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THE SPECTRUM OF CERTAIN SINGULAR SELFADJOINT DIFFERENTIAL OPERATORS

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CHAPTER I

INTRODUCTION

The motivation for this work is a statement by M. A. Naimark in [13] to the effect that one of the most important problems in the theory of differential operators is the question: In what way does the spectrum of selfadjoint extensions of symmetric operators depend on the behavior of the coefficients of the corresponding differential expressions? The nature of the spectrum, deficiency indices, and the expansion of functions with respect to eigenfunctions for singular differential operators are closely related subjects, and each will be investigated.

The initial development of the theory of singular differential operators can be found in the famous papers by H. Weyl in 1909 [20] and 1910 [21]. Included in these papers are the fundamental results for the singular operator of order two concerning the deficiency indices and the corresponding limit-point (Grenzpunktfall) and limit-circle (Grenzkreisfall) cases, the nature of the spectrum, and the expansion theory with respect to eigenfunctions.

Since the appearance of Weyl's papers the second order case has been the subject of researches by many authors. E. C. Titchmarsh has assembled many of the results prior to 1950 in a two-volume monograph [19]. More recent studies concerning the spectrum have been conducted by P. Hartman, C. R. Putnam, and A. Wintner [11,12,15,16,23].

Most of the above work has been centered on the operator

$$\tau = -\frac{d^2}{dx^2} + q(x)$$

on the interval $[0,\infty)$. Weyl showed that if $q(x) \to \infty$ as $x \to \infty$ then the spectrum is discrete. An extension of this result to operators of higher order obtained by I. M. Glazman is included in Chapter V.

In [12] the operator

$$\tau = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x) \qquad 0 \le x < \infty \tag{1}$$

is considered and a necessary and sufficient condition for a point to be in the continuous spectrum based on the behavior of $q(x)$ is obtained.

In [15] the connection between an operator being oscillatory and the discrete part of the spectrum is used to extend a theorem in [19]. In [16] a result concerning the independence of the continuous spectra and the boundary condition for the operator (1) is obtained.

Also in [19, Chapter 16] is included the following criterion concerning an operator with a weight function. For the operator

$$\tau = \frac{1}{m(x)}\left[-\frac{d^2}{dx^2} + q(x)\right] \qquad 0 \le x < \infty$$

it is shown that if $q(x)/m(x) \to \infty$ as $x \to \infty$ then the spectrum is discrete.

In 1953 A. M. Molchanov [13, page 245] found a necessary and sufficient condition for the discreteness of the spectrum for the operator

$$\tau = -\frac{d^2}{dx^2} + q(x) \qquad -\infty<x<\infty,$$

namely that

$$\lim_{x\to\pm\infty} \int_x^{x+a} q(t)dt = \infty$$

for each fixed $a > 0$.

The theory (expansion in eigenfunctions and nature of the spectrum) for differential operators of order higher than two has been investigated since 1950 by many mathematicians including M. S. P. Eastham [5], J. V. Baxley [2], M. A. Naimark [13], and I. M. Glazman [9,10]. Eastham compares operators with the Euler operator and his work is restricted to cases with singularity at infinity. Baxley utilizes the Friedrichs extension to study integer powers of the operator

$$\tau = \frac{1}{m(x)} \left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right] \qquad 0<x\leq 1.$$

He concludes that the spectrum is discrete if

$$\int_0^1 m(x) \int_x^1 1/p(t)dtdx = M < \infty,$$

a test which does not depend on $q(x)$ as did the pre-1950 studies listed above. Also a weight function $m(x)$ is included which makes the results more general.

The problem which we consider here is to find conditions on the coefficients of differential expressions of arbitrary even order, $2n$, and on arbitrary intervals, which ensure a compact inverse for the corresponding selfadjoint operators. In particular, the objective is to find conditions which apply to a class of operators for which neither Eastham's nor Baxley's tests yield results.

Similar to the approach adopted by Baxley, we use the Friedrichs extension to attack this problem, and hence, our work is restricted to semi-bounded operators.

In Chapter II we define the terminology to be used and present the basic properties of the spectrum, deficiency indices, and boundary conditions of formally selfadjoint differential operators. Also we present the splitting technique for studying the spectrum and a description of the Friedrichs extension.

In Chapter III the fundamental results of the paper are established by showing that if any one of the coefficients behaves according to certain criteria then the Friedrichs extension has a compact inverse. Also, in this case, the Friedrichs extension of any integer power of the symmetric operator will have a compact inverse.

In Chapter IV we obtain a boundary condition description of the Friedrichs extension for a class of operators which have a compact inverse. Also we present a class of operators of order $2n$ which illustrates the fact that the deficiency indices can take on any value between n and 2n inclusive.

In Chapter V we compare our results with those of Friedrichs, Baxley, and Eastham, and present some questions for further study.

CHAPTER II

BACKGROUND

It is well known [13, page 48] that any formally selfadjoint formal differential operator with real coefficients can be written in the form

$$\tau = \frac{1}{m(x)} \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} p_k(x) \frac{d^k}{dx^k} \tag{1}$$

The formal operator τ will be considered on an interval I with left endpoint a and right endpoint b. The coefficients $p_k(x)$, $k = 0,\ldots,n$ are assumed to be real and have sufficient differentiability on I, $p_n(x) > 0$ for x in I, and the functions $1/p_n(x), p_{n-1}(x),\ldots,p_o(x)$ are required to be Lebesgue-integrable on any closed subinterval $[\alpha,\beta]$ of (a,b). The function $m(x)$ is the weight function and hence must be positive and continuous on I.

Definition 2.1. The left endpoint a is regular if $a > -\infty$ and if the functions $1/p_n(x), p_{n-1}(x),\ldots,p_o(x)$ are Lebesgue-integrable in every interval $[a,\beta]$, $\beta < b$; otherwise we say the endpoint a is singular. Similarly we define the regularity and singularity of the right endpoint b.

Definition 2.2. The expression τ is regular if and only if both endpoints, a and b, are regular. Otherwise τ is said to be singular.

Theorems that establish the discreteness of the spectrum and the expansion of an arbitrary function in a Hilbert space with respect to the eigenfunctions of a regular ordinary differential operator may be found in many texts such as [3, Chapter 7] or [4, Chapter XIII]. The following treatment is concerned with singular operators.

In order to apply the abstract Hilbert-space theory of unbounded operators, we will require the domains of the operators which correspond to the formal operator τ to be subsets of a Hilbert space H.

Due to the lack of a commonly accepted terminology we present here the definitions of the terms used to describe the spectrum of a closed linear operator T, defined on the Hilbert space H.

Definition 2.3. A complex number λ is called a regularity point of the operator T if $(T-\lambda I)^{-1}$ exists and is bounded on all of H. The set of all regularity points is called the resolvent set.

Definition 2.4. The spectrum is the complement of the resolvent set relative to the complex plane.

Definition 2.5. The discrete spectrum is the set of points, λ, of the spectrum such that the closure of the range of $(T-\lambda I)$ is not all of H; i.e. the set of all eigenvalues.

Definition 2.6. The continuous spectrum is the set of points, λ, of the spectrum such that the range of $(T-\lambda I)$ is not closed.

Remark 2.1. The continuous spectrum as defined above is the same as the essential spectrum as defined by Dunford and Schwartz [4].

We will use $\mathcal{D}(\cdot)$ and $\mathcal{R}(\cdot)$ to denote the subsets of the Hilbert space H which are the domain and range, respectively, of an operator, and $(\cdot,\cdot)$ and $\|\cdot\|$ to denote the inner product and norm, respectively, on H.

Definition 2.7. If T_1 and T_2 are linear operators, T_1 is said to be an extension of T_2, and denoted by $T_2 \subset T_1$, if and only if $\mathcal{D}(T_2) \subset \mathcal{D}(T_1)$ and $T_1 f = T_2 f$ for every $f \varepsilon \mathcal{D}(T_2)$.

Definition 2.8. If $\mathcal{D}(T)$ is dense in H, then we define the Hilbert space adjoint of T to be the operator T^*, where $\mathcal{D}(T^*)$ consists of all g such that (Tf,g) is continuous for $f \varepsilon \mathcal{D}(T)$, and $T^* g = g^*$, where g^* is the unique point in H such that $(Tf,g) = (f,g^*)$ for every $f \varepsilon \mathcal{D}(T)$. In other words $(Tf,g) = (f,T^* g)$ for $f \varepsilon \mathcal{D}(T)$, $g \varepsilon \mathcal{D}(T^*)$.

Definition 2.9. A linear operator T defined on a Hilbert space H is said to be symmetric if

(a) $\mathcal{D}(T)$ is dense in H, and

(b) $f,\ g \varepsilon \mathcal{D}(T)$ implies $(Tf,g) = (f,Tg)$.

Definition 2.10. T is said to be selfadjoint if $T = T^*$.

Remark 2.2. The spectrum of a selfadjoint operator is a subset of the real numbers. For a selfadjoint operator the continuous spectrum is the collection of non-isolated points of the spectrum.

Remark 2.3. T^* is a closed operator. If T is symmetric, $T \subset T^*$ and the closure of T, $\bar{T}$, is given by $\bar{T} = T^{**}$.

Since our operators are differential operators, and therefore unbounded, the choice of domain of our operators is quite crucial. We will denote by $C^k(a,b)$ the class of all functions with k continuous derivatives on (a,b) and by $C_o^k(a,b)$ the class of all functions in $C^k(a,b)$ which also have compact support in (a,b).

Given a formal differential operator τ on (a,b) we denote by T_o the operator defined by $T_o u = \tau u$, where $u \varepsilon \mathcal{D}(T_o)$, and $\mathcal{D}(T_o) = C_o^k(a,b)$. We will refer to T_o as the minimal differential operator defined by τ. (The closure of T_o is referred to as the minimal operator by some authors).

Our goals are two-fold. First, find general conditions under which every selfadjoint operator T in Hilbert space, H, constructed from τ has a compact inverse. From this result will follow qualitative properties of the spectrum; i.e. it consists only of eigenvalues which accumulate only at infinity, and, in addition, the knowledge that the eigenfunctions are complete; i.e. every $f \varepsilon H$ can be expanded in a series of eigenfunctions and the series converge in the metric of H. Also, since T is a differential operator, each eigenvalue has finite multiplicity at most equal to the order of the operator.

Our second goal is, given the selfadjoint extension T, to describe $\mathcal{D}(T)$ in terms of the boundary conditions applied to $\mathcal{D}(T_o^*)$.

Deficiency Indices and Selfadjoint Extensions

The formal operator, τ, which we are studying has order 2n and will be defined on the interval [a,b). We assume that the left endpoint is regular and the right endpoint is singular.

Definition 2.11. Let T be a symmetric linear operator. Let R_λ denote the range of $(T-\lambda I)$ and let $N_\lambda = H - R_\lambda$ for a complex number λ. For $\lambda = i$, the dimension of N_i, d^+, is called the positive deficiency index and, similarly, the dimension of N_{-i}, d^-, is called the negative deficiency index. They are written in the form of an ordered pair (d^+,d^-).

The following known results are presented for completeness, and may be found in [4] and [13].

A closed symmetric linear operator T is selfadjoint if and only its deficiency indices are (0,0).

A symmetric operator T has selfadjoint extensions if and only if its deficiency indices are equal.

A symmetric differential operator with real coefficients has equal deficiency indices. Also, since the dimension of the manifold N_i is at most the order of the operator, the deficiency indices are finite.

The specific formal differential operator (1) which we will consider is such that T_o, defined above, is of order 2n, symmetric, and has real coefficients. Therefore the deficiency indices are equal, $d^+ = d^- = d$, and finite, $d \le 2n$, and hence, T_o has a selfadjoint extension.

Also, for the case of one singular endpoint as considered here, it is true that $n \le d$ (see [1, page 172]).

Boundary Conditions on Selfadjoint Extensions

In the general theory of selfadjoint extensions of symmetric differential operators, an arbitrary selfadjoint extension, T, of the symmetric operator T_o is characterized and described as a restriction of the adjoint operator T_o^* to a linear manifold $\mathcal{D}(T)$ such that $\mathcal{D}(T_o) \subset \mathcal{D}(T) \subset \mathcal{D}(T_o^*)$. The functions in $\mathcal{D}(T)$ are specified by applying appropriate boundary conditions to the functions in $\mathcal{D}(T_o^*)$.

Theorem 2.1 [4, Page 1238]. Let T_o be a symmetric operator with equal and finite deficiency indices $d = d^+ = d^-$, and let T be a selfadjoint extension of T_o. Then T is the restriction of T_o^* to the subspace of $\mathcal{D}(T_o^*)$ determined by a family of d linearly independent boundary conditions.

Continuous Spectrum of Selfadjoint Extensions

Let M and N be two subspaces of the Hilbert space H such that $M \subset N$.

Definition 2.12. The dimension of the subspace N modulo M, denoted by $\dim N(\mathrm{mod} M)$, is the largest number of linearly independent functions in N such that no non-trivial linear combination of them belongs to M.

Lemma 2.2. If T_o is a symmetric operator, $\mathcal{D}(\overline{T}_o)$, N_i, and N_{-i}, are closed orthogonal subspaces of (T_o^*) such that $\mathcal{D}(T_o^*) = \mathcal{D}(T_o) \oplus N_i \oplus N_{-i}$ (see [4, page 1227]).

Now, if T is a selfadjoint extension of T_o, then $\mathcal{D}(T) \subset \mathcal{D}(T_o^*)$. Therefore $\dim\mathcal{D}(T)\left[\mathrm{mod}\mathcal{D}(\bar{T}_o)\right] \le \dim(N_i) + \dim(N_{-i}) = d^+ + d^-$.

It is clear that the continuous spectrum of T_o is a subset of the continuous spectrum of T. However, we make further conclusions in the case d^+ and d^- are finite.

Theorem 2.3. All selfadjoint extensions of a closed symmetric operator with equal and finite deficiency indices have the same continuous spectrum.

Proof. It is sufficient to show that if λ is not in the continuous spectrum of T_o then λ is not in the continuous spectrum of T. Suppose λ is not in the continuous spectrum of T_o. Let $M_{o\lambda}$ be the subspace of eigenvectors of T_o associated with λ, and let $T_{o\lambda}$ be the restriction of T_o to $H - M_{o\lambda}$ for each λ. Note that if λ is a regular point of T_o then $T_{o\lambda} = T_o$. Then it follows that $(T_{o\lambda}-\lambda I)$ has an inverse for all λ. For the selfadjoint operator T, define M_λ and T_λ similarly.

In order to complete the proof we need the following lemma.

Lemma 2.4. The set of all λ such that $(T_{o\lambda}-\lambda I)^{-1}$ is not bounded is the continuous spectrum of T_o. Similarly for $(T_\lambda-\lambda I)^{-1}$ and T (see [10, page 9]).

The proof of the theorem will follow from the fact that the operator $(T_{o\lambda}-\lambda I)^{-1}$ is bounded if and only if the operator $(T_\lambda-\lambda I)^{-1}$ is bounded, which we now demonstrate.

First, it is clear that if $(T_\lambda-\lambda I)^{-1}$ is bounded then $(T_{o\lambda}-\lambda I)^{-1}$ is also. Now assume $(T_{o\lambda}-\lambda I)^{-1}$ is bounded and let h be an element in the range of $(T-\lambda I)$. Then $h = f + g$ where $f \varepsilon R(T_o-\lambda I)$ and g is in the finite dimensional complement of this range. Therefore,

$$\|(T_\lambda-\lambda I)^{-1}h\| = \|(T_\lambda-\lambda I)^{-1}f + (T_\lambda-\lambda I)^{-1}g\|$$

$$\leq \|(T_\lambda-\lambda I)^{-1}f\| + \|(T_\lambda-\lambda I)^{-1}g\|.$$

Now, let P be the projection operator onto the subspace $H - M_\lambda$; hence, $(T_\lambda-\lambda I)^{-1}f = P(T_{o\lambda}-\lambda I)^{-1}f$. Since g is in a finite dimensional subspace, $(T_\lambda-\lambda I)^{-1}$ is bounded there; i.e. there exists a number A such that $\|(T_\lambda-\lambda I)^{-1}g\| \leq A\|g\|$. Therefore

$$\|(T_\lambda-\lambda I)^{-1}h\| \leq \|(T_{o\lambda}-\lambda I)^{-1}f\| + A\|g\|$$

and $(T_\lambda-\lambda I)^{-1}$ is bounded follows from $(T_{o\lambda}-\lambda I)^{-1}$ is bounded.

Splitting Technique

Let τ be a formally symmetric formal differential operator defined on an interval I with endpoints a and b. Let T_0 be the minimal differential operator defined by τ on I.

Let c be any point in the interior of I, $a < c < b$. Let F_1 and F_2 be the minimal differential operators defined by τ on $I_1 = I \cap [a,c]$ and $I_2 = I \cap [c,b]$, respectively. Let $F_0 = F_1 \oplus F_2$, then

$$F_0 \subset T_0 \subset T_0^* \subset F_0^*$$

Let F_1' and F_2' be selfadjoint extensions of F_1 and F_2, respectively, and let $F = F_1' \oplus F_2'$. Then clearly F is selfadjoint and $F_0 \subset F$. If T is a selfadjoint extension of T_0, then $F \subset T$. Therefore $F_0 \subset T$ and both F and T are selfadjoint extensions of F_0. Hence, by Theorem 2.3, F and T have the same continuous spectra.

But, the discrete spectrum of F is the union of the discrete spectra of F_1' and F_2' and the continuous spectra of F is the union of the continuous spectra of F_1' and F_2'. Now suppose the interval $I = [a,b)$ so that $I_1 = I \cap [a,c] = [a,c]$. Then F_1' is a regular selfadjoint operator and hence, has a discrete spectrum; i.e. the continuous spectrum is empty. Therefore the continuous spectrum of T equals the continuous spectrum of F which is equal to the continuous spectrum of F_2'.

From this we see that if F_2' has a compact inverse and hence a discrete spectrum, then so does T. We conclude then that the discreteness of the spectrum of a selfadjoint operator is determined by the behavior of the coefficients in a neighborhood of the singular endpoint(s).

Also, if $I = (a,b)$ (i.e. both endpoints are singular), we can pick a point c, $a < c < b$, and consider the operators on $I_1 = (a,c]$ and $I_2 = [c,b)$ separately. Hence, it is sufficient to consider only the half-open, half-closed intervals in studying the relationship of the spectrum to the behavior of the coefficients of the formal differential operator.

Friedrichs Extension for Semi-Bounded Operators

We have seen that symmetric operators with equal and finite deficiency indices have selfadjoint extensions and by Theorem 2.1 the selfadjoint extensions can be described in terms of boundary conditions. We further characterize the class of operators which we study by the following definition.

Definition 2.13. A symmetric operator T is bounded below if there is a real number c such that $(Tu,u) \geq c(u,u)$ for all $u \varepsilon \mathcal{D}(T)$, and bounded above if there is a real number c such that $(Tu,u) \leq c(u,u)$ for all $u \varepsilon \mathcal{D}(T)$. If T is bounded below or above we say that T is semi-bounded.

Our studies are confined to symmetric operators bounded below by zero; i.e. $(Tu,u) \geq 0$. However, it is clear that if T is semi-bounded below, then for some constant α, $(T+\alpha I)$ is bounded below by one; i.e. $\big((T+\alpha I)u,u\big) \geq (u,u)$.

For the case of a symmetric semi-bounded operator, a particular selfadjoint extension having the same bound has been constructed by Friedrichs [6]. It is this extension, called the Friedrichs extension, that we will use in the following chapters. We present here Friedrichs' result and the description of the extension in terms of limits. Later we need this description to prove that the operator we study has a compact universe and to establish the boundary conditions.

Theorem 2.5. Every semi-bounded symmetric operator T_o, with domain $\mathcal{D}(T_o)$ dense in the Hilbert space H, has a semi-bounded selfadjoint extension T, with the same bound.

Indication of Proof. The proof may be found in [4,6,17], but we outline the proof here in order to describe the extension.

We assume without loss of generality that T_o is semi-bounded below by one,

$$(T_o u,u) \geq (u,u) > 0 \qquad u \varepsilon \mathcal{D}(T_o). \tag{2}$$

Define a new scalar product on $\mathcal{D}(T_o)$, $[\cdot,\cdot]$ by

$$[u,v] = (T_o u,v) = (u,T_o v),$$

and a new norm $[[\cdot]]$ by

$$[[u]] = [u,u]^{1/2} \geq (u,u)^{1/2} = \|u\|. \tag{3}$$

Thus $\mathcal{D}(T_o)$ is a normed linear space which in general is not complete. Let $\{u_n\}$ be a Cauchy sequence in $\mathcal{D}(T_o)$; that is

$$[[u_n - u_m]] \to 0 \qquad \text{for } m,n\to\infty.$$

If $\{u_n\}$ has no limit in $\mathcal{D}(T_o)$ assign an ideal limit element u. If we assign the same ideal limit element to equivalent Cauchy sequences, u is well-defined.

Let G be the space consisting of $\mathcal{D}(T_o)$ and the ideal limit elements. We now extend our scalar product and norm to all of G and hence, make G a Hilbert space.

If $\{u_n\}$ and $\{v_n\}$ are two Cauchy sequences such that $u_n \to u$ and $v_n \to v$ where either u, v, or both may be ideal elements of G, then $\lim_{n\to\infty}[u_n,v_n]$ exists and we define $[u,v]$ to be this limit. Similarly we extend the definition of $[[\cdot]]$ to G by

$$[[u]] = \lim_{n\to\infty}[[u_n]].$$

It is clear that G is a complete normed linear space, i.e. a Hilbert space, and $\mathcal{D}(T_o)$ is dense in G.

It follows from (3) that a Cauchy sequence in the new metric is also a Cauchy sequence in the original metric and, hence, converges to a well-defined element in H. Thus we can assign to each ideal element of G an element in H and consider G as a subspace of H. Then $\mathcal{D}(T_o) \subset G \subset H$.

Now define T by letting $\mathcal{D}(T) = G \cap \mathcal{D}(T_o^*)$ and

$$Tu = T_o^*u = \tau u \qquad \text{for } u\varepsilon\mathcal{D}(T).$$

Note that if $w\varepsilon\mathcal{D}(T_o)$ then $w\varepsilon\mathcal{D}(T_o^*)$ and $w\varepsilon G$; therefore, $w\varepsilon\mathcal{D}(T)$ and T is an extension of T_o, $T_o \subset T$.

For $u\varepsilon\mathcal{D}(T)$, $v\varepsilon\mathcal{D}(T_o)$, there exists a sequence $\{u_n\}\varepsilon\mathcal{D}(T_o)$, $u_n \to u$ such that

$$[u,v] = \lim_{n\to\infty}[u_n,v] = \lim_{n\to\infty}(T_ou_n,v)$$

$$= \lim_{n\to\infty}(u_n,T_ov) = (u,T_ov)$$

$$= (T_o^*u,v) = (Tu,v) \qquad (4)$$

Also, for $u\varepsilon\mathcal{D}(T)$, $v\varepsilon\mathcal{D}(T)$ and by continuity of the inner product

$$[u,v] = (Tu,v).$$

Hence, for u, $v\varepsilon\mathcal{D}(T)$

$$(Tu,v) = [u,v] = \overline{[v,u]} = \overline{(Tv,u)} = (u,Tv)$$

and, therefore, T is symmetric.

Also $(Tu,v) = \lim_{n\to\infty}(T_o u_n, v_n) \geq \lim_{n\to\infty}(u_n,v_n) = (u,v)$ implies that T is semi-bounded below by one.

It remains to show that T is selfadjoint. Let v be an arbitrary element of H. Then $(\cdot,v)$ is a continuous linear functional on G. Hence, by the Riesz Representation Theorem there exists an element $w\varepsilon G$ such that $(u,v) = [u,w]$ for all $u\varepsilon G$. But $[u,w] = (T_o u,w)$ for $u\varepsilon\mathcal{D}(T_o) \subset G$ and therefore $w\varepsilon\mathcal{D}(T_o^*)$. Hence $w\varepsilon\mathcal{D}(T) = G \cap \mathcal{D}(T_o^*)$. We have therefore,

$$(u,v) = [u,w] = (T_o u,w) = (u,Tw)$$

for all u in $\mathcal{D}(T_o)$ which is a dense subset of H. Therefore, $Tw = v$ and the range of T is the whole space H.

We conclude that the null space of T^* consists only of the zero element and hence T^* is one-to-one with range the entire space H, and, therefore $\mathcal{D}(T) = \mathcal{D}(T^*)$ and T is selfadjoint.

Corollary 2.6. If T is the Friedrichs extension of T_o and $u \varepsilon \mathcal{D}(T)$ there exists a sequence $\{u_n\}$, $u_n \varepsilon \mathcal{D}(T_o)$ such that

(a) $\|u_n - u\| \to 0$ and

(b) $(T_o u_n, u_n) \to (Tu, u)$

as $n \to \infty$.

Proof. Note that

$$\lim_{m\to\infty}[[u_m - u_n]] = [[u - u_n]]$$

and

$$\lim_{m\to\infty}[[u_m + u_n]] = [[u + u_n]].$$

Also, from the parellelogram law

$$([[u_m - u_n]])^2 + ([[u_m + u_n]])^2 = 2([[u_m]]^2 + [[u_n]]^2). \tag{5}$$

Now, we see that

$$\lim_{m,n\to\infty} ([[u_m + u_n]]) = 2[[u]]$$

which implies that

$$\lim_{n\to\infty}[[u + u_n]] = \lim_{n\to\infty}(\lim_{m\to\infty}[[u_m + u_n]]) = 2[[u]].$$

Now letting $m \to \infty$ in (5) we obtain

$$[[u-u_n]]^2 + [[u+u_n]]^2 = 2([[u]]^2 + [[u_n]]^2)$$

from which it follows that $\lim_{n\to\infty}[[u-u_n]]^2 = 0$. From (3) we see that $\lim_{n\to\infty}\|u-u_n\| = 0$ also.

Now from (4) we see that $\lim_{n\to\infty}(T_o u_n, v) = (Tu,v)$ for all $v\varepsilon\mathcal{D}(T_o)$, where $u_n \to u$, $\{u_n\}\varepsilon\mathcal{D}(T_o)$. In particular it is true for v equal to each of the u_i. Using a diagonalization argument we see that $\lim_{n\to\infty}(T_o u_n, u_n) = (Tu,u)$.

CHAPTER III

SUFFICIENT CONDITIONS FOR A COMPACT INVERSE

It can be shown [13, page 48] that any real formally selfadjoint formal differential operator is of even order 2n, and the general form of such an operator is

$$\tau = \frac{1}{m(x)} \sum_{k=o}^{n} (-1)^k \frac{d^k}{dx^k} p_k(x) \frac{d^k}{dx^k} \tag{1}$$

where the coefficients $p_k(x)$, $0 \le k \le n$, and the weight function $m(x)$ are real.

In what follows we will consider τ on an interval $a \le x < b$, where b may be finite or infinite, $p_k(x)$ is non-negative and in $C^{(k)}[a,b)$ for $0 \le k \le n$, and, for $x \varepsilon [a,b)$, $p_n(x) > 0$ and $m(x) > 0$. The endpoint b is possibly a singular endpoint. Our objective is to define linear operators corresponding to the formal operator τ and to study their adjoints and selfadjoint extensions. In particular, we wish to find answers to the question stated by Naimark [13, page 208]: In what way does the spectrum of selfadjoint extensions depend on the behavior of the coefficients of the corresponding differential expression τ.

As noted earlier the choice of a domain for an unbounded operator, and in particular for a differential operator, can be quite crucial to the nature of the spectrum. Accordingly we denote by $L^2(m)$ the collection of all measurable functions, u, defined on (a,b) for which

$$\int_a^b |u(x)|^2 m(x)dx < \infty.$$

We define a scalar product $(\cdot,\cdot)$ on $L^2(m)$ by

$$(u,v) = \int_a^b u(x)\overline{v(x)}m(x)dx$$

and the corresponding norm $\|\cdot\|$ by $\|u\| = (u,u)^{1/2}$. It then follows that $L^2(m)$ is a Hilbert space. The operators which we consider here will have as their domain a subset of $L^2(m)$.

Let T_o be the linear operator defined by $T_o u = \tau u$ for u in the domain $\mathcal{D}(T_o)$ which consists of those functions in $L^2(m)$ which are also in $C_o^\infty(a,b)$; i.e. each function in $\mathcal{D}(T_o)$ vanishes outside some compact subset of (a,b) where the compact subset may vary with the function. It follows that $\mathcal{D}(T_o)$ is dense in $L^2(m)$, and hence T_o is symmetric.

Since the coefficients of τ are real, T_o has equal deficiency indices and, since T_o is a differential operator, the deficiency indices are finite. Thus, T_o has a selfadjoint extension.

In addition, for $u \in \mathcal{D}(T_o)$, we see that

$$(T_o u,u) = \int_a^b m(x)[\tau u(x)]\overline{u(x)}dx$$

and, using integration by parts repeatedly, that

$$(T_o u,u) = \int_a^b \sum_{k=1}^n p_k(x)|u(x)|^2 dx. \tag{2}$$

Hence, that T_o is semi-bounded below by zero, $(T_o u,u) \geq 0$, follows from the assumption that $p_k(x) \geq 0$ for $0 \leq k \leq n$. Therefore, T_o has a Friedrichs extension.

Let k be an integer, $1 \leq k \leq n$. For this k define a sequence of functions $\{h_i\}_1^k$ as follows:

If, for each y in [a,b) it is true that $\int_y^b 1/p_k(t)dt$ is finite, then

$$h_1(x,k) = \left(\int_x^b 1/p_k(t)dt\right)^{1/2} \qquad \text{for } x\varepsilon[a,b). \tag{3a}$$

Otherwise

$$h_1(x,k) = \left(\int_a^x 1/p_k(t)dt\right)^{1/2} \qquad \text{for } x\varepsilon[a,b). \tag{3b}$$

If i is any integer $1 \leq i \leq k - 1$, and if, for each y in [a,b), it is true that $\int_y^b h_i(t,k)dt$ is finite, then

$$h_{i+1}(x,k) = \int_x^b h_i(t,k)dt, \qquad \text{for } x\varepsilon[a,b). \tag{4a}$$

Otherwise

$$h_{i+1}(x,k) = \int_a^x h_i(t,k)dt, \qquad \text{for } x\varepsilon[a,b) \tag{4b}$$

Theorem 3.1. Let T be the Friedrichs extension of T_o. If

$$\int_a^b m(x)[h_k(x,k)]^2 dx = M < \infty \tag{5}$$

for at least one integer k, $1 \le k \le n$, then T has a compact inverse, and in this case, every selfadjoint extension of T_o has a discrete spectrum.

Proof. We first prove two lemmas.

Lemma 3.2. If $u \varepsilon \mathcal{D}(T_o)$ and (5) is satisfied for some k, $1 \le k \le n$, then for $a \le x_1 < x_2 < b$,

$$|u(x_2) - u(x_1)| \le (T_o u,u)^{1/2} \left\{ \int_{x_1}^{x_2} 1/p_1(t)dt \right\}^{1/2} \qquad \text{if } k=1$$

and

$$|u(x_2) - u(x_1)| \le (T_o u,u)^{1/2} \left\{ \int_{x_1}^{x_2} h_{k-1}(t,k)dt \right\} \qquad \text{if } k>1.$$

Proof. For $u \varepsilon \mathcal{D}(T_o)$, it follows from the Schwartz inequality

$$\begin{aligned} |u^{(k-1)}(x_2) - u^{(k-1)}(x_1)|^2 &\le |\int_{x_1}^{x_2} u^{(k)}(t)dt|^2 \\ &\le \int_{x_1}^{x_2} p_k(t)[u^{(k)}(t)]^2 dt \cdot \int_{x_1}^{x_2} 1/p_k(t)dt \\ &\le (T_o u,u) \int_{x_1}^{x_2} 1/p_k(t)dt. \end{aligned}$$

Now if $k = 1$, the lemma is proven. If $k > 1$, set $x_1 = x$ and $x_2 = b$, or $x_1 = a$ and $x_2 = x < b$ depending on whether $h_1(x,k)$ is defined as in

(3a) or (3b), respectively. Since u is zero in a neighborhood of a and b, we get

$$|u^{(k-1)}(x)|^2 \le (T_o u,u)[h_1(x,k)]^2$$

$$|u^{(k-1)}(x)| \le (T_o u,u)^{1/2} h_1(x,k) \tag{6}$$

Using (6) above, we obtain

$$\begin{aligned} |u^{(k-2)}(x_2) - u^{(k-2)}(x_1)| &= \left|\int_{x_1}^{x_2} u^{(k-1)}(t)dt\right| \\ &\le \int_{x_1}^{x_2} (T_o u,u)^{1/2} h_1(t,k)dt \\ &= (T_o u,u)^{1/2} \int_{x_1}^{x_2} h_1(t,k)dt. \end{aligned}$$

Setting $x_1 = x$ and $x_2 = b$, or $x_1 = a$ and $x_2 = x < b$ for $h_2(x,k)$ defined as in (4a) or (4b), respectively,

$$|u^{(k-2)}(x)| \le (T_o u,u)^{1/2}\, h_2(x,k).$$

Continuing by induction, we get the inequalities

$$|u^{(k-j)}(x)| \le (T_o u,u)^{1/2}\, h_j(x,k), \qquad j=1,\ldots,k-1.$$

In particular for $j = k - 1$

$$|u'(x)| \le (T_o u,u)^{1/2}\, h_{k-1}(x,k).$$

Integration of both sides after multiplying by $m(x)$ leads to the conclusion of the lemma for $k > 1$.

Lemma 3.3. Let T be the Friedrichs extension of T_o. If $u \varepsilon \mathcal{D}(T)$ and (5) holds for some k, $1 \le k \le n$, then for $a \le x_1 < x_2 < b$,

(i) $$|u(x_2) - u(x_1)| \le (Tu,u)^{1/2} \left\{\int_{x_1}^{x_2} 1/p_1(t)\, dt\right\}^{1/2} \qquad \text{if } k=1,$$

and

$$|u(x_2) - u(x_1)| \le (Tu,u)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t,k)\, dt \qquad \text{if } 1<k\le n.$$

(ii) $(u,u) \le (Tu,u)M$.

Proof. For $u \varepsilon \mathcal{D}(T)$ we see from Corollary 2.6 that there exists a sequence $\{u_m\}$ with $u_m \varepsilon \mathcal{D}(T_o)$, $m=1,2,\ldots$, such that $\|u_m - u\| \to 0$ and $(T_o u_m, u_m) \to (Tu,u)$ as $m \to \infty$.

Since $(T_o u_m, u_m)$ is a convergent sequence it is bounded, that is $(T_o u_m, u_m) \le C$ for some positive number C, and all $m=1,\ldots$. From Lemma 3.2

$$|u_m(x_2) - u_m(x_1)| \le (T_o u_m, u_m)^{1/2} \left\{\int_{x_1}^{x_2} 1/p_1(t)\, dt\right\}^{1/2} \qquad \text{if } k=1,$$

and

$$|u_m(x_2) - u_m(x_1)| \le (T_o u_m, u_m)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t,k)\, dt \qquad \text{if } k>1.$$

In either case the functions $\{u_m\}$ are equicontinuous and uniformly bounded on compact subsets of $[a,b)$. By the Ascoli Theorem there is a subsequence of $\{u_m\}$ which converges uniformly on compact subsets of $[a,b)$. Restricting attention to this subsequence and taking limits in the last inequalities we get (i) for both $k = 1$ and $k > 1$.

Now if $h_1(x,1)$ is defined as in (3a) and $h_k(x,k)$ is defined as in (4a) for $k > 1$, let $x_1 = a$, and $x_2 = x$ in (i) to obtain for $k = 1$ or $1 < k \le n$,

$$|u(x)| \le (Tu,u)^{1/2}\, h_k(x,k).$$

If $h_1(x,1)$ is defined as in (3b) and $h_k(x,k)$ is defined as in (4b) for $k > 1$, then note that for each u_m in the subsequence there is a $x_2 < b$ such that $u_m(x_2) = 0$. Setting $x_1 = x$ we obtain for $k = 1$

$$\begin{aligned} |u_m(x)| &= |u_m(x_2)-u_m(x)| \\ &\le (T_o u_m,u_m)^{1/2} \left(\int_x^{x_2} 1/p_1(t)\, dt \right)^{1/2} \\ &\le (T_o u_m,u_m)^{1/2} \left(\int_x^{b} 1/p_1(t)\, dt \right)^{1/2} \\ &= (T_o u_m,u_m)^{1/2}\, h_1(x,1), \end{aligned}$$

and for $k > 1$

$$|u_m(x)| = |u_m(x_2) - u_m(x)|$$

$$\leq (T_o u_m, u_m)^{1/2} \int_x^{x_2} h_{k-1}(t,k)dt$$

$$\leq (T_o u_m, u_m)^{1/2} \int_x^{b} h_{k-1}(t,k)dt$$

$$= (T_o u_m, u_m)^{1/2} h_k(x,k).$$

Now take limits to obtain for this case also,

$$|u(x)| \leq (Tu,u)^{1/2} h_k(x,k).$$

Hence, in either case

$$|u(x)|^2 \leq (Tu,u)[h_k(x,k)]^2. \tag{7}$$

Multiplying by $m(x)$ and integrating the above we get (ii).

Proof of Theorem 3.1. Let λ be an eigenvalue of T associated with eigenvector u. Then $(Tu,u) = (\lambda u,u) \geq (1/M)(u,u)$. Hence, $\lambda \geq 1/M > 0$. Therefore all eigenvalues of T are positive and T^{-1} exists.

Let $\{Tu_n\}$ be a bounded sequence in the domain of T^{-1} and K be a number such that $\|Tu_n\| \leq K < \infty$, $n=1,\ldots$. Then $u_n \varepsilon \mathcal{D}(T)$ and, from Lemma 3.3 (ii),

$$\|u_n\|^2 = (u_n,u_n) \le M(Tu_n,u_n) \le M\|Tu_n\|\|u_n\|,$$

from which follows $\|u_n\| \le MK$, and $(Tu_n,u_n) \le MK^2$.

From Lemma 3.3 (i)

$$|u_n(x_2) - u_n(x_1)| \le (MK^2)^{1/2} \int_{x_1}^{x_2} h_{k-1}(t,k)\,dt$$

and, hence, $\{u_n\}$ is uniformly bounded and equicontinuous on compact subsets of [a,b). Using the Ascoli Theorem we get a subsequence $\{v_n\}$ which converges uniformly on compact subsets of [a,b) to a limit function u.

It remains to show that $\{v_n\}$ converges in $L^2(m)$. From (7) above

$$m(x)|v_n(x)|^2 \le (Tv_n,v_n)m(x)[h_k(x,k)]^2$$

$$\le MK^2 m(x)[h_k(x,k)]^2 \qquad n=1,\ldots.$$

The right side is integrable by hypothesis. The Lebesgue Dominated Convergence Theorem yields $u\varepsilon L^2(m)$ and $\|v_n\| \to \|u\|$. Hence, $\|v_n-u\| \to 0$.

Hence, T^{-1} is compact, and T has a discrete spectrum. It follows from Theorem 2.3 that every selfadjoint extension of T_o has a discrete spectrum.

Remark 3.1. Theorem 3.1 provides a test for a selfadjoint operator T to have a compact inverse and a discrete spectrum based on an analysis of the coefficients $p_k(x)$, $1 \le k \le n$. For a test based on $p_o(x)$ we present the following result found in [12, page 210].

Theorem 3.4. If $p_n(x)>0, p_{n-1}(x)\ge 0,\ldots,p_1(x)\ge 0$ and if

$$\lim_{x\to b} p_o(x) = \infty,$$

then every selfadjoint extension of T_o has a discrete spectrum.

If a formal operator τ is formally selfadjoint, then obviously so is any positive integer power of the operator. If τ is of order 2n, then τ^r is of order 2nr. Hence, τ^r can be written in the form of (1) with n changed to nr, where the coefficients are obtained from the coefficients of τ. However, if the coefficients for τ, $p_k(x)$, $k=0,1,\ldots,n$, satisfy the criteria stated at the beginning of this chapter, it does not follow that the coefficients for τ^r will necessarily satisfy these criteria.

For r a positive integer, define T_o^r by

$$T_o^r u = \tau^r u, \qquad u\varepsilon\mathcal{D}(T_o^r)$$

Where $\mathcal{D}(T_o^r)$ is the collection of functions in $L^2(m)$ which are also in $C_o^\infty[a,b)$. We shall also require $p_k(x)\varepsilon C^\infty(a,b)$ for $0 \le k \le n$, to be assured that τ^r is well defined.

It follows that $\mathcal{D}(T_o^r)$ is dense in $L^2(m)$. Also, it is routine to show by using integration by parts repeatedly that T_o^r is symmetric and semi-bounded below by zero. Hence, T_o^r has a Friedrichs extension which we denote by T_r. (Note: r is placed as a subscript rather than a superscript to distinguish T_r from T^r which would denote the rth power of T.)

For each integer k, $1 \leq k \leq n$, define the sequence of functions $\{h_i(x,k)\}$, $i=1,\ldots,k$, as in (3a), (3b), (4a), and (4b).

Theorem 3.5. If for at least one integer k, $1 \leq k \leq n$, it is true that

$$\int_a^b m(x)[h_k(x,k)]^2 dx = M < \infty, \tag{8}$$

then T_r has a compact inverse, and, in this case, every selfadjoint extension of T_o^r has a discrete spectrum.

We first prove some lemmas for $u \varepsilon \mathcal{D}(T_o^r)$, which is the same as the domain of T_o, $\mathcal{D}(T_o)$, and for $u \varepsilon \mathcal{D}(T_r)$.

Lemma 3.6. If $u \varepsilon \mathcal{D}(T_o^r)$ and (8) holds for some k, $1 \leq k \leq n$, then for $a \leq x_1 < x_2 < b$

(i) $|u(x)|^2 \leq M^{r-1}(T_o^r u,u)[h_k(x,k)]^2$ and

(ii) $$|u(x_2)-u(x_1)|^2 \leq M^{r-1}(T_o^r u,u)\left(\int_{x_1}^{x_2} 1/p_1(t)dt\right) \quad \text{if } k=1,$$

and

$$|u(x_2)-u(x_1)|^2 \leq M^{r-1}(T_o^r u,u)\left(\int_{x_1}^{x_2} h_{k-1}(t,k)dt\right)^2 \quad \text{if } k>1.$$

Proof. From Lemma 3.2 we have for $u \varepsilon \mathcal{D}(T_o^r)$

$$|u(x_2)-u(x_1)|^2 \le (T_o u,u)\left(\int_{x_1}^{x_2} 1/p_1(t)dt\right) \qquad \text{if } k=1,$$

and

$$|u(x_2)-u(x_1)|^2 \le (T_o u,u)\left(\int_{x_1}^{x_2} h_{k-1}(t,k)dt\right)^2 \qquad \text{if } k>1.$$

By appropriate choice of x_1 and x_2 depending on the definition of $h_k(x,k)$ we obtain

$$|u(x)|^2 \le (T_o u,u)[h_k(x,k)]^2. \tag{9}$$

Multiplying by m(x) and integrating we get

$$(u,u) \le (T_o u,u)M \tag{10}$$

and from the Schwartz inequality

$$\|u\| \le M\|T_o u\|. \tag{11}$$

If $T_o^r u$ has meaning then $T_o^s u \varepsilon \mathcal{D}(T_o)$ for $s < r$. In particular if r is an even integer $T_o^{r/2} u \varepsilon \mathcal{D}(T_o)$, and if r is odd $T_o^{(r+1)/2}u$ and $T_o^{(r-1)/2}u$ are in $\mathcal{D}(T_o)$.

Hence, from (10) and the symmetry of T_o, we obtain for r even

$$(T_o^r u,u) = (T^{r/2}u,T_o^{r/2}u) \le M(T_o^{(r/2)+1}u,T_o^{r/2}u)$$

$$= M(T_o^{r+1}u,u).$$

Similarly for r an odd integer and using (11)

$$(T_o^r u,u) = (T_o^{(r+1)/2}u,T_o^{(r-1)/2}u)$$

$$\le \|T_o^{(r+1)/2}u\| \, \|T_o^{(r-1)/2}u\|$$

$$\le M\|T_o^{(r+1)/2}u\|^2$$

$$= M(T_o^{r+1}u,u).$$

Therefore for consecutive values of r

$$(u,u) \le M(T_o u,u) \le M^2(T_o^2 u,u) \le \ldots \le M^r(T_o^r u,u). \qquad (12)$$

Using the second and last terms of the inequality we get

$$(T_o u,u) \le M^{r-1}(T_o^r u,u)$$

which when combined with (9) proves part (i) and when combined with Lemma 3.2 proves part (ii) of Lemma 3.6.

Lemma 3.7. If $u\in\mathcal{D}(T_r)$ and (8) holds for some k, $1 \le k \le n$, then for $a \le x_1 < x_2 < b$

$$\text{(i)}\quad |u(x_2)-u(x_1)|^2 \le M^{r-1}(T_r u,u) \int_{x_1}^{x_2} 1/p_1(t)dt \qquad \text{if } k=1,$$

and

$$|u(x_2)-u(x_1)|^2 \le M^{r-1}(T_r u,u) \left(\int_{x_1}^{x_2} h_k(t,k)dt \right)^2 \qquad \text{if } k>1.$$

and

$$\text{(ii)}\quad (u,u) \le (T_r u,u)M^r.$$

Proof. For $u\in\mathcal{D}(T_r)$, it follows from Corollary 2.6 that there is a sequence $\{u_n\}$ in $\mathcal{D}(T_o^r)$ such that $\|u_n-u\| \to 0$ and $(T_o^r u_n,u_n) \to (T_r u,u)$ as $n \to \infty$. From Lemma 3.6 (i) it follows that $\{u_n\}$ is uniformly bounded and equicontinuous on compact subsets of $[a,b)$. Hence, by the Ascoli Theorem there is a subsequence of $\{u_n\}$ which converges uniformly on compact subsets of $[a,b)$. Restricting attention to the subsequence and taking limits in the inequalities of Lemma 3.6 (i) and (ii) we obtain

$$|u(x)|^2 \le M^{r-1}(T_r u,u)[h_k(x,k)]^2 \tag{13}$$

and part (i) of Lemma 3.7. Multiplying (13) by m(x) and integrating we obtain part (ii) of Lemma 3.7.

Proof of Theorem 3.5. The proof parallels that of Theorem 3.1. Lemma 3.6 (ii) shows that T_r has only positive eigenvalues and hence, has an inverse. Suppose $\{T_r u_n\}$ is a bounded sequence in the domain of T_r^{-1}, and K is a number such that $\|T_r u_n\| \le K < \infty$ for n=1,2,.... Then

$$\|u_n\|^2 = (u_n,u_n) \le M^r(T_r u_n,u_n)$$

$$\le M^r\|T_r u_n\|\|u_n\|,$$

and

$$\|u_n\| \le M^r\|T_r u_n\| \le M^r K.$$

Using the Schwartz inequality we obtain

$$(T_r u_n,u_n) \le M^r K^2.$$

Hence from Lemma 3.6 (i)

$$|u_n(x_2)-u_n(x_1)|^2 \le M^{r-1}M^r K^2\left(\int_{x_1}^{x_2} 1/p_1(t)dt\right) \qquad \text{if } k=1,$$

and

$$|u_n(x_2)-u_n(x_1)|^2 \le M^{r-1}M^r K^2\left(\int_{x_1}^{x_2} h_{k-1}(t,k)dt\right)^2 \qquad \text{if } k>1.$$

Therefore, $\{u_n\}$ is uniformly bounded and equicontinuous on compact subsets of $[a,b)$, and, by the Ascoli Theorem, there exists a subsequence $\{v_n\}$ which converges uniformly on each compact subset of $[a,b)$ to a limit function u.

From (13) we have for each v_n

$$m(x)|v_n(x)|^2 \le M^{r-1}(T_r v_n, v_n) m(x)[h_k(x,k)]^2$$

$$\le M^{2r-1}K^2 m(x)[h_k(x,k)]^2.$$

The Lebesgue Dominated Convergence Theorem yields $u \varepsilon L^2(m)$ and $\|v_n\| \to \|u\|$. Hence $\|v_n - u\| \to 0$ as $n \to \infty$.

We conclude that T_r^{-1} is compact and, hence T_r has a discrete spectrum; again, using Theorem 2.3 it follows that every selfadjoint extension of T_o^r has a discrete spectrum.

The requirement that the domain of T_o consist of functions with compact support on (a,b) is quite strict. Also, since the elements of the domain of the Friedrichs extension are limits of elements in $\mathcal{D}(T_o)$, the boundary conditions on $\mathcal{D}(T)$ are quite strict at both endpoints as will be shown by some examples in the next chapter.

It is also true that we have been liberal in our hypotheses concerning the coefficients $p_k(x)$ in arriving at Theorems 3.1 and 3.5. In particular, with respect to defining the functions $h_i(x,k)$, if integrating toward the singular endpoint is not defined then we integrate away from the regular endpoint.

The question arises whether we can enlarge our domain of T_o and still obtain a compact inverse for the Friedrichs extension. And, if so, in what way and to what extent?

Our first step will be to relax the restriction on the functions in $\mathcal{D}(T_o)$ at the singular endpoint but maintain the restrictions on all their derivatives.

Let τ be the formal operator defined as in (1). Define T_o by

$$T_o u(x) = \tau u(x) \qquad \text{for } u(x)\varepsilon\mathcal{D}(T_o)$$

where $\mathcal{D}(T_o)$ is the collection of functions in $L^2(m)$ which are also in $C^\infty(a,b)$ and such that u'(x) has compact support in (a,b) and vanishes in a right neighborhood of x = a. Then u(x) is "free" at the right endpoint b.

It can be easily shown, by integration by parts, that T_o is semibounded below by zero and is symmetric. (Note that the domain of T_o contains the domain defined for Theorem 3.1, and hence is dense in $L^2(m)$.) Therefore T_o has a Friedrichs extension, T.

As before we define a sequence of functions $\{h_i(x,k)\}_1^k$ for each integer k, $1 \le k \le n$.

If it is true that for each y in [a,b) $\int_y^b 1/p_k(t)dt$ is finite, then

$$h_1(x,k) = \left(\int_x^b 1/p_k(t)dt\right)^{1/2} \qquad \text{for } x\varepsilon[a,b).$$

Otherwise

$$h_1(x,k) = \left(\int_a^x 1/p_k(t)dt\right)^{1/2} \qquad \text{for } x\varepsilon[a,b).$$

If i is any integer $1 \le i \le k-2$, and if it is true that for each y in [a,b), $\int_y^b h_i(t,k)dt$ is finite, then

$$h_{i+1}(x,k) = \int_x^b h_i(t,k)dt \qquad \text{for } x\varepsilon[a,b).$$

Otherwise

$$h_{i+1}(x,k) = \int_a^x h_i(t,k)dt \qquad \text{for } x\ [a,b).$$

Finally, if $i = k - 1$, and it is true that for each y in $[a,b)$, $\int_a^y h_{k-1}(t,k)dt$ is finite

$$h_{i+1}(x,k) = h_k(x,k) = \int_a^x h_{k-1}(t,k)dt \qquad \text{for } x\varepsilon[a,b).$$

Otherwise $h_k(x,k)$ is undefined.

Note that there is only one possible way to define $h_k(x,k)$ while $h_i(x,k)$ for $i < k$ there are two ways. In particular, $h_1(x,1)$ is defined only if $\int_a^y 1/p_1(t)dt$ is finite for each y in $[a,b)$.

The statements of Theorem 3.1 and Lemmas 3.2 and 3.3 remain the same as before. Also the proofs of Lemma 3.2 and Theorem 3.1 remain the same, but the proof of Lemma 3.3 must be modified only slightly as follows.

After taking limits in the results of Lemma 3.2 to obtain Lemma 3.3 (i) we have only one way to substitute for x_1 and x_2. Since $u_m(a) = 0$ for all $u_m \varepsilon \mathcal{D}(T_o)$, $u(a) = 0$ for all $u\varepsilon\mathcal{D}(T)$. Setting $x_1 = a$ and $x_2 = x$, we obtain inequality (7) again

$$|u(x)|^2 \leq (Tu,u)[h_k(x,k)]^2.$$

Integrating the above with the weight function m(x), we obtain (ii)

$$(u,u) \le M(Tu,u).$$

We have shown that the results of Lemma 3.3 and inequality (7) remain valid for functions in our modified domain $\mathcal{D}(T)$. Therefore Theorem 3.1 and its same proof hold.

The result of easing the conditions on $\mathcal{D}(T_o)$ is that $\mathcal{D}(T_o^*)$ becomes smaller. The question arises as to whether the Friedrichs extension of the new operator is different from the Friedrichs extension of the old operator.

Now suppose we enlarge our domain of T_o still further as follows: let $\mathcal{D}(T_o)$ consist of those functions u(x) in $L^2(m)$ which are also in $C^\infty(a,b)$ and such that $u^{(j)}(x)$ has compact support on (a,b), where j is an integer, $j \le n$, but u and all its derivatives vanish in a right neighborhood of x = a. Note that the cases j = 0 and j = 1 have already been examined.

Then, in order for the same proof of Theorem 3.1 to go through, the functions $\{h_i(x,k)\}_{i=k-j+1}^{k}$ can be defined in only one way, namely

$$h_i(x,k) = \int_a^x h_{i-1}(t,k)dt.$$

Also, in the proof of Lemma 3.3 (ii) we must select $x_1 = a$, and $x_2 = x$ since we can be certain only that u(a) = 0 for u in $\mathcal{D}(T)$.

CHAPTER IV

BOUNDARY CONDITIONS

In this chapter we will consider a class of differential operators which satisfy the criteria for a compact inverse and hence, have a discrete spectrum. First of all we will demonstrate that each of the class of 2nth order differential operators has deficiency indices (d,d) where d can assume each of the possible integers, $n \le d \le 2n$, depending on the choice of $m(x)$. Then the boundary condition description of the Friedrichs selfadjoint extension will be given for each case.

We have shown earlier that for a real symmetric differential operator of order 2n and one regular endpoint the deficiency indices must be equal, (d,d), and $n \le d \le 2n$. In 1921 W. Windau [22] and in 1938 D. Shin [18] concluded that the only possible value for d is n or 2n, corresponding to the limit-point and limit-circle cases of Weyl for the second order operator. In 1944 errors in these results were discovered and in 1950 I. M. Glazman [8] demonstrated by examples that any integer value of d between n and 2n can occur. In 1953 S. A. Orlov [14] presented other examples.

Consider the formal differential operator of order 2n

$$\tau = \frac{(-1)^n}{m(x)} \frac{d^n}{dx^n} p(x) \frac{d^n}{dx^n} \qquad 1 \le x < \infty$$

where $p(x) = x^r$ and $m(x) = x^s$.

Applying the definition of $h_1(x,k),\ldots,h_k(x,k)$ in Chapter III, we see that the definitions are significant only for $k = n$, and $h_1(x,n)$ is finite for x in $[1,\infty)$ if $r > 1$. Similarly, $h_k(x,n)$ is finite for x in $[1,\infty)$ if $r > 2k-1$. In particular $h_n(x,n)$ is finite for x in $[1,\infty)$ if $r = 2n - 1+\alpha$ where α satisfies $\alpha > 0$. Now applying Theorem 3.1, which requires that

$$\int_1^\infty m(x)[h_n(x,n)]^2 dx < \infty,$$

we see that $m(x) = x^s$ must be such that $s < \alpha-1$.

Define T_o by the method of Chapter III for Theorem 3.1. Then T_o is symmetric and semi-bounded below by 0 and, for the restrictions on $p(x)$ and $m(x)$ above, T_o has a selfadjoint extension T, which has a compact inverse.

It then follows that zero along with the entire negative semi-axis belongs to the same connected subset of the field of regularity of T_o and, hence, the deficiency indices are equal, (d,d), and equal to the dimension of the null space of T_o (see [1, page 92]).

To determine the number of solutions to $T_o u = 0$ we examine the solutions to $\tau u = 0$, and obtain the following set of $2n$ linearly independent solutions:

$$x^{n-r}, x^{n-r+1}, \ldots, x^{2n-r-1}, 1, x, \ldots, x^{n-1}.$$

Note that the requirement $r = 2n - 1 + \alpha$, $\alpha > 0$ implies that the first n functions have negative exponents.

The number of these solutions which are in $L^2(m)$, and hence, the deficiency indices of T_o, depend on the exact value of s as follows:

$$-1 \le s < \alpha - 1 \qquad d = n$$

$$-3 \le s < -1 \qquad d = n + 1$$

$$-5 \le s < -3 \qquad d = n + 2$$

$$\vdots \qquad\qquad \vdots$$

$$s < -(2n-1) \qquad d = 2n.$$

Since the deficiency indices depend on the weight function, $m(x) = x^s$, we are actually changing the Hilbert space in order to effect the change in the deficiency indices.

Using the values of r and s as restricted above we obtain the following boundary condition description of T.

Theorem 4.1. Let T be the Friedrichs extension of T_o defined as above. Then $u \varepsilon \mathcal{D}(T)$ if and only if $u \varepsilon \mathcal{D}(T_o^*)$ and u satisfies the following boundary conditions

(a) $u(1) = u'(1) = \ldots = u^{(n-1)}(1) = 0$

(b) $u^{(n-1)}(x) = 0(x^{(1-r)/2})$ as $x \to \infty$

$u^{(n-2)}(x) = 0(x^{(-r-1)/2})$ as $x \to \infty$

$\vdots \qquad \vdots$

$u(x) = 0(x^{(-r-2n+3)/2})$ as $x \to \infty$.

Proof. First we prove the necessity of the boundary conditions. From the description of the Friedrichs extension in Chapter II, we know that $u \varepsilon \mathcal{D}(T)$ implies that there exists a sequence $\{u_k\}$, $u_k \varepsilon \mathcal{D}(T_o)$ such that $\|u_k - u\| \to 0$ and $(T_o u_k, u_k) \to (Tu, u)$ as $k \to \infty$.

From the proof of Lemma 3.1, we know that for $1 \le x_1 < x_2 < \infty$ and $k=1,2,\ldots.$

$$|u_k^{(n-1)}(x_2) - u_k^{(n-1)}(x_1)|^2 \le (T_o u_k, u_k) \int_{x_1}^{x_2} 1/p(t)dt$$

$$\le C \int_{x_1}^{x_2} 1/p(t)dt \qquad (1)$$

where C does not depend on k since $(T_o u_k, u_k)$ is a convergent sequence. Also,

$$|u_k^{(n-2)}(x_2) - u_k^{(n-2)}(x_1)|^2 \le C\left(\int_{x_1}^{x_2} h_1(t,n)dt\right)^2$$

$$\vdots \qquad\qquad \vdots$$

$$|u_k(x_2) - u_k(x_1)|^2 \le C\left(\int_{x_1}^{x_2} h_{n-1}(t,n)dt\right)^2 .$$

Hence, each of the sequences $\{u_k\},\{u_k'\},\ldots,\{u_k^{(n-1)}\}$ is uniformly bounded and equicontinuous on compact subsets of $[1,\infty)$. Using Ascoli's Theorem and a diagonalization argument we find a subsequence of $\{u_k\}$ which converges, together with its derivatives up to and including order (n-1), uniformly on compact subsets of $[1,\infty)$. Hence, $u \varepsilon C^{(n-1)}[1,\infty)$

and $u_k^{(i)} \to u^{(i)}$ as $k \to \infty$ (restricting $u_k^{(i)}$ to the subsequence) uniformly on compact subsets of $[1,\infty)$.

Since $u_k^{(i)}(1) = 0$ for $k=1,2,\ldots$, $i=0,1,\ldots,(n-1)$, it follows that $u^{(i)}(1) = 0$ for $i=0,1,\ldots,(n-1)$. This proves part (a) of the theorem.

From (1) above, since $p(x) = x^r$, it follows that

$$|u_k^{(n-1)}(x_2)-u_k^{(n-1)}(x_1)|^2 \le C \int_{x_1}^{x_2} t^{-r}\, dt.$$

For each k, there is a point $x_2 < \infty$ such that $u_k^{(n-1)}(x_2) = 0$. Hence,

$$|u_k^{(n-1)}(x_1)|^2 \le C \int_{x_1}^{x_2} t^{-r}\, dt \le C \int_{x_1}^{\infty} t^{-r}\, dt.$$

Or

$$|u_k^{(n-1)}(x)|^2 \le C \int_{x}^{\infty} t^{-r}\, dt \qquad \text{for } x \text{ in } [1,\infty).$$

Taking limits as $k \to \infty$ (again restricting u_k to the subsequence) we see that for $u\varepsilon\mathcal{D}(T)$

$$|u^{(n-1)}(x)|^2 \le C \int_{x}^{\infty} t^{-r}\, dt = \frac{C}{r-1}\, x^{-r+1}.$$

Therefore $|u^{(n-1)}(x)| \le C' x^{(-r+1)/2}$, and repeated integration yields the boundary conditions of part (b) at the singular endpoint.

We now prove that the boundary conditions are sufficient. Since zero is not an eigenvalue of T, the image of T, $R(T)$, is the entire space $L^2(m)$.

Suppose $u \varepsilon \mathcal{D}(T_o^*)$ and satisfies the boundary conditions (a) and (b). Then there exists $v \varepsilon \mathcal{D}(T) \subset \mathcal{D}(T_o^*)$ such that $Tv = T_o^* u$. If we can show that $u = v$ then it follows that $u \varepsilon \mathcal{D}(T)$.

Let $w = u - v$, then $w \varepsilon \mathcal{D}(T_o^*)$. Also, by the proof of necessity above, v must satisfy the boundary conditions. Therefore w must satisfy them also.

Now $T_o^* w = T_o^* u - T_o^* v = Tv - Tv = 0$. Therefore w is in the null space of T_o and hence is a linear combination of the functions

$$x^{n-r}, x^{n-r+1}, \ldots, x^{2n-r-1},$$

and as many of the functions $1, x, \ldots, x^{n-1}$ as are in $L^2(m)$, which in turn depends on the value of s. We will assume, without loss of generality, that all of the functions are in the null space. Then

$$w(x) = c_1 x^{n-r} + c_2 x^{n-r+1} + \ldots + c_n x^{2n-r-1} + d_1 x^{n-1} + d_2 x^{n-2} + \ldots + d_{n-1} x + d_n .$$

Applying the boundary conditions (b) first we obtain

$$w^{(n-1)}(x) = \sum_{i=1}^{n} c_i [(n-r+i-1) \ldots (-r+i+1)] x^{-r+i} + d_1 (n-1)! .$$

$$\frac{w^{(n-1)}(x)}{x^{(1-r)/2}} = \sum_{i=1}^{n} c_i [(n-r+i-1) \ldots (-r+i+1)] x^{(2i-r-1)/2} + d_1 (n-1)! x^{(r-1)/2}$$

Therefore, $w^{(n-1)}(x) = 0(x^{(1-r)/2})$ as $x \to \infty$ implies $d_1 = 0$. Similarly applying the other boundary conditions at the singular endpoint we get

$d_1 = d_2 = \ldots = d_n = 0$, and

$$w(x) = c_1 x^{n-r} + c_2 x^{n-r+1} + \ldots + c_n x^{2n-r-1}.$$

Applying the boundary conditions at $x = 1$, we obtain the following homogeneous system of n equations in the n unknowns $c_1, c_2, \ldots, c_n$.

$$\begin{array}{rcl} c_1 + c_2 + \ldots + c_n &=& 0 \\ (n-r)c_1 + (n-r+1)c_2 + \ldots (2n-r-1)c_n &=& 0 \\ \vdots \qquad \vdots \qquad \vdots & & \\ [(n-r) \ldots (n-r-n+2)]c_1 + \ldots + [(2n-r-1) \ldots (n-r+1)]c_n &=& 0 \end{array}$$

The determinant of the coefficient matrix can be shown to be equivalent to the Vandermonde determinant by use of elementary row operations and, hence, is nonzero for all the cases considered here.

Therefore the system has only the trivial solution

$$c_1 = c_2 = \ldots = c_n = 0, \qquad \text{and} \qquad w = 0.$$

Hence $u = v$ and $u \varepsilon \mathcal{D}(T)$.

Note that if the deficiency indices are (d,d), then only the first (d-n) of the functions $1, x, \ldots, x^{n-1}$ are in $L^2(m)$ and, therefore, only (d-n) of the boundary conditions are needed at the singular end-point since each boundary condition was used to force one of the coefficients d_i equal to zero. Since there are always n boundary conditions at the regular endpoint, there are d boundary conditions necessary in all. This coincides with Theorem 2.1.

CHAPTER V

DISCUSSION OF RESULTS

In this chapter we will present some known results and compare them to our results in Chapter III. Also, we will present some questions for further study.

It will be assumed that the following operators are defined for the interval [a,b), i.e. the left endpoint a is regular and the right endpoint b is singular, b may be ∞.

Theorem 5.1 [*9*]. Let

$$\tau = (-1)^n \frac{d^{2n}}{dx^{2n}} + q(x).$$

If $\lim_{x \to b} q(x) = \infty$, then every selfadjoint extension of the minimal operator associated with τ has a discrete spectrum.

The following result is an extension of the above theorem and is found in [13, page 210].

Theorem 5.2. Let τ be a formal differential operator defined by

$$\tau = \sum_{k=o}^{n} (-1)^k \frac{d^k}{dx^k} p_k(x) \frac{d^k}{dx^k}$$

and suppose $p_n(x)>0, p_{n-1}(x)\geq 0, \ldots, p_1(x)\geq 0$ for x in [a,b).

If $\lim_{x\to b} p_o(x) = \infty$, then every selfadjoint extension of the minimal operator associated with τ has a discrete spectrum.

Our results in Theorem 3.1 give sufficient conditions for a discrete spectrum based on the coefficients. However, no conclusion was drawn based on the coefficient of the undifferentiated term, $p_o(x)$. Theorem 5.2 above provides such a test for the case of the weight function $m(x) = 1$. Theorems 3 and 4 and the Corollaries to Theorem 4 found in [13, pp. 211-214] provide other results concerning the continuous spectrum of differential operators of order higher than two.

In [7] Friedrichs presents a criterion for discrete spectrum of a second order differential operator as follows.

Let L be defined as follows,

$$L = -\frac{1}{r(x)}\left[\frac{d}{dx}\,p(x)\,\frac{d}{dx} - q(x)\right] \qquad \text{for } x \text{ in } (a,b)$$

where a may be $-\infty$ and b may be $+\infty$. Require that $p(x) > 0$ and $r(x) > 0$ on (a,b).

Let x_{-1}, x_o, x_1 be any points in (a,b) such that $a < x_{-1} < x_o < x_1 < b$. Define $h(x)$ as follows.

$$h(x) = \int_{x_o}^{x} \frac{dt}{p(t)} \qquad \text{for } x_1 \le x < b \qquad \text{if } \int_{x_o}^{b} \frac{dt}{p(t)} = \infty,$$

$$= \int_{x}^{b} \frac{dt}{p(t)} \qquad \text{for } x_1 \le x < b \qquad \text{if } \int_{x}^{b} \frac{dt}{p(t)} < \infty,$$

$$= \int_x^{x_o} \frac{dt}{p(t)} \qquad \text{for } a<x\le x_{-1} \qquad \text{if } -\int_b^{x_o} \frac{dt}{p(t)} = -\infty,$$

$$= \int_b^{x} \frac{dt}{p(t)} \qquad \text{for } a<x\le x_{-1} \qquad \text{if } -\int_b^{x_o} \frac{dt}{p(t)} > -\infty.$$

Theorem 5.3. Then the spectrum of L is discrete if

$$\frac{1}{r(x)}\left\{q(x) + \frac{1}{4p(x)[h(x)]^2}\right\} \to \infty \qquad \text{as } x\to b \text{ and as } x\to a.$$

Friedrichs then applies the criterion to the following operators and concludes a discrete spectrum in each case. (Note: $D = d/dx$).

1. $L = -D^2 + q$ where $b = \infty$ and $q(x)\to\infty$ as $x\to\infty$.
2. $L = -D^2 + q$ where (a,b) is a bounded interval and q is bounded below.
3. $L = -D(1-x^2)D$ $(a,b) = (-1,1)$.
4. $L = -e^{x}Dxe^{-x}D$ $(a,b) = (0,\infty)$.
5. $L = -\frac{1}{x} DxD + \frac{m^2}{x^2}$ $(a,b) = (0,1)$.
6. $L = -\frac{1}{x^{n-1}} Dx^{n-1}D$ $(a,b) =$ any finite interval.

If we apply Theorem 3.1 to examples 1, 3, and 6 and Theorem 5.2 to examples 2 and 5, we conclude a discrete spectrum also. Finally, if we make a change of variable in example 4, the Laguerre operator, to the familiar form

$$L = -DxD + \left(\frac{x}{4} - \frac{1}{2}\right) \qquad \text{where } b=\infty,$$

then Theorem 5.2 indicates a discrete spectrum for this problem also.

A more recent result of interest is that of M. S. P. Eastham [5]. Eastham considers the formal operator

$$\tau = \sum_{k=o}^{n} (-1)^k \frac{d^k}{dx^k} p_k(x) \frac{d^k}{dx^k} \qquad 0 \leq x < \infty$$

where the $p_k(x)$ are real-valued, $p_k(x) \varepsilon C^{(k)}[0,\infty)$ and $p_n(x) > 0$. Let μ denote the least limit point of the spectrum of any selfadjoint extension of the minimal operator associated with τ.

Theorem 5.4. Let

$$\lim_{x\to\infty} \inf\left(\frac{p_k(x)}{x^{2k}}\right) = \ell_k > -\infty.$$

If $\ell_k = 0$ for $\rho+1 \leq k \leq n$ and $\ell\rho \neq 0$, we assume that $p_k(x) \geq 0$ for x large enough and $\rho+1 \leq k \leq n$ and, if $\rho > 0$, that $\ell\rho > 0$. Then

$$\mu \geq [1^2 3^2 \ldots (2n-1)^2 4^{-n}]\ell_n + [1^2 3^2 \ldots (2n-3)^2 4^{-(n-1)}]\ell_{n-1} + \ldots + [1^2 4^{-1}]\ell_1 + \ell_0.$$

From this result it follows that if ℓ_k is equal to ∞ for at least one k, $k=0,1,\ldots,n$, then $\mu = \infty$ and the spectrum is entirely discrete.

The proof of Theorem 5.4 is based on a comparison to the Euler operator and requires $b = \infty$. However, the procedure used in Chapter III does not depend on comparison with known results, and Theorem 3.1

includes the case of a singularity at a finite endpoint. If $p_1(x) = (1+x)^{2+\alpha}$, $m(x) = 1$, $0 \leq x < \infty$, then it follows from both Theorem 3.1 and Theorem 5.4 that the spectrum is discrete if $\alpha > 0$. However, Theorem 5.4 indicates a discrete spectrum for $p_k(x) = (x+2)^{2k}\log(x+2)$, $0 \leq x < \infty$, and no conclusion can be drawn from Theorem 3.1.

Eastham's paper also contains an upper bound on μ based on $\limsup_{x\to\infty}(p_k(x)/x^{2k})$, $k=0,\ldots,n$. It is possible to construct examples of oscillatory $p_k(x)$ for which $\liminf_{x\to\infty}(p_k(x)/x^{2k})$ is finite and $\limsup_{x\to\infty}(p_k(x)/x^{2k})$ is infinite, and hence Eastham's results are inconclusive, but for which Theorem 3.1 implies a discrete spectrum.

The question arises concerning the result of Theorem 3.3 as to whether it is significant. Specifically, are there any formally self-adjoint operators which are positive integer powers of formally self-adjoint operators of lower order such that Theorem 3.3 can be applied to the coefficients of the lower order operator but Theorem 3.1 will not apply to the positive integer power of it? The answer to the question is in the affirmative as is seen from the following example.

Given that T_o is a symmetric semi-bounded operator (bounded below by zero), i.e. $(T_o u,u) \geq 0$ for all $u \epsilon \mathcal{D}(T_o)$, it follows from equation (12) in Chapter III that T_o^r is semi-bounded below by zero for all positive integers r. Note that the semi-boundedness of T_o in Chapter III results from the fact that

$$(T_o u,u) = \int_a^b \sum_{k=0}^{n} p_k(x)[u^{(k)}(x)]^2 dx$$

and the fact that $p_0, p_1, \ldots, p_n$ are all non-negative in $[a,b)$. However, it does not follow that the coefficients of T_o^2 are non-negative.

Example. Let $\tau u = -(x^3 u')' + xu$, $1 \le x < \infty$. Then $p_0 = x > 0$, and $p_1(x) = x^3 > 0$, on $[1,\infty)$. However,

$$\tau^2 u = (x^6 u'')'' - (-4x^4 u')' + (-2x^2)u$$

and $p_2(x) = x^6 > 0$, $p_1(x) = -4x^4 < 0$, and $p_0(x) = -2x^2 < 0$ for x in $[1,\infty)$.

Now let T_o be the minimal operator associated with τ on $[1,\infty)$. Then

$$h_1(x,1) = \left(\int_x^\infty \frac{1}{t^3}\, dt \right)^{1/2} = \frac{1}{\sqrt{2}\, x}$$

and

$$\int_1^\infty m(x)[h_1(x,1)]^2 dx = \frac{1}{2} < \infty.$$

Then the Friedrichs extension of T_o has a compact inverse by Theorem 3.1. However, if we let L_o be the minimal operator associated with τ^2 on $[1,\infty)$ then Theorem 3.1 does not apply since $p_0(x)$ and $p_1(x)$ are negative on $[1,\infty)$. However, Theorem 3.3 does apply to T_o^2, and it follows that Friedrichs extensions of T_o^2, in fact any selfadjoint operators associated with τ^2, have discrete spectra.

Theorem 3.1 provides sufficient conditions for a selfadjoint operator to have a discrete spectrum based on the behavior of any one coefficient provided the others are non-negative. It follows that any theorem which concludes that the continuous spectrum is non-empty must have hypotheses which place restrictions on all the coefficients to ensure that Theorem 3.1 does not apply.

Also, any necessary and sufficient condition for a discrete spectrum must impose restrictions on every coefficient. The necessary and sufficient condition given by A. M. Molchanov [13, page 245] concerns the formal operator

$$\tau = -D^2 + p_0(x) \qquad -\infty<x<+\infty,$$

and is based on the behavior of $p_0(x)$ as x tends to $+\infty$ and $-\infty$. However, the other coefficient $p_1(x)$ is restricted to equal to 1.

The results obtained lead to further questions concerning singular differential operators. First, we ask whether the results of Chapter III can be extended to multidimensional differential operators, which would have application to elliptic boundary-value problems of partial differential equations. Some results on this subject are found in [10].

In Chapter III we changed the domain of the symmetric operator by eliminating some restrictions at the singular endpoint. Then by reducing the possible ways we define the functions $h_i(x,k)$, we obtained similar results concerning the compact inverse of the Friedrichs self-adjoint extension. It is an open question whether the Friedrichs extensions of the various symmetric operators are identical or not. The answer to this question might be more accessible for those operators for which we can obtain a boundary condition description of the selfadjoint extension, as in Chapter IV.

The boundary condition description obtained in Chapter IV is for operators with only one non-zero coefficient. No general results are known for finding the specific boundary conditions for the case of more than one non-zero coefficient. One type of question concerning the general higher order operator is how do the boundary conditions depend on the coefficients. For example, in specifying a particular boundary condition how many of the coefficients must be involved so that the operator is selfadjoint.

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