THE METRIC OF THE EXPANDING UNIVERSE

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SUMMARY

It has recently been verified by Smoot et al. that the cosmic microwave background radiation is isotropic to 1 part in 3000. This remarkable degree of isotropy of the universe enables us to rule out certain models of the universe.

It is shown from the theory of Riemann spaces that isotropy implies homogeneity; furthermore, a line element can be derived for an isotropic space. The isotropy of the universe then suggests that its mass distribution is homogeneous and that the universe can be described by this isotropic line element (Robertson - Walker metric). The isotropic, homogeneous universe is marked by the feature of constant curvature of space which can be positive, zero or negative, corresponding to the geometry of a four-dimensional sphere, plane or hyperboloid, respectively.

The observational results of the cosmological redshift implying expansion of the universe can readily be accounted for by the Robertson-Walker metric. Assuming the validity of Einstein's theory of general relativity, a set of cosmological equations can be derived, known as Friedmann's equations. From these it follows that a static universe would be unstable which is in accordance with the observed expansion of the universe. Furthermore, the Friedmann model predicts that a universe of zero or negative curvature is open and will expand forever, while a universe of positive curvature is closed and will finally recontract.

The predictions made by the Friedmann model are compared with observational results. While cosmological observations are still fraught with uncertainties, it seems to be possible at present to construct a self consistent model for an expanding universe of negative curvature. However, the experimental data are still inconclusive, and there also remain open questions with the Friedmann model itself.

CHAPTER I

INTRODUCTION

"The heavens are empty and void of substance... The sun and moon and the company of stars float freely in the empty space, moving or standing still." This sentence has not been taken from the work of any author of post-Newtonian times, rather it is the remarkable description of the universe given 2000 years ago by the Chinese philosopher Hsüan Yeh (Gribbin, 1977). Not only did the Chinese give detailed descriptions of astronomical events and abandoned the notion of the earth being at the center of the universe long before Copernicus, but it seems they were already aware of the existence of other galaxies beside our own: "Heaven and earth are large, yet in the whole of empty space they are but as a small grain of rice... Empty space is like a kingdom and heaven and earth no more than a single individual person in that kingdom. How unreasonable it would be to suppose that besides the heaven and earth we can see, there are no other heavens and no other earths." If we only substitute "Milky Way" or "galaxy" as today's astronomical terms for "heaven," then these sentences by Têngu Mu give a remarkable modern statement. Furthermore the Chinese word for universe alone - yü-chou - can be taken as a 2000-year preview of Minkowski-space; for ÿu is space and chou is time, thus giving a literal translation of "space time."

What can Western cosmology offer in view of these early accomplishments by the Chinese? Cosmology in the West was widely left to speculation; observations were ignored or lagged far behind theory. As Weinberg (1977) pointed out, as late as in the 1950's the study of the early universe and its evolution was still widely regarded as "not the sort of thing to which a respectable scientist would devote his time." But things have changed since then. During the last decade a variety of observational data has been collected that allows us to compare cosmological theories with reality. Although many of these data are still fraught with uncertainties, even contradictions, they nevertheless enable us at least to rule out some theories that do not agree with observations.

Models

Since Einstein developed his theory of gravity* many models of the universe have been proposed none of which could at that time be affirmed or rejected by observations. Einstein himself developed a model in which the universe was closed and static. Soon thereafter, however, Friedmann (1922) showed that a static universe is necessarily unstable, and went on to develop equations for an expanding universe. Independent of Friedmann, Lemaître (1927) came to the same conclusion of a universe evolving from an initial singularity of infinitely high density and infinitely small radius. This model was later termed the "Big-Bang-Theory."

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*known as the General Theory of Relativity. The name can be explained by the fact that here Einstein generalized the special coordinate transformations of Special Relativity to include all (differentiable) coordinate transformations (principle of general covariance). From a modern point of view, we can justify the name by noting that General Relativity can be regarded as a gauge theory generated by the Poincaré group of Special Relativity.

Other proposed models include deSitter's exponentially expanding universe, and Lemaitre's original proposal of a universe that halts expansion to allow the formation of galaxies and then continues to expand. Models that do not rely on Einstein's equations include Gödel's rotating universe (Gödel, 1949) as well as Milne's Kinematic Relativity (Milne, 1948), which is based on the assumption that special relativity is valid on a global scale.

The best-known among the models not relying on Einstein's equations is the so-called "Steady-State-Theory" proposed by Bondi and Gold (1948) who postulate that - except for fluctuations and local evolutions - the universe has always been in the state we observe today. In other words, Bondi and Gold postulate that all points in space-time are equivalent. The observational evidence of the expansion of the universe precludes the conservation of energy as matter has to be created at a constant rate in order to keep the energy density constant. This violation of the energy conservation law, however, would be so small as to be hardly detectable on a local scale. The attractive feature of the Steady-State-Theory lies in the fact that it does not involve an initial singularity of doubtful physical properties. Therefore it became the main competitor of the Big-Bang Theory.

Of the many cosmological models proposed let us finally mention two that imply a very interesting result: a changing "constant" of gravity. Dirac (1937) was the first to note that the ratio of electrostatic to gravitational force between an electron and a proton is very nearly the same as the ratio of the present age of the universe

($t_o \cong 10^{10}$ years) to the time it takes a photon to cross an electron ($t_e \cong 10^{-23}$ sec):

$$\frac{e^2}{GMm} \cong \frac{t_o}{t_e} \qquad (1\text{-}1)$$

If we follow Dirac in postulating that this approximate equality is not a mere coincidence, but reflects a fundamental, though as yet unexplained truth, then we have to accept Dirac's conclusion that G decreases in time as the universe evolves, if masses and charges remain constant.

Dirac's theory inspired a number of attempts to formulate a field theory of gravitation in which the effective "constant" of gravity is some function of a scalar field. The most interesting and complete "scalar-tensor" theory of gravitation is that proposed by Brans and Dicke (1961). In this theory, the gravitational "constant" G is replaced with the reciprocal of a scalar field ϕ which is then incorporated into Einstein's equations to give an additional term.

We now proceed to apply Occam's razor to single out those theories of the universe that appear to be likely to give a correct description. In order to do so we shall discuss two observations that lie 130 years apart: the dark night sky leading to Olbers' paradox and the cosmic microwave background radiation.

Olber's Paradox

As we all know, our night sky is marked by an extreme paucity of radiation, in other words: it is rather dark at night. In 1826, Olbers asked the simple question, why it is dark at night when an uncountable number of stars shed their light onto the earth. As it turns out so often the simplest questions are the hardest to answer. The solution to the problem was found only a hundred years after Olbers posed the question. For Olbers showed that if there are infinitely many stars in the sky of a certain luminosity, then the radiation density received at the earth would be infinite. After allowing for the possibility that light from a star has been intercepted, it can be shown that the radiation density received on the earth must equal the average radiation density at the surface of a star.

Since this is obviously not true and no flaw is to be found in the calculations, we are then left to conclude that some of Olbers' assumptions were wrong. Let us restate and examine these assumptions:

i) space is infinite and obeys Euclid's geometry

ii) there is a uniform average distribution of stars in the universe

iii) each star has the same luminosity

iv) the universe is infinitely old

v) the known laws of physics apply.

Obviously, if space were finite, the amount of radiation received on earth could be reduced to the observed level. However, the present observations do not indicate any boundaries of the universe. If we drop the Euclidean property of space we are not able to resolve Olbers' paradox, because a change in the geometry of space only yields a change in the distance parameter. Assuming space still to be homogeneous, the non-Euclidean factor will appear both in the expression for the total luminosity of the stars and in the distance the light travels and will thus cancel, leaving us with the same result as before. Interstellar matter has been proposed to absorb part of the radiation, but absorption only heats up the material thus forcing it to reemit the light. If the universe were very young, say of age T, then only light from a distance less than cT would be able to reach us. While this proposal is preferable to the finiteness of space, it has perhaps a similar touch of artificiality, and we would be glad to drop it for a better explanation. Various authors proposed to change the laws for the propagation of light in such a way that the photons will lose part of their energy on their way through space. This is known as the "Tired-Light" hypothesis. Zeldovich (1963) showed that this idea would imply serious changes in Maxwell's equations for a static electric field. Experimental confirmation of Maxwell's equations, however, is such as to render the tired-light hypothesis highly improbable. But this line of thought brings us on the right track, because light traveling through space does lose part of its energy, if only by a completely different process: the expansion of the universe. As the source of the light moves away, the Doppler effect acts in such a way as to shift the energy of the light towards the red end of the spectrum. Since this red-shift of light from distant galaxies has actually been observed, we conclude then from Olbers' paradox that any cosmological theory must account for the expansion of the universe. Indeed, the existence of the red shift of the light from distant sources and the absence of blue shift is itself a strong argument in favor of expansion, as will be explained in greater detail later. Thus we can rule out models that propose a static universe.*

Note that a model proposing a final recontraction of the universe implies that someday the night will be as bright as the day, since the reverse process will bring about a blue-shift of photons.

The Cosmic Microwave Background Radiation

In 1965, A. A. Penzias and R. W. Wilson, while making radio-astronomical measurements using an antenna originally designed to receive signals reflected from the "Echo" satellites, quite accidentally discovered a strong background radiation in the microwave region. The radiation they measured at 7 cm wavelength could mostly be accounted for by sources such as atmospheric emission and ground emission. However, some residual radiation remained unaccounted for; and, more important, it was isotropic in character. Indeed, it was isotropic to such a high degree that it cannot possibly have originated from any sources close by; the only explanation seems to be that it is of cosmic origin, a cosmic microwave background (henceforth abbreviated as cmb).

The cmb can readily be explained within the framework of the Big Bang theory. As was pointed out by Gamow (1948), shortly after its origin the universe was radiation dominated and in a state of thermal equilibrium. The radiation would then have the Planck black body spectrum, and as the universe expands the temperature drops shifting

*For a different approach to solve the paradox, see for example Finlay-Freundlich (1957) or Harrison (1977). Their solutions of the problem, however, do not contradict an expanding universe.

the maximum of the black body spectrum to the millimeter range of wavelength. The Planckian form for the spectral distribution of the cmb has indeed been verified by Woody et al. (1975 and 1978).

The high degree of isotropy leads us to the assumption that the universe itself is highly isotropic on a large scale. Recently, Smoot et al. (1977) detected an anisotropy in the cmb that varies as the cosine of an angle $(\hat{\theta}, \hat{n})$ where $\hat{\theta}$, $\hat{n}$ are unit vectors, $\hat{\theta}$ varying over the sky and $\hat{n}$ giving the position (54° ± 10° lat, 245° ± 1.5° long) in galactic coordinates. This cosine component can be interpreted as due to the motion of the earth with respect to the radiation with a velocity of (390 ± 60) km/sec in the direction $\hat{n}$ towards the constellation Leo, in agreement with earlier reports by Rubin et al. (1976) of a motion of the earth as found by a different method. Excluding this cosine component, however, the cmb is isotropic to 1 part in 3000 thus confirming our assumption of the large scale isotropy of the universe.*

We saw that the Big Bang theory can explain the cmb as relic radiation from the early universe. The Steady State theory, however, encounters serious difficulties with the cmb as there is no origin of the universe within the framework of that theory. The Steady State theory explains the cmb in terms of very large numbers of sources emitting in the infrared and microwave regions. The difficulty with that explanation, however, is that the number of sources required to furnish such a high degree of isotropy must at least be as many as the number

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*Note the curious feature that the cmb gives us a "new aether drift" (Peebles, 1971). The cmb serves as a universal frame with respect to which every observer is able to determine his velocity.

of ordinary galaxies, a very unlikely assumption. That the cmb is really cosmic in origin and not emitted by a great number of sources within our galaxy has recently been shown by Lake and Partridge (1977) who detected "dips" in the cmb towards clusters of galaxies. When the cmb is "shadowed" by objects a billion light years away, then it must truly be of cosmic origin.

Thus, the existence of the cmb is the strongest piece of evidence against the validity of the Steady State theory, but it also serves as a crucial test for the Dirac cosmologies, independent of its true origin. It can be shown (Steigman, 1978) that a Planck spectrum is destroyed unless photons are conserved. Dirac's large-number hypothesis, however, requires creation of photons. Thus, either the present epoch is unique and the spectrum was not Planckian in the past nor will it remain Planckian in the future, or cosmologies in which photons are not conserved (as well as "tired-light" cosmologies) are unacceptable. To most, the first choice is an unacceptable violation of the Copernican principle.

In summary then, we conclude from Olbers' paradox and the red shift of light from distant sources that the universe must be expanding at present, while the cmb assures us that the universe is highly isotropic on a large scale. As we shall see in the next chapter, the assumption of isotropy alone will lead us to a global metric for the universe.

CHAPTER II

THE GLOBAL METRIC

Isotropy and Homogeneity

As we have seen in the foregoing chapter, observational evidence assures us that large scale isotropy of the universe is a plausible assumption. Obviously, the universe is not isotropic on a small scale like that of our galaxy or even the local group of galaxies. It is generally assumed now that galaxies tend to form groups, called clusters, with an average distance between the clusters of ~10 Mpc (1 parsec $\approxeq 3.09 \times 10^{16}$m). Whether or not these clusters themselves again form arrays of higher order called superclusters, is still uncertain (Murray et al., 1977; Darius, 1977). We shall therefore adopt the notion that the galactic clusters fill the universe as a fluid continuum of a highly idealized nature which is isotropic above a scale of ~10 Mpc.

In mathematical terms, regarding the universe as a Riemann space we are now going to show that the assumption of isotropy implies a space of constant curvature. A space of constant curvature has the same properties everywhere; in the case of General Relativity we know that curvature of space is caused by mass, and constant curvature is therefore caused by a homogeneous distribution of masses. Hence, isotropy implies homogeneity.

Spaces of Constant Curvature

Let us consider the implication of isotropy on a Riemann space (see e.g. Sexl and Urbanke, 1975; or Eisenhart, 1949). If no direction is distinguished in a point on the manifold X^3, then the Riemann curvature tensor $R_{ijk\ell}$ at the origin of a coordinate system has to be invariant under rotations. The most general form for a rotationally invariant tensor of rank four is given by

$$R_{ijk\ell} = a(x)g_{ij}g_{k\ell} + b(x)g_{ik}g_{j\ell} + c(x)g_{i\ell}g_{jk} \tag{2-1}$$

Since $R_{ijk\ell}$ has to be a tensor, no term involving $\varepsilon_{ijk\ell}$ will appear, as it is not a tensor but a tensor density. From the symmetry properties of the Riemann curvature tensor (App., eq. (A-14)) we have

$$a(x) = 0 \tag{2-2}$$

and

$$c(x) = -b(x) \tag{2-3}$$

so that the rotation invariant Riemann tensor reads

$$R_{ijk\ell} = b(x) \quad (g_{ik}g_{j\ell} - g_{i\ell}g_{jk}) \tag{2-4}$$

For the Ricci tensor we get the following

$$\begin{aligned} R_{jk} = R^{i}_{jki} = g^{i\ell}R_{ijk\ell} &= b(x) \ (g^{i\ell}g_{ik}g_{j\ell} - g^{i\ell}g_{i\ell}g_{jk}) \\ &= b(x) \ (g^{\ell}{}_{k}g_{j\ell} - 3g_{jk}) \\ &= -2b(x) \ g_{jk} \end{aligned} \tag{2-5}$$

and for the Riemann scalar

$$R = R_{jk}g^{jk} = -2b(x)\ g_{jk}g^{jk} = -6b(x) \tag{2-6}$$

In order to insert these expressions into the Bianchi identities

$$G^{m}{}_{k\|m} = 0 \tag{2-7}$$

we first have to calculate the Einstein tensor G^{m}_{k}:

$$\begin{aligned} G^{m}_{k} &= R^{m}_{k} - 1/2\ g^{m}_{k} R \\ &= g^{mj}R_{jk} - 1/2\ g^{m}_{k} R \\ &= -2b(x)\ g^{m}_{k} + 3b(x)g^{m}_{k} \\ &= b(x)g^{m}_{k}\ . \end{aligned} \tag{2-8}$$

Covariant differentiation of the Einstein tensor yields (App. eq. (A-16))

$$\begin{aligned} G^{m}{}_{k\|m} &= b(x)g^{m}{}_{k\|m} + b(x)_{|m}g^{m}_{k} \\ &= b(x)_{|k} \end{aligned} \tag{2-9}$$

Inserting into the Bianchi identity we finally have

$$b(x)_{|k} = 0 \tag{2-10}$$

and b is constant throughout the manifold X^3. Thus our result is that if the curvature R is independent of direction it is constant throughout the whole space, i.e. isotropy implies homogeneity.

Physically, this means that if we test the sky for homogeneity and find inhomogeneities on a large scale then we have to reconsider our assumption of isotropy. Indeed models have been proposed that are inhomogeneous and anisotropic. The present observations, however, do not give any reason to abandon our hypothesis. We shall return to this question in the fourth chapter.

The Isotropic Line Element

Since the Riemann tensor of a space of constant curvature shows a very simple form, we have reason to assume that the line element can also be written in a simple way. In order to show this we need the concept of conformity of two spaces.

Definition: Two spaces with metric tensors g_{kj} and $\tilde{g}_{kj}$ are said to be conformal, if

$$\tilde{g}_{kj} = \psi^{-2} g_{kj} \ , \qquad (2\text{-}11)$$

where $\psi = \psi(x)$.

If two spaces are conformal, then the following statement holds

$$\tilde{C}_{mk\ell} = C_{mk\ell} \qquad (2\text{-}12)$$

where $C_{mk\ell}$ is Weyl's conformal curvature tensor, given by

$$C_{mk\ell} = R_{mk||\ell} - R_{m\ell||k} + 1/4(g_{m\ell}R_{|k} - g_{mk}R_{|\ell})^{*} \qquad (2\text{-}13)$$

The reverse statement also holds true: if the Weyl tensors of two spaces are equal, then the two spaces are conformal (Eisenhart, 1960). Clearly, if $C_{mk\ell} = 0$ it follows that $R_{mk} = 0$ and the given space is flat.

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*Note that this is only true for spaces of dimension $n = 3$. For the case of higher dimension, Weyl's curvature tensor needs to be generalized.

Inserting (2-5) and (2-6) into (2-13) and noting (2-10), we then see that all spaces of constant curvature are conformal to Euclidean space. Therefore, the line element of a space of constant curvature can always be given in the simple form

$$g_{k\ell} = \psi^{-2}(x)\eta_{k\ell} \tag{2-11}$$

where $\eta_{k\ell}$ here denotes the metric of flat 3-space.

Inserting (2-11a) into the Riemann curvature tensor (App. eq. (A-13)), and into eq. (2-5) we have

$$\begin{aligned} R_{ijk\ell} = \psi^{-3}(\eta_{ik}\psi_{|j\ell} + \eta_{j\ell}\psi_{|ik} - \eta_{i\ell}\psi_{|jk} - \eta_{jk}\psi_{|i\ell}) \\ + \psi_{|m}\psi_{|n}\eta^{mn}(\eta_{jk}\eta_{i\ell} - \eta_{ik}\eta_{j\ell})\psi^{-4} \end{aligned} \tag{2-12}$$

$$R_{ijk\ell} = b(x)(\eta_{jk}\eta_{i\ell} - \eta_{ik}\eta_{j\ell})\psi^{-4} \tag{2-12a}$$

Equating (2-12) and (2-12a) yields

$$\begin{aligned} \psi(\eta_{ik}\psi_{|j\ell} + \eta_{j\ell}\psi_{|ik} - \eta_{i\ell}\psi_{|jk} - \eta_{jk}\psi_{|i\ell}) = -(b+\psi_{|m}\psi_{|n}\eta^{mn}) \\ (\eta_{jk}\eta_{i\ell} - \eta_{ik}\eta_{j\ell}) \end{aligned} \tag{2-13}$$

Now $i \neq j = k \neq \ell \neq i$ implies $\psi_{|ik} = 0$, hence ψ has to be of the form

$$\psi = \sum_{\ell} f_{\ell}(x_{\ell}) \tag{2-14}$$

Inserting (2-14) back into (2-13) yields for $j = k \neq i = \ell$

$$f_{j|jj} + f_{i|ii} = (b + \sum_{\ell} f^2_{\ell|\ell})\psi^{-1} \qquad (2\text{-}15)^*$$

Since the left hand side of (2-15) depends on x^j and x^k only, the right hand side, however, on all coordinates, it follows that the right hand side has to be equal to some constant c:

$$(b + \sum_{\ell} f^2_{\ell|\ell})\psi^{-1} = c \qquad (2\text{-}16)$$

from which $f_{j|jj} = 1/2\ c$ from (2-15) and therefore

$$f_j = 1/4\ c\ (x_j + a_j)^2 + d_j \qquad (2\text{-}17)$$

Now choosing the coordinate system such as to make $a_j = 0$ and setting $\sum_j d_j = d$, we have

$$\psi = \sum_j f_j = d + 1/4\ c \sum_j (x_j)^2 \qquad (2\text{-}18)$$

(2-16) then implies that b = cd, because $f_{j|j} = 1/2\ c\ x_j$ and

$$\frac{(b + 1/4\ c^2 \sum_j (x_j)^2)}{(d + 1/4\ c \sum_j (x_j)^2)} = c \qquad (2\text{-}19)$$

Let us now insert the calculated ψ in (2.18) into the metric (2-11a) to give the line element for a space of constant curvature

$$d\tau^2 = \psi^{-2}\eta_{k\ell}dx^k dx^\ell = \frac{\eta_{k\ell}dx^k dx^\ell}{(d^2 + 1/4\ bx^2)^2} \qquad (2\text{-}20)$$

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*no sum convention!

In order to be in accordance with the conventionally used notation, let us make the following substitutions $b = R^{-2}k$ and $x = d^{-1}uR$:

$$d\tau^2 = \frac{R\ d\sigma^2}{(1 + 1/4\ ku^2)^2} \qquad \text{(2-20a)}$$

where $d\sigma^2$ is the line element of flat 3-space with distance parameter u, e.g. in polar coordinates $d\sigma^2 = du^2 + u^2\ d\Omega^2$. The line element (2-20a) can be transformed (App., p. 79) into the following form

$$d\tau^2 = R^2\ [\frac{dr^2}{1 - kr^2} + r^2\ d\Omega^2] \qquad \text{(2-20b)}$$

Both forms, (2-20a) and (2-20b) compete in the literature. We shall mostly use (2-20b).

To summarize, let us note that (2-20) was derived for Riemann spaces, assuming only isotropy and rotational invariance, both reasonable assumptions to make about the universe. We also found that this isotropic space is necessarily homogeneous of constant curvature, and its metric is given by the isotropic line element (2-20).

Geometrical Interpretation

In order to gain a more intuitive understanding of the isotropic line element (2-20), let us consider a problem in pure four-dimensional differential geometry. But first of all, let us clarify the meaning of the symbols used in (2-20): we expressed the curvature parameter b in terms of R and k, where R denotes the radius of curvature of the isotropic 3-space, and k denotes the sign of the curvature. As we found b to be constant throughout the space, then so is R, and k takes on the values 1, 0, or -1, denoting positive, zero, or negative curvature of 3-space respectively.

In the case of flat 3-space ($k = 0$), note that (2-20) reduces to the usual line element of Euclidean space. This then leaves us to interpret the geometry of space for the cases $k = \pm 1$.

Let us now consider the problem of embedding a three-dimensional sphere and a three dimensional hyperboloid into four dimensional space. The three dimensional sphere can be embedded in a four dimensional Euclidean space, but the three dimensional hyperboloid cannot; but it can be embedded in a four dimensional Minkowski space.

The metric of four-space in these two cases is

$$ds^2 = k\,dw^2 + dx^2 + dy^2 + dz^2 \qquad (2\text{-}21)^*$$

($k = +1$: Euclidean, $k = -1$: Minkowskian)

If we restrict these coordinates to a hypersphere and hyper-hyperboloid respectively, the condition reads

$$kw^2 + x^2 + y^2 + z^2 = kR^2 \qquad (2\text{-}22)$$

with $k = +1$ denoting the equation for a hypersphere and $k = -1$ the equation for a hyper-hyperboloid.

Observe now that in order to transform to polar coordinates we have to introduce different sets of coordinates for the spherical and

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*Note that w is a fourth space dimension and has nothing whatsoever to do with time.

the hyperbolic case. As can easily be verified by inserting back into (2-22), the transformation equations are

$$w = R\, c_k(X) \qquad x = R\, s_k(X) \sin\theta \cos\phi$$
$$y = R\, s_k(X) \sin\theta \sin\phi \qquad z = R\, s_k(X) \cos\theta \tag{2-23}$$

where

$$s_k(X) = \begin{cases} \sin X & \text{for } k = +1 \\ \sinh X & \text{for } k = -1 \end{cases}, \quad c_k(X) = \begin{cases} \cos X & \text{for } k = +1 \\ \cosh X & \text{for } k = -1 \end{cases} \tag{2-24}$$

Thus we can transform the metric (2-21) into

$$\begin{aligned} ds^2 &= R^2[c_k(X)^2 dX^2 + k\, s_k(X)^2 + s_k(X)^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \\ &= R^2[dX^2 + s_k(X)^2\, d\Omega^2] \end{aligned} \tag{2-25}$$

Now setting

$$s_k(X) = r \tag{2-26}$$

we have

$$dr = c_k(X)\, dX \tag{2-27}$$

and

$$dX^2 = \frac{dr^2}{c_k(X)^2} = \frac{dr^2}{1 - kr^2} \tag{2-28}$$

Inserting (2-27), (2-28) into (2-26), we have finally

$$ds^2 = R^2[\frac{dr^2}{1 - kr^2} + r^2\, d\Omega^2] \tag{2-29}^*$$

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*Note that the apparent singularity in the denominator is entirely due to the choice of coordinates as can be seen by comparison with (2-25).

which is the isotropic line element (2-20b).

Thus we see that in the case of positive curvature the line element (2-20b) can be interpreted as the one of a three dimensional sphere embedded in four dimensional Euclidean space, while in the case of negative curvature (2-20b) gives the line element of a three dimensional hyperboloid embedded in four dimensional Minkowski space. The trivial case of $k = 0$ does not ascribe a curvature to space at all, and (2-20b) is the usual line element of three dimensional Euclidean space.

Time Evolution

So far, we have not mentioned time in our considerations of the metric of curved three-space. But as we saw in Chapter I, the universe is expanding, therefore we have to incorporate time-dependence into the isotropic line element (2-20), and the only way to do so is to let the radius R become a function of time. Any other choice of time-dependence would not preserve the form of the hyper-surface as time goes on and would thus destroy isotropy.

In order to write down now the space-time metric of the expanding isotropic universe, we still have to choose a time parameter. Three natural choices of a time parameter for the universe are possible:

i) the proper time t

ii) the expansion factor R

iii) an arc parameter η

The expansion factor R grows with time and can therefore serve to distinguish one phase of the expansion from another, and can consequently be regarded as a parametric measure of time in its own right.

In order to use R as a time parameter, however, the exact evolution of R with time has to be known and we would have to introduce the field equations and a pressure-density relation at this point which we would rather defer until later. Also the metric would take a more complicated form than with a choice of any of the other two time parameters.

The proper time t has the advantage of giving directly proper time elapsed since the start of the expansion. It is convenient at this point to introduce the concept of Gaussian (or comoving) coordinates. A comoving observer is at rest with respect to matter in his vicinity. Since our assumption of isotropy suggests that the mass distribution is homogeneous throughout the universe, we can synchronize a cosmic time t by instructing all comoving observers to use the density of matter in their vicinity as a measure of time. All cosmic observers therefore share the same cosmic time t.

Mathematically, we pick an initial spacelike hypersurface in four dimensional Minkowski space-time. We then place an arbitrary coordinate grid on it, erect geodesic world lines orthogonal to it, and give these world lines the coordinates

$$(x^1, x^2, x^3) = \text{constant}, \quad t = t_i + \tau \tag{2-30}$$

where τ is the proper time along the world line, beginning with $\tau = 0$ on the initial hypersurface. The metric then takes the Gaussian form ($c = 1$)

$$ds^2 = dt^2 = g_{ij}dx^i dx^j \tag{2-31}$$

and the coordinate time t measures proper time along the lines of constant x^i, i.e., $g_{tt} = 1$.

In Gaussian coordinates, the space-time metric of the expanding isotropic universe takes the form (insert (2-20b) into (2-31))

$$ds^2 = dt^2 - R(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 \, d\Omega^2\right] \tag{2-32}$$

(2-32) is known as the Robertson Walker line element (Robertson, 1932; Walker, 1932).

Let us briefly return to the question of the choice of time-parameters to discuss the remaining possibility of the arc parameter measure of time. During the interval of time dt, a photon traveling on a hypersurface with radius R(t) covers an arc measured in radians equal to

$$d\eta = \frac{dt}{R(t)} \tag{2-33}$$

The arc parameter η is then defined by the integral of this expression from the start of the expansion. Thus, small values of η mean early times, larger values mean later times, and the metric takes the form

$$ds^2 = A(\eta)^2 [d\eta^2 - \frac{dr^2}{1 - kr^2} - r^2 \, d\Omega^2] \ , \tag{2-32a}$$

where $A(\eta) = R(t)$.

It will sometimes be convenient to use (2-32a) instead of (2-32).

In summary then, we derived the Roberson-Walker line element of the universe under very general considerations. We assumed only isotropy of the universe, as well as local rotational invariance. All these assumptions seem to be well founded in experiment. So far, we have only given a metric of the expanding universe, but we have not in any way specified exactly how the universe evolves, whether the expansion is decelerating or going at a steady rate or even accelerating. In order to treat this question we will have to introduce a theory of gravity and derive cosmological field equations.

CHAPTER III

COSMOLOGY

As mentioned in the last chapter, the Robertson Walker metric was derived without the need for a theory of gravity. In order to develop cosmological theories and treat the exact behavior of the expanding universe, we need the gravitational field equations and a pressure-density for the assumed fluid continuum formed by the clusters of galaxies.

Before we come to these cosmological field equations, however, let us consider two implications from the Robertson Walker metric that can be drawn without the help of a theory of gravity and which are observable and have been confirmed by experiment: the red-shift of light from distant sources and the Hubble law.

Cosmological Redshift

Let an observer be placed at the origin $r = 0$ of a coordinate system, and let the light of a distant galaxy come in radially along the $-r$ direction, with θ and ϕ fixed. From the fact that light travels along null-geodesics with $ds^2 = 0$, it follows from (2-32) that

$$dt = -\frac{R(t)\,dr}{\sqrt{1-kr^2}} \tag{3-1}$$

If the light was emitted at $t = -\tau$ and is received at $t = 0$, we then have

$$\int_{-\tau}^{0} \frac{dt}{R(t)} = -\int_{\tau}^{0} \frac{dr}{\sqrt{1-kr^2}} = \begin{array}{ll} \sin^{-1} r & \text{for } k=1 \\ r & k=0 \\ \sinh^{-1} r & k=-1 \end{array} \tag{3-2}$$

Since at best terms quadratic in r are measurable, we shall expand (3-2) and neglect terms of higher order:

$$\int_{-\tau}^{0} \frac{dt}{R(t)} = r + 0(r^3) \tag{3-3}$$

Let ν_e be the frequency of the light when emitted, and ν_o the observed frequency. For the next light wave emitted at $t = -\tau + \frac{1}{\nu_e}$, we then have

$$\int_{-\tau+1/\nu_e}^{1/\nu_o} \frac{dt}{R(t)} = -\int_{o}^{r} \frac{dr}{\sqrt{1-kr^2}} \tag{3-4}$$

hence

$$\int_{-\tau}^{o} \frac{dt}{R(t)} = \int_{-\tau+1/\nu_e}^{1/\nu_o} \frac{dt}{R(t)} \tag{3-5}$$

which implies

$$\int_{o}^{1/\nu_o} \frac{dt}{R(t)} - \int_{-\tau}^{-\tau+1/\nu_e} \frac{dt}{R(t)} = 0 \tag{3-6}$$

Assuming R(t) to be constant over the short time intervals $\frac{1}{\nu_e}$ and $\frac{1}{\nu_o}$ we have

$$\frac{1}{R(-\tau)} \frac{1}{\nu_e} = \frac{1}{R(0)} \frac{1}{\nu_o} \tag{3-7}$$

$$\frac{\nu_o}{\nu_e} = \frac{R(-\tau)}{R(0)}$$

We now define a redshift parameter z as

$$1 + z = \frac{\nu_e}{\nu_o} = \frac{R(0)}{R(-\tau)} \tag{3-8}$$

When $z > 0$, spectral lines are redshifted, indicating an expanding universe, while for $z < 0$ spectral lines would be shifted toward the blue end of the spectrum, thus indicating a contracting universe. On the grounds of our interpretation of Olbers' paradox in Chapter I we will of course expect z to be greater than zero, confirming the expansion of the universe. A shift of spectral lines has indeed been observed.

It is important to note that the observed redshift is truly of cosmological origin. That it is not just due to the Doppler shift of receding objects, but is due to the overall expansion of the universe, is shown by the fact that virtually no blue shifted spectral lines are observed. If the shift in spectral lines were entirely due to the Doppler shift, we would expect as many objects moving away from us as moving towards us, on an average. Since this is not the case, we are confident that the redshift observed in all directions of the sky is of cosmological origin.

Many questions have been raised as to the nature of the universal expansion. It is important to note that only distances between clusters of galaxies and greater distances are subject to the expansion. Only on this gigantic scale of averaging does the notion of isotropy and homogeneity make sense. An atom or the distance between planets does not expand (with c, h, e, m remaining constant). To illustrate the expansion of the universe imagine a rubber balloon with pennies affixed to it: as the balloon is inflated, the pennies increase their separation from one another, but they themselves remain unchanged (Noerdlinger and Petrosian, 1971). It is also clear from this model that the velocity with which the pennies recede from one another should be proportional to the separation between them.

Hubble's Law

That the velocity of a receding galaxy is proportional to the distance, can be shown from the Roberson Walker metric. The velocity of a galaxy is usually expressed in terms of the observable quantity, the redshift. Let us therefore return to (3-8) and express the redshift in terms of the distance

$$D = r\,R(0) = r\,R_o \tag{3-9}$$

between the receding galaxy and the observer. Note that D gives the distance at the time $t = 0$, not at the time of the emission of the light.

The Taylor expansion of $R(t)$ is

$$\begin{aligned} R(-\tau) &= R_o - \tau\dot{R}_o + 1/2\,\tau^2\,\ddot{R}_o + \ldots \\ &= R_o(1 - \tau H_o - 1/2\,\tau^2 H_o^2 q_o) + \ldots \end{aligned} \tag{3-10}$$

where

$$H_o = \frac{\dot{R}_o}{R_o} \quad \text{and} \quad q_o = -\frac{\ddot{R}_o R_o}{\dot{R}_o^2} \tag{3-11}$$

Inserting (3-10) into (3-8), we have

$$1 + z \cong 1 + \tau H_o + \tau^2(1/2\ q_o + 1)H_o^2 \tag{3-12}$$

while (3-3) gives the relation between r and τ:

$$r \cong \int_{-\tau}^{o} \frac{dt}{R_t} \cong \int_{-\tau}^{o} \frac{dt}{R_o - t\dot{R}_o} \cong \frac{1}{R_o}\ (\tau + 1/2\tau^2 H_o) \tag{3-13}$$

where we inserted the Taylor expansion (3-10) for $R_t = R(t)$ performed the integration, and expanded the resulting $\ln(1 + \tau H_o)$ to second order in τ.

Now, setting $rR_o = D$ in (3-13), solving for τ and retaining only terms of 2nd order in D, we insert (3-13) into (3-12) to obtain

$$z \cong H_o D + 1/2\ D^2 H_o^2\ (1 + q_o) \tag{3-14}$$

which is the well-known redshift-distance relation to second order in D. The term linear in D had been proposed by Friedmann in 1922 and was later found by Hubble experimentally. Therefore, the linear part of (3-14) is known as the Hubble Law, and H_o is called the Hubble constant.

The redshift distance relation as derived from the Robertson-Walker metric predicts a deviation from the linearity of Hubble's Law that depends on the value of q_o. This parameter determines whether the universe is accelerating, steadily expanding (in which case Hubble's Law would be exact) or decelerating. Because of the negative sign in the definition (3-11), q_o is called the deceleration parameter. Its exact value has to be determined by specific cosmological theories based on gravitational field equations and can be checked by observations.

The redshift-distance relation (3-14) cannot be verified directly by experiment, because D itself cannot be measured for distant galaxies. Instead, one has to use the apparent luminosity ℓ of the source, which can be determined from the absolute luminosity L via

$$\ell = \frac{L}{4\pi D^2 \, (1+z)^2} \qquad (3\text{-}15)$$

The factor $4\pi D^2$ arises from the fact that the light emitted by the source forms a sphere of radius D at the time of observation, and the factor containing the redshift parameter z appears twice, because the emitted photons are redshifted by $(1+z)^{-1}$ and the time interval between emission and absorption is dilated by $(1+z)$ due to the cosmological expansion of the universe.

For historical reasons, astronomers use a scale for the apparent magnitude m of a source given by

$$m = \text{const} - 2.5 \log \ell \qquad (3\text{-}16)$$

from which

$$m = \text{const} - 2.5 \log L + 5 \log(1+z) + 5 \log D \qquad (3\text{-}17)$$

By squaring (3-14) and multiplying it by a factor $(1 + q_o)$, we can eliminate D from (3-14) and (3-17) to obtain

$$m = \text{const} - 2.5 \log LH_o^2 + 5 \log(1+z) + 5 \log z[1 - 1/2z(1+q_o)] \quad (3\text{-}18)$$

For small z we have the approximations

$$\log(1+z) = \frac{1}{\ln 10} z \tag{3-19}$$

and

$$\log[1 - 1/2\ z(1+q_o)] = -\frac{1}{\ln 10} 1/2\ z(1+q_o) \tag{3-20}$$

and therefore finally

$$m \cong \text{const} - 2.5 \log LH_o^2 + 5 \log z + 1.09\ [1 - q_o]z \quad , \tag{3-21}$$

known as the apparent magnitude - redshift relation. If the absolute luminosity L of a distant galaxy is known, then the constants H_o and q_o can be read off an m - log z diagram, known as the Hubble-diagram.

Cosmological Field Equations

All our results so far we derived without the help of a theory of gravity at all. We saw that the geometry of space-time is described by the Roberson Walker metric, and that the universe is necessarily expanding; both of the results above imply a cosmological redshift which has been confirmed to a high degree of accuracy.

Although we know that the universe is expanding, we do not yet know the quantitative law governing the expansion. It is here that various cosmological theories vary in giving alternative gravitational field equations to describe the exact time dependence of R(t), the expansion factor of the universe (only in the case of the hypersphere can we speak of R(t) as the "radius" fo the universe). As we found above that the Big Bang theory seems to comply best with the present observational data, we shall here adopt the approach of the Friedmann model, also known as the standard cosmological model, because it supplies a cosmological theory relying on simple, but reasonable assumptions.

The standard cosmological model assumes the correctness of Einstein's theory of gravity. The field equations are

$$R_{\mu\nu} - 1/2\, R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{3-22}$$

where $R_{\mu\nu}$ is the Riemann curvature tensor, $T_{\mu\nu}$ the energy momentum tensor, $\kappa = 8\pi G$ with G gravitational constant and Λ is the so-called cosmological constant. Ever since Einstein introduced the Λ-term into his equations, there has been a constant debate whether it should remain there or not. Einstein originally introduced the cosmological constant in order to construct a static model of the universe. Λ then acts as a repulsion force counteracting gravity in such a way that the universe remains in a stable condition. As we shall see later on, Friedmann showed that a static world model is unstable. Consequently, Einstein dropped the cosmological constant again, calling it "the biggest blunder in my life." Still, from the point of logic Λ has a perfect mathematical right to appear in Einstein's equations; other cosmologists like Eddington and Lemaitre even claimed a logical necessity for the cosmological constant, while Einstein considered it to be "gravely detrimental to the formal beauty of the theory" (Einstein, 1922). Whether or not the cosmological constant has to be included can only be decided by experiment, not by theory. For convenience and simplicity, we shall assume $\Lambda = 0$, and only at some points shall we return to a nonzero Λ to state its effect on the equations. Also, a vanishing cosmological constant seems to be slightly favored by observations.

Because of the assumed isotropy, $T_{\mu\nu}$ in (3-22) takes the simple form of the energy momentum tensor of an ideal fluid of average density ρ and average internal pressure p (see e.g. Weinberg, 1972, p. 48)

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p\, g_{\mu\nu} \tag{3-23}$$

where u_μ is the time-like unit vector tangential to the world lines of matter.

Inserting now the Robertson Walker metric (2-32) into Einstein's equations (3-22) is a lengthy but straightforward calculation (see e.g. Sexl and Urbanke, 1975). The resulting equations will be identities except for

$$\kappa\, \rho = 3\, \frac{k + \dot{R}^2}{R^2} - \Lambda \tag{3-24a}$$

$$\kappa\, p = -\, \frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} + \Lambda \tag{3-24b}$$

Let us first show some general consequences of these equations. If we multiply (3-24a) by R^3 and differentiate with respect to time, we obtain

$$\kappa\, (\dot{\rho R^3}) = 3k\dot{R} + 3\dot{R}^3 + 6\ddot{R}\dot{R}R + \Lambda(\dot{R^3}) \tag{3-25}$$

Multiplying (3-24b) by $3R^2\dot{R}$, and adding it to (3-25), we have

$$d(\rho R^3) + p\, d(R^3) = 0 \tag{3-26}$$

Since $M = \rho R^3$ measures the energy content of the element $d(R^3)$, and $p\, d(R^3) = p\, dV$ the work done against the pressure forces, we conclude that the energy balance is preserved under the cosmic evolution. Furthermore, since $d(\rho R^3) = dU$ is the internal energy and the first law of thermodynamics states that

$$dU = TdS - pdV \tag{3-27}$$

The equation (3-26) implies that $dS = 0$, i.e., the entropy is constant under the cosmic evolution. It is here that we can already see that there are no stable static solutions to the field equations. With the assumption $p \cong 0$ for the universe today which will be made plausible later, equations (3-24) read

$$\kappa\rho = \frac{3k}{R^2} - \Lambda \tag{3-24c}$$

$$\Lambda = \frac{k}{R^2} \tag{3-24d}$$

After eliminating $\frac{k}{R^2}$ from these equations, we obtain

$$\kappa\rho = 2\,\Lambda \tag{3-24e}$$

and reinserting into (3-24c), we have after rearranging

$$R^2 = \frac{2k}{\kappa\rho} \tag{3-24f}$$

from which it is clear that only the case $k = 1$ is permissible. It also follows from (3-24f), however, that this closed static Einstein model of the universe is unstable. Since (3-26) implies that $\rho R^3 =$ constant (for $p \cong 0$), a virtual increase in R would lead to a decrease in ρ, independent of the value of the cosmological constant, and therefore to an unstable behavior.

Thus, the cosmological constant does not serve the purpose for which Einstein originally introduced it and we shall henceforth set $\Lambda = 0$. We can therefore restate the equations (3-24a) and (3-24b) as

$$\kappa\rho = 3\,\frac{k + \dot{R}^2}{R^2}$$

$$\kappa p = -\,\frac{2R\ddot{R} + \dot{R}^2 + k}{R^2} \qquad (3\text{-}24)$$

Equation (3-24) yields a very interesting relation between the present density of the universe ρ_o, Hubble's constant H_o and the curvature parameter k. Rearrangement of (3-24a) and the definition of Hubble's constant (3-11) give

$$\rho_o = \frac{3}{\kappa}\,(H_o{}^2 + \frac{k}{R_o{}^2}) \qquad (3\text{-}28)$$

Thus a knowledge of accurate values for ρ_o and H_o would determine the sign of k. For k = 0,

$$\rho_c = \frac{3H_o^2}{\kappa} \qquad (3\text{-}29)$$

and ρ_c is known as the critical density. If the present density ρ_o exceeds ρ_c, then the universe is closed, if it is less than ρ_c, the universe is open.

Similarly, rearranging (3-24b) in terms of H_o, k and q_o with the help of the definitions (3-11), we obtain a relation between the present pressure in the universe p_o and H_o, q_o and k:

$$p_o = - \frac{1}{\kappa} [\frac{k}{R_o^2} + H_o^2 (1 - q_o)] \qquad (3\text{-}30)$$

In the case that $p_o << \rho_o$, we then have an interesting relation between k, H_o and q_o:

$$k = (2q_o - 1)H_o^2 R_o^2 \qquad (3\text{-}31)$$

Eliminating the curvature parameter k from equations (3-28) and (3-31), we obtain with the help of (3-29) the ratio of present to critical density of the universe, which is customarily denoted by Ω_o

$$\Omega_o = \frac{\rho_o}{\rho_c} = 2\, q_o \qquad (3\text{-}32)$$

In order to convert equations (3-24) into one differential equation for R, we need to make some assumptions about the dependence of pressure p and density ρ on R. If we regard the clusters of galaxies as particles of a gas, we can then apply the pressure-density relation of the kinetic theory of gases

$$\frac{p}{\rho} = \frac{\bar{v}^2}{3} \qquad (3\text{-}33)$$

where $\bar{v}$ is the root mean square velocity of the gas particles. The observed random motions of galaxies are in general much less than the velocity of light, i.e. $\bar{v} << 1$, except for the early history of the universe. Thus we assume a pressure density relation

$$p = 0 \quad \text{for the present universe, and} \tag{3-34a}$$

$$p = \frac{\rho}{3} \quad \text{for the early universe.} \tag{3-34b}$$

As a consequence of the conservation law of the energy (3-26), we have

$$\frac{d}{dt}(\rho R^3) + p\frac{d}{dt}(R^3) = 0 \tag{3-26a}$$

For the present universe, p = 0 and from (3-26a)

$$(\rho R^3) = a, \quad a = \text{constant} \tag{3-35}$$

so that ρ in terms of R will be given by

$$\rho = \frac{a}{R^3} \quad \text{for} \quad p = 0 \tag{3-36}$$

In the case of the early universe (3-34b), (3-26a) becomes

$$\frac{d}{dt}(\rho R^3) + \frac{\rho}{3}\frac{d}{dt}(R^3) = 0 \tag{3-37}$$

which admits the following solution for ρ in terms of R

$$\rho = \frac{b}{R^4} \quad b = \text{const.} \tag{3-38}$$

With these expressions for ρ, equation (3-24a) reads

$$\dot{R}^2 = \frac{\kappa a}{3R} - k \quad \text{for } p = 0 \tag{3-39a}$$

$$\dot{R}^2 = \frac{\kappa b}{3R^2} - k \quad \text{for } p = \frac{\rho}{3} \tag{3-39b}$$

Assuming the interaction between radiation and matter to be negligible we can combine equations (3-39a) and (3-39b) to get the so-called Friedmann Equation

$$\dot{R}^2 = \frac{\kappa b}{3R^2} + \frac{\kappa a}{3R} - k \tag{3-40}$$

The universe is called radiation dominated, if the first term gives the main contribution to the Friedmann equation, matter dominated, if the second term prevails, and curvature dominated, if k is greater than the other two terms. The Friedmann equation therefore gives a differential equation for the evolution of the expansion factor R, and thus describes the behavior of the expanding universe.

Our task now is to find solutions to the Friedmann equation. For small R the universe was radiation dominated and the Friedmann equation can be written

$$\dot{R}^2 = \frac{\kappa b}{3R^2} \tag{3-41}$$

with the solution

$$R = [\frac{4b\kappa}{3}]^{1/4} \ t^{1/2} \tag{3-42}$$

For large R (present era) we can neglect the first term on the right hand side of Friedmann's equation to obtain

$$\dot{R}^2 = \frac{a\kappa}{3R} - k \tag{3-43}$$

Obviously, solutions to (3-43) will depend on the value of the curvature parameter k.

In the case k = 0 we have the flat (Einstein - DeSitter) universe with the solution

$$R = [\frac{3a\kappa}{4}]^{1/3} t^{2/3} \tag{3-44}$$

Thus, the flat infinite universe will expand forever. We can determine the present age of the universe in terms of the Hubble constant by noting that the Hubble constant has the dimension of inverse time. Denoting the present time by t_o, we can calculate Hubble's constant with the help of (3-44)

$$H_o = \frac{\dot{R}_o}{R_o} = \frac{2}{3} t_o^{-1} \tag{3-45}$$

and the age of the universe is

$$t_o = \frac{2}{3} H_o^{-1} \tag{3-46}$$

The physical interpretation is shown in Figure 1. By obtaining expressions similar to (3-46) for the cases k = 1 and k = -1 and given an exact value of Hubble's constant, we can then compare the results for the age of the universe with values obtained by experimental methods.

Similarly, we can calculate q_o from (3-11) and (3-44) for the case k = 0

$$q_o = - \frac{R_o \ddot{R}_o}{\dot{R}_o^2} = 1/2 \tag{3-47}$$

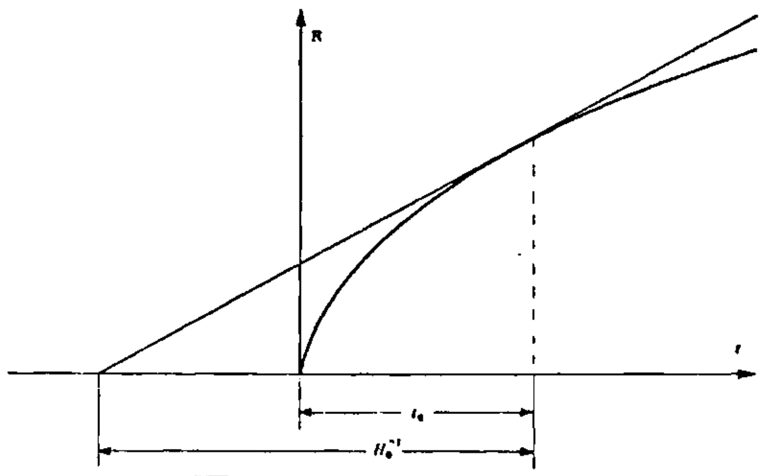


Figure 1. Physical Interpretation of the Hubble Constant H_o.

As above, we can then obtain q_o in a similar manner for the cases $k = 1$ and $k = -1$, and compare with observational data.

In the case of a closed universe ($k = +1$), the Friedmann equation for large R is

$$\dot{R}^2 = \frac{a\kappa}{3R} - 1 \tag{3-48}$$

with the solution

$$t = \frac{R_m}{2} \arccos[1 - 2\frac{R}{R_m}] - (R_m R - R^2)^{1/2} \tag{3-49}$$

where

$$R_m = \frac{a\kappa}{3} \tag{3-50}$$

is the maximum radius of the universe, as can be seen from (3-48) by setting $\dot{R} = 0$ and observing that $\ddot{R} < 0$ (see (3-24)). This maximum radius will be reached at a time $t_m = 1/2\,\pi R_m$, while $R = 0$ gives the correct answer $t = 0$ at the origin, and $t = 2t_m$ is the end of the universe. Thus (3-48) is the equation of a universe that evolves from an initial singularity to a maximum radius and then recontracts. Theoretically, the universe would start expanding again for $t > 2t_m$, but physically we cannot extrapolate beyond that point, as we cannot say what the universe was like for $t < 0$.

Note the curious feature that, since $a = \rho R^3$ = constant is proportional to the mass of the universe, the radius of the universe R with maximum

$$R_m \cong \frac{M\kappa}{3} \tag{3-51}$$

will always be smaller than the corresponding Schwarzschild radius

$$R_s = 2M\kappa \tag{3-52}$$

This result seems to be of purely formal value, however, and to say that nothing can leave the universe because it is a black hole appears to be a misconception of the intrinsic properties of expanding four-dimensional space, because there is no "outside" of the universe.

If we rewrite (3-49) by using the transformed metric (2-32a) we obtain

$$\begin{aligned} A(\eta) &= 1/2\ R_m(1 - \cos\eta) \\ t &= 1/2\ R_m(\eta - \sin\eta) \end{aligned} \tag{3-53}$$

which is the well known parametric representation of the cycloid.

In the case of the open universe k = -1, the Friedmann equation (3-43) becomes

$$\dot{R}^2 = \frac{a\kappa}{3R} + 1 \tag{3-54}$$

with the solution

$$t = [R(R_m + R)]^{1/2} - 1/2\ R_m \operatorname{arcosh}(1 + \frac{2R}{R_m}) \tag{3-55}$$

From (3-54) it is clear that there is no maximum R for $\dot{R} = 0$, and (3-55) shows that R = 0 gives t = 0. Thus, the open universe evolves from an initial singularity and keeps on expanding forever. In the transformed metric (2-32a), the solution takes the parametric form

$$A(\eta) = 1/2\ R_m(\cosh\eta - 1)$$
$$t = 1/2\ R_m(\sinh\eta - \eta) \tag{3-56}$$

Figure 2 shows a comparison of the various models of the universe. We have already computed the age of the universe in the case $k = 0$: $T_2 = 2/3\ H_o$. As the models $k = \pm 1$ do not yield explicit solutions for R, T_1 and T_2 are not that easily derived. It is clear, however, that $T_2 > 2/3\ H_o > T_1$. Also for $k = 0$, we already computed the deceleration parameter q_o. We can now obtain a more general expression, that includes all three possible values of k. Multiplying (3-24a) by 1/2, (3-24b) by 3/2 and adding these two equations, we obtain

$$1/2\ \kappa\rho\ R = -3\ddot{R} + \Lambda R \tag{3-57}$$

in the matter dominated case for which $p = 0$. Equation (3-57) shows that $\ddot{R} < 0$ for $\Lambda = 0$ regardless of k, thus the models of the Friedmann universe are decelerated, and necessarily $q_o > 0$. From (3-57) and the matter dominated Friedmann equation (3-43) we obtain

$$q_o = -\frac{R_o \ddot{R}_o}{\dot{R}_o^2} = \frac{\kappa\rho\ R_o^2}{6\left(\frac{a\kappa}{3R_o} - k\right)} \tag{3-58}$$
$$= \frac{1}{2}\left(1 - \frac{3kR_o}{a\kappa}\right)^{-1}$$

since $a = \rho R_o^3 = \text{const.}$ Hence

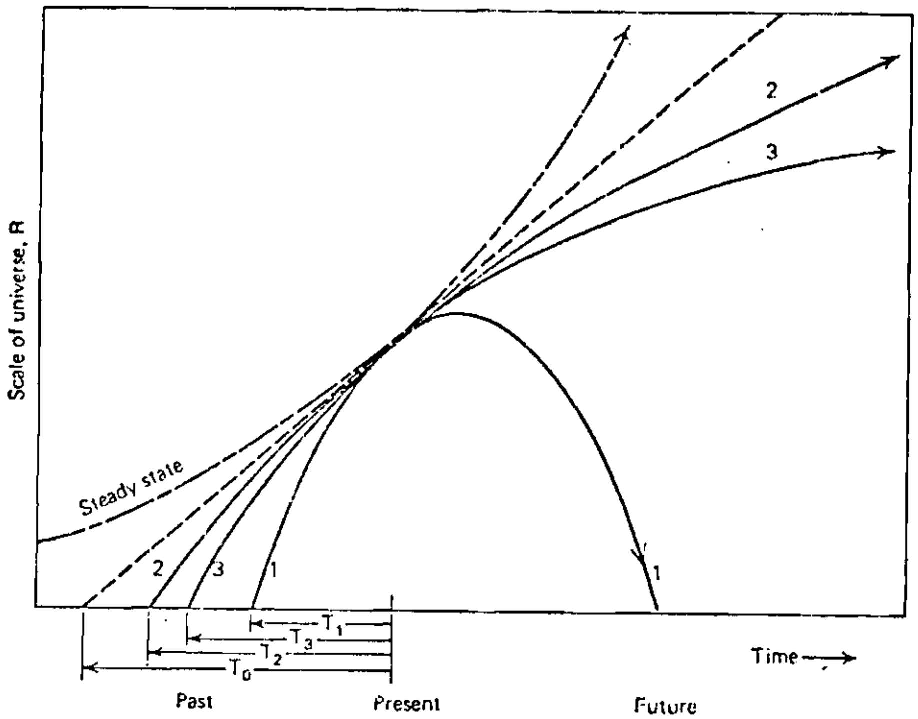


Figure 2. A plot of R(t), the scale of the universe, against time for various cosmological models. Curve 1 represents the class of solutions for closed universes, curve 2 represents the class for open universes, and curve 3 is the critical solution for the boundary between open and closed universes. The dashed-dotted line is the solution for the steady-state model. The light dashed line is the case for an empty universe.

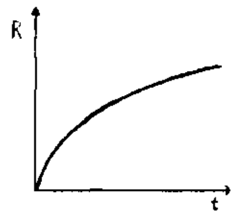
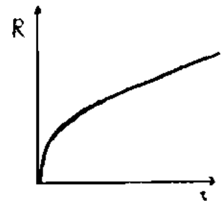
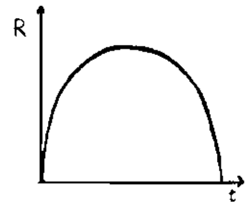
$$q_o \begin{cases} > 1/2 & \text{for } k = +1 \\ = 1/2 & \text{for } k = 0 \\ < 1/2 & \text{for } k = -1 \end{cases} \qquad (3\text{-}59)$$

In summary, Table 1 shows the predicted properties of the three possible solutions to the Friedmann model with $\Lambda = 0$. We did not include the Friedmann models with $\Lambda \neq 0$ in our discussion since we feel that a nonzero Λ serves only to complicate the equations without need. However, since it has not yet been proven nor disproven that $\Lambda = 0$, we include a table (Table 2) that cites the results for nonzero Λ. Λ_c is the value of the cosmological constant that leads to a static but unstable solution of the field equations; there are, however, two expanding solutions for $\Lambda = \Lambda_c$ beside the static one (Figure 6). Observe that for $0 < \Lambda < \Lambda_c$ there exists a solution of expanding and recontracting universes without an initial singularity, passionately advocated by Sir Arthur Eddington because in these models there is no need for a "Big Bang" with all its problems and difficulties (Figure 5).

Also of interest is the so-called Lemaître universe, where $\Lambda = \Lambda_c(1 + \varepsilon)$ and $\varepsilon << 1$. In this model there is a long time during which $R(t)$ is almost constant, and the time scale of the expansion is greatly stretched (Figure 7).

If we continue to assume the cosmological constant to be zero, the conclusion from this chapter then is that we need to determine three parameters experimentally in order to decide whether the universe is open and expanding, flat and expanding or closed and recontracting.

Table 1. Friedmann Models with $\Lambda = 0$

	Open	Critical	Closed
R(t)			
Future	Expand forever	Expand forever	Collapse
Curvature	$k < 0$ Hyperbolic	$k = 0$ Flat	$k > 0$ Spherical
Deceleration	$0 < q_o < 1/2$	$q_o = 1/2$	$q_o > 1/2$
Density parameter $\Omega_o = \rho_o/\rho_c$	$0 < \Omega_o < 1$	$\Omega_o = 1$	$\Omega_o > 1$
Age t_o	$1 > H_o t_o > 2/3$	$H_o t_o = 2/3$	$H_o t_o < 2/3$

Friedman models of the Universe. These are the three familiar cases that arise when no "cosmological-constant" term is included - the open Universe that expands forever, the closed Universe that collapses upon itself in a finite time, and the critical case between these two. R(t) is a scale factor of the Universe (a function of time t), and k, the curvature parameter, can take values -1, 0 and +1 as illustrated. Also q_o is the deceleration parameter Ω_o a dimensionless density parameter, and $H_o t_o$ is a dimensionless age parameter.

Table 2. Models with $\Lambda \neq 0$

	k = +1		k = 0	k = −1
$\Lambda < 0$	Figure 3		Figure 3	Figure 3
$\Lambda = 0$	Figure 3		Figure 4	Figure 4
$\Lambda > 0$	$0 < \Lambda < \Lambda_c$	Fig. 5	Figure 4	Figure 4
	$\Lambda > \Lambda_c$	Fig. 4		
$\Lambda = \Lambda_c$	Figure 6		-	-
$\Lambda = \Lambda_c(1+\varepsilon)$	Figure 7		-	-

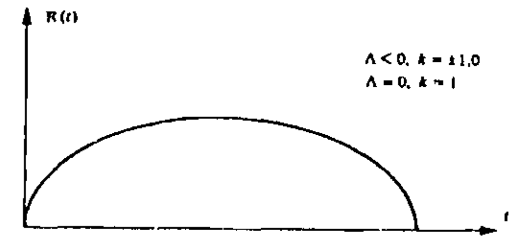


Figure 3. The Closed Universe.

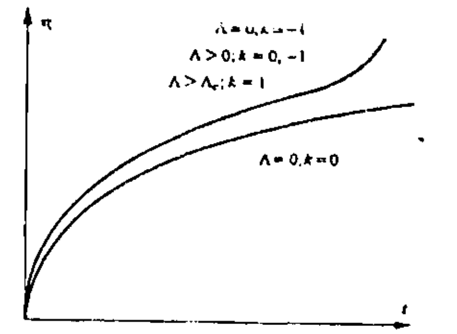


Figure 4. The Open Universe.

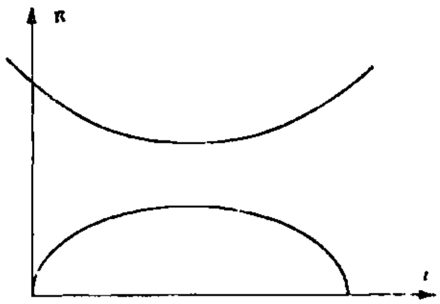


Figure 5. The Universe with $0 < \Lambda < \Lambda_c$.

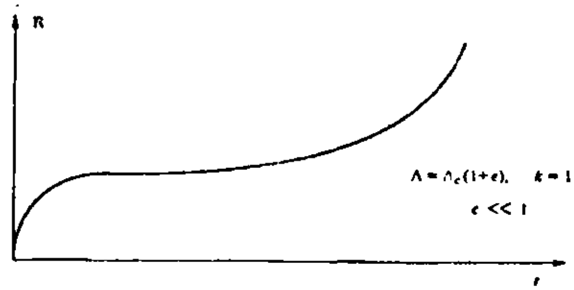


Figure 7. The Lemaître Universe.

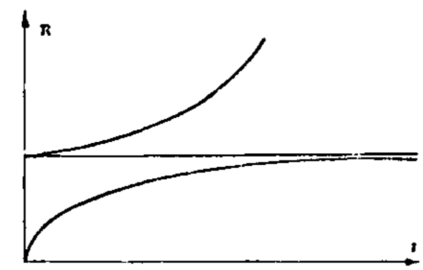


Figure 6. The Einstein Universe.

These parameters are the Hubble constant H_o, the deceleration parameter q_o and the mean mass density of the universe ρ_o. All other parameters like the expansion coefficient $R(t)$, and the curvature parameter k are not directly accessible to observation and have to be calculated in terms of the observable quantities H_o, q_o and ρ_o.

CHAPTER IV

OBSERVATIONS

Homogeneity

We have already discussed some observational results in Chapter I, mainly Olbers' paradox and the cosmic microwave background radiation (cmb). From these we concluded that we can assume that the universe is isotropic and expanding. These results served the purpose to rule out cosmological models in conflict with these observations and to concentrate on the Friedmann models which are firmly based on the assumption of isotropy and expansion.

On deriving the Robertson Walker metric for an isotropic Riemann space in Chapter II, we found that isotropy implies homogeneity. If we now test for inhomogeneities in the universe and find that the universe is not homogeneous, it then necessarily follows that we have to discard the assumption of isotropy. This would not only force us to look for a new explanation of the cmb, but it would mean that we have to abandon the Friedmann models of the universe altogether.

Clearly, the universe is not homogeneous on a small scale like the galaxy or the local cluster of galaxies. The condition of homogeneity will only be met on a sufficiently large scale when irregularities like galaxies and clusters of galaxies are smoothed over. This suggests an averaging length greater than 10 Mpc, the distance between clusters of galaxies.

One can test for homogeneity by counting galaxies as a function of limiting magnitude. This was done by Hubble in 1929 through 1936 (Hubble, 1934/36), and he did not encounter a clustering effect on the characteristic scale of 1000 Mpc with density contrast greater than a factor of two or so. Hubble even claimed to have found a systematic discrepancy between observed counts and the counts expected on a naive homogeneous static Euclidean model. This latter result, however, is now believed to have arisen from a systematic measuring error.

Some authors have claimed a tendency of clusters to form clusters themselves, termed superclusters, on scales up to 3000 Mpc (Abell, 1965; de Vaucouleurs 1970). While these observations are still debated, Seldner et al. suggested recently (Seldner et al., 1977) that galaxies are arranged in a hierarchy of clusters. Galaxies form clusters which form superclusters that in turn form clusters and so on. While hierarchical clustering has been proposed as long ago as 1761 by the mathematician Lambert, and again by Charlier in this century, Seldner et al. report that they found an upper limit to the clustering effect when the size of the cluster is about 60 million light years; beyond that range clustering is on the average comparatively weak.

While further observations have to be awaited to decide the validity of the conclusions made by Seldner et al., there is one major measurement that is in direct conflict with their results. The average mass of galaxies has a strong influence on the development of clustering, and the observed masses of galaxies are by a factor of 30 or so below the value required for the hierarchical clustering Seldner et al. claim to have found. We shall encounter this problem of "missing mass" later on. But while there we shall find a way to resolve the conflict, here it remains an open question. Hence for the time being we shall assume that galaxies are grouped in clusters. Whether or not clusters themselves tend to form superclusters,* remains uncertain, but even then the assumption of homogeneity would remain reasonable, if only on a larger scale.

The Hubble Constant

In 1929, Hubble discovered a linear relation between the distance and the velocity of a galaxy

$$v = H_o d \tag{4-1}$$

which agrees to first order with the result derived from the Robertson Walker metric in Chapter III. From nearby sources, Hubble (1936) then estimated the constant of proportionality H_o to be

$$H_o = 500 \text{ km sec}^{-1} \text{ Mpc}^{-1} \tag{4-2}$$

However, as was first shown by Baade (1952), major systematic corrections became necessary to the distance scale used by Hubble. It was then estimated that

$$H_o = 55 \pm 7 \text{ km sec}^{-1} \text{ Mpc}^{-1} \tag{4-3}$$

(Sandage and Tammann, 1971; Sandage, 1972), a value that became widely accepted.

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*See Darius (1977), Murray, et al. (1978).

Recently, Lynden-Bell (1977) determined the Hubble constant from super-luminal radiosources using a method that does not rely on the luminosity distance scale employed by Hubble and the other authors. He then came to the conclusion that

$$H_o = 110 \pm 10 \text{ km sec}^{-1}\text{Mpc}^{-1} \qquad (4\text{-}4)$$

Considering these uncertainties, we shall adopt a value

$$H_o = 100h \text{ km}^{-1}\text{sec}^{-1}\text{Mpc}^{-1} \qquad (4\text{-}5)$$

with

$$1/2 < h < 5/4 \qquad (4\text{-}6)$$

Several other estimates for H_o given by various authors lie within the range given by (4-5) (deVaucouleurs (1977); Tully and Fisher (1977)).

The Deceleration Parameter

Magnitude - Redshift Test

As an immediate consequence from the fact that the universe is described by the Robertson Walker metric we derived the magnitude redshift relation. This relation is a nonlinear extension of the Hubble law and permits to determine the deceleration parameter q_o through the relation

$$m = \text{const} - 2.5 \log LH_o^2 + 5\log z + 1.09(1 - q_o)z \qquad (3\text{-}21)$$

On deriving this formula we had made the problematic assumption that all sources have the same intrinsic luminosity. Sandage and Hardy (1973) have argued that the brightest galaxy in a cluster of galaxies may be used as a standard source. With this assumption Sandage has derived a value for q_o of

$$q_o = .96 \pm .4 \tag{4-7}$$

from a sample of galaxies reaching out to a redshift of z = .46 (see Figure 8).

There are, however, many possible sources of systematic error. The main source of a systematic error is the evolution of galaxies. Stellar evolution in elliptical galaxies is currently estimated to make their luminosity grow fainter at a rate

$$\frac{\partial \log L}{\partial \log t} = -1 \tag{4-8}$$

(Tinsley, 1975). Furthermore, we need to introduce two constants A and .65 into (3-21) in order to correct for the type of apparent magnitude used and to include the aperture correction (Gunn and Oke, 1975). Thus, the magnitude redshift relation can be rewritten as

$$m = \text{const} - 2.5 \log LH_o^2 + 5 \log z - 1.09(A + E + .65q_o)z \tag{4-9}$$

where

$$E = -\frac{1}{H_o t_o} \left(\frac{d \log L}{d \log t}\right)_o \tag{4-10}$$

With these corrections Gunn and Oke (1975) obtained a negative

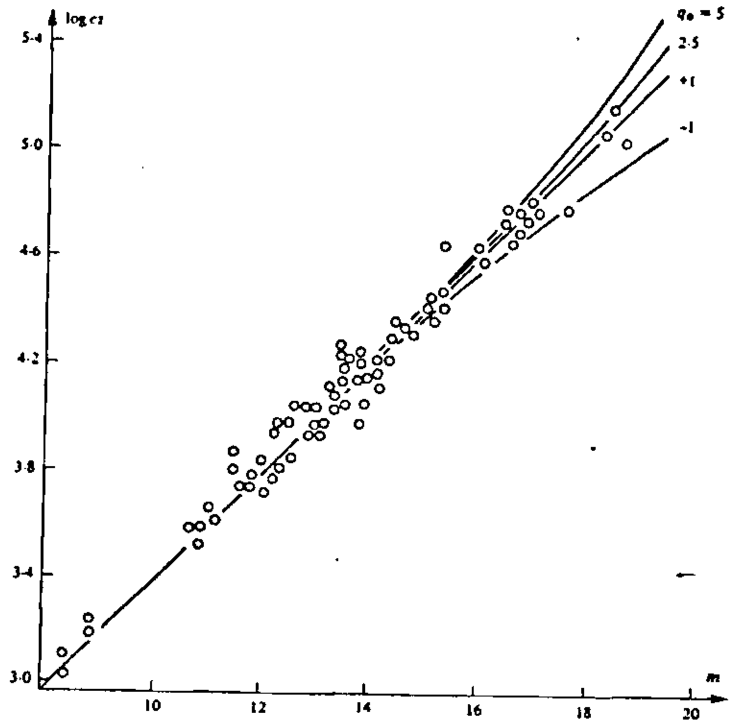

Figure 8. Hubble Diagram (after Sandage, 1972).

value for the deceleration parameter. This result then would imply that the universe is accelerating, a result which is in direct conflict with the standard Friedmann model where we expect the mass of the universe to slow down the expansion ($q_o > 0$, see (3-57)). This is why Gunn and Tinsley (1975) suggested a Friedmann model with positive cosmological constant Λ that would act as an additional repulsive force and thus lead to an accelerating universe. That in this model q_o will take negative values for a certain choice of Λ can be seen by using (3-57) with $\Lambda \neq 0$. We then have

$$q_o = -\frac{R_o \ddot{R}_o}{\dot{R}_o^2} = -\frac{\ddot{R}_o}{H_o R_o} \tag{4-11}$$

$$= \sigma_o - \frac{\Lambda}{3H_o^2}$$

where

$$\sigma_o = \frac{\kappa \rho_o}{6H_o^2} \tag{4-12}$$

is often introduced as a new dimensionless density parameter.

The revival of the cosmological constant, however, created some problems with density and age constraints (see Gunn and Tinsley, 1975), and an alternative explanation for the apparent negative value of q_o was therefore gladly accepted. Ostriker and Tremaine (1975) noted that clusters of galaxies can grow by accretion of smaller galaxies. Thus the luminosity evolution is complicated considerably, and the "swallowing effect" counteracts the dimming due to stellar evolution.

But the details of the dynamics of accretion cannot be computed yet due to a lack of data concerning the clusters and the "victim" - galaxies (Gunn and Tinsley, 1976). Also Chitre and Narlikar (1976) argued that intergalactac dust could affect the measurement of q_o in such a way that the estimate for q_o need to be revised upwards. A small positive value for q_o seems to be rather weakly favored. The uncertainties of the situation at present were recently summarized by Kristian et al. (1978):

"The use of the Hubble diagramm in cosmology now depends on a knowledge of brightness changes in galaxies, on the one hand, or of q_o from other evidence, on the other. For example, if it were known with certainty that there has been no significant change in elliptical galaxy luminosities during the last 4×10^9 years, then the present data are nearly good enough for one to say definitely that the universe is closed and finite. At the other extreme, if it were known with certainty from other evidence that the universe was nearly empty ($q_o \cong 0$), then the present data set the constraint that net galaxy luminosities have decreased by ~.5 mag during the last 5×10^9 years. It seems possible at present to construct a self consistent model with $q_o \cong 0$ that satisfies the known data, but the case is not yet settled."

The Angular Diameter Redshift Test

The second classical test of q_o is to measure the angular diameter of a source as a function of its redshift. A galaxy of constant proper size D was closer to us at a time t_e when it emitted photons which we receive at time t_o. Therefore it occupies a larger apparent angular extent than it would in a static universe.

The advantage of this test is that to employ it, it is not necessary to fully understand the luminosity evolution of galaxies. The test can be used on any distant source that has a well defined edge. Unfortunately, galaxies have kinks in their luminosity profiles only at very small radii. Also, the proper diameter of galaxies is not easy to define, hence this test has received less attention from optical astronomers than the magnitude redshift test. However, since there are virtually no evolutionary effects to be accounted for that complicate the magnitude redshift test, the angular diameter test will probably become much more important in the future.

Baum (1972) has devised a way to determine the angular diameters of photographic images of galaxies. His results (see Figure 9) indicate

$$q_o \cong 0.3 \pm 0.2 \tag{4-13}$$

in agreement with a low value for q_o as proposed above. Recently, however, Hickson (1977) employed 95 clusters of galaxies with redshifts ranging from 0.02 to 0.46 yielding a value of the deceleration parameter

$$q_o \cong -0.8 \pm 0.2 \tag{4-13a}$$

which is embarassingly low. Whether this is due to some yet unknown effects or whether we have to reintroduce the cosmological constant, remains to be seen. Further observational evidence is needed. In the light of the previous evidence, however, and considerations later on,

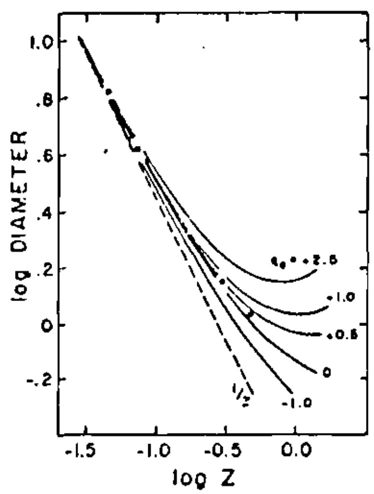

Figure 9. The Angular Diameter-redshift Relation. Theoretical curves for the standard model are shown for several values of q_o. The observations are from Baum (1972).

a negative value of q_o seems to be unlikely; but at the present stage of observational cosmology, (4-13) cannot that easily be dismissed.

General Relativity

In Chapter III we introduced Einstein's theory of gravity in order to derive the cosmological field equations for the Friedmann model. We have yet to verify this choice of Einstein's theory over various other theories of gravity. We already mentioned Dirac's theory with changing constant of gravity; furthermore there is a theory proposed by Brans and Dicke (1961), where a scalar component is added to the tensor formalism. As Thorne et al. (1971) have shown, all metric theories of gravity may be classified using nine parameters. By considering observational limits on each of the parameters they conclude that only a very restricted set of gravity theories including the Brans Dicke theory and, of course, General Relativity meet the observational tests.

Most of the data to test General Relativity are locally derived, either from laboratory experiments or observations of the earth and the solar system. The test with the longest history is the deflection of light by the sun. Using modern interferometric methods,the close agreement between the radio results and the value predicted by Einstein increases our confidence in General Relativity. It also sets an upper limit of 2% on the present value of the scalar component of the Brans Dicke theory, negligibly small from the point of view of cosmology.

We already mentioned a strong argument against Dirac's theory in Chapter I. His theory of gravity cannot be classified among the metric theories because he did not specify what field theory should replace Einstein's field theory. He predicted the rate of change of the gravitational "constant" to be

$$\left(\frac{\dot{G}}{G}\right)_o = -3H_o \cong -2.5 \times 10^{-10}\ \mathrm{yr}^{-1} \tag{4-14}$$

However, van Flandern (1975) and Dearborn and Schramm (1974) set an upper limit of about

$$\left(\frac{\dot{G}}{G}\right)_o = -10^{-11}\ \mathrm{yr}^{-1} \tag{4-15}$$

This plus the fact that in Dirac's theory $q_o = 2$ and the aforementioned argument of the conservation of the black body type spectrum leads us to assume that General Relativity is very likely to be the correct theory.

Furthermore let us note that the formula (3-24a) we used for the expansion rate is the 00-component of Einstein's field equations (doubled for convenience)

$$\frac{6\dot{R}^2}{R^2} + \frac{6k}{R^2} = 2\kappa\rho \tag{4-16}$$

After rearranging this reads

$$\left(\frac{dR}{dt}\right)^2 - \frac{1/3\ a\kappa}{R} = -\ k \tag{4-17}$$

where $a = \rho R^3$ denotes constant mass, see (3-35). As Gamov emphasized this is just to say that the kinetic energy in expansion almost exactly balances the gravitational potential energy which seems so simply as to be convincing. The only difference to Newton's classical notions is that the total energy is not an arbitrary constant any more, but is fixed to be

$$E_{tot} = -k \tag{4-18}$$

The Mean Mass Density of the Universe

The most direct way of determining the curvature of the universe is to somehow measure the average mass density and to compare it with the critical density, see eq. (3-28). Furthermore, the ratio of average to critical density would allow us to determine the deceleration parameter q_o, see eq. (3-32).

Since galaxies are the most conspicuous objects in the universe, let us first try to determine their contribution to the mass density. There are three methods to accomplish this task:

1. a dynamical analysis of the rotational velocities as function of distance from the galactic centers (for galaxies within about 15 Mpc)
2. the virial theorem

$$M = \frac{2\langle v^2\rangle}{G\langle d^{-1}\rangle} \tag{4-19}$$

   where $\langle v^2\rangle$ is the mean square velocity relative to the center of mass, $\langle d^{-1}\rangle$ the mean reciprocal separation between stars (for elliptical galaxies)
3. statistical analysis of relative velocities and separation for pairs of galaxies.

In all three methods the galactic mass is given by a formula of the form

$$M \alpha \frac{v^2 D}{G} \qquad (4\text{-}20)$$

where v is some characteristic internal velocity and D a characteristic dimension which is determined from the corresponding angular diameter δ and the redshift z by rewriting Hubble's law (small z)

$$D = \frac{z\delta}{H_o} \qquad (4\text{-}21)$$

The internal velocities are determined from the distribution in redshift around the average value z for the galaxy. The masses found in this way are described in terms of the mass to luminosity ratio $\frac{M}{L}$, where L is the absolute luminosity that was given in Chapter III by eq. (3-15) (with eq. (3-14) for small z)

$$L = 4\pi \ell z^2 H_o^{-2} \qquad (4\text{-}22)$$

From (4-20), (4-21) and (4-22) it follows that the mass to light ratio $\frac{M}{L}$ is proportional to the Hubble constant.

An overall mean $\frac{M}{L}$ ratio for all types of galaxies was estimated by Gott and Turner (1977) to be

$$\frac{M}{L} \cong 180\ h \qquad (4\text{-}23)$$

in units of the solar mass to light ratio $\frac{M_\odot}{L_\odot}$. Recently, Davis et al. (1978) argued for even higher values of the order of

$$\frac{M}{L} \cong 1000 - 1500 \text{ h} \qquad (4\text{-}24)$$

They maintain that the material responsible for this high value need not be associated with individual galaxies and refer to the hot gas recently observed in possible superclusters (Murray et al., 1978). Davis et al. also give an estimate of the luminosity density

$$\mathcal{L} = 1.2 \times 10^8 \text{ h } L_\odot \text{Mpc}^{-3} , \qquad (4\text{-}25)$$

needed to determine the mass density.

We can now compute the density parameter Ω by

$$\Omega = \frac{\rho}{\rho_c} = \mathcal{L} \left(\frac{M}{L}\right)\left(\frac{3H_o^2}{\kappa}\right)^{-1} , \qquad (4\text{-}26)$$

$$= 0.45 \pm 0.25$$

according to Davis et al. Note that this result is independent of the true value of H_o, as the uncertainty factor h cancels. Estimates by Gott and Turner (1976, 1977) as well as by Sargent and Turner (1977) that do not include hot intergalactic gas are considerably lower

$$\Omega \approx 0.1 \qquad (4\text{-}27)$$

so that (4-26) seems to give an upper bound. While these estimates are subject to uncertainties in estimating the mass to light ratios, the data seem to exclude $\Omega \geq 1$.

Let us recall that according to the standard cosmological model

$$\Omega = 2q_o \qquad (3\text{-}32)$$

so that the above estimates suggest for the deceleration parameter

$$0 < q_o < .35 \qquad (4\text{-}28)$$

which is in good agreement with the result for q_o given by Baum (1972) and the remark by Kristian et al. (1978) quoted above that a self consistent model with a value for q_o close to zero seems to be possible.

About a decade ago, estimates for Ω were even lower than the one given in (4-27), while it was generally agreed from magnitude redshift tests that $q_o \cong 1$, where luminosity evolution was not taken into account. Therefore, there was an embarassing disagreement between two observations. In order to remedy the situation, it was argued that the mass of the universe is not given by the mass of galaxies alone, but that there is "hidden mass" within galaxies and/or intergalactic space that would increase the value of Ω by two orders of magnitude. It cannot be precluded that large amounts of gaseous matter fill the universe, although one has to imagine quite exotic ways to hide the missing mass. It cannot reside in any electromagnetic background radiation, because most of the electromagnetic energy is stored in the cmb with an equivalent mass density so low as to be negligible, and the same is true for possible photon-, neutrino- and graviton radiation generated in the early universe. The possibility of a heavy population of black holes in the universe, or of neutral hydrogen gas filling all space cannot be ruled out, but will be difficult to detect. Even the intergalactic gas discovered by Murray et al. (1978) and accounted for by Davis et al. (1978) did not raise the value of Ω to the desired value to give a closed universe.

Thus, with the recently obtained correction to the value of q_o it seems to be more likely, although by no means settled, that the universe is open and expanding.

Cosmological Nucleosynthesis

It has long been thought that the only mechanism which accounts for the formation of elements has to be sought within stars. While this is true for the formation of heavier elements the remarkably uniform abundance of helium suggests that cosmological processes are involved in the formation of light elements. It has indeed been found that stellar nucleosynthesis produces, but also destroys deuterium to produce heavier elements so that the observed high amount of interstellar deuterium requires an environment of high energy, and to preserve it requires an enviromment of low density. The early stages of the Big Bang model can provide an environment that meets both of the above conditions.

In the primeval fireball there existed temperatures in excess of 10^{10} K. The "sea" of neutrons and protons was held in approximate equilibrium by the weak interaction

$$\begin{aligned} n &\longrightarrow p + e^- + \bar{\nu}_e \\ e^+ + n &\rightleftharpoons p + \bar{\nu}_e \\ \nu_e + n &\rightleftharpoons p + e^- \end{aligned} \qquad (4\text{-}29)$$

The free neutrons allow the reaction

$$n + p \rightleftharpoons D + \gamma \qquad (4\text{-}30)$$

to take place which is much faster than the proton-proton reaction in the sum

$$p + p \longrightarrow D + e^{+} + \nu_e \qquad (4\text{-}31)$$

because it does not involve weak interaction and there is no Coulomb barrier to overcome. At 10^9K and lower temperatures, the reaction (4-30) is favored toward the right and most of the free neutrons are absorbed in the formation of deuterium, which then in turn allows the formation of heavier elements, notably helium.

Most of the deuterium and He^3 burn to form He^4, and the fraction remaining depends inversely on the baryon density. Figure 10 shows the expected abundance as a function of the density. If we assume that the observed interstellar abundance of deuterium is equal to the primordial abundance allowing only for consumption of deuterium in stars, we arrive at a density ratio

$$\frac{\rho_o}{\rho_c} \cong 0.1 \qquad (4\text{-}32)$$

(Rogerson and York, 1973). This value lies within the range indicated above by measurements for q_o and Ω.

Figure 10 shows that the amount of helium depends only very weakly on the density parameter Ω. The helium abundance is restricted to the range of 20 - 30% by mass, which is in good agreement with the presently observed abundance of about 27% (D'Odorico et al., 1976).

In summary, observational evidence is not yet conclusive to yield positive proof for one world model, but recent data seem to suggest that the case of an open expanding universe is slightly favored.

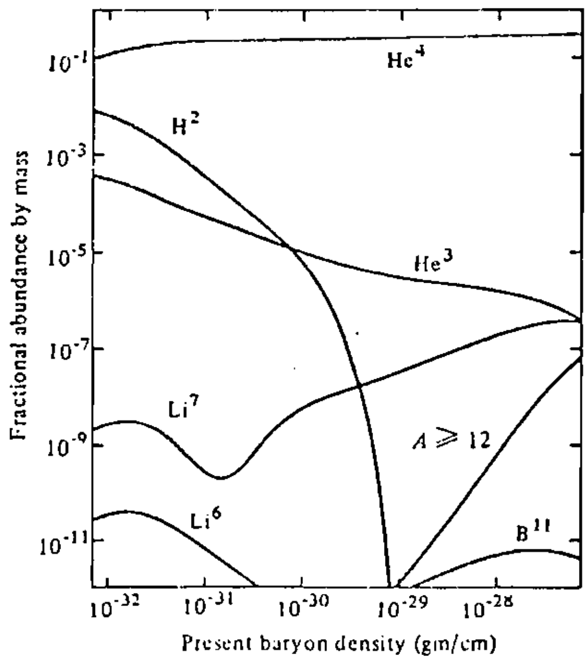

Figure 10. The Dependence of the Final Cosmic Mass Fractions of Various Nuclides on the Postulated Value of the Present Baryon Density, Based on the Present Photon Temperature of 2.7K,is Shown Here for Standard Big-Bang Models. (From Schramm and Wagoner, 1974).

CHAPTER V

SOME PROBLEMS IN COSMOLOGY

In the foregoing chapter we have tried to show that observations seem to indicate that the standard cosmological model is likely to give an adequate description of the evolution of the universe. Whether or not it really is the correct model, is still uncertain; but no observations flatly contradict the Friedmann models, and we pointed out several arguments that seem to rule out competing models of the universe. Which of the three possible cases of Friedmann models, the flat, the expanding or the recontracting universe, is the correct one, is also not yet decided; recent observations, however, seem to favor the case of an open, expanding universe.

Assuming that the standard model is indeed the correct one, there remain many problems and open questions concerning some of its details. First of all, there is no secure argument as to why the cosmological constant Λ should be zero. Clearly, beauty and simplicity of the theory cannot be accepted as the only arguments in favor of a vanishing cosmological constant. On the other hand, the fact that observations are consistent with this model increases our confidence that there is no need for a nonzero Λ. There is only one observation (Hickson, 1978) that would favor a small positive value of the cosmological constant, a result that is in disagreement with all other recent measurements.

The main problem with the Friedmann universe is clearly the assumption of ideal isotropy and the entailing ideal homogeneity. It is obvious that the universe is neither isotropic nor homogeneous on the local scale. The remarkable degree of isotropy of the cosmic microwave background radiation (cmb), on the other hand, compels us to conclude that the universe was indeed highly isotropic at the time when the cmb was last scattered, i.e., at the end of the radiation dominated period of the universe shortly after the Big Bang. Various attempts have been made, therefore, to resolve the apparent contradiction by introducing small density perturbations at an early epoch that would lead to an anisotropy in the cmb below the level detectable at present, and that would bring about inhomogeneities as seen in the universe today. Collins and Hawking (1973) have shown, however, that any such perturbations would either die away with time or grow very large. Also, it is by no means clear what the nature of these density perturbations should be, as the process of galaxy formation that is mainly responsible for inhomogeneities is not yet understood.

These considerations inevitably bring us to the outstanding feature of the Friedmann models: the initial singularity or Big Bang. It is not clear at all how one should deal with a state of infinite density and temperature, and zero radius. It was long hoped that the singularity would disappear or at least become "less singular" once the high symmetry of the model was relaxed. However, singularity theorems (see Hawking and Ellis (1973)) showed that symmetry was irrelevant in the proof of singularity.

When attempting to deal with the initial singularity, one would expect that at radii of less than 10^{-12} cm and times less than 10^{-43} sec (Planck time) quantum laws will play a major role. But no one knows for sure what a theory of quantum gravity would look like, let alone at such high temperatures and densities. Various attempts to quantize gravity have been proposed (see for example Misner, 1957; DeWitt, 1967), but none led to satisfactory results. The nonlinearity of Einstein's equations poses problems with the conventional field quantization procedure, and calculations of interaction amplitudes invariably lead to divergencies which apparently cannot be removed as in quantum electrodynamics (nonrenormalizability of the theory). Recently, however, a theory called supergravity based on fermion-boson symmetries has been proposed (Freedman et al., 1976) that is not only renormalizable but also undertakes to unify gravitation with the three other forces in nature. Interesting progress is to be expected in this field.

The initial singularity also poses problems as to the number and nature of initial conditions one should impose on this early stage of evolution. To quote Harrison (1974): "When postulating initial conditions we must always beware of falling into the trap that ensnared bishop Ussher in the seventeenth century, who declared that the universe was created complete in every detail in the year 4004 B.C. Those who later pointed out that fossils exist, which are much older, were told on good authority that the fossils were also created with the universe. The history of cosmology teaches us that when discussing the universe we must beware of resorting to initial conditions as an easy substitute for explanations."

This is precisely the situation we are in when noting that a special initial condition seems to be required by a problem arising from the isotropy of the cmb. Shortly after the Big Bang, at the time of the emission of the cmb, not all parts of the universe were causally connected, since the speed of light provides a boundary on the velocity of transmitting signals. The cmb we observe today comes from regions of the universe causally disjoint at the time of emission, yet the radiation shows the same temperature and degree of isotropy regardless of direction. As it is extremely unlikely that it happened by pure chance, there must have been a mechanism of "isotropization" at the early stages that assured equal conditions in causally not connected parts of the universe. Although some solutions to this problem have been suggested (Misner (1969): Mixmastermodel), the question is still under investigation.

There is still one more problem associated with the remarkable degree of isotropy of the cmb. As mentioned in the first chapter, it leads to the breakdown of the principle of relativity (Bergmann, 1969). The blackbody radiation is not invariant under Lorentz transformations. Hence it is isotropic in one reference frame only, which can then serve to represent a preferred "rest" frame. The outcome of experiments involving cmb would be different in all other inertial frames, and the exclusion of such experiments appears to be a rather artificial restriction on the theory of special relativity. It should be noted, however, that the existence of a distinguished "rest" frame does not invalidate special relativity as a whole; the mathematical apparatus to describe relative motions remains unchanged.

The preferred "rest" frame is seen to be the one where all comoving (fundamental) observers are located, because it is here that the cmb appears isotropic. However, only locally can each fundamental observer define a preferred "rest" frame. The Minkowski space of special relativity is only locally isomorphic to the Friedmann-Lobachevski space as described by the Robertson Walker metric (Davidson and Narlikar, 1966). It should therefore not be surprising that the Hubble law leads to superluminal recession velocities for observed redshifts $z > 1$. In order to restrict recession velocities to speeds less than the speed of light $v < 1$, it is common astronomical practice to employ the special relativistic Doppler formula in cases $z > 1$, although at the same time it is not doubted that special relativity is restricted to the local frame of reference. This could provide a clue to the anomalous behavior of some quasars at $z > 1$ (Blandford et al., 1977).* The Robertson Walker metric (in natural units)

$$ds^2 = c^2dt^2 + R(t)^2[\frac{dr^2}{1 - kr^2} + r^2\, d\Omega^2]$$

shows that the speed of light remains constant with respect to all fundamental observers, and it is the expansion factor $R(t)$ and the curvature of the universe that leads to apparent superluminal recession velocities (Prokhovnik, 1976). No artificial boundaries need be introduced to constrict all observed velocities to $v < c$. The only observational

*The irregular patterns in the redshift distribution of quasars is the reason why we did not include observations made on quasars. While they first appeared to violate Hubble's law, they now seem to be in agreement with the theory, but are still not well understood.

horizon arises from the constancy of the speed of light which renders part of the universe still unobservable (for a discussion of horizons see Rindler, 1956).

When Einstein developed his general theory of relativity he was deeply impressed by Mach's principle that states that "matter out there in the universe influences things here" and is thus meant to explain the concept of inertia (see Nightingale (1977) for a summary of the implications of Mach's principle). Since Mach's principle has never been unambiguously formulated in mathematical terms, many physicists tend to dismiss it as philosophy. The general form of the field equations convinced Einstein that he had succeeded in incorporating Mach's principle into a theory of gravity. It turned out later, however, that Einstein's equation admit more general solutions, for example Gödel's metric of a rotating universe, that do not satisfy Mach's principle. The Robertson Walker metric seems to obey this principle (Davidson and Narlikar, 1966), and in their search for a better theory of gravity many authors have been led to assume the validity of Mach's principal as a starting point (Brans and Dicke, 1961). Recently, Mach's principle has been formulated in terms of the Leibnitz group (Barbour and Bertotti, 1977) which then serves as basic mathematical framework for a cosmological theory. An interesting point of that theory is that not only the motions of bodies but also the strength of forces is determiend cosmologically. Calculations give the correct order of magnitude of the gravitational constant and the mass density, and there also arises a constant universal velocity the value of which is surprisingly close to the speed of light. Another surprising feature is that in this model based entirely on Mach's principle time is not an independent parameter.

The uniqueness of the "arrow of time" has long puzzled physicists. It is well known that Maxwell's equations are time symmetric. In calculating the radiation emitted by an accelerated charge there arise solutions that go backwards in time (advanced waves) as well as those that go forward in time (retarded waves). It is generally argued that the advanced solutions be discarded as they are not physical. However, Dirac (1938) showed that it is necessary to employ both solutions in order to express the empirically well-established formula for radiation damping in terms of a covariant electromagnetic field. Wheeler and Feynman (1945) took up his idea to extend it to explain the time asymmetry in the universe. They argued that the universe acts as an opaque box with a perfect absorber along the future light cone and an imperfect absorber along the past light cone in such a way that the advanced part is exactly cancelled while the retarded one comes out to be the one observed. Wheeler and Feynman conclude that in order for this mechanism to work, the universe needs to be closed (see also Davies, 1972) which would contradict the recent observational results. Measurements testing the future light cone to be a perfect absorber (Partridge, 1973), prove to be inconclusive, however, as was shown by Gott et al. (1974).

In conclusion, we can say that fascinating developments are to be expected in cosmology. The Friedmann models seem to provide the best model today, although they are by no means without problems. Present observations seem to favor an open and expanding universe, but the case is not definitely settled yet. Many cosmologists have favored a closed model of the universe on philosophical grounds because it appeals more to our sense of symmetry and simplicity. But after all, who says the universe needs be simple and symmetric? In terms of creation and destination a spherically closed universe does not make more sense than an open hyperbolical one.

APPENDIX

PART I. SOME RESULTS OF RIEMANNIAN GEOMETRY

The space of special relativity is four-dimensional Minkowski space. For general relativity one wants to generalize the space concept, and one therefore introduces the differentiable manifold. Roughly speaking, a manifold is a space that locally looks like $\mathbb{R}^n$. (For an exact definition, see e.g. Klingenberg, 1973.)

If a metric of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{A-1}$$

is defined on X^n, where x^μ are coordinates defined on X^n, then the differentiable manifold X^n is said to be a Riemann space, and $g_{\mu\nu}$ is the metric tensor of the Riemann space.

Let us restrict the following considerations to dimensions $n = 4$.

A scalar is defined as a quantity that is invariant under coordinate transformations in a Riemann space.

If a set of four quantities A^μ and B_μ respectively ($\mu,\nu = 0,1,2,3$) transforms under coordinate transformation $x^\mu \longrightarrow \bar{x}^\mu$ according to

$$\begin{aligned} \bar{A}^\nu &= \frac{\partial \bar{x}^\nu}{\partial x^\mu} A^\mu \\ \bar{B}_\nu &= \frac{\partial x^\mu}{\partial \bar{x}^\nu} B_\mu \end{aligned} \tag{A-2}$$

it is said to form a contravariant and covariant vector, respectively. Note the Einstein sum convention, that any two equal indices are to be summed over.

The product $A^{\mu} B_{\mu}$ is a scalar invariant.

The definition of covariant and contravariant vectors can be extended to tensors. A mixed tensor that is contravariant of a-th order and covariant of b-th order is defined by

$$\bar{T}^{\alpha_1 \ldots \alpha_a}_{\beta_1 \ldots \beta_b} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial \bar{x}^{\alpha_a}}{\partial x^{\mu_a}} \frac{\partial x^{\nu_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{\nu_b}}{\partial \bar{x}^{\beta_b}} T^{\mu_1 \ldots \mu_a}_{\nu_1 \ldots \nu_b} \tag{A-3}$$

A product of two tensors of the form $S_{\mu\nu} T^{\mu\nu}$ again is a scalar invariant.

For any mixed tensor of fourth order $T^{\mu\nu}_{\alpha\beta}$ the expression $T^{\mu\gamma}_{\alpha\gamma}$ is a tensor of second order, and the process is called contraction. T^{μ}_{μ} is an invariant scalar.

In general, the metric tensor is a symmetric covariant tensor of second order of determinant unity

$$g_{\mu\nu} = g_{\nu\mu} \quad \text{with} \quad g^{\mu\nu} g_{\mu\nu} = 4 \ (=n) \tag{A-4}$$

and we have

$$g^{\mu\gamma} g_{\gamma\nu} = \delta^{\mu}_{\nu} \tag{A-5}$$

where δ^{μ}_{ν} is the unit tensor (Kronecker delta).

Therefore the metric tensor can be used to raise and lower indices

$$A^{\mu}_{\nu} g_{\mu\alpha} = A_{\alpha\nu} \tag{A-6}$$

The derivative of a tensor is denoted by

$$A_{|\mu} = \frac{\partial A}{\partial x^{\mu}} \tag{A-7}$$

(here, A is a tensor of zero-th order, i.e. a scalar).

In order to let a vector become a tensor via the process of differentiation, it is necessary to add a term to retain the tensor properties as defined in (A-3). The resulting expression is then called the covariant derivative and is denoted by

$$A^{\mu}{}_{||\alpha} = A^{\mu}{}_{|\alpha} + \Gamma^{\mu}_{\alpha\beta}\, A^{\beta} \tag{A-8}$$

where

$$\Gamma^{\mu}_{\alpha\beta} = 1/2\ g^{\mu\gamma}\ (g_{\gamma\alpha|\beta} + g_{\gamma\beta|\alpha} - g_{\alpha\beta|\gamma}) \tag{A-9}$$

is known as the Christoffel symbol, which also serves to generalize the concept of straight lines in Riemannian space (geodesics):

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{ds}\frac{dx^{\beta}}{ds} = 0 \tag{A-10}$$

As can be seen from (A-9), in Euclidean (flat) space $\Gamma^{\mu}_{\alpha\beta} = 0$, since the metric tensor is a constant, and (A-10) reduces to the equation of a straight line in Euclidean space.

Covariant derivatives of vectors do not commute, instead

$$A_{\nu||\alpha||\beta} - A_{\nu||\beta||\alpha} = A_{\mu} R^{\mu}{}_{\nu\alpha\beta} \tag{A-11}$$

and $R^{\mu}{}_{\nu\alpha\beta}$ is the Riemann curvature tensor

$$R^{\mu}_{\nu\alpha\beta} = \Gamma^{\mu}_{\alpha\nu|\beta} - \Gamma^{\mu}_{\nu\beta|\alpha} + \Gamma^{\mu}_{\tau\beta}\,\Gamma^{\tau}_{\alpha\nu} - \Gamma^{\mu}_{\tau\alpha}\Gamma^{\tau}_{\beta\nu} \qquad \text{(A-12)}$$

or equivalently

$$g_{\mu'\mu}R^{\mu'}_{\nu\alpha\beta} = R_{\mu\nu\alpha\beta} = 1/2(g_{\mu\beta|\nu\alpha} + g_{\nu\alpha|\mu\beta} - g_{\nu\beta|\alpha\mu} - g_{\alpha\mu|\nu\beta}) \qquad \text{(A-13)}$$

$$+ g^{\gamma\delta}(\Gamma_{\gamma\nu\alpha}\Gamma_{\delta\mu\rho} - \Gamma_{\gamma\nu\beta}\Gamma_{\delta\mu\alpha})$$

Again, in Euclidean space the Christoffel symbols and the derivatives of the metric vanish and consequently the Riemann tensor is zero in flat space. The Riemann tensor shows the following symmetry properties

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = -R_{\mu\nu\beta\alpha} = R_{\alpha\beta\mu\nu} \qquad \text{(A-14)}$$

$$R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0$$

Two important quantities can be derived from the Riemann tensor via contraction

the Ricci tensor $\qquad R_{\nu\alpha} = R^{\mu}_{\nu\alpha\mu}$

the Riemann scalar $\qquad R = R_{\nu\alpha}g^{\nu\alpha}$ (A-15)

The following combination of these two tensors is known as the Einstein tensor

$$G^{\mu}_{\nu} = R^{\mu}_{\nu} - 1/2\ \delta^{\mu}_{\nu} R \qquad \text{(A-16)}$$

and it can be shown that the covariant derivative is equivalent to the Bianchi identities

$$R^{\mu}_{\nu\alpha\beta\|\gamma} + R^{\mu}_{\nu\beta\gamma\|\alpha} + R^{\mu}_{\nu\gamma\alpha\|\beta} = 0 \qquad \text{(A-17)}$$

so that

$$G^{\mu}_{\nu\|\gamma} = 0 \tag{A-18}$$

i.e., the Einstein tensor is divergenceless.

PART II: DERIVATION OF THE ALTERNATIVE FORM OF THE LINE ELEMENT

In order to transform the line element

$$d\tau^2 = R^2 \frac{(du^2 + u^2 d\Omega^2)}{(1 + 1/4\ ku^2)^2} \tag{2-20a}$$

to the form

$$d\tau^2 = R^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right] \tag{2-20b}$$

we use the following transformation

$$r^2 = U^{-2} u^2 \tag{A-19}$$

where we introduce the short notation

$$U = (1 + 1/4\ ku^2) \tag{A-20}$$

Differentiating (A-19), we have

$$dr^2 = \frac{U^2 - ku^2}{U^2} du^2 \tag{A-21}$$

which upon rearranging terms and using (A-19) becomes

$$\frac{dr^2}{1 - kr^2} = du^2 U^{-2} \qquad \text{(A-22)}$$

Inserting (A-19) and (A-22) into (2-20a), we immediately obtain the form (2-20b) of the isotropic line element.

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Abbreviations: MNRAS: Monthly Notices of the Royal Astronomical Society
AJ: Astrophysical Journal