_{22}r_{11}r_{21}r_{12} - r_{22}r_{11}r_{21}r_{12} + r_{21}r_{12}r_{21}r_{12}$$

$$d_{11}d_{22}^2 - d_{12} = (r_{11}r_{22} - r_{21}r_{12})^2 \geq 0 . \tag{5.22}$$

Thus

$$d_{12}^2 \leq d_{11}d_{22} .$$

If $d_{12}^2 < d_{11}d_{22}$ (5.22) implies that r_{22} and r_{21} are not both zero and therefore two linearly independent solutions correspond to $\tilde{\lambda}$. To show that the solution or solutions are in $L^2(-\infty,\infty)$ it will be assumed that only one solution corresponds to $\tilde{\lambda}$. The proof for the case where two solutions correspond to $\tilde{\lambda}$ is similar. (5.21) now becomes

$$\rho(\tilde{\lambda} + 0) - \rho(\tilde{\lambda} - 0) = \begin{matrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{matrix} = \begin{matrix} r_{11}^2 & r_{11}r_{12} \\ r_{11}r_{12} & r_{12}^2 \end{matrix}$$

and $d_{12}^2 = d_{11}d_{22}$.

$$h(t) = r_{11}\,\varphi_1(t,\tilde{\lambda}) + r_{12}\,\varphi_2(t,\tilde{\lambda}) \,.$$

Let $f(t) = h(t)$ for $|t| \leq a$ and $f = 0$ for $|t| > a$; the Parseval equality applied to f yields

$$\int_{-\infty}^{\infty} f(t) \cdot f(t)\, dt = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \bar{g}_j(\lambda)\, g_k(\lambda)\, d\rho_{jk}(\lambda)$$

where

$$g(\lambda) = \int_{-\infty}^{\infty} f(t) \cdot \varphi_j(t,\lambda)\, d\rho(\lambda)$$

$$\int_{-\infty}^{\infty} f(t) \cdot f(t)\, dt \geq \sum_{j,k=1}^{2} g_j(\tilde{\lambda})\, g_k(\tilde{\lambda})\, [\rho_{jk}(\tilde{\lambda} + 0) - \rho_{jk}(\tilde{\lambda} - 0)]$$

$$\int_{-a}^{a} f(t) \cdot f(t)\, dt \geq d_{11}\Big(\int_{-a}^{a} f \cdot \varphi_1\, dt\Big)^2$$

$$+ 2\, d_{12}\Big(\int_{-a}^{a} f \cdot \varphi_1\, dt\Big)\Big(\int_{-a}^{a} f \cdot \varphi_2\, dt\Big) + d_{22}\Big(\int_{-a}^{a} f \cdot \varphi_2\, dt\Big)^2.$$

$$r_{11}^2 \int_{-a}^{a} \varphi_1 \cdot \varphi_1\, dt + 2\, r_{11}r_{12} \int_{-a}^{a} \varphi_1 \cdot \varphi_2\, dt + r_{12}^2 \int_{-a}^{a} \varphi_2 \cdot \varphi_2\, dt \qquad (5.23)$$

$$\geq r_{11}^2\Big(\int_{-a}^{a} r_{11}\,\varphi_1 \cdot \varphi_1\, dt + \int_{-a}^{a} r_{12}\,\varphi_2 \cdot \varphi_1\, dt\Big)^2 +$$

$$+ \; 2\, r_{12} r_{11} \left(\int_{-a}^{a} r_{11}\, \varphi_1 \cdot \varphi_1 \, dt + \int_{-a}^{a} r_{12}\, \varphi_2 \cdot \varphi_1 \, dt \right)$$

$$\times \left(\int_{-a}^{a} r_{11}\, \varphi_1 \cdot \varphi_2 \, dt + \int_{-a}^{a} r_{12}\, \varphi_2 \cdot \varphi_2 \, dt \right)$$

$$+ \; r_{12}^2 \left(\int_{-a}^{a} r_{11}\, \varphi_1 \cdot \varphi_2 \, dt + \int_{-a}^{a} r_{12}\, \varphi_2 \cdot \varphi_2 \, dt \right)^2 .$$

By carrying out the operations indicated in (5.23) it may be seen that

$$\int_{-a}^{a} f(t) \cdot f(t) \, dt \;\geq\; \left(\int_{-a}^{a} f(t) \cdot f(t) \, dt \right)^2$$

which implies by the definition of f that

$$\int_{-a}^{a} h(t) \cdot h(t) \, dt \;\leq\; 1 ,$$

and since a is arbitrary h is therefore in the class $L^2(-\infty,\infty)$.

CHAPTER VI

SYSTEMS OF N EQUATIONS

Consider the problem

$$\pi_\delta : \qquad Lx = \ell x \qquad \beta_\delta x = M\, x(a) + N\, x(b) = 0$$

where

$$Lx = P_0 x' + P_1 x .$$

δ denotes the interval $[a, b]$, P_0 and P_1 are n-by-n matrices of continuous functions on $a \leq t \leq b$. Assume $L = L^+$, that is, $P_0 + P_0^* = 0$, $P_0' = P - P_1^*$. Assume M and N are n-by-n constant matrices such that for any $u, v \in c'[a, b]$ satisfying $Bu = Bv = 0$

$$\int_a^b Lu \cdot v \, dt = \int_a^b u \cdot Lv \, dt.$$

It has been shown in [5] that the eigenfunctions $\{x_{\delta j}\}$ of the self-adjoint problem π_δ form a complete orthonormal set.

Suppose that the interval under consideration is (c, d) where either c or d or both are infinite or else the coefficients in L have sufficiently singular behavior at one or both end points so that the treatment given in [5] does not apply. The Parseval equality for π_δ is

$$\int_a^b u(t) \cdot u(t) \, dt = \sum_{k=1}^{\infty} \left| \int_a^b u(t) \cdot x_{\delta k} \, dt \right|^2 \qquad (6.1)$$

where $u \in L^2(\delta)$. Let φ_j $(j = 1, 2, \ldots, n)$ be solutions of $Lx = \ell x$ which for some fixed c, $a < c < b_1$ satisfy

$$\varphi_{jk}(c, \ell) = \Delta_{jk}, \qquad (j, k = 1, 2, \ldots, n)$$

where Δ_{jk} is the Kronecker delta. Since the φ_j are independent solutions

$$x_{\delta k}(t) = \sum_{j=1}^{n} r_{\delta jk}\, \varphi_j(t, \lambda_{\delta k}) \tag{6.2}$$

where the $r_{\delta kj}$ are complex constants. Using (6.2) in (6.1) Parseval's equality becomes

$$\int_a^b u(t) \cdot u(t)\, dt = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \bar{g}_{\delta j}(\lambda)\, g_{\delta k}(\lambda)\, d\rho_{\delta jk}(\lambda)$$

where

$$g_{\delta j}(\lambda) = \int_{\delta} u(t) \cdot \lambda_j(t, \lambda)\, dt$$

and the matrix $\rho_\delta = (\rho_{\delta jk})$ consists of step functions with discontinuities at the eigenvalues. The jumps at the eigenvalues are given by

$$\rho_{\delta jk}(\lambda_\rho + 0) - \rho_{\delta jk}(\lambda_\rho - 0) = \sum r_{\delta mj}\, \bar{r}_{\delta mk}$$

where the sum is taken over all m such that $\lambda_{\delta m} = \lambda_{\delta \rho}$. The matrix ρ_δ has the properties (i) through (iii). The following theorem establishes the existence of at least one limiting matrix ρ as $\delta \to (c, d)$.

Theorem 6.1.--Let $\{\delta\}$ be a set of intervals tending to (c, d) and $\{\beta_\delta x = 0\}$ a corresponding set of self-adjoint boundary conditions. Then $\{\delta\}$ contains a sequence $\{\delta_j\}$ tending to (c, d) as $j \to \infty$ such that

$$\rho(\lambda) = \lim_{j\to\infty} \rho_{\delta j}(\lambda)$$

exists on $-\infty < x < \infty$. Moreover, the limit matrix ρ satisfies (i) through (iii).

Proof.--Taking into account the way $\rho_{\delta jk}$ is defined

$$\int_{-\mu}^{\mu} |d\rho_{\delta jk}| = \sum_{\ell}\sum_{m} |r_{\delta mj\ell}\, \overline{r}_{\delta mk\ell}| = \sum_{\ell}\sum_{m} |r_{\delta mj\ell}|\, |\overline{r}_{\delta mk\ell}|$$

where the first sum is taken over all ℓ such that

$$-\mu \leq \lambda_{\delta\ell} < \mu ,$$

$$(|r_{\delta mj\ell}| - |\overline{r}_{\delta mk\ell}|)^2 \geq 0 .$$

Therefore

$$2|r_{\delta mj\ell}|\, |\overline{r}_{\delta mk\ell}| \leq |r_{\delta mj\ell}|^2 + |\overline{r}_{\delta mk\ell}|^2. \tag{6.4}$$

Using (6.4) in (6.3) gives

$$2\int_{-\mu}^{\mu} |d\rho_{\delta jk}| \leq \int_{-\mu}^{\mu} d\rho_{\delta jj}(\lambda) + \int_{-\mu}^{\mu} d\rho_{\delta kk}(\lambda) . \tag{6.5}$$

In view of the Helly selection theorem it suffices to show there exists a continuous nonnegative function $H(\lambda)$ such that

$$|\rho_{\delta jk}(\lambda)| \leq H(\lambda) \tag{6.6}$$

and because of (6.5) it may be assumed $j = k$ in (6.6)

The functions φ_{jk} are continuous in (t,λ) and at $t = c$ are equal to δ_{jk}. Thus, given μ, there is an $h > 0$ such that

$$|\varphi_{jk}(t,\lambda) - \delta_{jk}| < \frac{1}{6} n^3 \tag{6.7}$$

for $c \leq t \leq c + h$ and $|\lambda| \leq \mu$. Let $\tilde{f}$ be a nonnegative scalar function of class c' on (c,d) vanishing outside of $(c, c+h)$ and normalized so that

$$\int_c^{c+h} f(t)\, dt = 1, \qquad j = 1, \ldots, n .$$

Define the vector function f_m to be

$$f_m(t) = f(t)\, \delta_m$$

where

$$\delta_m = \begin{pmatrix} \delta_{m1} \\ \vdots \\ \delta_{mn} \end{pmatrix} .$$

The Bessel inequality applied to f_ℓ gives

$$\int_c^{c+h} f_\ell \cdot f_\ell \, dt \geq \int_{-\mu}^{\mu} \sum_{j,k=1}^{n} \bar{g}_j(\lambda)\, g_k(\lambda)\, d\rho_{\delta jk}(\lambda) \tag{6.8}$$

where

$$g_k(\lambda) = \int_c^{c+h} f_\ell \cdot \varphi_k \, dt .$$

From (6.7) and (6.8)

$$g_k(\lambda) - \delta_{k\ell} = \int_c^{c+h} f_\ell \cdot \varphi_k \, dt - \int_c^{c+h} f_\ell \circ \delta_k \, dt$$

$$|g_k(\lambda) - \delta_{k\ell}| \le \frac{1}{6n^2} \frac{1}{6n^2} - \delta_{k\ell} \le g_k(\lambda) . \qquad (6.9)$$

Using (6.9) in (6.8)

$$\int_c^{c+h} f_\ell \cdot f_\ell \, dt \ge \int_{-\mu}^{\mu} \sum_{j,k=1}^{n} \left(\frac{1}{6n^2} - \delta_{j\ell}\right) \left(\frac{1}{6n^2} - \delta_{k\ell}\right) d\rho_{\delta jk}(\lambda)$$

$$\int_c^{c+h} f_\ell \circ f_\ell \, dt \ge \int_{-\mu}^{\mu} \frac{1}{36n^4} \sum_{j,k}^{n} d\rho_{\delta jk}(\lambda) - \frac{1}{6n^2} \sum_{k=1}^{n} d\rho_{\delta \ell k}(\lambda) - \frac{1}{6n^2} \sum_{j=1}^{n} d\rho_{\delta j\ell}(\lambda) + d\rho_{\delta \ell\ell}(\lambda)$$

$$\int_c^{c+h} f_\ell \cdot f_\ell \, dt \ge \int_{-\mu}^{\mu} d\rho_{\delta\ell\ell} - \int_{-\mu}^{\mu} \frac{1}{36n^4} \sum_{j,k=1}^{n} |d\rho_{\delta jk}(\lambda)| - \frac{1}{6n^2} \int_{-\mu}^{\mu} \sum_{k=1}^{n} |d\rho_{\delta \ell k}(\lambda)| - \frac{1}{6n^2} \int_{-\mu}^{\mu} \sum_{j=1}^{n} |d\rho_{\delta j\ell}(\lambda)|$$

$$\int_c^{c+h} f_\ell \circ f_\ell \, dt \ge \int_{-\mu}^{\mu} d\rho_{\delta\ell\ell} - \frac{1}{36n^4} \int_{-\mu}^{\mu} 2n \sum_{j=1}^{n} d\rho_{\delta jj} -$$

$$- \frac{1}{6n^2} \int_{-\mu}^{\mu} n d\rho_{\delta \ell \ell} - \frac{1}{6n^2} \int_{-\mu}^{\mu} \sum_{k=1}^{n} d\rho_{\delta kk}$$

$$- \frac{1}{6n^2} \int_{-\mu}^{\mu} n d\rho_{\delta \ell \ell} - \frac{1}{6n^2} \int_{-\mu}^{\mu} \sum_{j=1}^{n} d\rho_{\delta jj}$$

$$\int_{c}^{c+h} f_\ell \cdot f_\ell \, dt \geq \frac{1}{2} \int_{-\mu}^{\mu} d\rho_{\delta \ell \ell} - \left(\frac{1}{18n^3} + \frac{1}{6n^2} + \frac{1}{6n^2} \right) \int_{-\mu}^{\mu} \sum_{j=1}^{n} d\rho_{\delta jj}$$

$$\int_{c}^{c+h} f_\ell \cdot f_\ell \, dt \geq \frac{1}{2} \int_{-\mu}^{\mu} d\rho_{\delta \ell \ell} - \frac{1}{4n} \int_{-\mu}^{\mu} \sum_{j=1}^{n} d\rho_{\delta jj} .$$

Summing from $\ell = 1$ to $\ell = n$ gives

$$n \geq \frac{1}{4} \int_{-\mu}^{\mu} \sum_{\ell=1}^{n} d\rho_{\delta \ell \ell} .$$

Thus it has been shown that given a $\mu > 0$ there exists a $M(\mu) < \infty$ not depending on δ such that for $|\lambda| \leq \mu$

$$|\rho_{\delta jj}(\lambda)| \leq M(\mu) .$$

To obtain a continuous nonnegative function $H(\lambda)$ such that

$$|\rho_{\delta jj}(\lambda)| \leq H(\lambda) .$$

Choose $\mu = 1, 2, 3, \ldots$ and construct a continuous function $H(\lambda)$ such that $H(\lambda)$ increases as $|\lambda|$ increases and such that

$$H(0) = M(1)$$

$$H(\pm 1) = \sup\{M(1),\ M(2)\}$$

$$H(\pm n) = \sup\{M(k), \quad k = 1, 2, \ldots, n+1\} .$$

Define the space $L^2(\rho)$ as the set of all vector functions $g = (g_j)$ $j = 1, \ldots, n$, which are measurable with respect to ρ and such that

$$\|g\|^2 = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} g_k(\lambda)\, \bar{g}_j(\lambda)\, d\rho_{jk}(\lambda) < \infty .$$

Since the existence of a limiting matrix ρ has been established it is possible to prove the following expansion and completeness theorem in the same manner as the corresponding 2-by-2 theorem was proved.

Theorem 6.2.--Let ρ be any limit matrix given by Theorem 6.1. If $f \in L^2(c,d)$ there exists a vector $g \in L^2(\rho)$ such that if

$$g_{\delta j} = \int_{\delta} f(t) \cdot \varphi_j(t,\lambda)\, dt, \qquad \delta \subset (c,d)$$

then

$$\| g - g_{\delta} \| \to 0 \quad \text{as} \quad \delta \to (c,d) .$$

In terms of this g, the Parseval equality

$$\int_c^d f \cdot f\, dt = \| g \|^2 ,$$

and expansion

$$f(t) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{n} \varphi_j(t,\lambda)\, g_k(\lambda)\, d\rho_{jk}(\lambda) .$$

CHAPTER VII

HEAT CONDUCTION IN A COMPOSITE SOLID

Let Figure 1 represent a solid consisting of n long slender rods of the same unidorm cross section joined at a common point O.

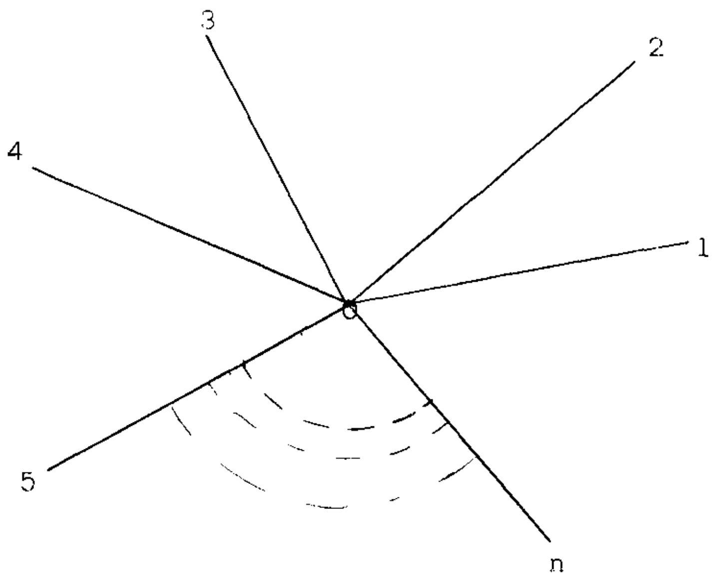

Figure 7.1

As an illustration of the possible applications to physical problems of the theory just developed, the problem of heat flow in the solid described by Figure 7.1 will be considered. A number of other physical problems involving a boundary value problem for systems of differential equations may be found in [8], [9], and [10].

Let the temperature in the solid be given by the vector function $U(x,t)$ where

$$U_j(x,t) = \text{the temperature in the } j^{th} \text{ rod}, \quad j = 1, \ldots, n$$

and x is the distance from the point O. Consider first the case where the rods are of length b and let the ends and lateral surfaces of the rods be insulated. This gives the boundary condition

$$\frac{\partial u}{\partial x}(b,t) = 0. \tag{7.1}$$

Since the rods meet at O, the temperature in all the rods at this point will be assumed to be the same and the second boundary condition

$$U_j(0,t) = U_k(0,t), \qquad k,j = 1, \ldots, n \tag{7.2}$$

is obtained. The last boundary condition arises from assuming the junction at O holds no heat; therefore the total flow of heat into the junction must be zero and thus

$$\sum_{j=1}^{n} - k\, A \frac{\partial u_j}{\partial x}(0,t) = 0 \tag{7.3}$$

where k is the thermal conductivity and A the cross sectional area of the rods.

Assuming the initial temperature of the solid to be given by

$$\begin{aligned} U(x,0) &= f(x), \qquad 0 < x < b \\ f_j(0) &= f_k(0), \qquad k,j = 1, \ldots, n \end{aligned} \tag{7.4}$$

and asking what is the temperature at any point in the solid at any time t, leads to the mathematical problem of solving the system of partial differential equations

$$\frac{\partial U_j}{\partial t} = \frac{C\rho}{k}\frac{\partial^2 U_j}{\partial x^2}, \qquad j = 1, \ldots, n \tag{7.5}$$

subject to the initial and boundary conditions

$$\frac{\partial u}{\partial x}(b,t) = 0, \qquad \sum_{j=1}^{n} - k A \frac{\partial U_j}{\partial x}(0,t) = 0,$$

$$U_j(0,t) = U_k(0,t), \quad k,j = 1, \ldots, n . \tag{7.6}$$

ρ is the density of the rods and C is the specific heat of the rods. ρ, C, and k will be assumed constant and the same for all the rods.

Separating variables in (7.5) leads to the problem of solving the system of ordinary differential equations

$$x_j'' + \ell^2 x_j = 0, \qquad j = 1, \ldots, n \tag{7.7}$$

subject to the initial and boundary conditions

$$\sum_{j=1}^{n} x_j(0) = 0, \qquad x_j(0) = x_k(0),$$

$$x_j'(b) = x_k'(b), \qquad j,k = 1, \ldots, n. \tag{7.8}$$

Now let

$$y_{2j-1} = x_j, \qquad \ell y_{2j} = x_j', \qquad j = 1, \ldots, n$$

and there results the problem

$$P_0 y' = \ell y, \qquad My(0) + Ny(0) \tag{7.9}$$

where P_o is a nonsingular 2n-by-2n skew Hermitian matrix and M and N are constant 2n-by-2n matrices.

The problem of heat conduction in the solid of Figure 7.1 where n = 2 certainly reduces to the problem of heat conduction in a single rod. However, this problem will be studied in detail since it adequately illustrates the method without being unnecessarily complicated. Consider the problem of heat conduction in the solid depicted by Figure 7.2.

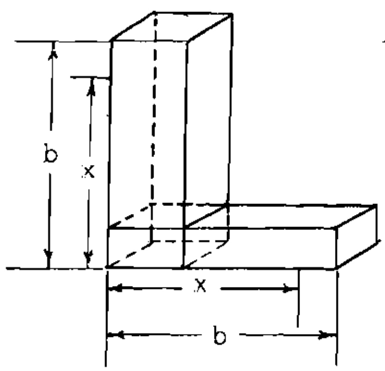

Figure 7.2

The solid may have any cross section, as has been mentioned before; however, a square cross section has been drawn in order to provide a simple illustration.

From what has already been done it is clear that the mathematical problem consisting of the system of partial differential equations

$$\frac{\partial u_j}{\partial t} = \frac{C\rho}{k}\frac{\partial^2 U}{\partial x^2}, \qquad j = 1,2 \tag{7.10}$$

subject to the initial and boundary conditions

$$\frac{\partial u_1}{\partial x}(0,t) + \frac{\partial u_2}{\partial x}(0,t) = 0, \qquad U_1(x,0) = f_1(x)$$

$$U_1(0,t) = U_2(0,t) \tag{7.11}$$

$$\frac{\partial u_1}{\partial x}(b,t) = \frac{\partial u_2}{\partial x}(b,t) = 0, \qquad U_2(x,0) = f_2(x)$$

results from considering the problem of heat conduction in the solid shown in Figure 7.2. Assuming

$$U_1(x,t) = x_1(x)\; T(t)$$

$$U_2(x,t) = x_2(x)\; T(t)$$

and separating variables in (7.10) leads to the system of ordinary differential equations

$$x_1'' + \ell^2 x_1 = 0, \qquad x_2'' + \ell^2 x_2 = 0, \qquad T' + \frac{c\rho\ell^2}{k} T = 0 \tag{7.12}$$

subject to the initial and boundary conditions

$$x_1'(0) + x_2'(0) = 0, \qquad x_1(0) - x_2(0) = 0, \qquad x_1'(b) = x_2'(b) = 0. \tag{7.13}$$

Now let $y_1 = x_1$, $\ell y_2 = x_1'$, $y_3 = x_2$, $\ell y_4 = x_2'$ and there results the problem

$$P_0 y' = \ell y \qquad My(0) + Ny(0) = 0 \tag{7.14}$$

where

$$P_o = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To show that the problem (7.11) is self-adjoint let R and S be vector functions of the clsss c' on $0 \leq t \leq b$ and satisfy the boundary conditions of (7.11). Green's formula gives

$$\int_0^b LR \cdot S\, dt - \int_0^b R \cdot LS\, dt = R_1(t)\, \overline{S}_2(t) - R_2(t)\, \overline{S}_1(t) - R_3(t)\, \overline{S}_4(t) - R_4(t)\, \overline{S}_3(t) \Big|_0^b$$

$$R_2(b) = R_4(b) = S_2(b) = S_4(b) = 0$$

which implies

$$R_1\overline{S}_2 - R_2\overline{S}_1 + R_3\overline{S}_4 - R_4\overline{S}_3 \Big|^b = 0$$

$$R_2(0) + R_4(0) = 0, \qquad R_1(0) - R_3(0) = 0,$$

$$S_2(0) + S_4(0) = 0, \qquad S_1(0) - S_3(0) = 0,$$

$$- R_2\overline{S}_1 - R_4\overline{S}_1 = 0, \qquad R_1\overline{S}_2 - R_3\overline{S}_2 = 0,$$

$$R_3\overline{S}_2 + R_3\overline{S}_4 = 0, \qquad R_4\overline{S}_1 - R_4\overline{S}_3 = 0 .$$

Addition of the above equations yields

$$R_1\overline{S}_2 - R_2\overline{S}_1 + R_3\overline{S}_4 - R_4\overline{S}_3 \Big|_0 .$$

Therefore the problem (7.11) is self-adjoint.

The first two equations of (7.10) imply that the solution of (7.11) has the form

$$y_1 = C_1 \sin \ell x + C_2 \cos \ell x$$

$$y_2 = C_1 \cos \ell x - C_2 \sin \ell x$$

$$y_3 = C_3 \sin \ell x + C_4 \cos \ell x$$

$$y_4 = C_3 \cos \ell x - C_4 \sin \ell x .$$

The boundary conditions give

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ \cos \ell b & -\sin \ell b & 0 & 0 \\ 0 & 0 & \cos \ell b & -\sin \ell b \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = 0 \qquad (7.15)$$

The determinant of the matrix of coefficients in (7.12) is $2 \sin \ell b \cos \ell b$. By setting this determinant equal to zero the eigenvalues ℓ are easily seen to be

$$\ell = \frac{n\pi}{2b}, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

If n is even then $C_1 = C_3 = 0$ and $C_2 = C_4$ so that corresponding to $\frac{n\pi}{2b}$ there is one eigenfunction ψ_n .

$$\psi_n(x) = \begin{cases} \cos \frac{n\pi}{2b} x \\ - \sin \frac{n\pi}{2b} x \\ \cos \frac{n\pi}{2b} x \\ - \sin \frac{n\pi}{2b} x \end{cases}$$

If n is odd then $C_2 = C_4 = 0$ and $C_1 = - C_3$ so that corresponding to $\frac{n\pi}{2b}$ there is the eigenfunction

$$\varphi_n(x) = \begin{cases} \sin \frac{n\pi}{2b} x \\ \cos \frac{n\pi}{2b} x \\ - \sin \frac{n\pi}{2b} x \\ - \cos \frac{n\pi}{2b} x \end{cases}$$

Corresponding to each eigenvalue $\ell = \frac{n\pi}{2b}$, $n = \pm 1, \pm 2, \ldots$ there is the eigenfunction $x_n(x)$ where

$$x_n(x) = \cos \frac{n\pi}{2} \psi_n(x) + \sin \frac{n\pi}{2} \varphi_n(x) .$$

Completeness of the sequence $\{x_n(x)\}$ in the space of vector functions with four components implies completeness of the sequence $\{\tilde{x}_n(x)\}$, where

$$\tilde{x}_n(x) = \cos\frac{n\pi}{2}\,\tilde{\psi}_n(x) + \sin\frac{n\pi}{2}\,\varphi_n(x)$$

$$\tilde{\psi}_n(x) = \begin{cases}\cos\frac{n\pi}{2b}x \\ \cos\frac{n\pi}{2b}x\end{cases} \qquad \tilde{\varphi}_n(x) = \begin{cases}\sin\frac{n\pi}{2b}x \\ -\sin\frac{n\pi}{2b}x\end{cases}$$

in the space of vector functions with two components.

The solution for the finite interval may be written as

$$U(x,t) = \int_{-\infty}^{\infty} e^{-\frac{C\rho\lambda^2 t}{k}}\, g_{b1}(\lambda)\,\tilde{\Psi}(x,\lambda)\,d\rho_{b1}(\lambda) + \int_{-\infty}^{\infty} e^{-\frac{C\rho\lambda^2 t}{k}}\, g_{b2}(\lambda)\,\tilde{\varphi}(x,\lambda)\,d\rho_{b2}(\lambda)$$

where

$$\tilde{\Psi}(x,\lambda) = \begin{cases}\cos\lambda x \\ \cos\lambda x\end{cases}, \qquad \tilde{\varphi}(x,\lambda) = \begin{cases}\sin\lambda x \\ -\sin\lambda x\end{cases}$$

$$g_{b1}(\lambda) = \int_0^b f(x)\cdot\tilde{\Psi}(x,\lambda)\,dx, \qquad g_{b2}(\lambda) = \int_0^b f(x)\cdot\tilde{\varphi}(x,\lambda)\,dx$$

and ρ_{b1} is an increasing step function with a jump of $1/b$ at $\lambda = \frac{n\pi}{2b}$, $n = 0, \pm 2, \pm 4, \ldots,$ and ρ_{b2} is a step sunction with a jump of $1/b$ at $\lambda = \frac{n\pi}{2b}$, $n = \pm 1, \pm 3, \pm 5, \ldots,$ $\rho_{bj}(0) = 0$, and $\rho_{bj}(\lambda + 0) = \rho_{bj}(\lambda)$. As $b \to \infty$ it is easily seen that $\rho_{bj} \to \rho_j$ where $\rho_j(\lambda) = \frac{\lambda}{\pi}$. As $b \to \infty$, $g_{b1} \to g_1$ and $g_{b2} \to g_2$ where

$$g_1(\lambda) = \int_0^\infty f(x) \cdot \tilde{\psi}(x,\lambda)\, dx \qquad g_2(\lambda) = \int_0^\infty f(x) \cdot \tilde{\varphi}(x,\lambda)\, dx .$$

Let f be of the class $L^2(0,\infty)$ then in view of theorem 6.2 $U(x,t)$ for the semi-infinite interval may be written

$$U(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\frac{C\rho\lambda^2 t}{k}} \left[\left(\int_0^\infty f \cdot \tilde{\psi}(x,\lambda)\, dx \right) \tilde{\psi}(x,\lambda) + \left(\int_0^\infty f \cdot \tilde{\varphi}(x,\lambda)\, dx \right) \tilde{\varphi}(x,\lambda) \right] .$$

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