СДІТ

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AUTOMATIC FEEDBACK CONTROL FOR ONE CLASS OF CONTACT PIEZOELECTRIC PROBLEMS

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In this paper we investigate the dynamics of solutions of the second order evolution inclusion with discontinuous interaction function which can be represented as the difference of subdifferentials. This case is actual for feedback automatic control problems. In particular, we concider mathematical model of contact piezoelectric process between a piezoelectric body and a foundation and for this problem investigate the long-term behavior of state function. We deduce a priory estimates for weak solutions of studied problem in the phase spase. The theorem on the existence of a global attractor for multi-valued semiflow generated by weak solutions of the problem and the structural properties of the limit sets is prooved. The main results of the paper were applied to the investigated piezoelectric problem.

INTRODUCTION AND PROBLEM FORMULATION

Let us consider a mathematical model which describes the contact between a piezoelectric body and a foundation. We formulate this problem as in [1].

Let $\mathbf{R}^d$ be a d-dimensional real linear space and $\mathbf{S}^d$ be the linear space of second order symmetric tensors on $\mathbf{R}^d$ with the inner product $\sigma:\tau=\sum_{ij}\sigma_{ij}\tau_{ij}$ and the corresponding norm $\|\tau\|^2_{\mathbf{S}^d}=\tau:\tau$, σ_{ij}, $\tau_{ij}\in\mathbf{S}^d$.

Let us consider a plane electro-elastic material which in its undeformed state occupies an open bounded domain $\Omega\subset\mathbf{R}^d$, $d=2$. This domain as a result of volume forces and boundary friction can contact with rigid or elastic support. Let the boundary of piezoelectric body Ω be Lipschitz continuous. Assume that the boundary Γ, on the one hand, consists of two disjoint measurable parts Γ_D and Γ_N, $m(\Gamma_D)>0$ and, on the other hand, consists of two disjoint measurable parts Γ_a and Γ_b, $m(\Gamma_a)>0$ (Figure). Suppose that the body is clamped on Γ_D, so the displacement field $u:Q\to\mathbf{R}^d$, $u=u(x,t)$, where $Q=\Omega\times(0,+\infty)$, vanishes there. Moreover, a surface traction of density g act on Γ_N, and the electric potential $\varphi:\Omega\to\mathbf{R}$ vanishes on Γ_a. The body Ω is lying on "support" medium, which introduce frictional effects. The interaction between the body and the support is described, due to the adhesion or skin friction, by a nonmonotone possibly multivalued law between the bonding forces and the corresponding displacements.

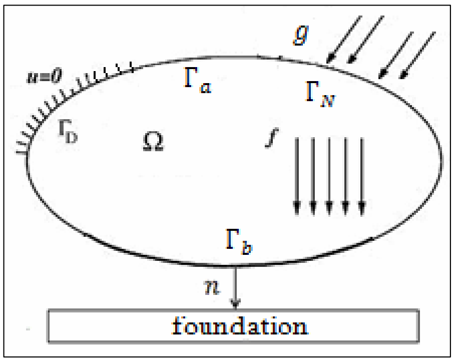

Figure. Partition of Γ

The body forces of density f consist of force f_e, which is prescribed external loading and force f_s which is the reaction of constrains introducing the skin effects, i.e. $f = f_e + f_s$. Here f_s is a possibly multivalued function of the displacement u.

To describe the contact between a piezoelectric body Ω and a foundation let us consider the basic piezoelectric equations: equation of motion, equilibrium equation, strain-displacement equation, equation of electric field-potential and other constitutive relations (see [1] and references therein).

We suppose that the process is dynamic. Let us set the constant mass density $\rho = 1$. Then we have the equation of motion for the stress field and the equilibrium equation for the electric displacement field respectively:

$$u_{tt} - \operatorname{Div}\sigma = f - \gamma u_t \quad \text{in } Q, \tag{1}$$

$$\operatorname{div} D = 0 \quad \text{in } Q,$$

where $\gamma \in L^\infty(\Omega)$ is nonnegative function of viscosity; $\sigma : Q \to \mathbf{S}^d$, $\sigma = (\sigma_{ij})$ is stress tensor; $D : \Omega \to \mathbf{R}^d$, $D = (D_i)$, $i, j = 1,2$ is the electric displacement field; $\operatorname{Div}\sigma = (\sigma_{ij,j})$ is the divergence operator for tensor valued functions; $\operatorname{div} D = (D_{i,i})$ is the divergence operator for vector valued. Equation (1) regulates the change in time of the mechanical state of the piezoelectric body.

The stress-charge form of piezoelectric constitutive relations describes the behavior of the material and are following:

$$\sigma = \mathsf{A}\varepsilon(u) - \mathsf{P}^{\mathrm{T}} E(\varphi) \quad \text{in } Q,$$

$$D = \mathsf{P}\varepsilon(u) + \mathsf{B}E(\varphi) \quad \text{in } Q,$$

where $\mathsf{A} : \Omega \times \mathbf{S}^d \to \mathbf{S}^d$ is a linear elasticity operator with the elasticity tensor $a = (a_{ijkl})$, $i, j, k, l = 1,2$; $\mathsf{P} : \Omega \times \mathbf{S}^d \to \mathbf{R}^d$ is a linear piezoelectric operator represented by the piezoelectric coefficients $p = (p_{ijk})$, $i, j, k = 1,2$; $\mathsf{P}^{\mathrm{T}} : \Omega \times \mathbf{R}^d \to \mathbf{S}^d$ is transpose to P operator represented by $\mathsf{P}^{\mathrm{T}} = (p^{\mathrm{T}}_{ijk}) = (p_{kij})$, $i, j, k = 1,2$, $\mathsf{B} : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ is a linear electric permittivity operator with the dielectric constants $\beta = (\beta_{ij})$, $i, j = 1,2$; $\varepsilon(u) = (\varepsilon_{ij}(u))$, $i, j = 1,2$ is the linear strain tensor; $E(\varphi) = (E_i(\varphi))$, the electric vector field.

The elastic strain-displacement and electric field-potential relations are given by

$$\varepsilon(u) = 1/2(\nabla u + (\nabla u)^{\mathrm{T}}) \quad \text{in} \ \ Q,$$

$$E(\varphi) = -\nabla \varphi \quad \text{in} \ \ Q.$$

We consider the reaction-displacement law of the form:

$$-f_s(x,t) \in \partial G_1(x, u(x,t)) - \partial G_2(x, u(x,t)) \quad \text{in} \ \ Q,$$

where $G_i : \Omega \times \mathbf{R}^d \to \mathbf{R}$, $i = 1,2$ are measurable in (x,u), convex in u for a.e. $x \in \Omega$ functionals; $\partial G_i(x,\cdot)$, $i = 1,2$ are their subdifferentials [2, Chapter 2].

Let u_0 be the initial displacement and u_1 be the initial velocity. The classical formulation of the mechanical model can be stated as follows: find a displacement field u on $\Omega \times \mathbf{R}^d$ and an electric potential φ on $\Omega \times \mathbf{R}$ such that:

$$\begin{gathered}
u_{tt} - \operatorname{Div} \sigma = f_e + f_s - \gamma u_t \quad \text{in} \ \ Q, \\
\operatorname{div} D = 0 \quad \text{in} \ \ Q, \\
\sigma = \mathsf{A}\varepsilon(u) - \mathsf{P}^{\mathrm{T}} E(\varphi) \ \ \text{in} \ \ Q, \\
D = \mathsf{P}\varepsilon(u) + \mathsf{B}E(\varphi) \ \ \text{in} \ \ Q, \\
-f_s(x,t) \in \partial G_1(x, u(x,t)) - \partial G_2(x, u(x,t)) \ \ \text{in} \ \ Q, \\
u = 0 \quad \text{on} \quad \Gamma_D \times (0,T), \ \ n = g \quad \text{on} \quad \Gamma_N \times (0,T), \\
\varphi = 0 \quad \text{on} \quad \Gamma_a \times (0,T), \ \ Dn = 0 \quad \text{on} \quad \Gamma_b \times (0,T), \\
u(0) = u_0, \ \ u_t(0) = u_1,
\end{gathered} \tag{2}$$

where n denotes the outward unit normal to Γ.

We now turn to the variational formulation of Problem (2). Let us consider the space $V = \{v \in H^1(\Omega;\mathbf{R}^d) : v = 0 \text{ on } \Gamma_D\} \subset H^1(\Omega;\mathbf{R}^d)$. Let $H = L^2(\Omega;\mathbf{R}^d)$, $\mathsf{H} = (\Omega;\mathbf{R}^d)$ be a Hilbert spaces equipped with the inner products $\langle u, v\rangle_H = \int_\Omega uv dx$ and $\langle \sigma, \tau\rangle_{\mathsf{H}} = \int_\Omega \sigma : \tau dx$ respectively. Then (V, H, V^*) be an evolution triple of spaces. Then $\langle u, v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathsf{H}}$, $\|v\|_V = \|\varepsilon(v)\|_{\mathsf{H}}$, $u, v \in V$ is the inner product and the corresponding norm on V. Therefore $(V, \|\cdot\|_V)$ is Hilbert space.

Assume that $G_i : \Omega \times \mathbf{R}^d \to \mathbf{R}$, $i = 1,2$, satisfies standard Carathéodory's conditions, and there exist $c^{(i)} \in L_1(\Omega)$ and $\alpha^{(i)} > 0$, and that $\| d^{(i)} \|_{\mathbf{R}^d} \le c^{(i)}(x) + \alpha^{(i)} \| u \|_{\mathbf{R}^d}$ for a.e. $x \in \Omega$ and any $u \in \mathbf{R}^d$, $d^{(i)} \in \partial G_i(x,u)$. Moreover, $\alpha^{(2)}$ is sufficiently small.

Let us set the following hypotheses for the constitutive tensors:

(i) $a = (a_{ijkl})$, $a_{ijkl} \in L^{\infty}(\Omega)$, $a_{ijkl} = a_{klij}$, $a_{ijkl} = a_{jikl}$, $a_{ijkl} = a_{ijlk}$, $a_{ijkl}(x)\tau_{ij}\tau_{kl} \ge \alpha\tau_{ij}\tau_{ij}$ for a.e. $x \in \Omega$, $\forall \tau = (\tau_{ij}) \in \mathbf{S}_{+}^{d}$, $\alpha > 0$;

(ii) $p = (p_{ijk})$, $p_{ijk} \in L^{\infty}(\Omega)$;

(iii) $\beta = (\beta_{ij})$, $\beta_{ij} = \beta_{ji} \in L^{\infty}(\Omega)$, $\beta_{ij}(x)\zeta_i\zeta_j \ge m_{\beta} \| \zeta \|_{\mathbf{R}^d}^2$ for a.e. $x \in \Omega$, $\forall \zeta = (\zeta_i) \in \mathbf{R}^d$, $m_{\beta} > 0$.

Without loss of generality let us consider $g \equiv 0$ and $f_e \equiv 0$. Following [1], we present Problem (2) in the generalized formulation:

$$\begin{cases} u_{tt}(t) + Bu_t(t) + Au(t) + \partial J_1(u(t)) - \partial J_2(u(t)) \ni \bar{0}, \text{ for a.e. } t, \\ u(0) = u_0, \ u_t(0) = u_1, \end{cases} \tag{3}$$

where $B : H \to V^*$; $A : V \to V^*$; $J_i : H \to R$, $i = 1,2$ are locally Lipschitz functionals, $J_i(u) := \int_{\Omega} G_i(x, u(x))dx$, $i = 1,2$. ∂J_i is the Clarke subdifferential for $J_i(\cdot)$, $i = 1,2$; $(V; H; V^*)$ is evolution triple.

Note that the parameters of Problem (3) satisfy following assumptions [1]:

- **Assumtion** (B): $B : H \to H$ be a linear symmetric such that there exists $\beta > 0$ such that $(Bv, v)_H = \beta\|v\|_H^2$ $\forall v \in H$;
- **Assumtion** (A): V is a Hilbert space; $A : V \to V^*$ be a linear, symmetric and there exists $c_A > 0$ such that $\langle Av, v \rangle_V \ge c_A\|v\|_V^2$ $\forall v \in V$;
- **Assumtion** (J): $J_i : H \to R$, $i = 1,2$ be the functions such that

(i) $J_i(\cdot)$, $i = 1,2$ are locally Lipschitz and regular (see Clarke [2]), i.e.:

- for any $x, v \in H$, the usual one-sided directional derivative $J_i'(x; v) = \lim_{t \to 0} \dfrac{J_i(x + tv) - J_i(x)}{t}$, $i = 1,2$, exists,

- for all $x, v \in H$, $J_i'(x; v) = J_i^{\circ}(x; v)$, where $J_i^{\circ}(x; v) = \overline{\lim_{y \to x, t \to 0}} \dfrac{J_i(y + tv) - J_i(y)}{t}$, $i = 1,2$;

(ii) for $i = 1,2$ there exists $c_i > 0$ such that

$$\|l\|_H \le c_i(1 + \|v\|_H), \ \forall l \in \partial J_i(v), \ \forall v \in H;$$

(iii) there exists $c_2 > 0$ such that

$$(l, v)_H \le \lambda\|v\|_H^2 + c_2, \ \forall l \in \partial J_2(v), \ \forall v \in H,$$

where $\partial J_i(u) = \{p \in H \mid (p, w)_H \le J_i^{\circ}(u; w) \ \forall w \in H\}$ denotes the Clarke subdifferentials of $J_i(\cdot)$, $i = 1,2$ at a point $u \in H$ (see Clarke [2] for details); $\lambda \in (0, \lambda_1)$, $\lambda_1 > 0$: $c_A\|v\|_V^2 \ge \lambda_1\|v\|_H^2$ $\forall v \in V$.

We define Hilbert space $X = V \times H$ as the phase space for Problem (3). Let $-\infty < \tau < T < +\infty$.

Definition. Let $-\infty < \tau < T < +\infty$. The function $(u(\cdot), u_t(\cdot))^T \in L_\infty[\tau, T; X]$ is called a weak solution for Problem (3) on $[\tau, T]$ if there exist $l_i \in L_2(\tau, T; H)$, $i = 1,2$, $l_i(t) \in \partial J_i(u(t))$ for a.e. $t \in (\tau, T)$ such that $\forall \psi \in V$, $\forall \eta \in C_0^\infty(\tau, T)$:

$$-\int_\tau^T (u_t(t), \psi)_H \eta_t(t) dt +$$

$$+\int_\tau^T \left[(u_t(t), \psi)_H + (u(t), \psi)_H + (l_1(t), \psi)_H - (l_2(t), \psi)_H\right] \eta(t) dt = 0.$$

Theorem 1.4 from [1] provides the existance of a weak solution of Problem (3) on $[\tau, T]$ with initial data

$$u(\tau) = a, \quad u_t(\tau) = b \tag{4}$$

for any $a \in V$, $b \in H$.

In the non-autonomous case the abstract existence results for such problem with nonmonotone skin effects are presented in [1]. The long-time behavior of all weak solutions for this problem with continuous interaction function is investigated by Ball in [3]. The solution dynamics for autonomous model when $J_2 \equiv 0$ is studied in [4], [5]. The particular scalar situation is considered in [6]. Here we consider the case of multidimensional laws with discontinuous interaction function which can be represented as the difference of subdifferentials, that is actual for feedback automatic control problems. The main purpose of this paper is to investigate the long-term behavior of state function, to study the structural properties of the limit sets and to deduce sufficient conditions that direct the system to the desired asymptotic level.

PROPERTIES OF SOLUTIONS

We consider a class of functions $W_\tau^T = C([\tau, T]; X)$. To simplify our conclusions from Assumptions (A), (B) we suppose that

$$(u, v)_V = \langle Au, v \rangle_V, \ \|v\|_V^2 = \langle Au, v \rangle_V, \ \beta(u, v)_H = (Bu, v)_H,$$

$$\beta \|v\|_H^2 = (Bv, v)_H \quad \forall u, v \in V. \tag{5}$$

Let us set $J(u) = J_1(u) - J_2(u)$, $u \in H$. Lebourgue's mean value theorem [2, Chapter 2] yields the existence of constants $c_3, c_4 > 0$ and $\mu \in (0, \lambda_1)$:

$$|J(u)| \le c_3(1 + \|u\|_H^2), \quad J(u) \ge -\frac{\mu}{2}\|u\|_H^2 - c_4 \quad \forall u \in H. \tag{6}$$

According to [7, Lemma 4.1, p. 78], [7, Lemma 3.1, p. 71] and [1, Theorem 1.4] the following existence result holds.

Lemma 1. For any $\tau < T$, $a \in V$, $b \in H$ Cauchy problem (3), (4) has a weak solution $(u, u_t)^T \in L_\infty(\tau, T; X)$. Moreover, each weak solution $(u, u_t)^T$ of Cauchy problem (3), (4) on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; X)$ and $u_{tt} \in L_2(\tau, T; V^*)$.

Let us consider the next denotations: $\forall \varphi_\tau = (a, b)^T \in X$ we consider $\mathsf{D}_{\tau,T}(\varphi_\tau) = \{(u(\cdot), u_t(\cdot))^T \mid (u, u_t)^T$ is a weak solution of (3) on $[\tau, T]$, $u(\tau) = a$, $u_t(\tau) = b\}$. Lemma 1 implies that $\mathsf{D}_{\tau,T}(\varphi_\tau) \subset C([\tau, T]; X) = W_\tau^T$.

Note that translation and concatenation of weak solutions are also the weak solutions.

Lemma 2. If $0 < \tau < T < +\infty$, $\varphi_\tau \in X$, $\varphi(\cdot) \in \mathsf{D}_{\tau,T}(\varphi_\tau)$, then $\psi(\cdot) = \varphi(\cdot + s) \in \mathsf{D}_{\tau-s,T-s}(\varphi_\tau)$ $\forall s$. If $\tau < t < T$, $\varphi_\tau \in X$, $\varphi(\cdot) \in \mathsf{D}_{\tau,t}(\varphi_\tau)$ and $\psi(\cdot) \in \mathsf{D}_{t,T}(\varphi_\tau)$, then

$$\theta(s) = \begin{cases} \varphi(s), & s \in [\tau, t], \\ \psi(s), & s \in [t, T] \end{cases} \text{ belongs to } \mathsf{D}_{\tau,T}(\varphi_\tau).$$

Proof. The proof is trivial.

Let $\varphi = (a, b)^T \in X$ and

$$\mathsf{V}(\varphi) = \frac{1}{2}\|\varphi\|_X^2 + J_1(a) - J_2(a). \tag{7}$$

Then we have the next lemma.

Lemma 3. Let $-\infty < \tau < T < +\infty$, $\varphi_\tau \in X$, $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in \mathsf{D}_{\tau,T}(\varphi_\tau)$. Then $\mathsf{V} \circ \varphi : [\tau, T] \to R$ is absolutely continuous function, and for a.e. $t \in (\tau, T)$ $\frac{d}{dt}\mathsf{V}(\varphi(t)) = -\beta\|u_t(t)\|_H^2$.

Proof. Let $-\infty < \tau < T < +\infty$, $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in W_\tau^T$ be an arbitrary weak solution of (3) on (τ, T). As $\partial J_i(u(\cdot)) \subset L_2(\tau, T; H)$, $i = 1,2$ then from [7, Lemma 4.1, p. 78] and [7, Lemma 3.1, p. 71] we get that the function $t \to \|u_t(t)\|_H^2 + \|u(t)\|_V^2$ is absolutely continuous, and for a.e. $t \in (\tau, T)$:

$$\frac{1}{2}\frac{d}{dt}\left[\|u_t(t)\|_H^2 + \|u(t)\|_V^2\right] = (u_{tt}(t) + Au(t), u_t(t))_H =$$

$$= -\beta\|u_t(t)\|_H^2 - (l_1(t), u_t(t))_H + (l_2(t), u_t(t))_H, \tag{8}$$

where $l_i(t) \in \partial J_i(u(t))$, $i = 1,2$ for a.e. $t \in (\tau, T)$ and $l_i(\cdot) \in L_2(\tau, T; H)$.

As $u(\cdot) \in C^1([\tau, T]; H)$ and $J_i : H \to R$, $i = 1,2$ are regular and locally Lipschitz, due to [5, Lemma 2.16] we obtain that for a.e. $t \in (\tau, T)$ there exist $\frac{d}{dt}(J_i \circ u)(t)$, $i = 1,2$. Moreover, $\frac{d}{dt}(J_i \circ u)(\cdot) \in L_1(\tau, T)$, $i = 1,2$ and for a.e.

$t \in (\tau, T)$, $\forall\, p \in \partial J_i(u(t))$, $i = 1,2$, $\frac{d}{dt}(J_i \circ u)(t) = (p, u_t(t))_H$, $i = 1,2$. In particular, for a.e. $t \in (\tau, T)$ $\frac{d}{dt}(J_i \circ u)(t) = (l_i(t), u_t(t))_H$, $i = 1,2$. Taking into account (8) we finally obtain the necessary statement.

The lemma is proved.

Lemma 4. Let $T > 0$. Then any weak solution of Problem (3) on $[0, T]$ can be extended to a global one defined on $[0, +\infty)$. For any $\varphi_0 \in X$ and $\varphi \in \mathsf{D}(\varphi_0)$ the next inequality holds $\forall\, t > 0$:

$$\|\varphi(t)\|_X^2 \le \frac{\lambda_1 + 2c_3}{\lambda_1 - \mu}\|\varphi(0)\|_X^2 + \frac{2(c_3 + c_4)\lambda_1}{\lambda_1 - \mu}, \tag{9}$$

where for an arbitrary $\varphi_0 \in X$ let $\mathsf{D}(\varphi_0)$ be the set of all weak solutions (defined on $[0, +\infty)$) of problem (3) with initial data $\varphi(0) = \varphi_0$.

Proof. The statement of this lemma follows from Lemmas 1–3, conditions (5), (6) and from the next estimates:

$$\forall\, \tau < T,\ \forall\, \varphi_\tau \in X,\ \forall\, \varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in \mathsf{D}_{\tau,T}(\varphi_\tau),$$

$$\forall\, t \in [\tau, T]\ \ 2c_3 + \left(1 + \frac{2c_3}{\lambda_1}\right)\|u(\tau)\|_V^2 + \|u_t(\tau)\|_H^2 \ge 2\mathsf{V}(\varphi(\tau)) \ge 2\mathsf{V}(\varphi(t)) =$$

$$= \|u(t)\|_V^2 + \|u_t(t)\|_H^2 + 2J(u(t)) \ge \left(1 - \frac{\mu}{\lambda_1}\right)\|u(t)\|_V^2 + \|u_t(t)\|_H^2 - 2c_4.$$

The lemma is proved.

Now let us provide the continuity property for the weak solutions of the main problem in the weak topologies of the phase and the extended phase spaces.

Theorem 1. Let $\tau < T$, $\{\varphi_n(\cdot)\}_{n \ge 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (3) on $[\tau, T]$ such that $\varphi_n(\tau) \to \varphi_\tau$ weakly in X, $n \to +\infty$, and let $\{t_n\}_{n \ge 1} \subset [\tau, T]$ be a sequence such that $t_n \to t_0$, $n \to +\infty$. Then there exists $\varphi \in \mathsf{D}_{\tau,T}(\varphi_\tau)$ such that up to a subsequence $\varphi_n(t_n) \to \varphi(t_0)$ weakly in X, $n \to +\infty$.

Proof. Let $\tau < T$, $\{\varphi_n(\cdot) = (u_n(\cdot), u_n'(\cdot))\}_{n \ge 1} \subset W_\tau^T$ be an arbitrary sequence of the weak solutions of (3) on $[\tau, T]$, and $\{t_n\}_{n \ge 1} \subset [\tau, T]$ such that

$$\varphi_n(\tau) \to \varphi_\tau \ \text{ weakly in } X,\ \ t_n \to t_0,\ \ n \to +\infty. \tag{10}$$

According to Lemma 4 we have that $\{\varphi_n(\cdot)\}_{n \ge 1}$ is bounded on $W_\tau^T \subset L_\infty(\tau, T; X)$. Therefore there exists a subsequence $\{\varphi_{n_k}(\cdot)\}_{k \ge 1} \subset \{\varphi_n(\cdot)\}_{n \ge 1}$ such that

$$\begin{array}{llll} u_{n_k} \to u & \text{weakly star in} & L_\infty(\tau,T;V), & k\to+\infty, \\ u'_{n_k} \to u' & \text{weakly star in} & L_\infty(\tau,T;H), & k\to+\infty, \\ u''_{n_k} \to u'' & \text{weakly star in} & L_\infty(\tau,T;V^*), & k\to+\infty, \\ l_{n_k,i} \to l_i & \text{weakly star in} & L_\infty(\tau,T;H), & k\to+\infty, \\ u_{n_k} \to u & \text{in} & L_2(\tau,T;H), & k\to+\infty, \\ u_{n_k}(t) \to u(t) & \text{in } H \text{ for a.e.} & t\in[\tau,T], & k\to+\infty, \\ u'_{n_k}(t) \to u'(t) & \text{in } V^* \text{ for a.e.} & t\in(\tau,T), & k\to+\infty, \\ Bu'_{n_k} \to Bu' & \text{weakly star in} & L_2(\tau,T;H), & k\to+\infty, \\ Au_{n_k} \to Au & \text{weakly star in} & L_2(\tau,T;V^*), & k\to+\infty, \end{array} \tag{11}$$

where $l_{n,i} \in L_2(\tau,T;H)$ be such that

$$u''_n(t) + Bu'_n(t) + l_{n,1}(t) - l_{n,2}(t) + Au_n(t) = F,$$

$$l_{n,i}(t) \in \partial j_i(u_n(t)), \text{ for a.e. } t\in(\tau,T), \ n\ge 1, \ i=1,2.$$

Since ∂j_i, $i=1,2$ is demiclosed, the following inclusion holds:

$$l_i(\cdot) \in \partial j_i(u(\cdot)), \ i=1,2, \text{ where } \varphi := (u,u_t) \in \mathcal{D}_{\tau,T}(\varphi_\tau) \subset W_\tau^T.$$

For a fixed $h\in V$ formula (11) implies that the sequence of real functions $(u_{n_k}(\cdot),h)$ is uniformly bounded and equicontinuous one. According to (9), (11) and the density of V in H we obtain that $u'_{n_k}(t_{n_k}) \to u'(t_0)$ weakly in H and $u_{n_k}(t_{n_k}) \to u(t_0)$ weakly in V as $k\to+\infty$.

The theorem is proved.

Theorem 2. Let $\tau < T$, $\{\varphi_n(\cdot)\}_{n\ge1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of (3) on $[\tau,T]$ such that $\varphi_n(\tau) \to \varphi_\tau$ strongly in X, $n\to+\infty$, then up to a subsequence $\varphi_n(\cdot) \to \varphi(\cdot)$ in $C([\tau,T];X)$, $n\to+\infty$.

Proof. The proof follows from [4, Theorem 2] and Lemma 3.

MAIN RESULTS

Now let us examine the long-time behavior of all weak solutions of the main problem as time $t\to+\infty$. For this purpose let us define the m-semiflow $\mathcal{G}$ as

$$\mathcal{G}(t,\xi_0) = \{\xi(t) \mid \xi(\cdot) \in \mathcal{D}(\xi_0)\}, \ t\ge 0. \tag{12}$$

Denote the set of all nonempty (nonempty bounded) subsets of X by $P(X)$ $(\beta(X))$. Note that the multivalued map $\mathcal{G}: R_+ \times X \to P(X)$ is a *strict m-semiflow*, i.e. (see Lemma 2): $\mathcal{G}(0,\cdot) = \mathrm{I}d$ (the identity map), $\mathcal{G}(t+s,x) = \mathcal{G}(t,\mathcal{G}(s,x))$ $\forall x\in X$, $t,s\in R_+$. Further $\varphi\in\mathcal{G}$ will mean that $\varphi\in\mathcal{D}(\xi_0)$ for some $\xi_0\in X$.

We recall, that the m-semiflow G is asymptotically compact if for any sequence $\{\varphi_n\}_{n\ge 1}\in \mathsf{G}$ $\{\varphi_n(0)\}_{n\ge 1}$ is bounded, and for any sequence $\{t_n\}_{n\ge 1}$: $t_n \to +\infty$ as $n\to\infty$, the sequence $\{\varphi_n(t_n)\}_{n\ge 1}$ has a convergent subsequence.

Let us consider a family $\mathsf{K}_+ = \bigcup_{u_0\in X}\mathsf{D}(u_0)$ of all weak solutions of inclusion (3) defined on $[0,+\infty)$. Note that K_+ is *translation invariant one*, i.e. $\forall u(\cdot)\in\mathsf{K}_+$, $\forall h\ge 0$ $u_h(\cdot)\in\mathsf{K}_+$, where $u_h(s)=u(h+s)$, $s\ge 0$. On K_+ we set the *translation semigroup* $\{T(h)\}_{h\ge 0}$, $T(h)u(\cdot)=u_h(\cdot)$, $h\ge 0$, $u\in\mathsf{K}_+$. In view of the translation invariance of K_+ we conclude that $T(h)\mathsf{K}_+\subset\mathsf{K}_+$ as $h\ge 0$. On K_+ we consider a topology induced from the Fréchet space $C^{\mathrm{loc}}(R_+;X)$. Note that

$$f_n(\cdot)\to f(\cdot) \ \text{ in } \ C^{\mathrm{loc}}(R_+;X) \Leftrightarrow \forall M>0$$

$$\Pi_M f_n(\cdot)\to \Pi_M f(\cdot) \ \text{ in } \ C([0,M];X),$$

where Π_M is the restriction operator to the interval $[0,M]$ [8, p.179]. We denote the restriction operator to $[0,+\infty)$ by Π_+.

Let us consider autonomous inclusion (3) on the entire time axis. Similarly to the space $C^{\mathrm{loc}}(R_+;X)$ the space $C^{\mathrm{loc}}(R;X)$ is endowed with the topology of local uniform convergence on each interval $[-M,M]\subset R$ (cf. [8, p. 180]).

A function $u\in C^{\mathrm{loc}}(R;X)\cap L_\infty(R;X)$ is said to be *a complete trajectory* of inclusion (3) if $\forall h\in R$ $\Pi_+ u_h(\cdot)\in\mathsf{K}_+$ [8, p. 180].

Let K be a family of *all complete trajectories* of inclusion (3). Note that $\forall h\in R$, $\forall u(\cdot)\in\mathsf{K}$ $u_h(\cdot)\in\mathsf{K}$. We say that the complete trajectory $\varphi\in\mathsf{K}$ is *stationary* if $\varphi(t)=z$ for all $t\in R$ for some $z\in X$.

Following [9, p. 486] we denote the set of the rest points of G by $Z(\mathsf{G})$. We remark that $Z(\mathsf{G})=\{(\bar 0,u)\,|\,u\in V,\ A(u)+\partial J_1(u)-\partial J_2(u)\ni\bar 0\}$. Assumptions (A) and (J) provide that the set $Z(\mathsf{G})$ is bounded in X. Lemma 3 implies the existence of a Lyapunov type function [9, p.486] for m-semiflow G.

We consider construction presented in Ball [9], Melnik and Valero [10]. We recall that the set A is said to be *a global attractor* for G if: (i) $\mathsf{A}\subset\mathsf{G}(t,\mathsf{A})$, $\forall t\ge 0$; (ii) A is attracting set, i.e.

$$\operatorname{dist}(\mathsf{G}(t,C),\mathsf{A})\to 0,\ \ t\to+\infty,\ \ \forall\, C\in\beta(X), \tag{13}$$

where $\operatorname{dist}(D,E)=\sup_{d\in D}\inf_{e\in E}\|d-e\|_X$ is the Hausdorff semidistance; (iii) for any closed set $Y\subset H$ satisfying (13), we have $\mathsf{A}\subseteq Y$. The global attractor is *invariant* if $\mathsf{A}=\mathsf{G}(t,\mathsf{A})$, $\forall t\ge 0$.

Provide the main result of this paper.

Theorem 3. The *m*-semiflow G has the invariant compact in the phase space X global attractor A. For each $\psi \in \mathsf{K}$ the limit sets

$$\alpha(\psi) = \{z \in X \mid \psi(t_j) \to z \text{ for some sequence } t_j \to -\infty\},$$

$$\omega(\psi) = \{z \in X \mid \psi(t_j) \to z \text{ for some sequence } t_j \to +\infty\}$$

are connected subsets of $Z(\mathsf{G})$ on which V is constant. If $Z(\mathsf{G})$ is totally disconnected (in particular, if $Z(\mathsf{G})$ is countable), the limits $z_- = \lim_{t\to-\infty} \psi(t)$, $z_+ = \lim_{t\to+\infty} \psi(t)$ exist and z_-, z_+ are the rest points; furthermore, $\varphi(t)$ tends to a rest point as $t \to +\infty$ for every $\varphi \in \mathsf{K}_+$.

Proof. According to Theorems 1, 2 and [3, Theorem 2.7] we need to provide that m-semiflow G is asymptotically compact.

Let $\xi_n \in \mathsf{G}(t_n, v_n)$, $v_n \in C \in \beta(X)$, $n \ge 1$, $t_n \to +\infty$, $n \to +\infty$. Let us check the precompactness of $\{\xi_n\}_{n\ge1}$ in X. In order to do that without loss of the generality it is sufficiently to extract a convergent in X subsequence from $\{\xi_n\}_{n\ge1}$. From Lemma 4 we obtain that there exist such $\{\xi_{n_k}\}_{k\ge1}$ and $\xi \in X$ that $\xi_{n_k} \to \xi$ weakly in X, $\|\xi_{n_k}\|_X \to a \ge \|\xi\|_X$, $k \to +\infty$. Show that $a \le \|\xi\|_X$.

Let us fix an arbitrary $T_0 > 0$. Then for rather big $k \ge 1$ $\mathsf{G}(t_{n_k}, v_{n_k}) \subset \mathsf{G}(T_0, \mathsf{G}(t_{n_k} - T_0, v_{n_k}))$. Hence, $\xi_{n_k} \in \mathsf{G}(T_0, \beta_{n_k})$, where $\beta_{n_k} \in \mathsf{G}(t_{n_k} - T_0, v_{n_k})$ and $\sup_{k\ge1} \|\beta_{n_k}\|_X < +\infty$ (see Lemma 4). From Theorem 1 for some $\{\xi_{k_j}, \beta_{k_j}\}_{j\ge1} \subset \{\xi_{n_k}, \beta_{n_k}\}_{k\ge1}$, $\beta_{T_0} \in X$ we obtain:

$$\xi \in \mathsf{G}(T_0, \beta_{T_0}), \quad \beta_{k_j} \to \beta_{T_0} \quad \text{weakly in } X, \quad j \to +\infty. \tag{14}$$

From the definition of G we set: $\forall j \ge 1$ $\xi_{k_j} = (u_j(T_0), u_j'(T_0))^T$, $\beta_{k_j} = (u_j(0), u_j'(0))^T$, $\xi = (u_0(T_0), u_0'(T_0))^T$, $\beta_{T_0} = (u_0(0), u_0'(0))^T$, where $\varphi_j = (u_j, u_j')^T \in C([0,T_0]; X)$, $u_j' \in L_2(0,T_0; V^*)$, $l_j \in L_\infty(0,T_0; H)$,

$$u_j''(t) + Bu_j'(t) + Au_j(t) + l_{j,1}(t) - l_{j,2}(t) = \bar{0}, \quad l_{j,i}(t) \in \partial J_i(u_j(t)),$$

$$i = 1,2 \quad \text{for a.e.} \quad t \in (0,T_0).$$

Let for each $t \in [0,T_0]$

$$I(\varphi_j(t)) := \frac{1}{2}\|\varphi_j(t)\|_X^2 + J_1(u_j(t)) - J_2(u_j(t)) + \frac{\beta}{2}(u_j'(t), u_j(t)).$$

Then, in virtue of [5, Lemma 2.16], [7, Lemma 4.1, p.78] and [7, Lemma 3.1, p.71], $\dfrac{dI(\varphi_j(t))}{dt} = -\beta I(\varphi_j(t)) + \beta \mathsf{H}(\varphi_j(t))$, for a.e. $t \in (0,T_0)$, where

$$\mathsf{H}(\varphi_j(t)) = J_1(u_j(t)) - \frac{1}{2}(l_{j,1}(t), u_j(t)) - J_2(u_j(t)) + \frac{1}{2}(l_{j,2}(t), u_j(t)).$$

From (9), (14) we have that there exists $\bar{R} > 0$: $\forall j \geq 0 \quad \forall t \in [0, T_0]$ $\|u'_j(t)\|_H^2 + \|u_j(t)\|_V^2 \leq \bar{R}^2$. Moreover,

$$\begin{gathered} u_j \to u_0 \quad \text{weakly in } L_2(0,T_0;V),\ j \to +\infty, \\ u'_j \to u'_0 \quad \text{weakly in } L_2(0,T_0;H),\ j \to +\infty, \\ u_j \to u_0 \quad \text{in } L_2(0,T_0;H),\ j \to +\infty, \\ l_{j,i} \to l_i \quad \text{weakly in } L_2(0,T_0;H),\ j \to +\infty, \\ u''_j \to u''_0 \quad \text{weakly in } L_2(0,T_0;V^*),\ j \to +\infty, \\ \forall t \in [0,T_0] \quad u_j(t) \to u_0(t) \quad \text{in } H,\ j \to +\infty. \end{gathered} \tag{15}$$

For any $j \geq 0$ and $t \in [0, T_0]$

$$I(\varphi_j(t)) = I(\varphi_j(0))e^{-\beta t} + \int_0^t \mathsf{H}(\varphi_j(s))e^{-\beta(t-s)}ds,$$

in particular

$$I(\varphi_j(T_0)) = I(\varphi_j(0))e^{-\beta T_0} + \int_0^{T_0} \mathsf{H}(\varphi_j(s))e^{-\beta(T_0-s)}ds.$$

From (15) and [5, Lemma 2.16] we have

$$\int_0^{T_0} \mathsf{H}(\varphi_j(s))e^{-\beta(T_0-s)}ds \to \int_0^{T_0} \mathsf{H}(\varphi_0(s))e^{-\beta(T_0-s)}ds,\ j \to +\infty.$$

Therefore,

$$\overline{\lim_{j\to+\infty}} I(\varphi_j(T_0)) \leq \overline{\lim_{j\to+\infty}} I(\varphi_j(0))e^{-\beta T_0} + \int_0^{T_0} \mathsf{H}(\varphi_0(s))e^{-\beta(T_0-s)}ds =$$

$$= I(\varphi_0(T_0)) + \left[\overline{\lim_{j\to+\infty}} I(\varphi_j(0)) - I(\varphi_0(0))\right]e^{-\beta T_0} \leq I(\varphi_0(T_0)) + \varsigma e^{-\beta T_0},$$

where ς does not depend on $T_0 > 0$. On the other hand, from (15) we have

$$\overline{\lim_{j\to+\infty}} I(\varphi_j(T_0)) \geq$$

$$\geq \frac{1}{2}\overline{\lim_{j\to+\infty}} \|\varphi_j(T_0)\|_X^2 + J_1(u_0(T_0)) - J_2(u_0(T_0)) + \frac{\beta}{2}(u'_0(T_0), u_0(T_0)).$$

Therefore, we obtain: $\frac{1}{2}a^2 \leq \frac{1}{2}\|\xi\|_X^2 + \varsigma e^{-\beta T_0}$ $\forall T_0 > 0$. Thus, $a \leq \|\xi\|_X$.

The Theorem is proved.

APLICATION

Let us apply main Theorem 3 to Problem (2).

Corollary. Under listed above assumptions on parameters of Problem (2) all statements of Theorem 3 for m-semiflow G defined in (12) hold.

In particular, for any $\overline{u} \in \overline{V}$ such that $A\overline{u} \in H$ there exist such functionals G_1 and G_2 such that Assumption (J) holds and $Z(\mathsf{G}) = \{\overline{u}\}$.

CONCLUSIONS

For one class of feedback automatic control problems in sence of the global attractor theory the dynamics of solutions is investigated. In particular, we concider the mathematical model of contact piezoelectric problem with discontinuous interaction function which can be represented as the difference of subdifferentials.

A priory estimates for weak solutions of studied problem in the phase spase are deduced. This contributes to obtain the existence of the weak solutions and their properties.

The existence of global attractor for generated multi-valued semiflow is proved. The structural properties of the limit sets are studied. These results are applied to the considered piezoelectric problem. Thus, it became possible to forecast the long-term behavior of state function and to direct the investigated system to the desired asymptotic level.

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