SOME MIXED BOUNDARY VALUE PROBLEMS IN ELASTODYNAMICS

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SOME MIXED BOUNDARY VALUE

PROBLEMS IN ELASTODYNAMICS

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SUMMARY

We use an integral equation-variational procedure to characterize important quantities occurring in two distinct mixed boundary value problems of elastodynamics. In the first problem we obtain a low frequency approximation for the displacement of a flat rigid punch of elliptic cross-section undergoing vertical oscillations on the surface of an elastic half-space. In the second problem we construct a stationary functional for the scattering cross-section for the diffraction, by a plane longitudinal wave, of a flat crack of arbitrary shape situated in an infinite elastic medium. Results are obtained for a circular crack.

CHAPTER I

INTRODUCTION

We consider two mixed boundary value problems in elastodynamics and approximate their solutions by a variational technique. The first problem utilizes an established variational formula, whereas in the second problem we develop a new variational formula.

Both of our mixed boundary value problems can be analyzed in terms of a linear, homogeneous, isotropic, elastic medium occupying a half-space D, $z > 0$, with the plane $z = 0$ as the boundary B. The boundary will be divided into two parts, B_1 and B_2. The portion B_1 will be a bounded region, and $B_2 = B - B_1$. Each part will have different boundary conditions prescribed.

Our first problem will be referred to as a "punch" problem, in which a rigid punch, of elliptical cross-section and with flat base, is forced to oscillate in contact with the boundary region B_1. The oscillations will have harmonic time dependence $e^{-i\omega t}$. Keeping the load on the punch of the form $P_0e^{-i\omega t}$, where P_0 is constant, we approximate the amplitude of oscillation of the punch by means of a variational technique. We describe this problem more precisely in Chapter III.

Previously in three dimensions, punch problems, for several different modes of excitation, have been successfully attacked only when the punch is circular and flat-based. A brief summary of this work is given in [12].

Our second problem will be referred to as a "crack" problem, in which two half-spaces of the type described are attached at the unbounded portion B_2 of the boundary B. In the region B_1, there is an infinitesimal separation, called a crack. We allow a plane longitudinal wave, which is normally incident on the crack with harmonic time dependence $e^{-i\omega t}$, to be scattered by the crack, and we calculate an approximation to the far-field scattering cross-section [1], again using a variational technique. More details will be discussed in Chapter IV. Previously, in three dimensions, the only dynamic crack problems that have been successfully attacked are those for the circle involving some kind of symmetry.

Mixed boundary value problems lead to integral equations. In the problems for the circular punch and the circular crack so far discussed, the inherent symmetry has enabled the two-dimensional integral equations to be reduced to those of one-dimension. By means of various devices, it has also been possible to obtain one-dimensional (finite) Fredholm integral equations of the second kind which have either been solved numerically or approximately in terms of expansions involving an appropriate non-dimensional frequency parameter. However, when the shape of the punch or crack is an ellipse the corresponding two-dimensional integral equations cannot be further reduced. For this reason we do not attempt an analytic attack directly on the integral equation, but rather we obtain an approximate solution through a variational technique.

CHAPTER II

BASIC PRELIMINARIES

For our elastodynamic problems the governing equations of motion of the half-space D is a system of elliptic partial differential equations. In order to motivate the initial steps of our attack, we indicate some comparisons with a more familiar elliptic partial differential equation, namely the Helmholtz equation in an exterior domain. We then develop some preliminaries that are common to both our problems.

In cartesian coordinates, the wave equation has the form

$$\Phi_{xx}(P) + \Phi_{yy}(P) + \Phi_{zz}(P) - \frac{1}{c^2}\Phi_{tt}(P) = 0, \quad P \in D, \tag{1}$$

where $P = (x, y, z)$ is position, t is time, and $c > 0$ is a constant. Assuming a solution with harmonic time dependence

$$\Phi(x,y,z,t) = u(P)e^{-i\omega t}, \tag{2}$$

the wave equation in D, that is in the half-space $z > 0$, becomes

$$[u_{xx}(P) + u_{yy}(P) + u_{zz}(P) + \frac{\omega^2}{c^2}u(P)]e^{-i\omega t} = 0, \tag{3}$$

which yields the Helmholtz equation

$$\nabla^2 u(P) + k^2 u(P) = 0, \quad P \in D, \tag{4}$$

where $\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$, the Laplacian of u, and $k = \frac{\omega}{c}$. When the frequency $\omega > 0$, i.e. $k > 0$, we will henceforth refer to this problem as the dynamic problem.

A fundamental singularity of the Helmholtz equation (4) is a function S(P,Q), for the source point $P \in D$, where

(i) S(P,Q) satisfies (4) in D, with respect to Q, except at P, and

(ii) S(P,Q) has the form $\frac{1}{4\pi R} e^{ikR} + s(P,Q)$,

with R = R(P,Q) denoting the distance between P and Q, and s(P,Q) any regular solution of (4). A regular solution is understood to satisfy an appropriate radiation condition.

The following Green's identity [4, p. 270], which holds for regular solutions of (4), is the basis for all that follows:

$$u(P) = \int_B \left[S(P,Q) \frac{\partial u(Q)}{\partial n_Q} - u(Q) \frac{\partial S(P,Q)}{\partial n_Q} \right] dA_Q \, , \; P \in D \, , \tag{5}$$

where S(P,Q) is an arbitrarily chosen, but fixed, fundamental singularity of the Helmholtz equation. We note that the imposition of an appropriate radiation condition on u will guarantee the uniqueness of certain classes of boundary value problems.

By specifying particular types of behavior on the boundary, we can produce particular fundamental singularities, called fundamental solutions, which generate alternative representations for solutions of (4) satisfying specific boundary conditions. To this end, we define the fundamental solutions G(P,Q) and N(P,Q), called the Green's function and Neumann function, respectively, to satisfy the boundary conditions

$$G(P, Q) = 0, \quad Q \in B, \quad P \in D \, , \tag{6}$$

and

$$\frac{\partial N(P,Q)}{\partial n_Q} = 0, \quad Q \varepsilon B, \quad P \varepsilon D . \qquad (7)$$

It can be shown [4, pp. 14, 50] that these fundamental solutions are symmetric in P and Q; i.e. G(P,Q) = G(Q,P), etc. Inserting the Green's function into (6), we obtain

$$u(P) = - \int_B \frac{\partial G(P,Q)}{\partial n_Q} u(Q) \, dA_Q, \quad P \varepsilon D, \qquad (8)$$

for the displacement u(P) when u is known on the boundary B. This is called the Dirichlet problem. The Neumann function in (6) yields the solution of the Neumann problem, where the normal derivative of displacement $\frac{\partial u}{\partial n}$ is known on the boundary. This solution is

$$u(P) = \int_B N(P,Q) \frac{\partial u(Q)}{\partial n_Q} \, dA_Q , \quad P \varepsilon D + B , \qquad (9)$$

where we are able to extend the range of P to include the boundary, due to the weak singularity in N(P,Q) when $P \varepsilon D \to P_0 \varepsilon B$.

If we let $k \to 0$ in the Neumann problem for the Helmholtz equation in an exterior domain, the solution is still given by (9). This similarity in integral representations for the Neumann problem associated with the Helmholtz and Laplace's equations in exterior domains led Stallybrass [23] to suspect that the corresponding integral representations associated with the dynamic* and equilibrium equations of

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*We will henceforth refer to a dynamic elasticity problem as one in which the time dependence is $e^{-i\omega t}$.

elasticity would have a common structure.

An integral representation due to Simigliana [24] was available for the displacement field for the equilibrium, or elastostatic, case. That representation, in rectangular cartesian coordinates, is

$$u_\alpha(P) = \int_B u_\alpha^i(P,Q)T_i(Q)dA_Q, \quad P \in D + B , \tag{10}$$

for the displacement field $u_\alpha(P)$ generated in an elastic medium occupying a semi-infinite domain D, by tractions $T_i(Q)$ applied on the bounding surface B of the region, where Q is a point on B and $T_i = \tau_{ij}n_j$, τ_{ij} being the components of the stress tensor, and n_j being the components of the outward unit normal to B. We can interpret the singular function $u_\alpha^i(P, Q)$ as the component of displacement in the x_α-direction at P, due to a concentrated surface force of unit magnitude in the x_i-direction at Q. Using the Betti Reciprocal Theorem, the analogue in elasticity of the appropriate Green's Theorem for the Helmholtz equation, Stallybrass [23] was able to show that in the dynamic case, if the fundamental singularities $u_\alpha^i(P,Q)$ are interpreted properly and the time factor $e^{-i\omega t}$ is omitted,* the displacement components $u_\alpha(P)$ have the same form as in (10). He interpreted $u_\alpha^i(P,Q)$ as the component of displacement in the x_α-direction at P, due to a concentrated harmonically oscillating surface force of unit magnitude in the x_i-direction at Q, together with an appropriate radiation condition. In this representation the fundamental solutions play the role of the Neumann function for the Helmholtz equation in an exterior domain.

In the problems we will discuss, the half-space D: z > 0 will have no tangential stress on its boundary B, so only T_3 in (10) will

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*The time factor $e^{-i\omega t}$ will be omitted in subsequent expressions.

be non-zero, and the outward unit normal to B is $\vec{n} = (0, 0, -1)$. Since $T_3(Q) = -\tau_{33}(Q)$, our only interest is with the displacement $u_3(P)$ and the fundamental solution $u_3^3(P, Q)$. For notational convenience, we replace $u_3(P)$ by $w(P)$, and $\tau_{33}(Q)$ by $\tau(Q)$. The fundamental solution $u_3^3(P, Q)$, for harmonic time dependence, is derivable from Ewing, Jardetzky, and Press [9], who quote the classical result given by Lamb [13]. For $P = (x,y,z)$ and $Q = (\hat{x}, \hat{y}, 0)$, this is given by

$$u_3^3(P, Q) = \frac{-1}{2\pi\mu} \int_0^\infty \frac{\xi\sqrt{\xi^2-h^2}\left[(k^2-2\xi^2)e^{-z\sqrt{\xi^2-h^2}} + 2\xi^2 e^{-z\sqrt{\xi^2-k^2}}\right]}{F(\xi)} J_0(\xi R)d\xi \quad (11)$$

where $h^2 = \rho_0 \dfrac{\omega^2}{\lambda + 2\mu}$, $k^2 = \rho_0 \dfrac{\omega^2}{\mu}$,

$$F(\xi) = (2\xi^2 - k^2)^2 - 4\xi^2 \sqrt{\xi^2 - h^2}\sqrt{\xi^2 - k^2}, \quad (12)$$

R is the distance from P to Q, ρ_0 is the density of the medium, and λ and μ are the Lamé constants, which define the material properties of the medium. Then the vertical displacement $w(P)$, for $P \epsilon B$, i.e. $z = 0$, becomes

$$w(P) = -\int_B u_3^3(P,Q)\tau(Q)dA_Q \quad (13a)$$

or

$$w(P) = \frac{k}{8\pi\mu} \int_B \left[\int_0^\infty M(t)J_0(ktR)dt\right] \tau(Q)dA_Q \quad (13b)$$

where we let $\xi = kt$, $\gamma = h/k < 1$, and

$$M(t) = \frac{t\sqrt{t^2 - \gamma^2}}{(t^2 - \frac{1}{2})^2 - t^2\sqrt{t^2 - \gamma^2}\,\sqrt{t^2 - 1}} . \qquad (14)$$

We show in Appendix I that for axially-symmetric problems, representation (13b) reduces to the form

$$w(r) = \frac{k}{8\pi\mu} \int_B \left[\int_0^\infty M(t)J_0(ktr)J_0(kt\rho)dt\right]\tau(\rho)dA_Q \qquad (15)$$

where $P = (r, \theta, 0)$ and $Q = (\rho, \phi, 0)$ in cylindrical coordinates.

It is obvious that there is ambiguity in the integral representations (13b) and (15) due to the square root functions in $M(t)$ and the fact that $M(t)$ has simple poles on the real axis at $\pm s$, where $s > 1$ [20]. Furthermore, to interpret the infinite integral properly, we must satisfy an appropriate radiation condition. To guarantee convergence of the integral in (12), we need the square roots to be positive for ξ large and positive, since $z \geq 0$.

Thus, we take the branch cuts for $M(t)$ from the branch points $\pm\gamma$ and ± 1 along the real axis outward to infinity, as shown in Figures 1 and 2.

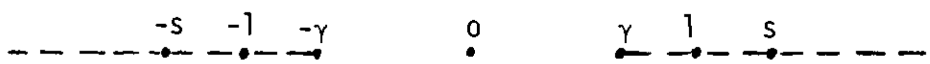

Figure 1. Branch Cuts for $(t^2 - \gamma^2)^{1/2}$.

Figure 2. Branch Cuts for $(t^2 - 1)^{1/2}$.

The mathematical analysis of wave propagation associated with the Helmholtz equation must necessarily allow for both incoming and outgoing waves. However, when a source point is located within the domain D and there is no reflection of waves at infinity, a means of eliminating the incoming waves in the general mathematical solution must be introduced. In the case of the Helmholtz equation in an exterior domain, Sommerfeld [21, p. 189], motivated by physical considerations, developed conditions to be imposed on the solution which would accomplish that. This has come to be known as the Sommerfeld radiation condition. In dynamic problems in elasticity, an appropriate radiation condition which generalizes that of Sommerfeld is the choice of the proper Riemann surface for contour integration.

The principle of limiting absorption [22, p. 261] expediates this choice. By allowing a small, positive viscosity in the medium in which the waves propagate, the energy of the wave decreases as the wave proceeds away from the source. This is distinct from the natural decay in amplitude of the wave due to increased surface area of the wave front. Mathematically, this viscosity moves the branch points and poles off the real axis, so that integration along this axis can be carried out with no ambiguity. Then we allow the viscosity to decrease to zero, and it becomes apparent how the contour must be deformed to avoid the branch points and pole and maintain the radiation condition. We illustrate this technique with the Helmholtz equation where the function requiring interpretation is $\sqrt{x^2 - k^2}$. Upon completion, we will simply present the analogous result for dynamic elasticity problems.

The wave equation with positive viscosity in the medium takes the form

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} + \varepsilon\frac{\partial\Phi}{\partial t} , \quad \varepsilon > 0 , \tag{16}$$

where ε is a measure of the viscosity. Upon separating variables, $\Phi = u(P)e^{-i\omega t}$, we again get

$$\nabla^2 u + k^2 u = 0 , \quad k^2 = (\tfrac{\omega}{c})^2 + i\varepsilon\omega. \tag{17}$$

We let $k = k_1 + ik_2$, with $k_1 > 0$, and require that the fundamental singularity $\frac{1}{R}e^{ikR}$, where R is the distance from the source of the waves, produce outgoing waves and diminish as R gets large. Since

$$\frac{1}{R}e^{i(k_1 + ik_2)R}\, e^{-i\omega t} = \frac{1}{R}e^{-k_2 R}\, e^{-i(\omega t - k_1 R)} \tag{18}$$

we require $k_2 > 0$. The term $\omega t - k_1 R$ characterizes outgoing waves. Thus, the branch point is shifted up, off the real axis, and the contour of integration proceeds along the real axis, below the branch point, as in Figure 3. Now letting the viscosity factor ε go to zero forces the branch point down onto the real axis. To avoid it and main-

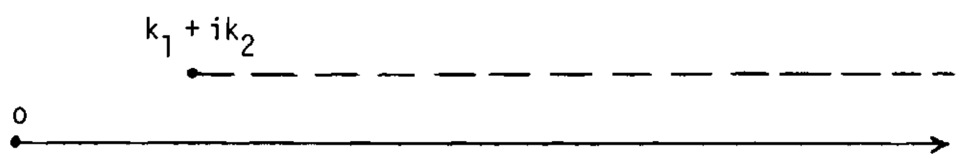

Figure 3. Shifted Branch Cut.

tain the necessary properties of the solution, we must deform the contour to go below the branch point and along the lower edge of the branch cut, as in Figure 4.

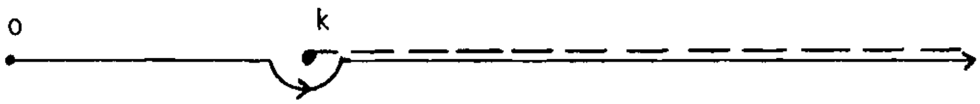

Figure 4. Deformed Contour for Helmholtz Equation.

In a similar fashion, the principle of limiting absorption is also very convenient in connection with dynamic problems of elasticity associated with exterior domains. We choose our contour from 0 to ∞ below the branch points and pole, as in Figure 5, and interpret integrals of the sort $\int_0^\infty M(t)E(t)dt$, for continuous functions E(t), as follows:

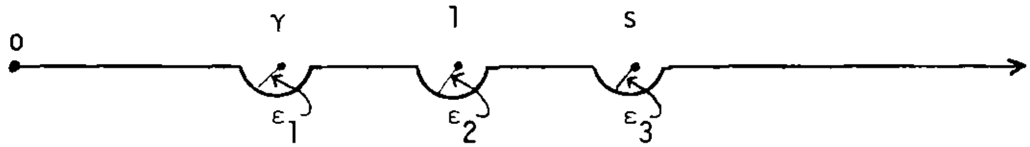

Figure 5. Deformed Contour for Elastodynamic Problems.

$$\int_0^\infty M(t)E(t)dt = \lim_{\varepsilon_1\to 0} \int_0^{\gamma-\varepsilon_1} M(t)E(t)dt + \lim_{\varepsilon_1,\varepsilon_2\to 0} \int_{\gamma+\varepsilon_1}^{1-\varepsilon_2} M(t)E(t)dt \qquad (19)$$

$$+ \lim_{\varepsilon_2,\varepsilon_3\to 0} \int_{1+\varepsilon_2}^{s-\varepsilon_3} M(t)E(t)dt + R(s) + \lim_{\varepsilon_3\to 0} \int_{s+\varepsilon_3}^{\infty} M(t)E(t)dt$$

$$= \int_0^\gamma M(t)E(t)dt + \int_\gamma^1 M(t)E(t)dt + P\int_1^\infty M(t)E(t)dt + R(s)$$

where R(s) is the contribution of the pole, and all other contributions from detours are zero.

In order to evaluate the last infinite integral in (19), we will integrate over a contour in the lower half-plane, and a proper continuation of our square root functions $\sqrt{t^2 - \gamma^2}$ and $\sqrt{t^2 - 1}$ is necessary. As previously mentioned, for t positive and large, the values of these functions must be positive. The analytic continuation of these below the branch cuts, for t real, becomes

$$\alpha(t) = \begin{cases} -\sqrt{t^2 - \gamma^2}\,, & t < -\gamma \\ -i\sqrt{\gamma^2 - t^2}\,, & |t| \le \gamma \\ \sqrt{t^2 - \gamma^2}\,, & t > \gamma \end{cases} \qquad \beta(t) = \begin{cases} -\sqrt{t^2 - 1}\,, & t < -1 \\ -i\sqrt{1 - t^2}\,, & |t| \le 1 \\ \sqrt{t^2 - 1}\,, & t > 1\,. \end{cases} \tag{20}$$

We define the following convenient notation for t in the lower half-plane. Let

$$\alpha(t) = \begin{cases} -\alpha\,, & t < -\gamma \\ -\alpha\,, & |t| \le \gamma \\ \alpha\,, & t > \gamma \end{cases} \qquad \beta(t) = \begin{cases} -\beta\,, & t < -1 \\ -\beta\,, & |t| \le 1 \\ \beta\,, & t > 1\,. \end{cases} \tag{21}$$

In line with this notation, we define the Rayleigh function

$$f(t) = (t^2 - \tfrac{1}{2})^2 - t^2\alpha(t)\beta(t) \tag{22}$$

and use the abbreviations

$$\begin{aligned} f &= (t^2 - \tfrac{1}{2})^2 - t^2\alpha\beta\,, \quad t \notin [\gamma, 1] \\ \hat{f} &= (t^2 - \tfrac{1}{2})^2 + t^2\alpha\beta\,, \quad t \in [\gamma, 1]\,. \end{aligned} \tag{23}$$

We consolidate these definitions in the following diagram:

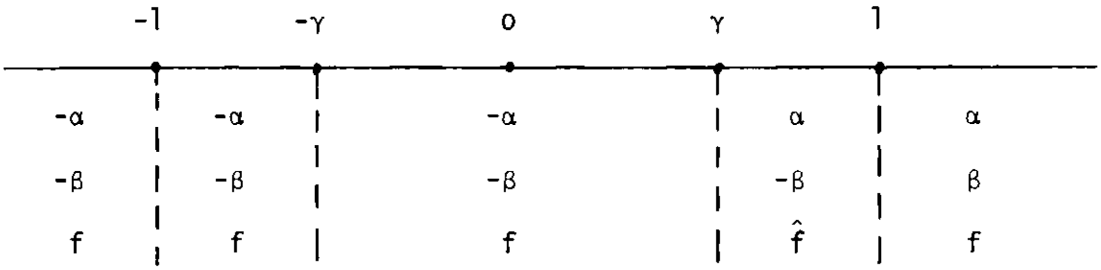

Figure 6. Square Roots and Rayleigh Function.

These definitions will be used in connection with the contour Γ in Figure 7 for evaluation of the infinite integrals that occur subsequently.

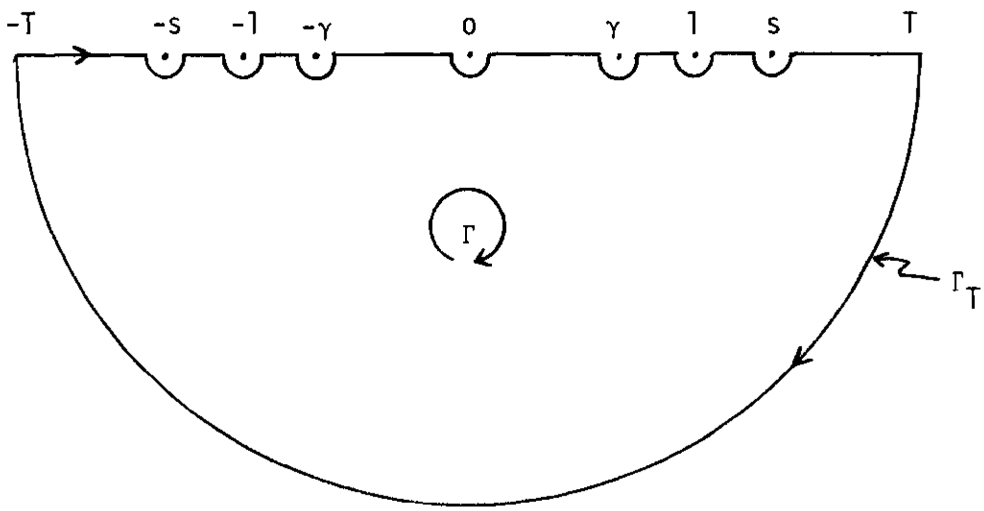

Figure 7. The Contour Γ.

Our method will be to obtain approximate solutions to problems by means of a variational technique. The fact that an integral equation can be associated with a variational principle was stated as early as 1884 by Volterra [25].

Consider the integral equation

$$w(P) = \int_B \tilde{K}(P, Q)\tau(Q)dA_Q, \quad P \in B, \tag{24}$$

where w(P) is a known continuous function, $\tau(Q)$ is an unknown integrable function, and the kernel $\tilde{K}(P, Q)$ is an integrable function, symmetric in P and Q. This Fredholm integral equation is the Euler-Lagrange equation for the problem of finding a stationary value for the linear functional

$$F_1[\hat{\tau}] = 2 \int_B \hat{\tau}(Q)w(Q)dA_Q - \int_B \int_B \tilde{K}(P,Q)\hat{\tau}(P)\hat{\tau}(Q)dA_Q dA_P \tag{25}$$

where $\hat{\tau}$ is an arbitrary integrable function having a singularity of the same order as τ [6]. We shall refer to $\hat{\tau}$ as an admissible or trial function.

If we define the inner product $\langle u,v\rangle = \int_B u(Q)v(Q)dA_Q$ and the linear operator $L[\hat{\tau}] = \int_B \tilde{K}(P,Q)\hat{\tau}(Q)dA_Q$, then equations (24) and (25) become

$$w = L[\tau] \tag{26}$$

and

$$F_1[\hat{\tau}] = 2\langle w,\hat{\tau}\rangle - \langle L[\hat{\tau}],\hat{\tau}\rangle . \tag{27}$$

By the stationary principle [22, p. 357], F_1 has a unique stationary value at $\hat{\tau} = \tau$, and

$$F_1[\tau] = \langle w, \tau\rangle \tag{28}$$

Furthermore, if $\hat{\tau} - \tau = O(\varepsilon)$, then $F_1[\hat{\tau}] - F_1[\tau] = O(\varepsilon^2)$. That is, when the error between $\hat{\tau}$ and τ is small, say of order $O(\varepsilon)$, then the error between $F_1[\hat{\tau}]$ and $F_1[\tau]$ is of order $O(\varepsilon^2)$.

Consider the trial function

$$\hat{\tau} = c^* \tau^* \tag{29}$$

where τ^* is a fixed, but arbitrary, admissible function and c^* is a parameter. Since F_1 is stationary at the exact solution τ, we determine the best value for c^* by setting the first derivative of F_1 with respect to c^* equal to zero. We have

$$\begin{aligned} F_1[c^*\tau^*] &= 2\langle w, c^*\tau^*\rangle - \langle L[c^*\tau^*], c^*\tau^*\rangle \\ &= 2c^*\langle w,\tau^*\rangle - (c^*)^2\langle L[\tau^*], \tau^*\rangle \end{aligned} \tag{30}$$

and $\frac{d}{dc^*} F_1[c^*\tau^*] = 0$ when

$$c^* = \frac{\langle w, \tau^*\rangle}{\langle L[\tau^*], \tau^*\rangle} . \tag{31}$$

So,

$$F_1[c^*\tau^*] = \frac{\langle w,\tau^*\rangle^2}{\langle L[\tau^*], \tau^*\rangle} . \tag{32}$$

If we define a second functional by the right side of equation (32)

$$F_2(\hat{\tau}) = \frac{\langle w, \hat{\tau}\rangle^2}{\langle L[\hat{\tau}], \hat{\tau}\rangle} , \quad \hat{\tau} \neq 0 , \tag{33}$$

we see that F_2 is scalar invariant, in that $F_2[c\hat{\tau}] = F_2[\hat{\tau}]$ for any scalar $c \neq 0$, and that F_2 has the same stationary value as F_1.

Using the notation $\left\{<w, \tau>\right\}$ as the variational approximation to the exact value of $<w, \tau>$, we have

$$\left\{<w, \tau>\right\} = \frac{<w, \tau^*>^2}{<L[\tau^*], \tau^*>} \tag{34}$$

for any admissible τ^*. Recall that we will require τ^* to have singularities of the same order as those possessed by τ. We now seek a systematic method for determining good admissible functions for use in our functional. Since the order of singularity in the dynamic problem is the same as that in the corresponding equilibrium problem, we will use $\tau^* = \tau_S$, the exact solution of (24) when the frequency $\omega = 0$, omitting any unnecessary multiplicative constants. For this trial function, we will call (34) our first variational approximation.

This variational method was used by Stallybrass [23] to set up approximate solutions to a class of punch problems. He used the torsion problem for the circular punch as a test case, since there is an exact solution available [16], and found his first approximation to be accurate in a range of the non-dimensional parameter ka from 0.0 to 2.0. We mention in passing that his second variational approximation was extremely accurate even beyond this range.

CHAPTER III

THE PUNCH PROBLEM

We first consider the punch problem in which a rigid punch, with elliptical cross-section and flat base, is forced to oscillate in contact with an elastic medium occupying a half-space. Using our variational principle (34), we have a scalar invariant functional whose stationary value will be shown to be proportional to the amplitude of oscillation of the punch.

With reference to rectangular cartesian coordinates, the governing system of partial differential equations is

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \rho_0\omega^2 u_i = 0 \quad \text{in } D \tag{35}$$

$(i,j = 1,2,3)$, where D is the domain, u_i is the i^{th} component of displacement, λ and μ are the Lamé constants, ρ_0 is the density of the medium, ω is the frequency of excitation,* commas indicate partial differentiation, and a repeated subscript indicates summation over the range 1, 2, 3. The boundary $z = 0$ of the half-space $z > 0$ will be denoted by B, which is divided into two parts B_1 and B_2 such that

(i) on B_1, the region under the punch, we prescribe all tangential stresses to be zero, which corresponds to the idealized case of no friction under the punch, and the normal component of displacement $w(P) = u_3(P) = C$, and

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*Recall that the time dependence $e^{-i\omega t}$ is omitted in all subsequent expressions.

(ii) on B_2, the remaining portion of B, all components of surface traction will vanish.

The integral representation (13a) for $P \varepsilon B_1$, using the boundary conditions (i) and (ii) and the notation $\tau(Q) = \tau_{33}(Q)$, gives

$$C = -\int_{B_1} u_3^3(P, Q)\tau(Q)dA_Q \tag{36}$$

which is an integral equation for $\tau(Q)$, $Q \varepsilon B_1$. Set

$$\langle u, v\rangle = \int_{B_1} u(Q)v(Q)dA_Q \tag{37}$$

and

$$L[\hat{\tau}] = -\int_{B_1} u_3^3(P, Q)\hat{\tau}(Q)dA_Q\,. \tag{38}$$

Then based on (34) and (36), with (37) and (38), we can obtain a variational approximation for C. If we use $\tau^* = \tau_s$, the exact static solution, our first variational approximation becomes

$$\left\{C \int_{B_1} \tau(Q)dA_Q\right\} = \frac{\left[C \int_{B_1} \tau_s(Q)dA_Q\right]^2}{-\int_{B_1}\left[\int_{B_1} u_3^3(P,Q)\tau_s(Q)dA_Q\right]\tau_s(P)dA_P} \tag{39}$$

where

$$u_3^3(P, Q) = \frac{-k}{8\pi\mu}\int_0^\infty M(t)J_0(ktR)dt, \quad P, Q \varepsilon B_1. \tag{40}$$

The integral $-\int_{B_1} \tau(Q)dA_Q$ is the load P_0 on the punch, which will be maintained constant.

The exact static-case stress was obtained by Green and Sneddon [11] in their investigation of the distribution of stress due to the indentation of a half-space by a flat-ended elliptical punch. To within a multiplicative constant, it is

$$\tau_s = \left[1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right]^{-1/2} \tag{41}$$

for the ellipse B_1: $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ with $0 < a \leq b$. This yields

$$\int_{B_1} \tau_s(Q)dA_Q = 2\pi ab \; . \tag{42}$$

We obtain our first variational approximation to C from (39) as

$$C = \frac{-k\,P_o}{32\pi^3\mu a^2 b^2} \int_{B_1} \left\{ \int_{B_1} \left[\int_0^\infty M(t)J_o(ktR)dt \right] \tau_s(\hat{x},\hat{y})d\hat{x}d\hat{y} \right\} \tau_s(x,y)dxdy \tag{43}$$

where $R = \sqrt{(x-\hat{x})^2 + (y-\hat{y})^2}$. Interchanging the order of integration, we have

$$C = \frac{-kP_o}{32\pi^3\mu a^2 b^2} \int_0^\infty M(t)dt \int_{B_1} \int_{B_1} J_o(ktR)\tau_s(\hat{x},\hat{y})\tau_s(x,y)d\hat{x}d\hat{y}\,dxdy. \tag{44}$$

Consider

$$I(t) = \int_{B_1} \int_{B_1} J_o(ktR)\tau_s(\hat{x},\hat{y})\tau_s(x,y)d\hat{x}d\hat{y}\,dxdy \; . \tag{45}$$

Realizing that

$$J_o(ktR) = \mathrm{Re}\left\{H_o^{(1)}(ktR)\right\} \tag{46}$$

and [see Appendix II]

$$H_o^{(1)}(ktR) = \frac{-i}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i[v(x-\hat{x})+w(y-\hat{y})]}}{v^2+w^2-k^2t^2}\,dvdw\,, \tag{47}$$

we see that upon interchanging orders of integration

$$I(t) = \mathrm{Re}\left[\frac{-i}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dvdw}{v^2+w^2-k^2t^2}\int_{B_1}\int_{B_1} e^{i[v(x-\hat{x})+w(y-\hat{y})]}\tau_s(\hat{x},\hat{y})\tau_s(x,y)d\hat{x}d\hat{y}\,dxdy\right]. \tag{48}$$

We let

$$L = \int_{B_1}\int_{B_1} e^{i[v(x-\hat{x})+w(y-\hat{y})]}\tau_s(\hat{x},\hat{y})\tau_s(x,y)d\hat{x}d\hat{y}\,dxdy \tag{49}$$

$$= \int_{B_1} e^{i(vx+wy)}\tau_s(x,y)dxdy \cdot \int_{B_1} e^{-i(v\hat{x}+w\hat{y})}\tau_s(\hat{x},\hat{y})d\hat{x}d\hat{y}$$

$$= \left|\int_{B_1} e^{i(vx+wy)}\tau_s(x,y)dxdy\right|^2$$

where $\tau_s(x,y) = \sqrt{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}$, $0 < a \leq b$.

Introducing elliptic polar coordinates (ξ, ϕ), defined by

$$x = a\xi\cos\phi \quad \text{and} \quad y = b\xi\sin\phi\,, \tag{50}$$

we have

$$\int_{B_1} \frac{e^{i(vx+wy)}}{\sqrt{1 - (\frac{x^2}{a^2} + \frac{y^2}{b^2})}} dxdy = ab \int_0^1 \frac{\xi\, d\xi}{\sqrt{1 - \xi^2}} \int_0^{2\pi} e^{i(A\cos\phi + B\sin\phi)} d\phi \quad (51)$$

where $A = va\xi$ and $B = wb\xi$. Now

$$\int_0^{2\pi} e^{i(A\cos\phi + B\sin\phi)} d\phi = \int_0^{2\pi} e^{iA\cos\phi}[\cos(B\sin\phi) + i\sin(B\sin\phi)]d\phi, \quad (52)$$

and

$$\int_0^{2\pi} e^{iA\cos\phi} \sin(B\sin\phi)d\phi = 0 \quad (53a)$$

since the integrand is odd about $\phi = \pi$. From [2, p. 82 (17)],

$$\int_0^{2\pi} e^{iA\cos\phi} \cos(B\sin\phi)d\phi = 2\pi J_o(\sqrt{A^2 + B^2}) . \quad (53b)$$

Thus, from (49)

$$L = 4\pi^2 a^2 b^2 \left| \int_0^1 \frac{\xi J_o(\xi\sqrt{a^2v^2 + b^2w^2})}{\sqrt{1 - \xi^2}} d\xi \right|^2 \quad (54)$$

$$= 4\pi^2 a^2 b^2 \frac{\sin^2(\sqrt{a^2v^2 + b^2w^2})}{a^2v^2 + b^2w^2}$$

using [3, p. 7 (5)].

We then require, from (48)

$$I(t) = Re\left[-i4a^2b^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin^2(\sqrt{a^2v^2 + b^2w^2})dvdw}{(v^2 + w^2 - k^2t^2)(a^2v^2 + b^2w^2)}\right]. \quad (55)$$

The value obtained from this integration is complex, since we must deform the contour to avoid the pole. This deformation must be similar to that indicated in Figure 4, in accordance with the radiation condition. Furthermore since we require the real part in (55), we need only the purely imaginary part of the double integral, which comes from the deformation. Letting

$$av = g\cos\theta \quad \text{and} \quad bw = g\sin\theta \ , \tag{56}$$

then $dvdw = \frac{1}{ab}\, gdgd\theta$, and

$$I(t) = \text{Im}\left[4ab \int_0^{2\pi} \frac{a^2 d\theta}{1-\varepsilon^2\sin^2\theta} \int_0^{\infty} \frac{\sin^2(g)}{g(g^2-g_0^2)}\, dg\right] \tag{57}$$

where $\varepsilon^2 = 1 - (\frac{a}{b})^2$ is the eccentricity of the ellipse, and $g_0^2 = \frac{a^2k^2t^2}{1-\varepsilon^2\sin^2\theta}$.

This form has only a simple pole at $g = g_0$. Looping under this pole as in Figure 4, we obtain

$$\text{Im}\left[\int_0^{\infty} \frac{\sin^2(g)dg}{g(g^2-g_0^2)}\right] = \frac{\pi \sin^2(g_0)}{2g_0^2} \ , \tag{58}$$

and from (57)

$$I(t) = \frac{2\pi ab}{k^2t^2} \int_0^{2\pi} \sin^2(g_0(\theta))d\theta \ . \tag{59}$$

From (44), (45), and (59) we have

$$C = \frac{-P_0}{16\pi^2\mu abk} \int_0^{2\pi} d\theta \int_0^{\infty} \frac{M(t)}{t^2} \sin^2(Gt)\, dt \tag{60}$$

where $G = ka(1 - \varepsilon^2 \sin^2\theta)^{-1/2}$. To evaluate the infinite integral in (60), we define its value as in (19) and consider the integral

$$\int_\Gamma \frac{M(z)}{z^2} [1 - e^{-i2Gz}]dz \qquad (61)$$

where Γ is the contour in Figure 7. Since the integrand is analytic in and on Γ, expression (61) is identically zero by Cauchy's Theorem. Upon letting the radii of the indentations around zero and the branch points go to zero, we see their contributions vanish; and when $T \to \infty$, the integrand on Γ_T is of order $O(T^{-2})$, so that contribution also goes to zero. The result is

$$R(-s) + P\int_{-\infty}^{0} \frac{-\alpha}{tf} F(t)dt + \int_0^\gamma \frac{-\alpha}{tf} F(t)dt \qquad (62)$$

$$+ \int_\gamma^1 \frac{\alpha}{t\hat{f}} F(t)dt + P\int_1^\infty \frac{\alpha}{tf} F(t)dt + R(s) = 0$$

where $F(t) = 1 - e^{-i2Gt}$, f and $\hat{f}$ are defined in (23), and $R(-s)$ and $R(s)$ are the contributions around the poles $-s$ and s, respectively. In the first integral, we replace t by $-t$ and obtain

$$P\int_1^\infty \frac{\alpha}{tf} \sin^2(Gt)dt = \frac{\pi\alpha(s)}{2sf'(s)} \sin(2Gs) + \frac{i}{2}\int_0^\gamma \frac{\alpha}{tf} \sin(2Gt)dt \qquad (63)$$

$$- \frac{1}{4}\int_\gamma^1 \frac{\alpha}{tf\hat{f}} [fF(t) + \hat{f}F(-t)]dt \ ,$$

using some trigonometric identities and the fact that $f'(-s) = -f'(s)$. Now by (19)

$$\int_0^\infty \frac{M(t)}{t^2}\sin^2(Gt)dt = \frac{-i}{2}\left[\int_0^1 K(t)\left\{\frac{1-e^{i2Gt}}{t^2}\right\}dt - \frac{\pi\alpha(s)}{f'(s)}\left\{\frac{1-e^{i2Gs}}{s}\right\}\right] \quad (64)$$

where

$$K(t) = \begin{cases} \dfrac{t\sqrt{\gamma^2 - t^2}}{(t^2-\frac{1}{2})^2 + t^2\sqrt{\gamma^2-t^2}\sqrt{1-t^2}}, & 0 \le t \le \gamma \\[2ex] \dfrac{t^3(t^2-\gamma^2)\sqrt{1-t^2}}{(t^2-\frac{1}{2})^4 + t^4(t^2-\gamma^2)(1-t^2)}, & \gamma < t \le 1 . \end{cases} \quad (65)$$

We expand the function $1 - e^{i2Gt}$ in (64) in powers of Gt, and (60) becomes

$$C = \frac{P_0(1-\nu)}{8\pi^2\mu bka}\int_0^{2\pi}\begin{bmatrix} GI_0 - \frac{2}{3}G^3I_2 + \frac{2}{15}G^5I_4 - \frac{4}{315}G^7I_6 + \dots \\ +i\left\{G^2I_1 - \frac{1}{3}G^4I_3 + \frac{2}{45}G^6I_5 - \frac{1}{315}G^8I_7 + \dots\right. \end{bmatrix}d\theta \quad (66)$$

where $G = ka(1 - \epsilon^2\sin^2\theta)^{-1/2}$, ν is Poisson's ratio, and

$$I_n = \frac{1}{2(1-\nu)}\left[\int_0^1 K(t)t^{n-1}dt - \frac{\pi\alpha(s)s^n}{f'(s)}\right], \quad n = 0,1,2,\dots . \quad (67)$$

These I_n are the same quantities that occurred in Robertson [17] and Gladwell [10] in connection with the oscillation of a punch with a flat circular base.* We note that $I_0 = \pi$.

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*As observed by Gladwell, the values of I_n, as given by Robertson, are inaccurate beyond the second decimal place. They are recalculated in Appendix III.

We use [12] and [5, (220.00), (220.01)] to evaluate

$$\int_0^{2\pi} G^n(\theta)d\theta = 4(ka)^n \int_0^{\pi/2} (1 - \varepsilon^2 \sin^2\theta)^{-n/2} d\theta, \; n = 1,2,\ldots,8. \quad (68)$$

For n odd, the integrals are elliptic functions and require numerical integration. Letting

$$\begin{aligned}
\hat{G}_0\left(\frac{b}{a}\right) &= \frac{2}{\pi}\int_0^{\pi/2} (1 - \varepsilon^2 \sin^2\theta)^{-1/2} d\theta \\
\hat{G}_1\left(\frac{b}{a}\right) &= \frac{4}{3\pi^2}\int_0^{\pi/2} (1 - \varepsilon^2 \sin^2\theta)^{-3/2} d\theta \\
\hat{G}_2\left(\frac{b}{a}\right) &= \frac{4}{15\pi^2}\int_0^{\pi/2} (1 - \varepsilon^2 \sin^2\theta)^{-5/2} d\theta \\
\hat{G}_3\left(\frac{b}{a}\right) &= \frac{8}{315\pi^2}\int_0^{\pi/2} (1 - \varepsilon^2 \sin^2\theta)^{-7/2} d\theta
\end{aligned} \quad (69)$$

where $\varepsilon^2 = 1 - (a/b)^2$, our first variational approximation for the ellipse is

$$C = \frac{P_0(1-\nu)}{4\mu b}\left[\hat{G}_0(q) - \hat{G}_1(q)I_2\kappa^2 + \hat{G}_2(q)I_4\kappa^4 - \hat{G}_3(q)I_6\kappa^6 + \ldots + i\left\{\frac{q}{\pi} I_1\kappa - \frac{q(1+q^2)}{6\pi} I_3\kappa^3 + \frac{q(3+2q^2+3q^4)}{180\,\pi} I_5\kappa^5 - \frac{q(5+3q^2+3q^4+5q^6)}{5040\,\pi} I_7\kappa^7 + \ldots\right.\right. \quad (70)$$

where $\kappa = ka$ and $q = b/a$. The values of the $\hat{G}_n$ are given for several ratios q in Table 1. Now let

$$\frac{4\mu bC}{P_o(1-\nu)} = f_1 + if_2 \ . \tag{71}$$

As a representative value of ν, we use $\nu \simeq \frac{1}{4}$, i.e. $\gamma^2 \approx \frac{1}{3}$, and graph f_1 and f_2 for $\gamma^2 = .33$ for several values of q in Figure 8 and Figure 9. We graph $(f_1^2 + f_2^2)^{1/2}$ in Figure 10.

When the ellipse degenerates to a circle, we have

$$C = \frac{P_o(1-\nu)}{4\mu a}\left[1 - \frac{2}{3\pi} I_2\kappa^2 + \frac{2}{15\pi} I_4\kappa^4 - \frac{4}{315\pi} I_6\kappa^6 + \ldots \right. \\ \left. + i\left\{\frac{1}{\pi} I_1\kappa - \frac{1}{3\pi} I_3\kappa^3 + \frac{2}{45\pi} I_5\kappa^5 - \frac{1}{315\pi} I_7\kappa^7 + \ldots \right. \right. . \tag{72}$$

To check this result, we use (15) for axially-symmetric problems, with B_1 defined by $\rho \leq a$, in polar coordinates. With the static stress

$$\tau_S(\rho) = a(a^2 - \rho^2)^{-1/2}, \quad \rho \leq a \ , \tag{73}$$

we have

$$C = \frac{-kP_o}{32\pi^3\mu a^4} \int_{B_1} \left\{\int_{B_1}\left[\int_0^\infty M(t)J_0(ktr)J_0(kt\rho)dt\right]\tau_S(\rho)\rho d\rho d\phi\right\}\tau_S(r)rdrd\theta$$

$$= \frac{-kP_o}{8\pi\mu a^2} \int_0^\infty M(t)\left[\int_0^a \frac{\rho}{\sqrt{a^2 - \rho^2}} J_0(kt\rho)d\rho\right]^2 dt \tag{74}$$

$$= \frac{-P_o}{8\pi\mu\kappa a} \int_0^\infty M(t)\left[\frac{\sin(\kappa t)}{t}\right]^2 dt$$

using [3, p. 7(5)], and $\kappa = ka$. This last integral arose in equation (60), and from (64) and (67), replacing G by κ, our first variational approximation for the circle becomes

$$C = \frac{P_0(1-\nu)}{4\mu a}\left[1 - \frac{2}{3\pi} I_2\kappa^2 + \frac{2}{15\pi} I_4\kappa^4 - \frac{4}{315\pi} I_6\kappa^6 + \ldots \right. \tag{75}$$
$$\left. + i\left\{\frac{1}{\pi} I_1\kappa - \frac{1}{3\pi} I_3\kappa^3 + \frac{2}{45\pi} I_5\kappa^5 - \frac{1}{315\pi} I_7\kappa^7 + \ldots,\right.\right.$$

which is identical to (72). Furthermore, we have compared this result for the circle with that obtained from results given by Robertson [17] in equations (5.1) and (4.6). The results agree exactly up to the term in (our) κ^3. Further terms were not calculated.

Although we have only discussed a first variational approximation, second and higher order variational approximations for the displacement of the flat elliptical punch could be obtained in a manner analogous to that discussed by Stallybrass in [23] for the torsional oscillations of a circular punch.

Table 1. Values for the $\hat{G}_n$.

q	1	2	3	4
$\hat{G}_0$	1.00000	1.37288	1.60977	1.78330
$\hat{G}_1$	.21221	.65443	1.35415	2.31780
$\hat{G}_2$	.04244	.35860	1.60057	4.85003
$\hat{G}_3$	.00404	.10669	1.08020	5.85811

Table 2 and Table 3 were calculated from the power series expansion of our approximation. The column headings "κ^n" indicate that terms in the expansion through the power κ^n were used. We compared the results in an attempt to determine the range of accuracy, as is indicated in Figures 8 and 9. The range for both f_1 and f_2 for $\kappa = ka$ diminishes as the ratio $q = b/a$ increases.

An approximation for the displacement of a punch of arbitrary shape which has been used in the past is the displacement of circular punch with an equivalent base. That is, the circular punch has the same area as the punch being considered. In view of this, we make such an approximation for various ellipses, and compare this with our variational approximation in Table 4. In equation (71) we ignore multiplicative constants and let

$$C^* = \frac{(f_1^2 + f_2^2)^{1/2}}{b} .$$

For an ellipse with axes of length a and b ($a < b$) and area πab, an equivalent circular punch must have radius $r = \sqrt{ab}$. We simply note that for the range of κ and b/a in Table 4 the displacement for the circular punch is greater than that for the equivalent elliptical punch.

Table 2. Values for f_1 with $\gamma^2 = .33$

q	1		2		3		4	
terms / κ	κ^6	κ^4	κ^6	κ^4	κ^6	κ^4	κ^6	κ^4
0.0	1.000	1.000	1.373	1.373	1.610	1.610	1.783	1.783
0.1	.995	.995	1.358	1.358	1.578	1.578	1.730	1.730
0.2	.980	.980	1.313	1.313	1.488	1.489	1.583	1.584
0.3	.956	.956	1.241	1.241	1.353	1.355	1.377	1.389
0.4	.923	.923	1.148	1.149	1.189	1.201	1.150	1.216
0.5	.882	.882	1.039	1.043	1.013	1.059	.913	1.165
0.6	.833	.834	.919	.933	.834	.973	.614	1.366
0.7	.779	.780	.796	.830	.644	.993	.079	1.972
0.8	.720	.723	.672	.748	.404	1.182		
0.9	.658	.664	.548	.704	.032	1.609		
1.0	.594	.605	.421	.714				
1.1	.529	.549	.281	.800				
1.2	.464	.497	.108	.983				
1.3	.401	.454						
1.4	.339	.422						
1.5	.278	.405						

Table 3. Values for f_2 with γ^2 = .33

q	1		2		3		4	
terms / κ	κ^7	κ^5	κ^7	κ^5	κ^7	κ^5	κ^7	κ^5
0.0	.000	.000	.000	.000	.000	.000	.000	.000
0.1	.079	.079	.157	.157	.233	.233	.307	.307
0.2	.156	.156	.306	.306	.445	.445	.568	.568
0.3	.230	.230	.441	.441	.617	.617	.749	.752
0.4	.300	.300	.556	.557	.739	.743	.840	.866
0.5	.364	.364	.648	.649	.809	.826	.846	.969
0.6	.422	.422	.714	.718	.827	.888	.747	1.190
0.7	.472	.473	.753	.765	.793	.973		
0.8	.514	.515	.766	.796	.693	1.150		
0.9	.547	.549	.755	.823	.477	1.520		
1.0	.572	.575	.718	.861	.036	2.217		
1.1	.587	.593	.654	.932				
1.2	.593	.604	.553	1.064				
1.3	.591	.610	.397	1.293				
1.4	.581	.612						
1.5	.562	.613						

Table 4. Comparison of C^* for Ellipse and Circle of Equal Area

κ	a = 1, b = 2	r = $\sqrt{2}$	a = 1, b = 3	r = $\sqrt{3}$	a = 1, b = 4	r = 2
0.0	.687	.707	.537	.577	.446	.500
0.1	.684	.706	.532	.576	.439	.499
0.2	.674	.702	.518	.573	.421	.496
0.3	.659	.695	.497	.568	.392	.492
0.4	.638	.687	.467	.561	.356	.486
0.5	.613	.675	.432	.551	.311	.477
0.6	.582	.661	.392	.539		
0.7	.548	.644	.341	.526		
0.8	.510	.626				
0.9	.497	.605				
1.0	.416	.583				

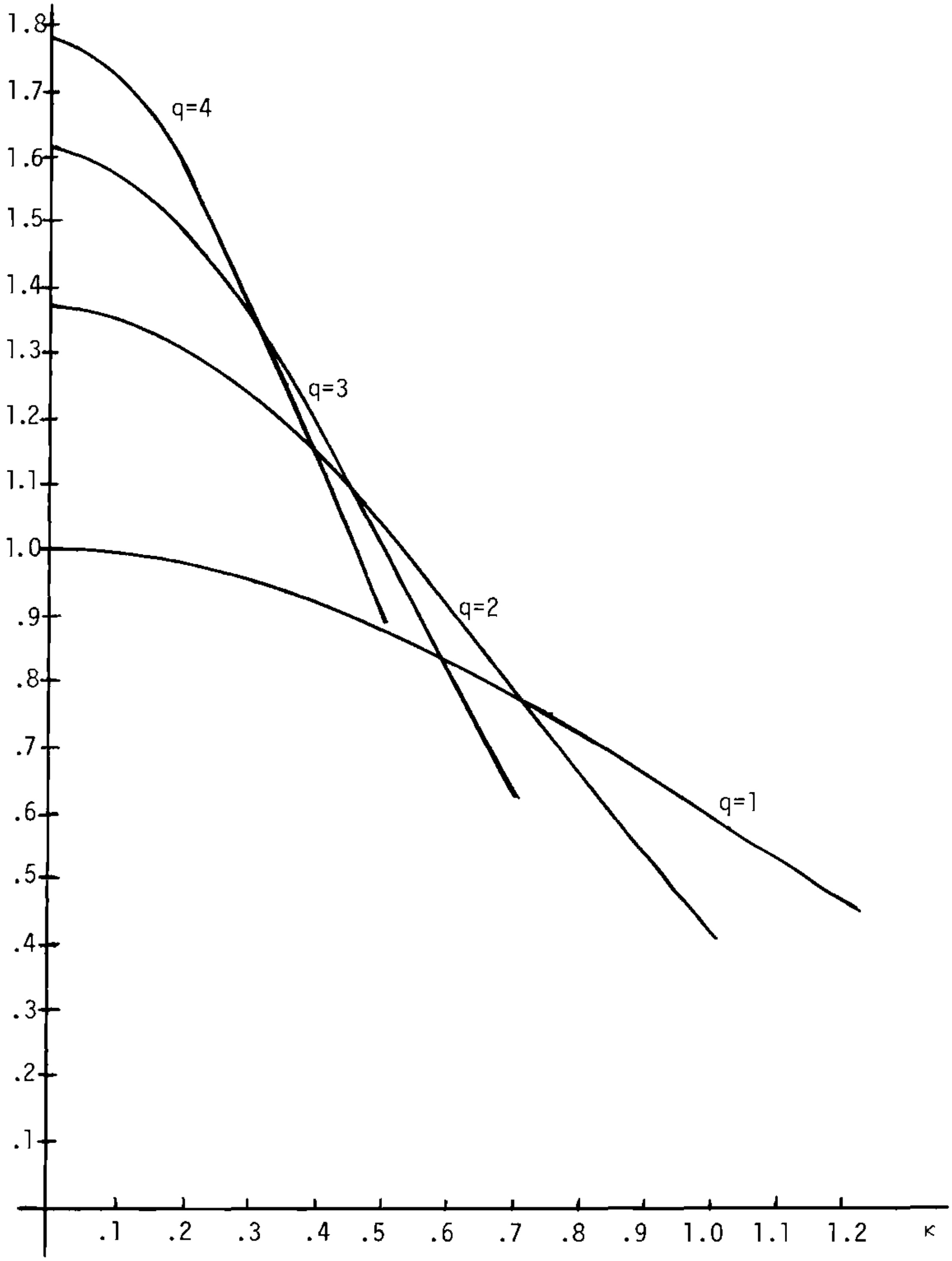

Figure 8. Graph of f_1.

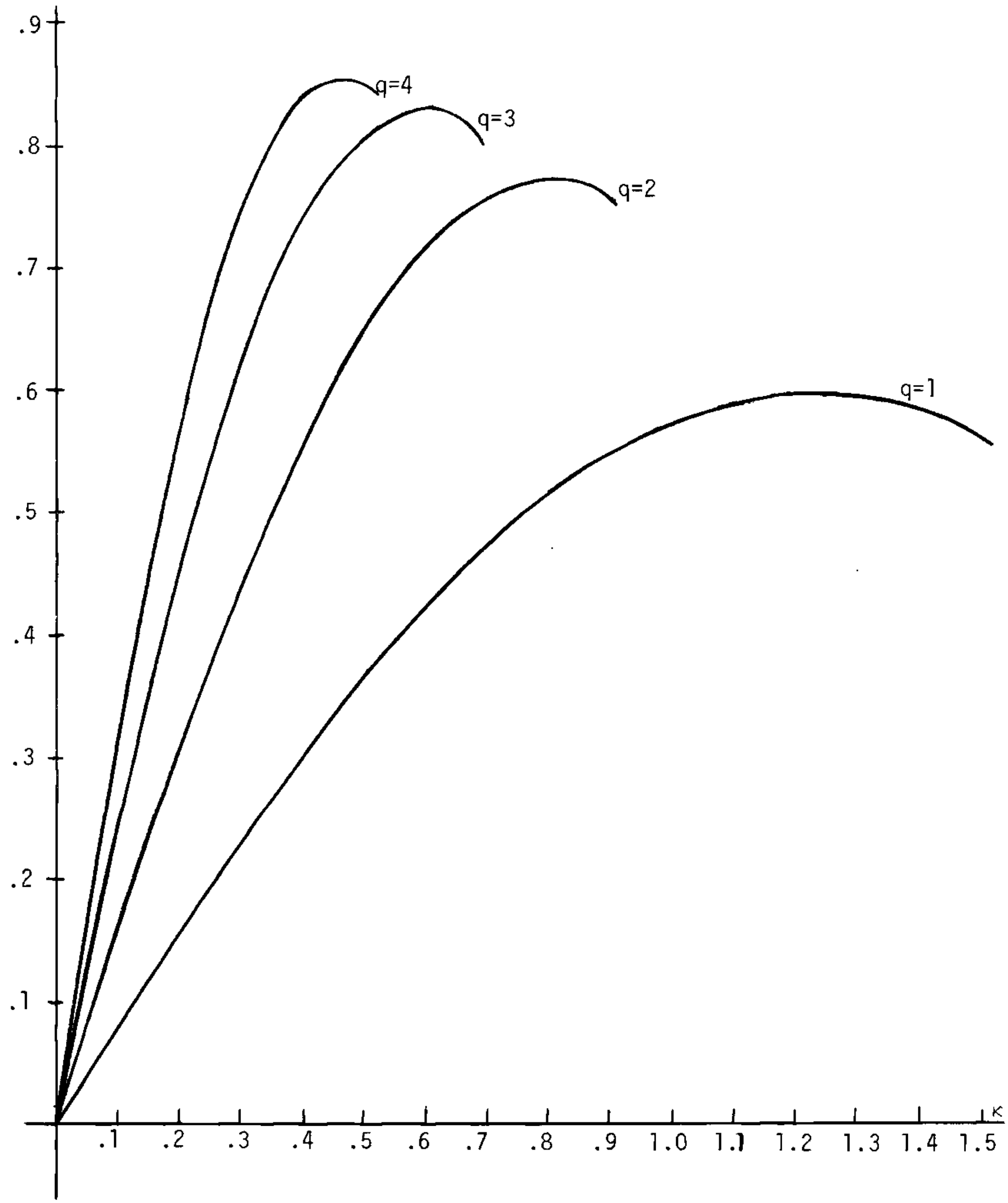

Figure 9. Graph of f_2.

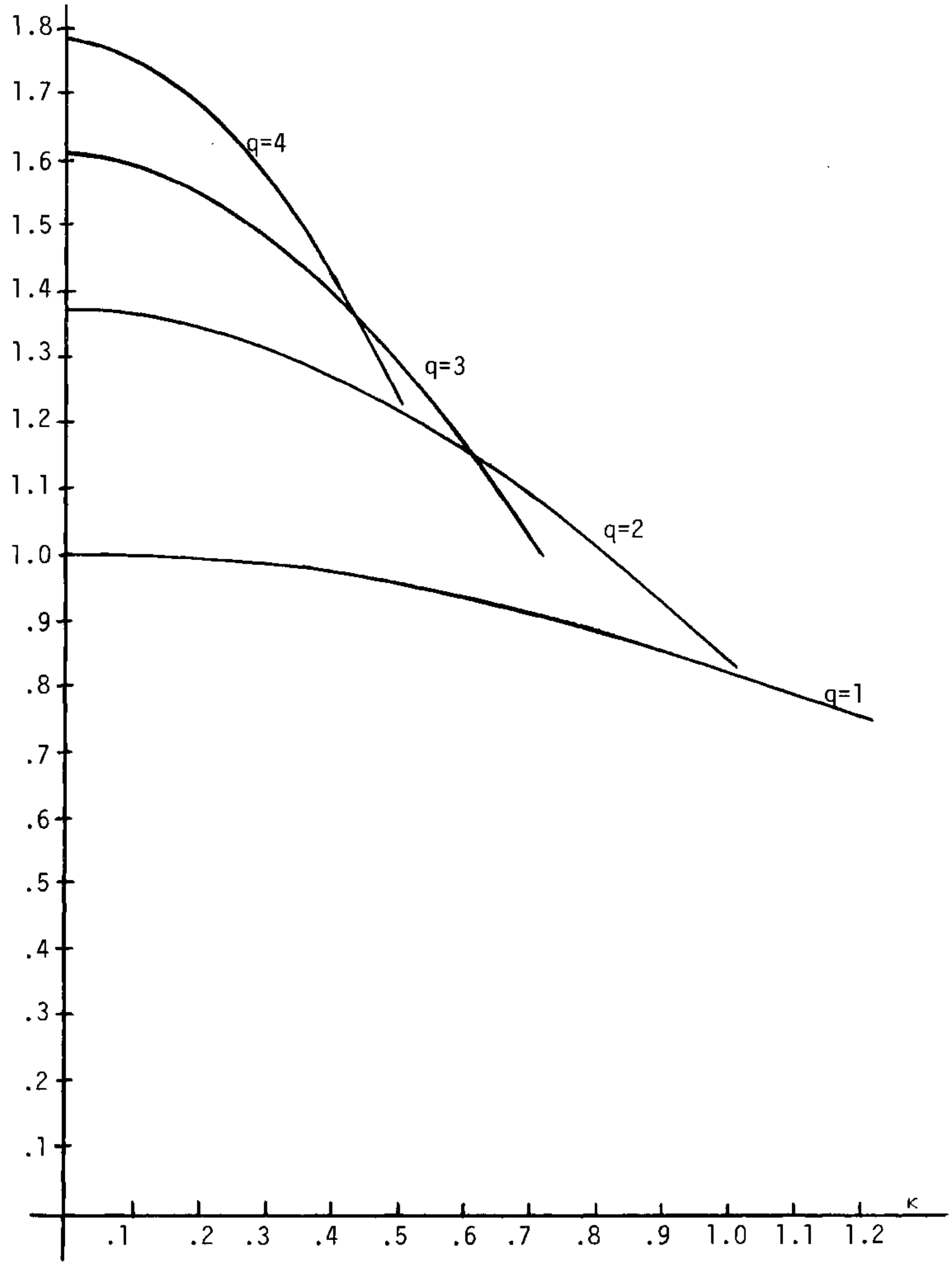

Figure 10. Graph of $(f_1^2 + f_2^2)^{1/2}$.

CHAPTER IV

THE CRACK PROBLEM

When a plane wave encounters a crack in an infinite elastic medium, the wave is scattered. A measure of the effect of this disturbance is the scattering cross-section, which is defined to be the ratio of the average rate at which energy is scattered by the obstacle to the average rate at which the energy of the incident wave crosses a unit of area perpendicular to the direction of propagation of the wave. Barratt and Collins [1] have shown that this cross-section for a plane harmonic wave is accessible from an appropriate far-field amplitude of the scattered waves in the direction of propagation of the incident wave.

We choose our crack to be in the plane $z = 0$, and the incident harmonic plane wave will emanate from $z = -\infty$ and proceed in the positive z-direction, i.e. normally incident on the crack. Considering the plane $z = 0$ to be the boundary B, our boundary conditions become*

(i) on B_1, the region which the crack occupies, all tangential stresses are zero, and the normal stress is $\tau(Q) = -p_0$, a constant, and

(ii) on B_2, the remaining portion of B, by symmetry all tangential stresses are zero; also the vertical displacement is $w(P) = 0$, $P \in B_2$.

---

*Recall that the time dependence $e^{-i\omega t}$ is omitted.

Now if we write the displacement w for large positive z as

$$w \sim g(\theta,\ \phi)\Big|_{(0,0)} \cdot \frac{e^{ihz}}{z}\ ,\ z \to \infty \tag{76}$$

where $\theta = 0$, $\phi = 0$ is the angle of propagation of the wave relative to the positive z-axis, Barratt and Collins [1, (2.25)] show that the scattering cross-section $\sum$ is

$$\sum = \frac{4\pi}{h}\ \mathrm{Im}[g(0,\ 0)]\ . \tag{77}$$

Our technique will be to set up a variational formula which yields a quantity proportional to g(0, 0). Furthermore, our formula will be applicable to cracks of arbitrary shape.

From equations (10) and (11), the vertical displacement w at a field point P = (0, 0, z) is

$$w(P) = \frac{1}{2\pi\mu}\int_B \left\{\int_0^\infty \frac{\xi\sqrt{\xi^2-h^2}\left[(k^2-2\xi^2)e^{-z\sqrt{\xi^2-h^2}} + 2\xi^2 e^{-z\sqrt{\xi^2-k^2}}\right]}{F(\xi)} J_0(\xi R)d\xi\right\}\tau(Q)dA_Q \tag{78}$$

where $Q = (\hat{x},\ \hat{y},\ 0)$, $R = \sqrt{\hat{x}^2 + \hat{y}^2 + z^2}$, and the infinite integral is along the contour in Figure 5. We require the scattered displacement for $z >> 1$ in the form (76). The only part dependent on z is contained in the brackets $\{\ \}$. We split this into two integrals

$$w_1(z) = \int_0^\infty \frac{\xi\sqrt{\xi^2-h^2}(k^2-2\xi^2)e^{-z\sqrt{\xi^2-h^2}}}{F(\xi)}\ J_0(\xi R)d\xi \tag{79}$$

and

$$w_2(z) = \int_0^\infty \frac{2\xi^3\sqrt{\xi^2-h^2}\ e^{-z\sqrt{\xi^2-k^2}}}{F(\xi)} J_0(\xi R)d\xi. \tag{80}$$

For the asymptotic analysis of w_1, split the integral at $\xi = h$. For $0 \leq \xi \leq h$, replace ξ by $h\sqrt{1-v^2}$, and denote the integral by $w_1^1(z)$. For $\xi > h$, we denote the integral by $w_1^2(z)$. Then

$$w_1^1(z) = -ih^3 \int_0^1 \frac{v^2(k^2 - 2h^2 + 2h^2v^2)}{F(h\sqrt{1-v^2})} J_0(hR\sqrt{1-v^2})e^{ihvz}dv \tag{81a}$$

and

$$w_1^2(z) = \int_h^\infty \frac{\xi\sqrt{\xi^2-h^2}\ (k^2 - 2\xi^2)}{F(\xi)} J_0(\ R)\ e^{-z\sqrt{\xi^2-h^2}}\ d\xi \tag{81b}$$

where the contour for $w_1^2(z)$ must be as in Figure 5. The square roots in (79) - (81) are defined as in (21) - (23).

Similarly, split the integral for w_2 at $\xi = k$. For $0 \leq \xi \leq k$, replace ξ by $k\sqrt{1-v^2}$ and call the integral $w_2^1(z)$. For $\xi > k$, denote the integral by $w_2^2(z)$. Then

$$w_2^1(z) = 2k^4 \int_0^1 \frac{v(1-v^2)\sqrt{k^2 - h^2 - k^2v^2}}{F(k\sqrt{1 - v^2})} J_0(kR\sqrt{1-v^2})e^{ikvz}dv \tag{82a}$$

and

$$w_2^2(z) = \int_k^\infty \frac{2\xi^3\sqrt{\xi^2-h^2}}{F(\xi)} J_0(\xi R)e^{-z\sqrt{\xi^2-k^2}}d\xi\ . \tag{82b}$$

We treat the integrals for $w_1^1(z)$ and $w_2^1(z)$ first. The integrand in $w_1^1(z)$ has no poles, since F has no zeros within the interval. Thus,

$$w_1^1(z) = -ih^3 \int_0^1 A(v) J_0(hR\sqrt{1-v^2}) e^{ihvz} dv \tag{83}$$

where

$$A(v) = \frac{v^2(k^2 - 2h^2 + 2h^2v^2)}{F(h\sqrt{1-v^2})}, \tag{84}$$

which is real in [0, 1], and $R = \sqrt{\rho^2 + z^2}$, $\rho^2 = \hat{x}^2 + \hat{y}^2$. Using (47), we have

$$w_1^1(z) = -ih^3 \int_0^1 A(v) e^{ihvz} \mathrm{Re}\left\{\frac{-i}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i(x\rho+yz)}}{x^2+y^2-h^2+h^2v^2} dxdy\right\} dv^*$$

$$= -\frac{ih^3}{\pi^2} \int_0^1 A(v) e^{ihvz}\, \mathrm{Im}\left\{\int_0^{2\pi} \int_0^{\infty} \frac{e^{ir(\rho\cos\theta + z\sin\theta)}}{r^2 - h^2 + h^2v^2} r dr d\theta\right\} dv,$$

where $x = r\cos\theta$, $y = r\sin\theta$,

$$= -\frac{ih^3}{\pi^2} \int_0^1 A(v) \left\{ \begin{aligned} &\mathrm{Im} \int_0^{\infty}\int_0^{2\pi} \frac{\cos(hvz) e^{ir(\rho\cos\theta + z\sin\theta)}}{r^2 - h^2 + h^2v^2} r d\theta dr \\ &+ i\,\mathrm{Im} \int_0^{\infty}\int_0^{2\pi} \frac{\sin(hvz) e^{ir(\rho\cos\theta + z\sin\theta)}}{r^2 - h^2 + h^2v^2} r d\theta dr \end{aligned} \right\} dv$$

---

*All integrals must be in accordance with the radiation condition.

$$= -\frac{ih^3}{2\pi^2}\begin{cases} \text{Im} \int_0^\infty rB(r,z)dr \int_0^1 \frac{A(v)(e^{ihvz} + e^{-ihvz})}{r^2 - h^2 + h^2v^2} dv \\ -i\text{Re} \int_0^\infty rB(r,z)dr \int_0^1 \frac{A(v)(e^{ihvz} - e^{-ihvz})}{r^2 - h^2 + h^2v^2} dv \end{cases} \tag{85}$$

where, by (52),

$$B(r,z) = \int_0^{2\pi} e^{ir(\rho \cos\theta + z \sin\theta)} d\theta = 2\pi J_0(rR), \quad R = \sqrt{\rho^2 + z^2}\,. \tag{86}$$

Using a standard technique, as in Erdelyi [8], we have

$$\int_0^1 \frac{A(v)e^{\pm ihvz}}{r^2 - h^2 + h^2v^2} dv = \left(\frac{1}{k^2r^2}\right) \frac{e^{\pm ihz}}{\pm ihz} + o(z^{-1}), \quad z >> 1\,, \tag{87}$$

making

$$w_1^1(z) \sim -\frac{2\gamma^2}{\pi} \frac{e^{ihz}}{z} \text{Im} \int_0^\infty \frac{J_0(rR)}{r} dr, \quad z >> 1, \tag{88}$$

where $\gamma = h/k$, and the infinite integral is actually over a contour that goes below the pole $r = 0$.

Now consider

$$\int_{\hat{\Gamma}} \frac{H_0^{(2)}(tR)}{t} dt \tag{89}$$

where $H_0^{(2)}$ is the Hankel function of the second kind, and $\hat{\Gamma}$ is a contour like Γ in Figure 7 with only a branch cut along the negative real axis due to the logarithmic singularity in the Hankel function, and only one indentation at t = 0. Since the integrand is analytic in and on $\hat{\Gamma}$, (89) is identically zero, and by Jordan's lemma,

$$\int_{-\infty}^{-\varepsilon} \frac{H_0^{(2)}(rR)}{r} dr + \int_{\varepsilon}^{\infty} \frac{H_0^{(2)}(rR)}{r} dr = -i\int_{-\pi/2}^{0} H_0^{(2)}(\varepsilon e^{i\theta}R)d\theta. \qquad (90)$$

Since $H_0^{(2)}(e^{-i\pi}rR) = -H_0^{(1)}(rR)$, upon replacing r by $e^{-i\pi}r$ in the first integral in (90), we have

$$\text{Im} \int_{-\infty}^{-\varepsilon} \frac{H_0^{(2)}(rR)}{r} dr = \text{Im} \int_{\varepsilon}^{\infty} \frac{H_0^{(1)}(rR)}{r} dr , \qquad (91)$$

and (90) becomes

$$2 \int_{\varepsilon}^{\infty} \frac{J_0(rR)}{r} dr = -i \int_{-\pi/2}^{0} H_0^{(2)}(\varepsilon e^{i\theta}R)d\theta. \qquad (92)$$

Taking the limit as $\varepsilon \to 0$, and evaluating only the imaginary part, we have

$$\begin{aligned} \text{Im} \int_{0}^{\infty} \frac{J_0(rR)}{r} dr &= \text{Im}\left[-\frac{\pi i}{2} \lim_{r\to 0} r\left\{\frac{J_0(rR)}{r} + i\,\frac{Y_0(rR)}{r}\right\}\right] \qquad (93) \\ &= -\frac{\pi}{2} \lim_{r\to 0} \left\{J_0(rR) + \arg(r)J_0(rR)\right\} \\ &= -\frac{\pi}{2} , \end{aligned}$$

and for large positive z,

$$w_1^1(z) \sim \gamma^2 \frac{e^{ihz}}{z} . \qquad (94)$$

Following similar steps, we have

$$w_2^1(z) \sim o(z^{-1}), \quad z >> 1 , \tag{95}$$

since, in the evaluation of an integral like (87), the integrand is zero at both end points.

We now treat $w_1^2(z)$ and $w_2^2(z)$. Both integrals require a contour like that in Figure 5, however, starting at $\xi = h$ and $\xi = k$, respectively. They can be handled by the method of steepest descent. In $w_1^2(z)$, for $z >> 1$,

$$J_o(\xi R) \sim J_o(\xi z) \sim \sqrt{\frac{2}{\pi \xi z}} \cos\ (\xi z - \frac{\pi}{4}) \tag{96}$$

where $\sqrt{\xi}$ requires a branch cut along the negative real axis. Thus, (81b) becomes

$$w_1^2(z) \sim z^{-1/2} \int_h^\infty \tilde{A}(\xi)[e^{i(\xi z - \pi/4)} + e^{-i(\xi z - \pi/4)}] e^{-z\sqrt{\xi^2 - h^2}} d\xi \tag{97}$$

where

$$\tilde{A}(\xi) = \frac{\sqrt{\xi}\ \sqrt{\xi^2 - h^2}\ (k^2 - 2\xi^2)}{\sqrt{2\pi}\ F(\xi)} . \tag{98}$$

Split (97) into two integrals

$$b_1(z) = z^{-1/2} \int_\Omega \tilde{A}(\xi) e^{-i\pi/4}\ e^{z(i\xi - \sqrt{\xi^2 - h^2})} d\xi \tag{99a}$$

and

$$b_2(z) = z^{-1/2}\int_\Omega \tilde{A}(\xi)e^{i\pi/4}\, e^{z(-i\xi - \sqrt{\xi^2 - h^2})}d\xi \qquad (99b)$$

where Ω is the contour indicated in Figure 11. In (99a) the integrand has one saddle point, $\xi = h/\sqrt{2}$, and the path P_1 of steepest descent through this point is also indicated in Figure 11, where

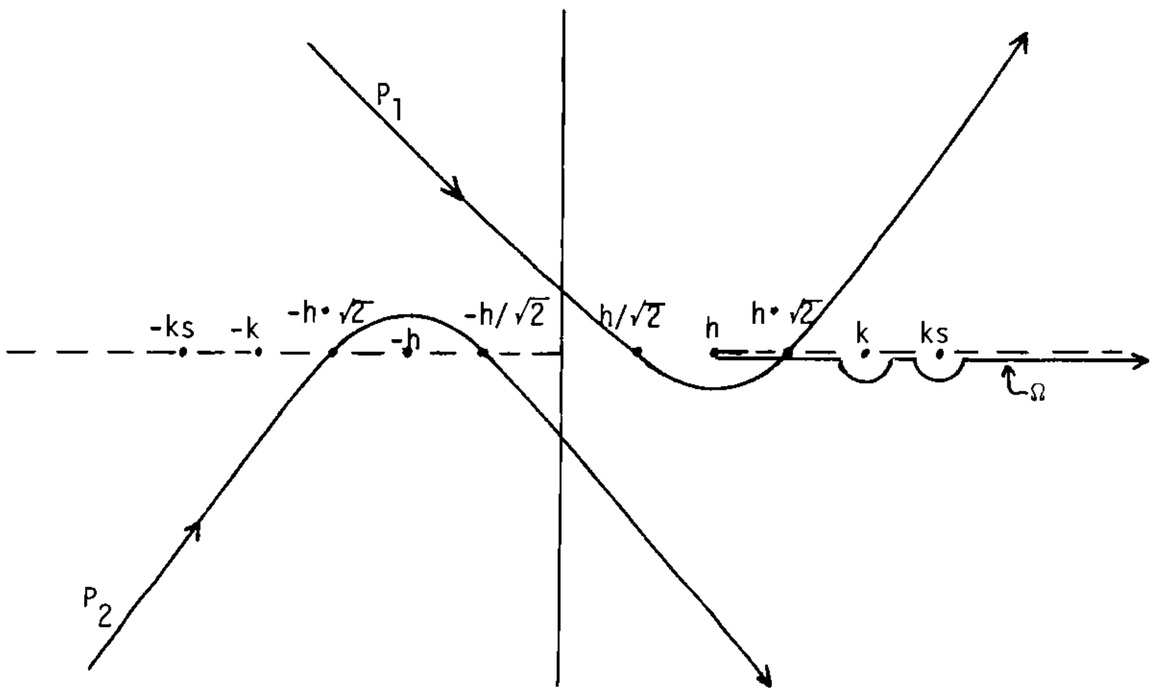

Figure 11. Steepest Descent Paths for $w_1^2(z)$.

the dashed lines indicate branch cuts due to square root functions in $\tilde{A}(\xi)$. In order to deform the contour Ω into P_1, we must cross two branch points, $\xi = h$ and $\xi = k$, and the pole $\xi = ks$. In doing so, we pick up contributions of order $O(z^{-3/2})$ at the branch points and the residue from the pole, which decays exponentially as $z \to \infty$. From [7], the saddle point contribution is

$$b_1(z) \sim \frac{-h^2(k^2 - h^2)}{2\sqrt{2}\ F(h/\sqrt{2})} \cdot \frac{e^{ihz\sqrt{2}}}{z} . \tag{100}$$

The integrand of $b_2(z)$ has one saddle point, $\xi = -h/\sqrt{2}$, and the steepest descent path P_2 is indicated in Figure 11. In order to deform Ω into P_2, we cross two branch points, $\xi = 0$ and $\xi = -h$. Their contributions are $O(z^{-3/2})$, and the contribution from the saddle point to $b_2(z)$ is the negative of (100). Thus, for $z >> 1$,

$$w_1^2(z) = o(z^{-1}) . \tag{101}$$

In the same manner, we find that for $z >> 1$

$$w_2^2(z) = o(z^{-1}) . \tag{102}$$

Combining (94), (95), (101), and (102), we have

$$w(P) \sim \left[\frac{\gamma^2}{2\pi\mu} \int_B \tau(Q)dA_Q\right] \frac{e^{ihz}}{z} \tag{103}$$

for $P = (0, 0, z)$ and $z >> 1$. By (76), we can now determine that

$$g(0,0) = \frac{\gamma^2}{2\pi\mu} \int_B \tau(Q)dA_Q . \tag{104}$$

We now proceed to set up our variational approximation so that the quantity we approximate is proportional to g(0,0) in (104). Using the boundary conditions and (10) we have

$$\int_{B_2} u_3^3(S,\ Q)\tau(Q)dA_Q = p_o \int_{B_1} u_3^3(S,\ Q)dA_Q,\ S \in B_2, \tag{105}$$

a Fredholm integral equation of the first kind; the only unknown being $\tau(Q)$ for $Q \in B_2$.

We define

$$L[v] = \int_{B_2} u_3^3(S,\ Q)v(Q)dA_Q,\ S \in B_2\ , \tag{106}$$

and

$$\langle v_1, v_2 \rangle = \int_{B_2} v_1(Q)v_2(Q)dA_Q\ . \tag{107}$$

Applying the variational formula (34) with (105), we obtain

$$\left\{\int_{B_2} \tau(S)\ p_o \int_{B_1} u_3^3(S,\ Q)dA_Q dA_S\right\} = \frac{p_o^2\left[\int_{B_2} \tau^*(S) \int_{B_1} u_3^3(S,\ Q)dA_Q dA_S\right]^2}{\int_{B_2}\int_{B_2} u_3^3(S,\ Q)\tau^*(Q)\tau^*(S)dA_Q dA_S}\ . \tag{108}$$

We set

$$V = p_o \int_{B_2} \tau(S) \int_{B_1} u_3^3(S,\ Q)dA_Q dA_S \tag{109}$$

$$= p_o \int_{B_2} \tau(S) \int_B u_3^3(S,Q)dA_Q dA_S - p_o \int_{B_2} \tau(S) \int_{B_2} u_3^3(S,Q)dA_Q dA_S\ .$$

In Appendix IV, we show that $\int_B u_3^3(S, Q)dA_Q$ is a constant, independent of S. We let

$$\hat{c} = \int_B u_3^3(S, Q)dA_Q = \frac{i\gamma}{\mu k}, \tag{110}$$

and (109) becomes

$$\begin{aligned} V &= p_o\hat{c} \int_{B_2} \tau(S)dA_S - p_o\int_{B_2}\int_{B_2} u_3^3(Q, S)\tau(S)dA_S dA_Q \\ &= p_o\hat{c} \int_{B_2} \tau(S)dA_S - p_o^2\int_{B_2}\int_{B_1} u_3^3(S, Q)dA_Q dA_S \end{aligned} \tag{111}$$

where we have used the symmetry of $u_3^3(S, Q)$ and equation (105) and interchanged S and Q.

In equation (104), our interest is in the integral

$$\begin{aligned} \int_B \tau(Q)dA_Q &= \int_{B_2} \tau(S)dA_S + \int_{B_1} (-p_o)dA_Q \\ &= \frac{V}{p_o\hat{c}} + \frac{p_o}{\hat{c}} \int_{B_2}\int_{B_1} u_3^3(S, Q)dA_Q dA_S - p_o\cdot\text{area}(B_1) \end{aligned} \tag{112}$$

using (111). Furthermore,

$$\begin{aligned} \int_{B_2}\int_{B_1} u_3^3(S, Q)dA_Q dA_S &= \left[\int_B - \int_{B_1}\right]\int_{B_1} u_3^3(S, Q)dA_Q dA_S \\ &= \hat{c}\cdot\text{area}(B_1) - \int_{B_1}\int_{B_1} u_3^3(S, Q)dA_Q dA_S. \end{aligned} \tag{113}$$

Thus, inserting (113) into (112), we have the approximation

$$\left\{\int_B \tau(Q)dA_Q\right\} = \frac{1}{p_o\hat{c}}\left\{V\right\} - \frac{p_o}{\hat{c}}\int_{B_1}\int_{B_1} u_3^3(S, Q)dA_Q dA_S \, . \tag{114}$$

From (114), (109) and (108), using $\tau^* = \tau_S$, the exact static-case stress, we obtain our first variational approximation

$$\left\{\int_B \tau(Q)dA_Q\right\} = \frac{p_o}{\hat{c}}\left[\frac{\left[\int_{B_2} \tau_S(S) \int_{B_1} u_3^3(S, Q)dA_Q dA_S\right]^2}{\int_{B_2}\int_{B_2} u_3^3(S, Q)\tau_S(S)\tau_S(Q)dA_Q dA_S} - \int_{B_1}\int_{B_1} u_3^3(S, Q)dA_Q dA_S\right] \tag{115}$$

which is valid for a crack of arbitrary shape.

We now specialize the problem to a circle. In the plane z = 0, the static-case stress in the exterior of a cricle of radius a is given by Sneddon [19]. Omitting multiplicative constants, we have

$$\tau_S(S) = \frac{a}{\sqrt{r^2 - a^2}} - \sin^{-1}(\tfrac{a}{r}) \, , \quad r > a \, . \tag{116}$$

For axially-symmetric problems, with $S = (r, \theta, 0)$ and $Q = (\rho, \phi, 0)$ in polar coordinates, the integrals in (115) reduce to [see Appendix I]

$$V_1 = \int_{B_2} \tau_S(S) \int_{B_1} u_3^3(S,Q)dA_Q dA_S$$

$$= \frac{k}{8\pi\mu}\int_0^{2\pi} d\theta \int_a^{\infty} \tau_S(r)rdr \int_0^{2\pi} d\phi \int_0^{a} \rho d\rho \int_0^{\infty} M(x)J_0(krx)J_0(k\rho x)dx$$

$$= \frac{\pi k}{2\mu} \int_0^\infty M(x)dx \int_0^a \rho J_0(k\rho x)d\rho \int_a^\infty \tau_S(r) r J_0(krx)dr$$

$$= \frac{\pi k a^4}{2\mu} \int_0^\infty M(x)\left[\frac{J_1(\kappa x)}{\kappa x}\right]\left[\frac{\pi}{2}\frac{J_1(\kappa x)}{\kappa x} + \frac{\{\kappa x \cdot \cos(\kappa x) - \sin(\kappa x)\}}{(\kappa x)^2}\right]dx \quad (117)$$

where $\kappa = ka$, and we used [3, p. 7(5), p. 18(5)].

$$V_2 = \int_{B_2}\int_{B_2} u_3^3(S, Q)\tau_S(S)\tau_S(Q)dA_Q dA_S$$

$$= \frac{\pi k}{2\mu} \int_0^\infty M(x)\left[\int_a^\infty \tau_S(r) r J_0(krx)dr\right]^2 dx$$

$$= \frac{\pi k a^4}{2\mu} \int_0^\infty M(x)\left[\frac{\pi}{2}\frac{J_1(\kappa x)}{\kappa x} + \frac{\{\kappa x \cdot \cos(\kappa x) - \sin(\kappa x)\}}{(\kappa x)^2}\right]^2 dx . \quad (118)$$

$$V_3 = \int_{B_1}\int_{B_1} u_3^3(S, Q)dA_Q dA_S$$

$$= \frac{\pi k}{2\mu} \int_0^\infty M(x)\left[\int_0^a r J_0(krx)dr\right]^2 dx$$

$$= \frac{\pi k a^4}{2\mu} \int_0^\infty M(x)\left[\frac{J_1(\kappa x)}{\kappa x}\right]^2 dx . \quad (119)$$

The evaluation of V_1, V_2, and V_3 requires the evaluation of the following three integrals:

$$A_1 = \int_0^\infty M(x) \left[\frac{J_1(\kappa x)}{\kappa x}\right]^2 dx$$

$$A_2 = \int_0^\infty M(x) \left[\frac{J_1(\kappa x)\{\kappa x \cdot \cos(\kappa x) - \sin(\kappa x)\}}{(\kappa x)^3}\right] dx \tag{120}$$

$$A_3 = \int_0^\infty M(x) \left[\frac{\kappa x \cdot \cos(\kappa x) - \sin(\kappa x)}{(\kappa x)^2}\right]^2 dx$$

For A_1, we use

$$J_1(z) = \frac{2}{\pi} \int_0^1 \frac{t \sin(zt)}{\sqrt{1 - t^2}} dt . \tag{121}$$

Then

$$A_1 = \frac{4}{\pi^2 \kappa^2} \int_0^1 \frac{t\,dt}{\sqrt{1 - t^2}} \int_0^1 \frac{\tau\,d\tau}{\sqrt{1 - \tau^2}} \int_0^\infty \frac{M(x)\sin(\kappa xt)\sin(\kappa x\tau)}{x^2} dx . \tag{122}$$

We use the contour Γ in Figure 7 to evaluate the infinite integral. For $t < \tau$, we consider

$$\int_\Gamma \frac{M(z)\sin(\kappa zt)e^{-i\kappa z\tau}}{z^2} dz , \tag{123}$$

which is zero by Cauchy's Theorem. Letting the radii of the indentations go to zero and $T \to \infty$, and using the notation of (21) and (23), we have

$$P \int_1^\infty \frac{\alpha}{xf} \sin(\hat{x}t)\sin(\hat{x}\tau)dx = i \int_0^\gamma \frac{\alpha}{xf} \sin(\hat{x}t)\sin(\hat{x}\tau)dx$$
$$+ \frac{i}{2} \int_\gamma^1 \frac{\alpha}{xf\hat{f}} [\hat{f} \sin(\hat{x}t)e^{i\hat{x}\tau} - f \sin(\hat{x}t)e^{-i\hat{x}\tau}]dx$$
$$+ \frac{\pi\alpha(s)}{sf'(s)} \sin(\hat{s}t) \sin(\hat{s}\tau) \tag{124}$$

where $\hat{x} = \kappa x$, $\hat{s} = \kappa s$, and f, $\hat{f}$ are defined in (23). Thus, by (19), for $t < \tau$

$$\int_0^\infty \frac{M(x)\sin(\hat{x}t)\sin(\hat{x}\tau)}{x^2}dx = -\int_0^1 \frac{K(x)\sin(\hat{x}t)e^{i\hat{x}\tau}}{x^2}dx + \frac{\pi\alpha(s)}{sf'(s)}\sin(\hat{s}t)e^{i\hat{s}\tau} \tag{125}$$

where K(x) is given by (65).

In order to carry out the next two integrations, we must expand the functions in (125) in powers of t and τ, and integrate term-by-term. The result is

$$A_1 = \frac{-8(1-\nu)}{\pi^2\kappa^2}\Big[.66667 I_0\kappa - i(.61685)I_1\kappa^2 - .35556 I_2\kappa^3 - i(.15421)I_3\kappa^4 + .05418 I_4\kappa^5 + \dots \tag{126}$$

where the I_n are defined in (67).

For the evaluation of A_2, consider

$$\int_\Gamma \frac{M(z)H_1^{(2)}(\kappa z)[\kappa z\cdot\cos(\kappa z) - \sin(\kappa z)]}{z^3}dz, \tag{127}$$

with $H_1^{(2)}$ chosen since it decays exponentially on Γ_T as $T \to \infty$, and Γ as in Figure 7 with an additional branch cut along the negative real axis due to the logarithmic nature of the Hankel function. Similar steps as in the evaluation of A_1 lead us to

$$A_2 = -\frac{i}{\kappa^2}\left[\int_0^1 \frac{K(x)H_1^{(1)}(\hat{x})h(\hat{x})dx}{x^2} - \frac{\pi\alpha(s)}{sf'(s)} H_1^{(1)}(\hat{s})h(\hat{s})\right] \tag{128}$$

where $H_1^{(1)}$ is the Hankel function of the first kind, $\hat{x} = \kappa x$, $\hat{s} = \kappa s$, and

$$h(\hat{x}) = \cos(\hat{x}) - \frac{\sin(\hat{x})}{\hat{x}} . \tag{129}$$

Due to the logarithmic nature of $H_1^{(1)}(\hat{x})$ at $x = 0$, we are unable to expand A_2 in a power series about $x = 0$. Our expansion necessarily contains logarithmic terms and is given by

$$A_2 = \frac{2(1-\nu)}{\kappa^2}\Big[.21221\ I_0\kappa - .10610\ I_2 \ln(\kappa)\kappa^3 + i(.16667)I_2\kappa^3 + .04413\ I_2\kappa^3 - .10610\ L_2\kappa^3 + .02387\ I_4 \ln(\kappa)\kappa^5 - i(.03750)I_4\kappa^5 - .02389\ I_4\kappa^5 + .02387\ L_4\kappa^5 + \ldots \tag{130}$$

where

$$L_n = \frac{1}{2(1-\nu)}\left[\int_0^1 K(x)x^{n-1}\ln(x)dx - \frac{\pi\alpha(s)}{f'(s)} s^n \ln(s)\right], \quad n \geq 1. \tag{131}$$

The evaluation of A_3 is carried out in a manner similar to A_1, and we obtain

$$A_3 = \frac{1}{\kappa^2}\left[\int_0^1 \frac{K(x)P(\hat{x})}{x^2}\,dx - \frac{\pi\alpha(s)}{sf'(s)} P(\hat{s})\right] \tag{132}$$

where

$$P(\hat{x}) = \frac{-i}{2\hat{x}^2}\left[1 + \hat{x}^2 + e^{i2\hat{x}}\{-1 + 2\hat{x}i + \hat{x}^2\}\right]. \tag{133}$$

Expanding $P(\hat{x})$ in a power series, we have

$$A_3 = \frac{2(1-\nu)}{\kappa^2}\,[-.33333I_0\kappa - .13333I_2\kappa^3 - i(.11111)I_3\kappa^4 + .05714I_4\kappa^5 + \ldots]. \quad (134)$$

The values of V_1, V_2, and V_3 then become

$$V_1 = \frac{-\pi a^2(1-\nu)}{\mu k}\Big[+ .21221\ I_0\kappa + .39270I_1\kappa^2 + .10610I_2 \ln(\kappa)\kappa^3 - .27049I_2\kappa^3 - i(.16667)I_2\kappa^3 + .10610L_2\kappa^3 - i(.09817)I_3\kappa^4 - .02387I_4 \ln(\kappa)\kappa^5 + .05839I_4\kappa^5 + i(.03750)I_4\kappa^5 - .02387L_4\kappa^5 + \ldots \quad (135)$$

$$V_2 = \frac{-\pi a^2(1-\nu)}{\mu k}\Big[+ .33333I_0\kappa + i(.61685)I_1\kappa^2 + .33333I_2 \ln(\kappa)\kappa^3 - .36087I_2\kappa^3 - i(.52360)I_2\kappa^3 + .33333L_2\kappa^3 - i(.04310)I_3\kappa^4 - .07500I_4 \ln(\kappa)\kappa^5 + .07210I_4\kappa^5 + i(.11781)I_4\kappa^5 - .07500L_4\kappa^5 + \ldots \quad (136)$$

$$V_3 = \frac{-\pi a^2(1-\nu)}{\mu k}\Big[27019I_0\kappa + i(.25000)I_1\kappa^2 - .14410I_2\kappa^3 - i(.06250)I_3\kappa^4 + .02196I_4\kappa^5 + \ldots \quad (137)$$

Then the variational approximation (115) becomes

$$\left\{\int_B \tau(S)dA_S\right\} = -\frac{ip_o\mu k}{\gamma}\left[\frac{V_1^2}{V_2} - V_3\right] \tag{138}$$

and our approximation to the scattering cross-section for the circle is

$$\begin{aligned} \sum &= \frac{4\pi}{h}\,\mathrm{Im}\{g(0,0)\} = \frac{4\pi}{h}\cdot\frac{\gamma^2}{2\pi\mu}\,\mathrm{Im}\left\{\int_B \tau(S)dA_S\right\} \\ &= -2p_o\cdot\mathrm{Re}\left\{\frac{V_1^2}{V_2} - V_3\right\} \\ &= \frac{4p_o a^3}{3(1-\gamma^2)\mu}\Big[1 + .12732 I_2\kappa^2 - .05457 I_4\kappa^4 \\ &\quad + \{.05876 - .01946\ln(\kappa) - .02533\ln^2(\kappa)\} I_2^2\kappa^4 \\ &\quad - \{.01946 + .05066\ln(\kappa)\} I_2 L_2\kappa^4 - .02533 L_2^2\kappa^4 + \ldots \end{aligned} \tag{139}$$

where we have used the fact that

$$\frac{1}{1-\gamma^2} = 2(1-\nu)\ . \tag{140}$$

We compare result (139) with that of Robertson [18, (8.11)]* and note that the first two terms are identical; that is, up through κ^2. The results are given in Table 5, where the correction, mentioned in the footnote, was used for Robertson's result.

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*Robertson's equation was misprinted. The κ^4-term should read "-108," instead of "+108."

Table 5. The Scattering Cross-Section for $a = 1$, $p_0 = 1$, $\mu = 1$

κ	Scherer's	Robertson's
0.0	1.990	1.990
0.1	1.996	1.996
0.2	2.014	2.014
0.3	2.046	2.045
0.4	2.094	2.088
0.5	2.164	2.146
0.6	2.259	2.218
0.7	2.383	2.308
0.8	2.542	2.415
0.9	2.737	2.544
1.0	2.969	2.694

A P P E N D I C E S

APPENDIX I

VERTICAL DISPLACEMENT ON B FOR AXIALLY-SYMMETRIC PROBLEMS

For axially-symmetric problems, i.e. where the stress $\tau_{33}(Q)$ for Q on the boundary B is dependent only on the distance from Q to the origin, we can simplify representation (13b). Letting $Q = (\rho, \phi, 0)$ and $P = (r, \theta, 0)$ in cylindrical coordinates, we have from (13b)

$$w(P) = \frac{k}{8\pi\mu} \int_B \left[\int_0^\infty M(t) J_0(ktR)\,dt\right] \tau(\rho)\,dA_Q$$

where $R = \sqrt{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}$. Upon changing orders of integration, we have

$$w(P) = \frac{k}{8\pi\mu} \int_0^\infty \tau(\rho)\rho\,d\rho \int_0^\infty M(t)\,dt \int_0^{2\pi} J_0(ktR)\,d\phi \ .$$

From [13, p. 21(3a)] we have

$$J_0(ktR) = \sum_{n=0}^{\infty} \varepsilon_n J_n(ktr) J_n(kt\rho) \cos(n(\theta - \phi))$$

where $\varepsilon_0 = 1$, $\varepsilon_n = 2$ for $n \geq 1$. Thus,

$$\int_0^{2\pi} J_0(ktR)\,d\phi = \sum_{n=0}^{\infty} \varepsilon_n J_n(ktr) J_n(kt\rho) \int_0^{2\pi} \cos(n(\theta - \phi))\,d\phi \ .$$

Since

$$\int_0^{2\pi} \cos(n(\theta - \phi))d\phi = \begin{cases} 2\pi, & n = 0 \\ 0, & n = 1,2,\ldots \end{cases} ,$$

then

$$\int_0^{2\pi} J_0(ktR)d\phi = 2\pi J_0(ktr)J_0(kt\rho) ,$$

which gives

$$\begin{aligned} w(P) &= \frac{k}{8\pi\mu} \cdot 2\pi \int_0^\infty \tau(\rho)\rho d\rho \int_0^\infty M(t)J_0(ktr)J_0(kt\rho)dt \\ &= \frac{k}{8\pi\mu} \int_0^{2\pi} d\phi \int_0^\infty \tau(\rho)\rho d\rho \int_0^\infty M(t)J_0(ktr)J_0(kt\rho)dt , \end{aligned}$$

since there is no ϕ dependence. We note this expression is independent of θ. So the displacement at P is dependent only on the distance from P to the origin. The result can be put in the form

$$w(r) = \frac{k}{8\pi\mu} \int_B \left[\int_0^\infty M(t)J_0(ktr)J_0(kt\rho)dt \right] \tau(\rho)dA_Q .$$

APPENDIX II

INTEGRAL REPRESENTATION FOR $H_0^{(1)}(tR)$

Consider the integral

$$I = \int_{-\infty}^{\infty} e^{ivx}\, dv \int_{-\infty}^{\infty} \frac{e^{iwy}}{w^2 - (t^2 - v^2)}\, dw$$

and let

$$J = \int_{-\infty}^{\infty} \frac{e^{iwy}}{w^2 - (t^2 - v^2)}\, dw\ .$$

The integral J has simple poles at $\pm w_0 = \pm\sqrt{t^2 - v^2}$. In order to satisfy the radiation condition for the factor $e^{-i\omega t}$, we must detour around the poles as shown in Figure 12. For $y > 0$, we can close in the upper

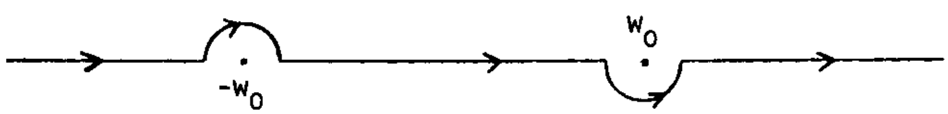

Figure 12. The Contour Λ .

half-plane by Λ^+, Figure 13, and apply Jordan's lemma to get

$$J = \frac{\pi i e^{iyw_0}}{w_0}\ , \quad y > 0\ .$$

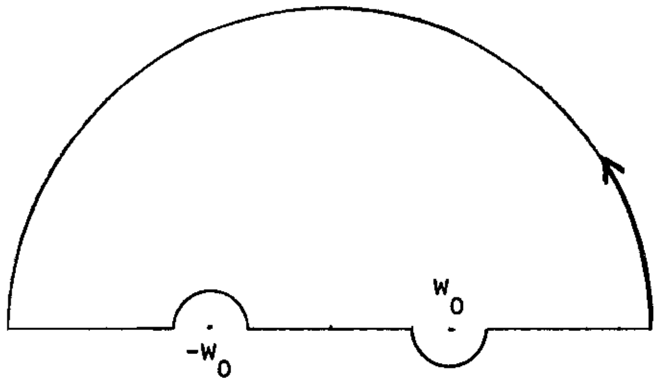

Figure 13. The Contour Λ^+.

When $y < 0$, we enclose the lower half-plane, and consolidate the results for all y as

$$J = \frac{\pi i e^{i|y|w_0}}{w_0} ,$$

giving

$$I = \pi i \int_{-\infty}^{\infty} \frac{e^{i(vx + |y|\sqrt{t^2 - v^2})}}{\sqrt{t^2 - v^2}} dv .$$

This representation for I has branch points at $v = \pm t$, and we require a contour similar to that in Figure 12. Furthermore we need

$$\sqrt{t^2 - v^2} = i\sqrt{v^2 - t^2} \quad \text{for } |v| > t$$

so that the integral converges. Introducing polar coordinates for x and $|y|$, we have

$$x = R \sin \phi \qquad |y| = R \cos \phi$$

where $0 \leq R < \infty$ and $0 \leq \phi \leq \pi/2$. We also change the variable of integration by letting

$$v = t \sin(z)$$

where $z = z_1 + iz_2$, which maps the v-plane into any strip in the z-plane of the form $-\pi/2 + n\pi \leq z_1 \leq \pi/2 + n\pi$, n an integer. We choose the strip $-\pi/2 \leq z_1 \leq \pi/2$. Then we require a contour in the z-plane equivalent to Λ_z, as shown in Figure 14, in order to have

$$\sqrt{t^2 - v^2} = t \cos(z) \ .$$

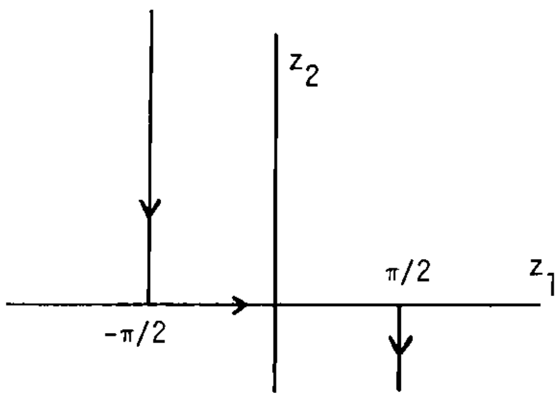

Figure 14. The Contour Λ_z.

Then

$$I = \pi i \int_{\Lambda_z} \frac{e^{itR(\sin(z)\sin(\phi) + \cos(z)\cos(\phi))}}{t \cos(z)} t \cos(z) dz$$

$$= \pi i \int_{\Lambda_z} e^{itR\cos(z-\phi)} \, dz \ ,$$

which now has no branch points.

Now let $\hat{z} = z - \phi$, and

$$I = \pi i \int_{\Lambda_{\hat{z}}} e^{itR\cos(\hat{z})} d\hat{z} ,$$

where $\Lambda_{\hat{z}}$ is a contour equivalent to $\Lambda_{\hat{z}}$ shifted by ϕ. For convergence, all we require is $\mathrm{Im}[\cos(\hat{z})] > 0$. Since

$$\cos(\hat{z}) = \cos(\hat{z}_1 + i\hat{z}_2) = \cos(\hat{z}_1)\cosh(\hat{z}_2) - i\sin(\hat{z}_1)\sinh(\hat{z}_2),$$

when $\hat{z}_2 > 0$, $\sinh(\hat{z}_2) > 0$, and we need $\sin(\hat{z}_1) > 0$. We choose $-\pi < \hat{z}_1 < 0$. When $\hat{z}_2 < 0$, $\sinh(\hat{z}_2) < 0$, and we require $\sin(\hat{z}_1) > 0$. Here we choose $0 < \hat{z}_1 < \pi$. Thus, we can use any curve originating in the section $-\pi < \hat{z}_1 < 0$, $\hat{z}_2 > 0$ and continuing into the section $0 < \hat{z}_1 < \pi$, $\hat{z}_2 < 0$, which does not go outside these regions. In particular, the curve Λ_z satisfies these conditions. That is, the value of I is independent of ϕ.

The result is

$$I = \pi i \int_{\Lambda_z} e^{itR\cos z} dz .$$
$$= \pi^2 i \, H_0^{(1)}(tR) ,$$

using Sommerfeld [19, p. 89] for this representation of $H_0^{(1)}$. Therefore,

$$H_0^{(1)}(tR) = -\frac{i}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i(vx+wy)}}{w^2 + v^2 - t^2} \, dw dv$$

where $R = \sqrt{x^2 + y^2}$.

APPENDIX III

CALCULATIONS OF I_n's AND L_2

I_n

n	$\gamma^2 = 0.0$	$\gamma^2 = .25$	$\gamma^2 = .33$	$\gamma^2 = .40$	$\gamma^2 = .50$
1	2.62445	2.45896	2.48021	2.53501	2.68875
2	2.35618	2.29073	2.35233	2.46090	2.74889
3	2.22035	2.26303	2.37031	2.53678	2.97081
4	2.15983	2.30028	2.45665	2.68693	3.28886
5	2.14502	2.37669	2.58571	2.88743	3.68605
6	2.15983	2.48100	2.74722	3.12949	4.16016
7	2.19518	2.60767	2.93660	3.41018	4.71524
8	2.24573	2.75379	3.15201	3.72940	5.35897

L_2

2	-.18909	-.07709	-.03333	.02209	.16000

APPENDIX IV

CALCULATION OF $\int_B u_3^3(S,Q)dA_Q$

Method I: From (13a), the vertical displacement at S is

$$w(S) = -\int_B u_3^3(S,Q)\tau_{33}(Q)dA_Q, \; S \varepsilon B.$$

We require the stress $\tau_{33}(Q) = -1$ for $Q \varepsilon B$. That is,

$$\hat{c} = \int_B u_3^3(S,Q)dA_Q \;, \quad S \varepsilon B$$

is the displacement in the z-direction due to the stress $\tau_{33}(Q) = -1$ for all $Q \varepsilon B$, when all tangential stress is zero.

We consider the following dynamic punch problem:* A rigid circular punch of radius a is attached to a half-space D with boundary B where z = 0. If P = (r, θ, 0) in cylindrical coordinates, define $B_1 = \{P: 0 \le r \le 1\}$ and $B_2 = B - B_1$. The boundary conditions will be

(i) on B_1, $w(P) = c_1$, a constant, and all tangential stresses are zero.

(ii) on B_2, all stresses are zero.

We will show that as the frequency of oscillation $\omega \to \infty$, the stress distribution approaches a constant on B_1. When $\omega \to \infty$, the non-dimensional parameter $ka \to \infty$. We can accomplish the latter either by holding

---

*Recall $e^{-i\omega t}$ is omitted.

a constant and letting $k \to \infty$, or by holding k constant and letting $a \to \infty$. In both cases, the solution must be the same by uniqueness of the solution for the boundary value problem.

We set up the first situation as a pair of dual integral equations, which we borrow from Robertson [16, p. 548 (3.3), (3.4)], setting a = 1:

$$-\frac{k^2}{4(1-n)}\int_0^\infty \frac{\sqrt{y^2-h^2}\ B(y)\ J_0(ry)dy}{(y^2-\frac{1}{2}k^2)^2 - y^2\sqrt{y^2-h^2}\ \sqrt{y^2-k^2}} = c_1 \qquad 0\le r\le 1$$

$$\int_0^\infty B(y)J_0(ry)dy = 0 \qquad r>1.$$

In the first integral we use the fact that $\int_0^\infty = \lim_{N\to\infty}\int_0^N$ and approximate the following functions asymptotically for $k >> N$ and $0 \le y < N$:

$$\sqrt{y^2-h^2} = -i\ \sqrt{h^2-y^2}\ \sim -ih$$

$$\sqrt{y^2-k^2} = -i\ \sqrt{k^2-y^2}\ \sim -ik$$

$$(y^2-\frac{1}{2}k^2)^2 - y^2\ \sqrt{y^2-h^2}\ \sqrt{y^2-k^2} \sim \frac{k^4}{4}\ .$$

Then, as $k \to \infty$

$$\int_0^\infty \frac{\sqrt{y^2-h^2}\ B(y)\ J_0(ry)dy}{(y^2-\frac{1}{2}k^2)^2 - y^2\sqrt{y^2-h^2}\ \sqrt{y^2-k^2}} \sim \lim_{N\to\infty}\int_0^N -\frac{4ih}{k^4}B(y)\ J_0(ry)dy$$

$$\sim -\frac{4ih}{k^4}\int_0^\infty B(y)J_0(ry)dy\ ,$$

and the dual integral equations become

$$\int_0^\infty B(y)J_0(ry)dy = \begin{cases} -\dfrac{i(1-n)k}{\gamma}c_1 , & 0 \le r \le 1 \\ 0 , & r > 1 . \end{cases}$$

The stress distribution is then

$$\tau_{33}(r) = -\frac{\mu}{1-n}\int_0^\infty B(y)J_0(ry)dy$$

$$= \frac{i\mu k}{\gamma}\begin{cases} c_1 , & 0 \le r \le 1 \\ 0 , & r > 1 , \end{cases}$$

which shows that $\tau_{33}(Q)$ is constant, for $Q \in B_1$.

Returning to the original boundary value problem, we now hold k constant and let $a \to \infty$. Then region B_1 is given by $r \ge 0$, and by uniqueness of solutions, we must still have the stress constant on B_1. That is

$$\tau_{33}(r) = \frac{i\mu k}{\gamma}c_1 , \quad r \ge 0 .$$

In order to have $\tau_{33}(r) = -1$ for $r \ge 0$, we must choose

$$c_1 = \frac{i\gamma}{\mu k} ,$$

which is the displacement in the z-direction due to the stress $\tau_{33}(Q) = -1$ on the entire boundary. Hence,

$$\hat{c} = c_1 = \frac{i\gamma}{\mu k} .$$

Method II: Since $\hat{c}$ is the displacement in the z-direction due to the stress $\tau_{33}(Q) = -1$, for $Q \varepsilon B$, using representation (15) we have

$$\hat{c} = -\frac{k}{8\pi\mu}\int_B\left[\int_0^\infty M(t)J_0(ktr)J_0(kt\rho)dt\right]\rho d\rho d\phi$$

$$= -\frac{k}{8\pi\mu}\int_0^{2\pi} d\phi\int_0^\infty \rho d\rho\int_0^\infty M(t)J_0(ktr)\left[\frac{2}{\pi}\int_1^\infty \frac{\sin(kt\rho x)}{\sqrt{x^2-1}}\,dx\right]dt$$

using [13, p. 27] to replace $J_0(kt\rho)$. Letting $u = ktx$ and interchanging orders of integration, we have

$$\hat{c} = -\frac{k}{2\pi\mu}\int_1^\infty \frac{dx}{\sqrt{x^2-1}}\int_0^\infty M(t)J_0(ktr)dt\int_0^\infty \rho\sin(u\rho)d\rho\ .$$

The interpretation of $\int_0^\infty \rho\sin(u\rho)d\rho$ comes from distribution theory. Letting $\delta(u)$ be a delta function [20, p. 8], we have

$$\int_{-\infty}^\infty e^{-i\rho u}\frac{d}{du}\delta(u)du = -\int_{-\infty}^\infty \frac{d}{du}[e^{-i\rho u}]\delta(u)du = i\rho\ .$$

We apply a Fourier transform to get

$$\frac{d}{du}\delta(u) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{i\rho u}(i\rho)d\rho$$

$$= \frac{i}{2\pi}\left[\int_{-\infty}^\infty \rho\cos(\rho u)d\rho + i\int_{-\infty}^\infty \rho\sin(\rho u)d\rho\right]\ .$$

Since $\rho\cdot\cos(\rho u)$ is odd in ρ and $\rho\cdot\sin(\rho u)$ is even in ρ,

$$\frac{d}{du}\delta(u) = -\frac{1}{\pi}\int_0^\infty \rho\sin(\rho u)d\rho\ .$$

Recalling that u = ktx, we then have

$$\hat{c} = \frac{k}{2\mu} \int_1^\infty \frac{dx}{\sqrt{x^2 - 1}} \int_0^\infty M(t)J_0(ktr) \frac{d}{du} \delta(u)dt$$

$$= \frac{k}{2\mu} \int_1^\infty \frac{1}{\sqrt{x^2 - 1}} \left[- \frac{1}{2} \left\{ g_+'(0) + g_-'(0) \right\} \right] dx ,$$

using [20, p. 24], where $g(u) = M(\frac{u}{kx})J_0(\frac{ur}{x})$ and $' \equiv \frac{d}{du}$. Since $g(u) = 0$ for $u \leq 0$, $g_-'(0) = 0$; and $g_+'(0) = - \frac{4i\gamma}{k^2x^2}$. Thus,

$$\hat{c} = \frac{i\gamma}{\mu k} \int_1^\infty \frac{1}{x^2\sqrt{x^2 - 1}} dx$$

$$= \frac{i\gamma}{\mu k} .$$

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VITA

Stephen Edwin Scherer was born January 3, 1944, in Coral Gables, Florida. He completed his primary and secondary education in the public schools in South Miami and Coral Gables, Florida, and entered the University of Miami as a freshman in September, 1962. He transferred to the Georgia Institute of Technology in September, 1964, and received the degree of Bachelor of Science in Applied Mathematics in June, 1967. He continued at Georgia Tech as a graduate student with the aid of a National Science Foundation training fellowship, and received the degree of Master of Science in Applied Mathematics in August, 1969. During this time he served intermittently as a graduate teaching assistant in mathematics.

In September, 1969, Mr. Scherer was appointed an Instructor the School of Mathematics at Georgia Tech, at which time he entered the Ph.D. program in mathematics.

He is married to the former Beverly Ann Lester of Coral Gables, Florida. The Scherers have one child, Lisa Michelle Scherer, born January 6, 1973.