Introduction to the Special Issue on Partial Differential Equations and Geometry-Driven Diffusion in Image Processing and Analysis

I. What Is This Special Issue About?

THE USE of partial differential equations (PDE's) and curvature driven flows in image analysis has become an interest raising research topic in the past few years. Let Φ_0: $\mathbf{R}^2 \to \mathbf{R}$ represent a gray-level image, where $\Phi_0(x, y)$ is the gray-level value. Introducing an artificial time t, the image deforms in a partial differential evolution equation according to

$$\frac{\partial \Phi}{\partial t} = \mathcal{F}[\Phi(x, y, t)] \tag{1}$$

where $\Phi(x, y, t)$: $\mathbf{R}^2 \times [0, \tau) \to \mathbf{R}$ is the evolving image, $\mathcal{F}$: $\mathbf{R} \to \mathbf{R}$ is an operator that characterizes the given algorithm, and the image Φ_0 is the initial condition.[1] The solution $\Phi(x, y, t)$ of the differential equation gives the processed image at scale t. In the case of vector-valued images, a system of coupled PDE's of the form of (1) is obtained.

The same formalism can be applied to planar curves (boundaries of planar shapes), where Φ is a function from $\mathbf{R}$ to $\mathbf{R}^2$, or surfaces, functions from $\mathbf{R}^2$ to $\mathbf{R}^3$. In this case, the operator $\mathcal{F}$ must be restricted to the curve, and all isotropic motions can be described as a deformation of the curve or surface in its normal direction, with velocity related to its principal curvature(s). In more formal terms, a flow of the form

$$\frac{\partial \Phi}{\partial t} = \mathcal{F}(\kappa_i)\vec{\mathcal{N}} \tag{2}$$

is obtained, where κ_i are the principal curvatures and $\vec{\mathcal{N}}$ is the normal to the curve or surface Φ. A tangential velocity can be added as well, which may help the analysis but does not affect the geometry of the flow.

PDE's can be obtained from variational problems. Assume a variational approach to an image processing problem formulated as

$$\arg\{\mathrm{Min}_\Phi \mathcal{U}(\Phi)\}$$

where $\mathcal{U}$ is a given energy. Let $\mathcal{F}(\Phi)$ denote the Euler–Lagrange derivative (first variation). Since under general assumptions, a necessary condition for Φ to be a minimizer of $\mathcal{U}$ is that $\mathcal{F}(\Phi) = 0$, the (local) minima may be computed via the steady state solution of the equation

$$\frac{\partial \Phi}{\partial t} = \mathcal{F}(\Phi)$$

where t is again an "artificial" time parameter. PDE's obtained in this way have been used already for quite some time in computer vision and image processing, and the literature is large. The most classical example is the Dirichlet integral

$$\mathcal{U}(\Phi) = \int |\nabla \Phi|^2(x)\, dx$$

which is associated with the heat equation

$$\frac{\partial \Phi}{\partial t}(t, x) = \Delta \Phi(x).$$

More recently, extensive research is being done on the direct derivation of evolution equations which are not necessarily obtained from the energy approaches. This is in fact the case for a number of curvature equations of the form (2).

Clearly, when introducing a new approach to a given research area, one must justify its possible advantages. Using PDE's and the curve/surface flows in image analysis leads to model images in a continuous domain. This simplifies the formalism, which becomes grid independent and isotropic. The understanding of discrete local nonlinear filters is facilitated when one lets the grid mesh tend to zero and, thanks to an asymptotic expansion, rewrite the discrete filter as a partial differential operator.

Conversely, when the image is represented as a continuous signal, PDE's can be seen as the iteration of local filters with an infinitesimal neighborhood. This interpretation of PDE's allows one to unify and classify a number of the known iterated filters, as well as to derive new ones. Actually, Alvarez *et al.* [1] classified all the PDE's that satisfy several stability requirements for imaging processing such as locality and causality. (As pioneered in [31], future research might give up the locality requirement.)

Further, the PDE formulation is very natural in order to combine algorithms. If two different image processing schemes are given by

$$\frac{\partial \Phi}{\partial t} = \mathcal{F}_1[\Phi(x, y, t)], \quad \frac{\partial \Phi}{\partial t} = \mathcal{F}_2[\Phi(x, y, t)]$$

then they can be combined as $\partial\Phi/\partial t = \alpha\mathcal{F}_1 + \mathcal{F}_2$, where $\alpha \in \mathbf{R}^+$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ above are the corresponding Euler–Lagrange operators of two energy minimization problems with energies $\mathcal{U}_1$ and $\mathcal{U}_2$, the flow above minimizes the energy $\alpha\mathcal{U}_1 + \mathcal{U}_2$.

Another important advantage of the PDE approach is the possibility of achieving high accuracy and stability, with the help of the extensive available research on numerical analysis. Of course, when considering PDE's for image processing and numerical implementations, we are dealing with derivatives of nonsmooth signals, and the right framework must be defined. The theory of *viscosity solutions* [6] provides a framework for

Publisher Item Identifier S 1057-7149(98)01752-7.

[1] $\mathcal{F}$ typically depends on the image and the first and second spatial derivatives.

rigorously employing a partial differential formalism, in spite of the fact that the image may be not smooth enough to give a classical sense to the first and second derivatives involved in the PDE. Last but not least, this area has a quite unique level of formal analysis, giving the possibility to provide not only successful algorithms but also useful theoretical results like existence and uniqueness of solutions.

II. Some Background

It is difficult to write the history of a topic, and is not our intention to do it. We just want to mention several contributions that, in our opinion, have had a major impact in this area. A larger number of relevant references can be found in the book edited by ter Haar Romeny [28].

Ideas on the use of PDE's in image processing go back at least to Gabor [10], and a bit more recently, to Jain [13]. However, we believe that the field really took off thanks to the independent works of Koenderink [17] and Witkin [37]. (Koenderink initiated the Utrecht scale-space school, e.g., [9].) These researchers rigorously introduced the notion of *scale-space*, that is, the representation of images simultaneously at multiple scales. Their seminal contribution is to a large extent the basis of most of the research in PDE's for image processing. In their work, the multiscale image representation is obtained by Gaussian filtering. This is equivalent to deforming the original image via the classical heat equation, obtaining in this way an isotropic diffusion flow. In the late 1980's, Hummel noted that the heat flow is not the only parabolic PDE that can be used to create a scale-space, and indeed argued that an evolution equation that satisfies the maximum principle will define a scale-space as well. Maximum principle appears to be a natural mathematical translation of *causality*.

Perona and Malik's [26] work on anisotropic diffusion has been one of the most influential papers in the area. They proposed to replace Gaussian smoothing, which is equivalent to isotropic diffusion via the heat flow, by a directional diffusion that preserves edges. Their work opened a number of theoretical and practical questions that continue to occupy the PDE image processing community, e.g., [2], [28]. In the same framework, the seminal works of Osher and Rudin on shock filters [25] and Rudin *et al.* [29] on total variation decreasing methods explicitly stated the importance and the need for understanding PDE's for image processing applications. We should also point out that about at the same time, Price *et al.* published a very interesting paper on the use of Turing's reaction-diffusion theory for a number of image processing problems [27].

As we have noted, many of the PDE's used in image processing and computer vision are based on moving curves and surfaces with curvature based velocities. In this area, the level-set numerical method developed by Osher and Sethian [24] was very influential (see also the early development in the level-sets methodology for mean curvature motion in [21]). The idea is to represent the deforming curve, surface, or image, as the level-set of a higher dimensional hypersurface. This technique, not only provides more accurate numerical implementations, but also solves topological issues that were very difficult to treat before. The representation of objects as level-sets (zero-sets) is of course not completely new to the computer vision and image processing communities, since it is one of the fundamental techniques in mathematical morphology [35]. This morphological approach is actually the one adopted in [1] to classify all contrast invariant PDE's.

Another key contribution in the PDE formalism has been the general segmentation framework developed by Mumford and Shah [20]. Their work has unified a large number of image segmentation approaches, and opened as well a large number of theoretical and practical problems (see [19]).

Next in [16], Kimia *et al.* introduced curve evolution methods into computer vision for a computational theory of planar shape. (For some of the key mathematical works in curvature driven flows upon which this work is founded (see [11], [12], and [24] and the references therein). They defined a "reaction-diffusion" scale-space that allows one to smooth shapes as well as to employ the theory of shocks for a hierarchy of parts combining an anisotropic smoothing effect with a morphological one.

Finally, the work of Terzopoulos *et al.* on active contours for image segmentation [14] indirectly also had an important impact on the PDE's community. This work has subsequently been extended by a number of authors using geometric PDE's.

It should be noted that a number of the above approaches rely quite heavily on a large number of mathematical advances in differential geometry for curve evolution [12] and in viscosity solutions theory for curvature motion (see e.g., Evans and Spruck [7]).

Of course, the frameworks of PDE's and geometry driven diffusion have been applied to many problems in image processing and computer vision, since the seminal works mentioned above. Examples include continuous mathematical morphology, invariant shape analysis, shape from shading, segmentation, object detection, optical flow, stereo, image denoising, image sharpening, contrast enhancement, and image quantization. The interested reader is referred to [28] and [30], as well as the papers in this special issue, for an extensive list of references.

III. Contents of the Special Issue

One of the interesting theoretical and practical questions in this area is the study of PDE's that are invariant to camera transformations. This work was initiated by Alvarez *et al.* [1] and by Olver *et al.* [22], [23], [33], [34]. Extensions were introduced by Faugeras [8]. Dibos presents in this issue a possible alternative to deal with the high number of derivatives involved in projective invariant flows.

Following the importance of anisotropic diffusion equations as introduced by Perona and Malik [26] and extended by many others, the numerical implementation of such equations became a central research topic. This is the subject of the paper by Acton and the one by Weickert *et al*.

An alternative model for anisotropic diffusion is presented in the paper by Cottet and El Ayyadi. They introduce interesting concepts like time-delays and relations to neural networks.

Following the work by Sapiro and Ringach [32] on vector-valued diffusion and energy minimization, Blomgren and Chan present a study of total variation minimization for vector images with special attention to color data.

In two similar independent works, Sochen *et al.* and Yezzi propose to treat images as high-dimensional surfaces (graphs), and process them based on projected curvature motion flows. The approach can be embedded in a Riemannian geometry framework. These works are as well motivated by [5], [32], and [36].

Chambolle *et al.* analyze the relations between wavelets based algorithms for image enhancement and energy minimization based ones.

Faugeras and Keriven propose a PDE model to compute 3-D shape from stereo images.

Chan *et al.* consider the Yanowitz–Bruckstein method for image thresholding, and give it a variational formulation.

Carmona and Zhong investigate the use of high-order structure, obtained for example from the image Hessian, in anisotropic diffusion.

Without any doubt, one of the most interesting results in the past years of computer vision and image processing is the concept of active contours or snakes introduced by Terzopoulos *et al.* [14]. This is the topic of two papers in this special issue. Xu and Prince extend the potential energy that attracts the deforming contours to the scene object, making it more effective. Siddiqi *et al.* deal with curve evolution based active contours. They add a new term to the works described in [4] and [15], which are extensions of [3] and [18].

Chan and Wong investigate the numerical implementation of algorithms for blind deconvolution based on total variation techniques. They present fast and accurate solutions. (You and Kaveh have independently proposed the same model for blind deconvolution in their ICIP'96 paper.)

Caselles *et al.* present an axiomatic approach for image interpolation. They show all the possible (level-set based) image interpolation algorithms satisfying a number of natural requirements and compare them.

Teboul *et al.* present an extended analysis of edge preserving regularization techniques, based on ideas form the Mumford–Shah [20] segmentation technique and extensions introduced by Ambrosio and Tortorelli.

Moisan presents a fast numerical curve evolution scheme that preserves the affine invariance in the affine heat flow.

Based on robust statistics, and the theory of influence functions, Black *et al.* show how to design the stopping term in the Perona–Malik flow and how to explicitly introduce coherence in the outliers. The anisotropic diffusion flow they propose has nontrivial steady state solutions, even without the addition of a data term.

Most of the research in scale-space theory was done for still images. Guichard extends this theory to movies.

Perona's work deals with the regularization of orientation maps. Since orientations are defined periodically, there is a need for a special design of diffusion flows and their numerical implementation. Perona addresses this issue motivated by physical analogies.

ACKNOWLEDGMENT

The guest editors wish to thank Prof. A. Bovik, Prof. M. Kaveh, Prof. D. Munson, and Prof. M. Vetterli for proposing and supporting this special issue. We could not produce this special issue without the help of the anonymous reviewers, who responded to a very demanding schedule. A number of reviewers reviewed up to five papers. We deeply thank all of them. Since they do deserve a lot of the credit for the success of this project, we list them here: L. Alvarez, L. Ambrosio, G. Aubert, C. Ballester, Banham, M. Barlaud, P. Blomgren, C. Bouman, V. Caselles, F. Catte, A. Chambolle, T. Chan, L. Cohen, T. Coll, G. Cottet, R. Deriche, M. El Ayyadi, O. Faugeras, D. Fleet, L. Florack, J. Froment, J. Garding, G. Gerig, R. Ginosar, F. Guichard, H. Guo, S. Kalitzin, R. Keriven, S. Kichenassamy, R. Kimmel, C. Koch, G. Koepfler, R. Kresch, M. Luettgen, R. Malladi, D. Marimont, J. Marroquin, T. McInerney, L. Moisan, J.-M. Morel, P. Mulet, D. Mumford, P. Olver, S. Osher, E. Pauwels, I. Pollak, D. Ringach, B. Romeny, Y. Rubner, A. Salden, F. Santosa, G. Sapiro, M. Schneider, C. Schnorr, J. Shah, D. Shaked, T. Shiota, R. Stevenson, D. Strong, A. Tannenbaum, P. Teo, J. Weickert, L. Vese, C. Vogel, J. Weber, R. Whitaker, Y.-L. You, S. Zampieri.

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