PERFORMANCE IMPROVEMENTS THROUGH FLEXIBLE WORKFORCE

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PERFORMANCE IMPROVEMENTS THROUGH FLEXIBLE WORKFORCE

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SUMMARY

This dissertation is concerned with increasing the efficiency of systems with cross-trained workforce and finite storage spaces. Our objective is to maximize production rate and minimize setup costs (if they exist). More specifically, we determine effective cross-training strategies and dynamic assignment policies for flexible servers in production lines with finite buffers.

First, we study the assignment of flexible servers to stations in tandem lines with service times that are not necessarily exponentially distributed. Our goal is to achieve optimal or near-optimal throughput. For systems with infinite buffers, it is already known that the effective assignment of flexible servers is robust to the service time distributions. We provide analytical results for small systems and numerical results for larger systems that support the same conclusion for tandem lines with finite buffers. In the process, we propose server assignment heuristics that perform well for systems with different service time distributions. Our research suggests that policies known to be optimal or near-optimal for Markovian systems are also likely to be effective when used to assign servers to tasks in non-Markovian systems.

Next, we identify optimal server assignment policies in under-staffed lines with finite buffers. Our objective is to maximize the production rate. We study systems with different flexibility structures and deterministic or exponential service times. Our results show that, when the service times are deterministic, the production rate of the line with full server flexibility can be obtained with partial flexibility, and we also determine the critical skills required in order to achieve this. Furthermore, we observe that the optimal server assignment policy for Markovian systems with small buffer sizes is either of priority- or threshold-type, depending on the flexibility

structure. Our numerical results imply that when the optimal assignment policies for Markovian systems with small buffer sizes are employed in Markovian systems with larger buffer sizes, near-optimal throughput of the fully flexible systems can be achieved even with partial cross-training. Moreover, our numerical results provide guidance about the choice of the best flexibility structure in Markovian lines.

Finally, we study the dynamic assignment of flexible servers to stations in the presence of setup costs that are incurred when servers move between stations. The goal is to maximize the long-run average profit. First, we prove the optimality of "multiple threshold" policies for systems with small buffer sizes; i.e., we show that it is optimal to move servers between stations when the number of jobs in the system reaches certain thresholds that depend on the current locations of the servers. Then, we investigate how the optimal server assignment policy for such systems depends on the magnitude of the setup costs. Finally, we perform numerical experiments that support the conjecture that multiple threshold policies are also optimal for systems with larger buffer sizes.

CHAPTER I

INTRODUCTION

This thesis is concerned with performance improvement in tandem lines. Such systems exist in various industries, and companies perpetually need to identify new strategies to increase their efficiency and stay competitive. It is important to include the effective training and management of workforce among such strategies. In this work, we study effective ways of using cross-training to increase the production rate and profit of the systems under consideration.

Cross-training is a widely used strategy for adding flexibility to systems. It requires less effort and resources compared to structural changes like modifying the layout of the factory or the order of production. Furthermore, it has various advantages for companies because it increases productivity and responsiveness, decreases costs, and can even increase workers' job satisfaction. However, once companies resort to using agile workforce, they should also determine the most effective ways of operating the workforce. Our goal in this thesis is to address this issue.

Queueing theory has been successfully used to model manufacturing systems. Queueing models provide a good representation of discrete material flow, and they are capable of capturing most situations that are observed in real-life (e.g., blockage, setups, or failures). Contrary to most previous applications of queueing systems that assume stationary servers, in this study we consider servers that are capable of working at different stations (i.e., cross-trained servers). More specifically, we are interested in improving the performance of manufacturing systems via effective use of the cross-trained workforce. One possible way of employing flexible workforce is to permanently pool several stations into a single station and then assigning a group of cross-trained workers to this station. By contrast, we study how to dynamically assign servers to stations in order to increase the system performance.

In this thesis, we study tandem manufacturing systems with flexible servers. We assume that there is an infinite supply of jobs in front of the first station, infinite room for completed jobs after the last station, and buffers of finite sizes between the stations. We further assume that several servers can work on the same job, and their service rates are additive if they are collaborating. Note that allowing finite buffers is necessary to consider more realistic representations of actual systems and their operations. For example, finite buffers occur frequently in manufacturing environments due to physical constraints, and sometimes they are used to control work-in-process (e.g., due to a desire for "just-in-time" processing). Having finite buffers also makes our problem quite difficult, because the tools used in dealing with infinite-buffered systems (e.g., fluid and diffusion limits) cannot be applied in our problem.

Our first goal is to determine the dynamic server assignment policies that maximize the long-run average throughput in non-Markovian tandem systems. More specifically, we want to show that the effective assignment of flexible servers is robust to the service time distributions. Considering service time distributions other than the exponential distribution is important to cover real-life situations where service times are unlikely to have the memoryless property. At the same time, this is a more difficult problem because Markov chain theory mostly does not apply in our case. Given the difficulty associated with rigorously analyzing non-Markovian systems with finite buffers, we document the robustness of effective server assignment policies to service time distributions by providing analytical results for small systems and numerical results for larger systems. More specifically, we determine the optimal server assignment policy for systems with two stations and two or three servers to the extent possible, and support the robustness of the optimal server assignment policy to service time distributions with extensive numerical experiments for other systems. We also identify heuristic server assignment policies for systems with more than two stations, show that these policies have good long-run average throughput performance, and conclude that the performance of the heuristics is robust to the service time distributions.

The second part of this thesis considers understaffed tandem lines (i.e., lines with more tasks than servers) with the objective of maximizing the throughput. A commonly used strategy for addressing problems with understaffed lines is "task partitioning," which involves grouping the tasks and assigning each server to one group of tasks, taking into account each server's capabilities. Server flexibility yields improved performance compared to strict task-portioning because strict task-partitioning corresponds to a subset of the policy space over which we solve our optimization problem (see Section 4.3.1). Rather than strictly partitioning tasks and assigning servers to them, in this thesis we consider various cross-training structures ranging from full flexibility to zone-training, and using combinations of dedicated and flexible servers. Partial cross-training strategies are especially important in industries where it is costly or not possible to have fully flexible servers, such as when each task requires extensive training or when the number of tasks is large compared to the number of available servers. In this work, we focus on tandem lines with two flexible servers and three stations because lines with more servers and stations become analytically intractable. Our goal is to identify both the optimal server assignment policy and the critical skills needed for the effective operation of fully and partially flexible systems. In the process, we also show that when the objective is to maximize the throughput, most (sometimes all) of the benefits of full flexibility can be obtained with partial flexibility.

The final part of this thesis studies dynamic assignment policies for flexible servers in tandem lines with setups. We assume that a revenue is obtained each time a job leaves the system and that there is a cost associated with server movements. Our objective is to maximize the long-run average profit. To the best of our knowledge, our work is the first that incorporates setups for a tandem system with finite buffers. Incorporating positive switching costs is a more realistic representation of actual systems, because server movements often cause some efficiency loss in real life. However, the inclusion of setup costs also complicates the analysis due to the necessity of keeping track of all server locations in the state space. Hence, most of our results concentrate on systems with two stations and two flexible servers because of the complexity associated with analyzing larger finite buffered systems. Our goal is to determine the structure of the optimal policy analytically and numerically for various systems, and to study how the optimal server assignment policy changes with the setup costs.

The remainder of this thesis is organized as follows. In Chapter 2, we review the previous research about flexible servers. In Chapter 3, we provide our results supporting the robustness of effective server assignment policies to service time distributions. In Chapter 4, we study how partially flexible servers should be dynamically assigned to stations in understaffed tandem lines. In Chapter 5, we consider the server assignment problem in the presence of setup costs. In Chapter 6, we describe the contributions of this research and our future directions. Finally, we provide the details of the proofs of the results in Chapters 3, 4, and 5 in Appendices A, B, and C, respectively.

CHAPTER II

LITERATURE REVIEW

Queues with flexible servers can be used to model various manufacturing and service systems. In this chapter, we provide a review of the work on effective flexible workforce management. First, we study the permanent assignment of flexible servers to stations in Section 2.1, concentrating on pooling in queueing systems. Next, we review the research where servers can be dynamically assigned to stations in Section 2.2.

2.1 Permanent Workforce Assignment

Some of the earlier works on the server assignment problem consider the permanent assignment of workers to stations. In a closed queueing network with identical servers, Shantikumar and Yao [62, 63] study the optimal assignment of servers to stations. They show that the optimal policy assigns more servers to stations with higher workload and they propose heuristic assignment policies that also appear to attain the optimal throughput in systems with two stations. Hillier and So [42] study the permanent assignment of servers in overstaffed tandem systems with small buffer sizes, and show that the interior stations should be given priority when assigning the extra workforce. Hillier and So [43] consider the workload allocation problem together with the server allocation problem in a tandem line. They show that unbalanced workload and server allocations result in higher throughput than the corresponding balanced system. Moreover, they show that when maximizing throughput is the goal, the most effective way of unbalancing the line involves assigning all extra servers and the highest workload to the end stations. In earlier work, Hillier and Boiling [41] show that unbalancing the line may increase the throughput, and that more workload should be assigned to the end stations. They refer to this workload allocation as the bowl phenomenon. Andradóttir, Ayhan, and Down [10] study tandem lines with two stations, and identify the optimal permanent assignment of servers to stations. When there are two servers who are equally skilled at all tasks, they show that the faster server should be assigned to the slower station. Finally, Yamazaki, Sakasegawa, and Shanthikumar [73] consider a tandem line with no buffers between the stations. They show that the two slowest stations should be placed at the two ends of the line, and that the arrangement of the rest of the stations should be made according to the bowl phenomenon.

One strategy to effectively manage a cross-trained workforce is to pool several stations together permanently and to assign servers to the pooled stations as a team. In an earlier work on parallel queues, Kleinrock [51] shows that pooling $M/M/1$ queues decrease the average waiting time for the customers. In a system with parallel stations and general interarrival distributions, Smith and Whitt [65] show that pooling decreases average waiting time if the service times have the same distribution but it may be disadvantageous otherwise.

Benjaafar [24] shows that pooling in a queueing network with homogeneous customers and workload allocation decreases the average waiting time and provides bounds on the performance improvement in the pooled system. However, he shows that if there are multiple customer classes, this conclusion is not always correct. Mandelbaum and Reiman [54] also study complete and partial pooling in Jackson networks. They show that complete pooling always helps in tandem systems, but it may deteriorate the system performance for more general queueing networks. Buzacott [27] shows that complete pooling of stations in a tandem line is beneficial, especially in the presence of high processing time variability among the tasks. Argon and Andradóttir [13] study the effects of partial pooling in tandem lines. They show that pooling the stations at the beginning or end of the line result in higher system throughput, but this is not always correct for the intermediate stations. Furthermore, they show that in a line with balanced workload, pooling the central stations is even better than pooling the stations near the beginning or end of the line.

2.2 Dynamic Workforce Assignment

We classify the dynamic server assignment literature with respect to the system configuration. In Section 2.2.1, we review research on systems with parallel stations, in Section 2.2.2 we study tandem lines, and finally in Section 2.2.3 we consider systems with more general network configurations.

2.2.1 Parallel Systems

The majority of the research about systems with parallel stations is related to service systems. More specifically, call centers have been an area of interest for various researchers, see, e.g., Gans, Koole, and Mandelbaum [32] and Akşin, Armony, and Mehrotra [5] for recent reviews.

Most papers concerned with parallel queueing systems have the objective of minimizing the holding cost. The majority of these papers assume infinite buffers and analyze the system in heavy traffic, although some study a clearing system and use Markov decision process (MDP) model. For a system with two stations in parallel, one dedicated, and one flexible server, Bell and Williams [22] show that in heavy traffic, the asymptotically optimal policy that minimizes the holding cost never idles the dedicated server, and is of threshold type for the flexible server. Ahn, Duenyas, and Zhang [3] consider a system similar to that of Bell and Williams [22] with the modification that there are no arrivals. They show that the optimal policy might be exhaustive or it might have a switching curve depending on the problem parameters. Harrison and López [40] are interested in a system with an arbitrary number of job classes, fully trained servers, and linear holding costs. They study the associated Brownian control problem and find a condition that leads to a heavy traffic resource pooling. Using these results they conjecture an asymptotically optimal assignment policy. Mandelbaum and Stolyar [55] propose and prove the asymptotic optimality of the generalized $c\mu$-rule in the same system when the holding costs are convex and increasing. Bell and Williams [23] also study the same system and prove the asymptotic optimality of the dynamic threshold policy proposed in Williams [69].

Some recent works address server assignment issues that are observed in call centers. We now review a few such papers. Bhulai and Koole [26] study a call center where the average waiting time of the incoming calls has to be kept below a limit, whereas outgoing calls do not have any waiting time limit. They study the call blending problem; i.e., how to assign the staff dynamically to incoming and outgoing calls. Gans and Zhou [33] analyze the same problem using different solution techniques, and provide the optimal staffing rule when the service rates for the different customer types can be different (Bhulai and Koole [26] provide a heuristic policy in this case). Gurvich, Armony, and Mandelbaum [38] are interested in call centers with multiple customer classes and fully flexible servers. They assume customers have different service level requirements and provide asymptotically optimal staffing and assignment policies. Armony and Magleras [14, 15] study a call center with a call-back option and flexible servers. They identify asymptotically optimal routing and staffing rules that guarantee that the maximum waiting time in the queue is not exceeded when the customers use the call-back option. Bassamboo, Harrison, and Zeevi [21] study a call center where arrivals to the system occur according to an arrival rate that varies randomly over time, and customers abandon the system if they wait too long. They present a staffing and routing algorithm that asymptotically minimizes the staffing costs and penalty costs associated with abandonments. Finally, Gans and Zhou [34] study the routing problem in a call center that outsources some of its operations. They show that for call centers with high outsourcing requirements, routing structures with minimal coordination between the client and vendor may result in high service levels.

Several papers focus on comparing the benefits of partial flexibility with full flexibility in parallel queues, and here we review a few of them. Jordan and Graves [48] study a setting with multiple products and plants, and show that most of the demand can be satisfied even with partially flexible plants (as opposed to plants that can produce all the products), as long as the assignment of products to plants is done well. Graves and Tomlin [35] study a similar problem in multi-stage supply chains, and show that partial flexibility structures (chaining, to be more specific) in each stage are sufficient, and that there is no need to coordinate the flexibility structures in different stages. Sheikhzadeh, Benjaafar, and Gupta [64] study the assignment of products to machines in a plant and consider operational issues such as finite storage spaces, setup times, work-in-process (WIP), inventory levels, and manufacturing lead-time. Their work also supports the conclusion that most of the benefits of full flexibility can be obtained by partial flexibility (using chaining structures). Gurumuthi and Benjaafar [37] study a parallel service system with flexible servers. They show that asymmetric server allocations are generally better than chaining structures if the servers are heterogeneous and different customer types have different demand rates. Wallace and Whitt [68] study routing and server assignment in a call center, and show that most of the benefits of full flexibility can be reached even with one additional skill per agent. Note that Hopp, Tekin, and Van Oyen [45] and Andradóttir, Ayhan, and Down [10] provide similar results related to the benefits of partial flexibility in tandem systems, and they will be reviewed in Section 2.2.2.

2.2.2 Tandem Systems

Tandem systems are mostly employed in modeling of flow lines in production environments. The papers that will be cited here have different objectives, including cost minimization, throughput maximization, and line balancing.

Most papers on flexible servers in tandem lines focus on the cost minimization problem in systems with two stations. Ahn, Duenyas, and Zhang [2] characterize the necessary and sufficient conditions for assigning all servers to the same station when there are two flexible servers and no external arrivals. When there is only one flexible server and no dedicated servers, Iravani, Posner, and Buzacott [47] show that the optimal policy in the second stage is greedy (e.g., the server never idles at the second stage as long as there are jobs there, but may move to the first stage even if second stage is not empty), and if the holding cost in the second stage is greater than or equal to the one of the first stage, the optimal policy is also exhaustive (i.e., the server works at the second stage until it becomes empty). Ahn, Duenyas, and Lewis [1] assume that there are two flexible servers, no dedicated servers, and Poisson arrivals. They study the cases where the flexible servers collaborate on the same job and where they do not collaborate but can work on separate jobs at the same station in both finite and infinite horizon models. They present conditions under which the optimal policy is exhaustive. Kaufman, Ahn, and Lewis [49] search for the optimal server assignment policy when there are Poisson arrivals, no dedicated servers, and the number of flexible servers varies randomly. They show that under both the discounted and average cost criteria, there exists an exhaustive policy that is optimal.

The papers reviewed in the preceding paragraph consider systems with no dedicated servers. Other works address the cost minimization problem for systems with both dedicated and flexible workforce. Rosberg, Varaiya, and Walrand [60] consider a tandem line with Poisson arrivals, flexible workforce that can be assigned either to the first station or to the second station, and dedicated workforce at the second station. They find that the policy that minimizes the total expected or average cost has a switching structure (when the number of jobs in front of the first station exceeds a threshold whose value depend on the number of jobs in the second queue, it is optimal to use all the flexible service effort at the first station; otherwise it is optimal to idle the first station). With no exogenous arrivals, one dedicated server at each station, and one flexible server, Farrar [31] studies the assignment of the flexible server that minimizes the holding cost until all the initial jobs depart from the system. He shows that the optimal control policy is transition monotone (i.e., after a service completion at any station, the optimal service rate at that station does not increase, and the optimal service rate at the other station does not decrease). Wu, Lewis, and Veatch [72] consider a clearing system with dedicated servers at both stations and flexible servers capable of working at either station. They show that there exists a transition monotone policy that minimizes the holding cost both with and without failures of dedicated servers. Under some additional assumptions, Wu, Down, and Lewis [71] show the same result for the system with external arrivals.

Some papers consider the effects of setups in tandem systems with infinite buffers between the stations. Duenyas, Gupta, and Olsen [29] consider a tandem line with a single flexible server and positive setup times when the server switches between the stations. They partially characterize the optimal policy and develop effective heuristic assignment policies. Iravani, Posner, and Buzacott [47] study a two-stage tandem queue with a flexible server, and identify the policy that minimizes the total holding and setup costs. Sennott, Van Oyen, and Iravani [61] consider a tandem line with a dedicated server at each station and one moving server. They allow positive setup costs and setup times, and provide recommendations on how to use the moving server more effectively.

Several papers focus on line balancing via server flexibility. Bartholdi and Eisenstein [17] show that the “bucket brigade” policy results in a stable partition of work if the work is infinitely divisible, servers are ordered from slowest to fastest, and the service times are deterministic. Bartholdi, Bunimovich, and Eisenstein [16] study the asymptotic behavior that may be observed in bucket brigades with two or three workers. Bartholdi, Eisenstein, and Foley [19] study the performance of the bucket brigade policy when the work consists of discrete tasks whose service requirements have an exponential distribution. Zavadlav, McClain, and Thomas [74] perform simulations for systems with more servers than stations and conclude that when the servers are fully cross-trained, all servers will be busy the same fraction of the time under the assignment policy they propose. Ahn and Righter [4] characterize some properties of the optimal policy for general tandem lines in order to achieve a balanced line. They consider different objective functions like maximizing the job completion process and minimizing holding costs. Moreover, they show that the optimal policy is often last-buffer-last-served (LBLS) or first-buffer-first-served (FBFS). Ostolaza, McClain, and Thomas [57] are interested in a tandem system where there are both dedicated tasks that must be done at a particular station and shared tasks that can be done at either of two consecutive stations. They propose and test some heuristics that result in higher throughput, as well as a balanced workload among stations. McClain, Thomas, and Sox [56] study dynamic load balancing in systems with small buffer sizes. Hopp, Tekin, and Van Oyen [45] study the capacity balancing problem for a line with equal number of workers and stations under a CONWIP (constant work-in-process) policy and show that a skill-chaining strategy with two skills per worker outperforms a "cherry picking" strategy in which some workers are cross-trained at bottleneck stations, especially in systems with high variability and low WIP.

With respect to maximizing the steady-state throughput of tandem lines with finite buffers, Andradóttir, Ayhan, and Down [7] show that any nonidling policy is optimal when the service rate only depends on the server or the station. They also identify the optimal server assignment policy for Markovian systems with two stations, two flexible servers, a finite buffer between the stations, and arbitrary service rates. Moreover, they propose heuristic server assignment policies that yield near-optimal throughput for larger systems. Andradóttir and Ayhan [6] are interested in Markovian systems with two stations and more than two flexible servers. They identify the optimal server assignment policy for systems with three servers. For systems with more than three servers, they conjecture the structure of the optimal policy and support their conjecture with extensive numerical results. Andradóttir, Ayhan, and Down [11] consider the effects of server failures in the same settings. Andradóttir, Ayhan, and Down [10] study tandem lines with two stations, and dedicated and flexible servers. They show how the optimal server assignment policy and throughput change depending on the number of flexible and dedicated servers in the system. They also give examples showing that for systems with moderate to large buffer sizes, most of the benefits of the flexibility can be obtained even with a single flexible server.

2.2.3 General Queueing Networks

Less work has been done on server assignment for systems with structures other than parallel and tandem. Such queueing networks can be used to model complex systems, such as wafer production plants, where different types of products do not go through the processing steps in the same order.

Hajek [39] considers systems with two stations where both external arrivals and arrivals from the other station are possible. He proves the existence of an optimal switching curve for finite horizon or long-run average cost problems. Tassiulas and Bhattacharya [66] present a non-preemptive dynamic assignment policy and provide some necessary conditions for this policy to achieve stability in a general queueing networks. Andradóttir, Ayhan, and Down [8] also study general queueing networks and they allow positive switchover times. They specify the maximal capacity and propose server assignment policies that have capacity arbitrarily close to the maximal capacity even in the existence of positive setup times. Their results also show the robustness of the effective server assignment policies to the service time distribution for the infinite-buffered tandem systems. Andradóttir, Ayhan, and Down [9] consider a similar problem with the modification that both servers and stations can fail. They specify the maximal capacity for the network, provide server assignment algorithms that perform arbitrarily close to the maximal capacity, and determine when server flexibility can compensate for the failures. Dai and Lin [28] consider a more general class of queueing networks called Stochastic Processing Networks. In their model, some tasks may require multiple servers and materials may be split up or joined with other materials. Under certain conditions, they show the throughput optimality of maximum pressure policies that allocate the service effort according to the service rates, buffer sizes, and network structure.

CHAPTER III

ROBUSTNESS OF EFFECTIVE SERVER ASSIGNMENT POLICIES TO SERVICE TIME DISTRIBUTIONS

3.1 Introduction

We consider a tandem line with $N \geq 2$ stations and $M \geq 1$ flexible servers. We assume that there is an infinite supply of jobs in front of the first station, infinite room for completed jobs after the last station, and a finite buffer of size $0 \leq B_j < \infty$ between stations $j-1$ and j, where $j \in \{2, \ldots, N\}$. The line operates under the manufacturing blocking mechanism, and travel times of the servers and setup times at the stations are assumed to be negligible. Let μ_{ij} denote the deterministic rate with which server $i \in \{1, \ldots, M\}$ works at station $j \in \{1, \ldots, N\}$. We assume that $\sum_{j=1}^{N} \mu_{ij} > 0$ for $i \in \{1, \ldots, M\}$ (because the other case is equivalent to a system with a smaller number of servers) and $\sum_{i=1}^{M} \mu_{ij} > 0$ for $j \in \{1, \ldots, N\}$ (because otherwise all policies have zero throughput). Several servers are allowed to work together on the same job, in which case their service rates are additive. Service times at each station $j \in \{1, \ldots, N\}$ are independent and identically distributed (i.i.d.) with mean $0 < m(j) < \infty$, and service times at different stations are independent.

Our work belongs to the set of papers dealing with throughput maximization in tandem lines with finite buffers. However, unlike the earlier work, we focus on non-Markovian systems. Iravani, Buzacott, and Posner [46] show that the effective assignment of a flexible workforce is robust to the arrival process when the objective is to minimize holding and setup costs. By contrast, we show the robustness of server assignment policies to service time distributions when the objective is to maximize throughput. Our results complement corresponding results for queueing networks with infinite buffers obtained by Andradóttir, Ayhan, and Down [8, 9], and suggest that when capacity is the primary concern, effective server assignment policies are robust with respect to the form of underlying service distributions, even in the finite buffer setting.

The remainder of the chapter is organized as follows. In Section 3.2, we formulate our problem and provide some general results about lines with two stations that will be used later. In Section 3.3, we show that the server assignment policy proven to be optimal for Markovian tandem lines with two stations and two servers in Andradóttir, Ayhan, and Down [7] is also optimal both for deterministic systems if the buffer size is arbitrary and for systems with general service times if the buffer size is zero. In Section 3.4, we show that the optimal policy for tandem lines with two stations and three servers is a threshold-type policy (as the optimal policy for Markovian systems with two stations and three servers, see Andradóttir and Ayhan [6]) if the service times are deterministic and the buffer size is arbitrary or if the service times follow a general distribution and the buffer size is zero. In Section 3.5, we propose heuristic server assignment policies for larger systems that appear to perform well for a broad range of problems, and provide the results of numerical experiments that show that policies that work well for Markovian systems also appear to be effective for non-Markovian systems. In Section 3.6, we make some concluding remarks.

3.2 Problem Formulation and Preliminary Results

Let the state space S of the system be chosen to capture the number of jobs at each station and the status (operating, starved, or blocked) of each station. Decision epochs are the service completion times at any station, so that decisions are made when changes to the state of the system are observable. Consequently, we will restrict ourselves to the set Π of all Markovian stationary deterministic policies corresponding to the state space S. We are interested in finding a policy in the set Π that maximizes the long-run average throughput. However, we also show that sometimes the best policy in Π is also optimal over all possible ways of assigning servers to stations.

Specifically, for systems with two stations and intermediate buffer of size $B_2 = B$, we use the stochastic process $\{X(t) : t \geq 0\}$ to keep track of the number of jobs that have already been processed at station 1 and are either waiting for service or being processed at station 2. Hence, the state space is $S = \{0, 1, \ldots, B+2\}$, with $X(t) = 0$ if a job is being processed at station 1 and station 2 is starved at time $t \geq 0$; $X(t) = s \in \{1, \ldots, B+1\}$ if there are jobs being processed at both stations 1 and 2 and $s-1$ jobs waiting to be processed in the intermediate buffer at time $t \geq 0$; finally, $X(t) = B+2$ if station 1 is blocked, B jobs are waiting to be processed in the buffer, and one job is being processed at station 2 at time $t \geq 0$.

In order to solve our optimization problem, we will identify the optimal action in each state. For any $s \in S$, A_s denotes the set of allowable actions at state s. Possible actions are idling a server or assigning the server to station 1 or 2. Note that $\pi = (d)^\infty$ for every policy π in Π, where the corresponding decision rule d is a $(B+3)$-dimensional vector, with $d(s) \in A_s$ for all $s \in S$.

We now provide two preliminary results about the structure of the optimal policy for tandem lines with two stations.

Lemma 3.2.1 *When $N = 2$, there exists an optimal policy that assigns all servers to station 2 if station 1 is blocked, and to station 1 if station 2 is starved.*

Proof: Let π be any policy that idles any of the servers when $s = B+2$. Now compare π with the policy π' that assigns all the servers to station 2 when $s = B+2$ and agrees with π otherwise. The only difference between these two policies is the transition time from state $B+2$ to state $B+1$, and this transition time is never longer for π'. Consequently, the number of departures is never smaller under π', and hence there exists an optimal policy that does not idle any servers when $s = B+2$. A similar logic follows when $s = 0$. $\square$

Lemma 3.2.2 *When $M = N = 2$, there exists an optimal policy that does not idle any of the servers when station 1 is not blocked, and station 2 is not starved.*

Proof: If both servers are idled in state $s \in \{1, \ldots, B+1\}$, then s is an absorbing state for $\{X(t)\}$ and the long-run average throughput is zero. Consequently, at least one of the two servers should be assigned to one of the stations in state s. Suppose that one server is assigned to station 1 and the other server is left idle (the case where one server is assigned to station 2 is similar). Then the only possible transition is from state s to state $s+1$. The transition time from s to $s+1$ is never longer if we assign both servers to station 1, which implies that the number of departures is never smaller. Thus, assigning both servers to the same station is never worse than assigning one to that station and idling the other. $\square$

Finding the optimal server assignment policy for finite queueing systems with general service time distributions is a very difficult task in general, even when $M = N = 2$. However, the following result facilitates the analysis for systems with two stations and a buffer of size 0 between the stations. Hence, some of the theoretical results in this paper are restricted to the case with $N = 2$ and $B = 0$, with numerical results supporting our robustness conclusion for larger systems.

Lemma 3.2.3 *When $N = 2$ and $B = 0$, the optimal policy minimizes the expected time between successive visits of $\{X(t)\}$ to state 1.*

Proof: When the buffer size is zero, the state space becomes $S = \{0, 1, 2\}$. Every time the process hits state 1, the process restarts regeneratively and there is exactly one departure from the system between every two successive visits to state 1. Hence, the long-run average throughput is the reciprocal of the expected time between two visits to state 1 by the renewal reward theorem, and any policy that minimizes this expected time maximizes the throughput. $\square$

We conclude this section with a lemma that provides an upper bound on the throughput of our system, and will be used for identifying the optimal policy for deterministic systems.

Lemma 3.2.4 *The maximal capacity of a tandem line with outside arrivals and infinite buffers between the stations is an upper bound on the throughput of the corresponding tandem line with infinite amount of raw material in front of the first station and finite buffers between the stations.*

Proof: Consider the following "allocation" linear program (LP) with decision variables λ and $\{\delta_{ij}\}$ for a system with N stations in tandem and M flexible servers:

$$\left.\begin{array}{ll} \max & \lambda \\ \text{s.t.} & \sum_{i=1}^{M} \delta_{ij} \frac{\mu_{ij}}{m(j)} \geq \lambda, \text{ for all } j \in \{1, \ldots, N\}, \\ & \sum_{j=1}^{N} \delta_{ij} \leq 1, \text{ for all } i \in \{1, \ldots, M\}, \\ & \delta_{ij} \geq 0, \text{ for all } i \in \{1, \ldots, M\}, \ j \in \{1, \ldots, N\}. \end{array}\right\} \quad (1)$$

In this LP, λ can be interpreted as capacity, and δ_{ij} where $i \in \{1, \ldots, M\}$, $j \in \{1, \ldots, N\}$, can be interpreted as the long-run proportion of time server i is assigned to station j.

Let λ^* denote the optimal value of λ for this LP. Then λ^* is the maximal capacity of the infinite-buffered version of our tandem line with an outside arrival process that satisfies some stochastic assumptions (we refer to this system as "System 1") as shown by Andradóttir, Ayhan, and Down [8]. We use the stochastic process $B(t) = \{X(t), V(t), Y(t)\}$ to model this system, where $X(t)$ is the vector showing the number of jobs either in service or waiting for service at each station at time t, $V(t)$ is the residual interarrival time to the system at time t, and $Y(t)$ is the vector of residual service requirements at each station at time t.

Let T^π denote the long-run average throughput under policy $\pi \in \Pi$ in our system with finite buffers and infinite amount of raw material in front of the first station (we refer to this system as "System 2"). Assume that there exists a policy $\pi \in \Pi$ such that $T^\pi = \bar{\lambda} > \lambda^*$. Note that the state space of System 1 is $\mathbb{N}^N \times \mathbb{R} \times \mathbb{R}^N$. and that the state space of System 2 is $S \times \mathbb{R}^N$, where $S \subseteq \mathbb{N}^{N-1}$ (we disregard the first element of the state vector since the number of jobs in front of the first station is infinite in System 2, and we do not need to keep track of a residual interarrival time to the system). Let π' be a policy for System 1 such that $\pi'(s_1, s, v, y) = \pi(s, y)$ for $s_1 \in \{0, 1, \ldots\}$, $s \in S$, $v \in \mathbb{R}$, and $y \in \mathbb{R}^N$ (without loss of generality assume that π idles servers rather than assigning them to stations that are blocked). Furthermore, define $\pi'(s_1, s, v, y)$ for $s_1 \in \{0, 1, \ldots\}$ $s \notin S$, $v \in \mathbb{R}$, and $y \in \mathbb{R}^N$ such that System 1 will enter a state (s_1, s', v', y') in finite time where $s' \in S$, $v' \in \mathbb{R}$, and $y' \in \mathbb{R}^N$ (for example, if $s = (s_2, \ldots, s_N) \notin S$, choose π' such that all servers are assigned to the station $2 \leq i \leq N$ that is closest to the end of the line among the stations where s'_i is not feasible given that $s'_{i+1}, \ldots, s'_N$ are feasible in System 2).

Now, use the policy π' for System 1 with an outside arrival rate of $\lambda \in (\lambda^*, \bar{\lambda})$. It follows from part (ii) of Theorem 1 of Andradóttir, Ayhan, and Down [8] that the system is not stable and that the buffer space in front of the first station will not be empty almost surely after a finite amount of time (because π' guarantees that the number of jobs in the buffers between the stations will be bounded above for all $t \geq 0$). Hence the throughput of System 1 under π' will be equal to the throughput $\bar{\lambda}$ of System 2 under π. But this is a contradiction because the throughput of System 1 can not exceed λ (the departure rate cannot exceed the arrival rate) and we assumed that $\lambda < \bar{\lambda}$. Hence there cannot exist a policy $\pi \in \Pi$ with $T^\pi > \lambda^*$, and the result follows. □

3.3 Lines with Two Stations and Two Flexible Servers

In this section we consider the special case of a tandem line with two stations and two flexible servers. Since we can relabel the servers if necessary, we assume that $\frac{\mu_{11}}{\mu_{12}} \geq \frac{\mu_{21}}{\mu_{22}}$

without loss of generality (we use the convention $c/0 = \infty$, for all $c \geq 0$, throughout this thesis). This implies that $\mu_{11} > 0$ and $\mu_{22} > 0$ under our assumptions on the service rates. When there are two servers, we represent the actions by $a_{\sigma_1\sigma_2}$, where $\sigma_i \in \{0, 1, 2\}$ for $i \in \{1, 2\}$. We use $\sigma_i = 0$ when server i is idle and $\sigma_i = j \in \{1, 2\}$ when server i is working at station j.

Andradóttir, Ayhan, and Down [7] show that for the corresponding Markovian system, it is optimal to assign server 1 to station 1 and server 2 to station 2 when neither station is blocked or starved, and both servers to station 1 (station 2) when station 2 (station 1) is starved (blocked). Our goal is to generalize this result to arbitrary service time distributions to the extent possible. The outline of this section is as follows. We first consider the case when the service times are deterministic in Section 3.3.1, then the case with general service times in Section 3.3.2.

3.3.1 Deterministic Service Times

When the service times are deterministic, we prove the following theorem by showing that the long-run average throughput of the finite-buffered system under the proposed policy is equal to "the maximal capacity" of the system, which is defined as a tight upper bound on the set of arrival rates for which the infinite-buffered version of the system (with outside arrivals) is stable.

Theorem 3.3.1 *For a system with two stations in tandem, two flexible servers, a finite intermediate buffer of arbitrary size, and deterministic service times, the optimal policy is identical to the one of the corresponding Markovian system. Moreover, the optimal throughput is equal to the maximal capacity of the system, regardless of the size of the intermediate buffer.*

Proof: Let $u_1 = m(1)$ and $u_2 = m(2)$ denote the service times at stations 1 and 2, respectively. Now consider the allocation LP (1) with $M = N = 2$. Lemma 3.2.4 shows that the maximal capacity λ^* of the line with outside arrivals and infinite buffers is an upper bound on the throughput of our finite-buffered tandem line with infinite amount of raw material in front of the first station.

Lemma A.1.2 specifies the long-run average throughput if $\frac{u_1}{\mu_{11}} \leq \frac{u_2}{\mu_{22}}$ and the policy described in the theorem is used. Similarly, Lemma A.1.3 specifies the long-run average throughput if $\frac{u_1}{\mu_{11}} > \frac{u_2}{\mu_{22}}$ and the policy described in the theorem is used. Lemma A.1.1 shows that the throughput is equal to λ^* in both cases. Hence, the policy of the theorem is optimal, and the proof is complete. □

3.3.2 General Service Times

We consider a system with general service times and zero buffer between the stations. The following theorem follows from sample path arguments and Lemma 3.2.3.

Theorem 3.3.2 *For a system with two stations in tandem, two flexible servers, zero buffer between the two stations, and service times having general distributions, the optimal policy is the same as the one of the corresponding Markovian system.*

Proof: Let the services times at stations 1 and 2 have the cumulative distribution functions (CDF's) F_1 and F_2, respectively. Lemmas 3.2.1 and 3.2.2 show that we can use the following action space in order to determine the optimal policy:

$$A_s = \begin{cases} a_{11} & \text{for } s = 0, \\ \{a_{11}, a_{12}, a_{21}, a_{22}\} & \text{for } s = 1, \\ a_{22} & \text{for } s = 2. \end{cases}$$

Let $E_{\sigma_1\sigma_2}$ denote the expected time between two consecutive visits to state 1 when action $a_{\sigma_1\sigma_2}$ is selected in state 1. Then,

$$\begin{aligned} E_{12} &= \int_0^\infty \int_0^{\frac{u_2\mu_{11}}{\mu_{22}}} C_1(u_1,u_2)dF_1(u_1)dF_2(u_2) + \int_0^\infty \int_{\frac{u_2\mu_{11}}{\mu_{22}}}^\infty C_3(u_1,u_2)dF_1(u_1)dF_2(u_2), \\ E_{21} &= \int_0^\infty \int_0^{\frac{u_2\mu_{21}}{\mu_{12}}} C_2(u_1,u_2)dF_1(u_1)dF_2(u_2) + \int_0^\infty \int_{\frac{u_2\mu_{21}}{\mu_{12}}}^\infty C_4(u_1,u_2)dF_1(u_1)dF_2(u_2), \\ E_{11} &= E_{22} = \int_0^\infty \int_0^\infty C_5(u_1,u_2)dF_1(u_1)dF_2(u_2), \end{aligned}$$

where $C_i(u_1, u_2)$, for $i \in \{1, \ldots, 5\}$, are defined in Appendix A.1. Lemma A.1.4 now implies that $E_{12} \leq \min\{E_{11}, E_{21}, E_{22}\}$, and hence, Lemma 3.2.3 shows that it is optimal to use action a_{12} in state 1. □

3.4 Lines with Two Stations and Three Flexible Servers

In this section, we consider the special case of a tandem line with two stations and three flexible servers. By relabeling the servers if necessary, we can assume without loss of generality that $\frac{\mu_{11}}{\mu_{12}} \geq \frac{\mu_{21}}{\mu_{22}} \geq \frac{\mu_{31}}{\mu_{32}}$. This implies that $\mu_{11} > 0$ and $\mu_{32} > 0$ under our assumptions on the service rates.

In this case, $a_{\sigma_1\sigma_2\sigma_3}$ denotes the possible actions in each state, where for $i \in \{1, 2, 3\}$, $\sigma_i = 0$ when server i is idle and $\sigma_i = j$ when server i is working at station $j \in \{1, 2\}$. We use the following lemmas in this section; their proofs are provided in Appendix A.3. Note that Lemma 3.4.1 generalizes the result of Lemma 3.2.2 to $M = 3$, but under the assumption that $B = 0$.

Lemma 3.4.1 *It is optimal not to idle any of the servers in any state for a system with two stations in tandem, three flexible servers, and zero buffer between the two stations.*

Lemma 3.4.2 *Any policy that uses actions a_{111} or a_{222} in state 1 cannot be optimal for a system with two stations in tandem, three flexible servers, and zero buffer between the two stations.*

Andradóttir and Ayhan [6] show that when the service times have an exponential distribution, the optimal policy is as follows:

- when Station 2 is starved (i.e., in state 0), assign all servers to Station 1;
- when neither of the stations is blocked or starved (i.e., in states $s \in \{1, \ldots, B+1\}$), assign Servers 1 and 2 to Station 1, Server 3 to Station 2 if $s < s^*$; and Server 1 to Station 1, Servers 2 and 3 to Station 2 if $s \geq s^*$;

- when Station 1 is blocked (i.e., in state $B+2$), assign all servers to Station 2;

where the threshold $s^* \in S\backslash\{0\}$ depends on the problem parameters (see Theorem 3.1 in [6]). We prove similar results for systems with deterministic service time distributions and arbitrary buffer size, as well as systems with general service time distributions and zero buffer size. This section is organized as follows. In Section 3.4.1 we consider the system with deterministic service times, and in Section 3.4.2 we study the system with general service times.

3.4.1 Deterministic Service Times

When the buffer size is bigger than zero, the theorem below shows that the optimal policy is of threshold type, and specifies the value of the threshold. Furthermore, it shows that if the service time $u_1 = m(1)$ at station 1 is large (small) relative to the service time $u_2 = m(2)$ at station 2, then more service effort should be given to station 1 (2), and indicates how the comparison of u_1 and u_2 should depend on the service rates.

Theorem 3.4.1 *For a system with two stations in tandem, three flexible servers, a buffer of positive size between the two stations, and deterministic service times* $u_1, u_2 \in \mathbb{R}^+$ *at stations 1, 2, respectively, the following policy is optimal:*

- *when Station 2 is starved (i.e., in state 0), assign all servers to Station 1;*
- *when neither of the stations is blocked or starved (i.e., in states* $s \in \{1, \ldots, B+1\}$*),*
  - *if* $\frac{u_1}{u_2} \leq \frac{\mu_{11}}{\mu_{22}+\mu_{32}}$, *assign Server 1 to Station 1, Servers 2 and 3 to Station 2;*
  - *if* $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$, *assign Servers 1 and 2 to Station 1, Server 3 to Station 2 if* $s < s^*$*; and Server 1 to Station 1, Servers 2 and 3 to Station 2 if* $s \geq s^*$, *where* s^* *can be chosen to be any state in* $\{2, \ldots, B+1\}$*;*

– *if* $\frac{u_1}{u_2} > \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$*, assign Servers 1 and 2 to Station 1, Server 3 to Station 2;*

- *when Station 1 is blocked (i.e., in state* $B+2$*), assign all servers to Station 2.*

Moreover, the optimal throughput is equal to the maximal capacity of the system, regardless of the size of the intermediate buffer (as long as it is positive).

Proof: Lemma A.2.2 specifies the long-run average throughput when $\frac{u_1}{u_2} \le \frac{\mu_{11}}{\mu_{22}+\mu_{32}}$ and the policy described in the theorem is used. Similarly, Lemma A.2.3 specifies the long-run average throughput when $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \le \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$ and the policy described in the theorem is used. Finally, Lemma A.2.4 specifies the long-run average throughput when $\frac{u_1}{u_2} > \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$ and the policy described in the theorem is used. Lemma A.2.1 shows that the throughput is equal to the maximal capacity λ^* of the line in all three cases. Hence, the optimality of the policy of the theorem follows from Lemma 3.2.4, and the proof is complete. □

Theorem 3.4.1 shows that if the service times are deterministic, then the maximal capacity of infinite-buffered systems can be achieved for any $B > 0$. Note, however, that when $B = 0$ and $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \le \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$, there is no way to implement the policy of Theorem 3.4.1, because this policy requires two adjacent states (s^*-1 and s^*) where both stations are operating, and when $B = 0$, the state space is $\{0,1,2\}$ and the only state where both stations are operating is state 1. Consequently, it is not always possible to attain the maximal capacity if the buffer size between the two stations is zero, and a different approach is needed to determine the optimal policy when $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \le \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$ and $B = 0$.

More specifically, in the case of deterministic service times, we can exactly determine the time between two visits to state 1 for all possible policies in Π if the buffer size is zero, and then use Lemma 3.2.3 to find the optimal policy. Let

$$r = \frac{(\mu_{11}+\mu_{21})(\mu_{11}\mu_{22}-\mu_{12}\mu_{21}+\mu_{11}\mu_{32}-\mu_{12}\mu_{31})}{(\mu_{22}+\mu_{32})(\mu_{11}\mu_{32}-\mu_{12}\mu_{31}+\mu_{21}\mu_{32}-\mu_{22}\mu_{31})}.$$

The following theorem identifies the optimal dynamic server assignment policy if the intermediate buffer between the stations is of size zero.

Theorem 3.4.2 *For a system with two stations in tandem, three flexible servers, zero buffer between the two stations, and deterministic service times* $u_1, u_2 \in \mathbb{R}^+$ *at stations 1, 2, respectively, the following policy is optimal:*

- *when Station 2 is starved (i.e., in state 0), assign all servers to Station 1;*
- *when neither of the stations is blocked or starved (i.e., in state 1), assign Servers 1 and 2 to Station 1, Server 3 to Station 2 if* $\frac{u_1}{u_2} > r$*, and Server 1 to Station 1, Servers 2 and 3 to Station 2 if* $\frac{u_1}{u_2} \leq r$*;*
- *when Station 1 is blocked (i.e., in state 2), assign all servers to Station 2.*

Moreover, the optimal throughput is equal to the maximal capacity of the system if $\frac{u_1}{u_2} \leq \frac{\mu_{11}}{\mu_{22}+\mu_{32}}$ *or* $\frac{u_1}{u_2} > \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$.

Proof: When $\frac{u_1}{u_2} \leq \frac{\mu_{11}}{\mu_{22}+\mu_{32}}$ or $\frac{u_1}{u_2} > \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$, the optimality of the policy specified in the theorem can be proved as in Theorem 3.4.1. Next, we will determine the optimal policy when $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$. Lemmas 3.2.1, 3.4.1, and 3.4.2 show that the following action space is suitable:

$$A_s = \begin{cases} a_{111} & \text{for } s = 0, \\ \{a_{112}, a_{122}, a_{121}, a_{211}, a_{212}, a_{221}\} & \text{for } s = 1, \\ a_{222} & \text{for } s = 2. \end{cases}$$

Lemma A.3.1 shows that $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} \leq r \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$. When $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \leq r$, Lemma A.3.2 shows that it is optimal to use action a_{122} in state 1. When $r < \frac{u_1}{u_2} \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$, Lemma A.3.3 shows that it is optimal to use action a_{112} in state 1. This completes the proof. □

3.4.2 General Service Times

Recall that F_1 and F_2 denote the CDF's of the service times at stations 1 and 2, respectively. Let $E_{\sigma_1\sigma_2\sigma_3}$ denote the expected time between two visits to state 1 if action a_{111} is used in state 0, action $a_{\sigma_1\sigma_2\sigma_3}$ is used in state 1, and action a_{222} is used in state 2. Then,

$$\begin{aligned} E_{112} &= \int_0^\infty \int_0^{\frac{u_2(\mu_{11}+\mu_{21})}{\mu_{32}}} \widehat{C}_1(u_1,u_2)dF_1(u_1)dF_2(u_2) \\ &\quad + \int_0^\infty \int_{\frac{u_2(\mu_{11}+\mu_{21})}{\mu_{32}}}^\infty \widehat{C}_2(u_1,u_2)dF_1(u_1)dF_2(u_2), \\ E_{122} &= \int_0^\infty \int_0^{\frac{u_2\mu_{11}}{\mu_{22}+\mu_{32}}} \widehat{C}_3(u_1,u_2)dF_1(u_1)dF_2(u_2) \\ &\quad + \int_0^\infty \int_{\frac{u_2\mu_{11}}{\mu_{22}+\mu_{32}}}^\infty \widehat{C}_4(u_1,u_2)dF_1(u_1)dF_2(u_2), \end{aligned}$$

where the functions $\widehat{C}_i(u_1,u_2)$, for $i \in \{1,2,3,4\}$, are defined in Appendix A.3. The following theorem now identifies the optimal policy when the service times have a general distribution and the buffer size is zero.

Theorem 3.4.3 *For a system with two stations in tandem, three flexible servers, zero buffer between the two stations, and service times coming from general distributions, the optimal policy is a threshold policy like the one for the corresponding Markovian system. In other words, the following policy is optimal:*

- *when Station 2 is starved (i.e., in state 0) assign all servers to Station 1;*
- *when neither of the stations is blocked or starved (i.e., in state 1), assign Servers 1 and 2 to Station 1, Server 3 to Station 2 if $E_{112} \le E_{122}$, and Server 1 to Station 1 and Servers 2 and 3 to Station 2 if $E_{122} < E_{112}$;*
- *when Station 1 is blocked (i.e., in state 2), assign all servers to Station 2.*

Proof: Using the same notation as above, let us define

$$
\begin{aligned}
E_{121} &= \int_0^\infty \int_0^{\frac{u_2(\mu_{11}+\mu_{31})}{\mu_{22}}} \widehat{C}_5(u_1,u_2)dF_1(u_1)dF_2(u_2) \\
&\quad + \int_0^\infty \int_{\frac{u_2(\mu_{11}+\mu_{31})}{\mu_{22}}}^\infty \widehat{C}_6(u_1,u_2)dF_1(u_1)dF_2(u_2), \\
E_{211} &= \int_0^\infty \int_0^{\frac{u_2(\mu_{21}+\mu_{31})}{\mu_{12}}} \widehat{C}_7(u_1,u_2)dF_1(u_1)dF_2(u_2) \\
&\quad + \int_0^\infty \int_{\frac{u_2(\mu_{21}+\mu_{31})}{\mu_{12}}}^\infty \widehat{C}_8(u_1,u_2)dF_1(u_1)dF_2(u_2), \\
E_{212} &= \int_0^\infty \int_0^{\frac{u_2\mu_{21}}{\mu_{12}+\mu_{32}}} \widehat{C}_9(u_1,u_2)dF_1(u_1)dF_2(u_2) \\
&\quad + \int_0^\infty \int_{\frac{u_2\mu_{21}}{\mu_{12}+\mu_{32}}}^\infty \widehat{C}_{10}(u_1,u_2)dF_1(u_1)dF_2(u_2), \\
E_{221} &= \int_0^\infty \int_0^{\frac{u_2\mu_{31}}{\mu_{12}+\mu_{22}}} \widehat{C}_{11}(u_1,u_2)dF_1(u_1)dF_2(u_2) \\
&\quad + \int_0^\infty \int_{\frac{u_2\mu_{31}}{\mu_{12}+\mu_{22}}}^\infty \widehat{C}_{12}(u_1,u_2)dF_1(u_1)dF_2(u_2),
\end{aligned}
$$

where the functions $\widehat{C}_i(u_1,u_2)$, $i \in \{5,\ldots,12\}$, are defined in Appendix A.3. Lemma 3.2.1 shows that it is optimal to use action a_{111} in state 0 and action a_{222} in state 2. Hence, only the optimal action in state 1 needs to be determined. Lemmas 3.4.1 and 3.4.2 show that, it is sufficient to consider actions $a_{112}, a_{122}, a_{121}, a_{211}, a_{212}$, and a_{221} in state 1. We use Lemma 3.2.3 to compare different actions in state 1.

It is shown in the proofs of Lemmas A.3.3 and A.3.4 that for all $u_1, u_2 \in \mathbb{R}^+$, we have

$$\max\{\widehat{C}_1(u_1,u_2), \widehat{C}_2(u_1,u_2)\} \le \min\{\widehat{C}_7(u_1,u_2), \widehat{C}_8(u_1,u_2), \widehat{C}_{11}(u_1,u_2), \widehat{C}_{12}(u_1,u_2)\}.$$

Consequently, it is clear that $E_{112} \le \min\{E_{211}, E_{221}\}$, and actions a_{211} and a_{221} cannot be optimal. Furthermore,

$$
\begin{aligned}
E_{112} = &\int_0^\infty \int_0^{\frac{u_2\mu_{21}}{\mu_{12}+\mu_{32}}} \widehat{C}_1(u_1,u_2)dF_1(u_1)dF_2(u_2) + \int_0^\infty \int_{\frac{u_2\mu_{21}}{\mu_{12}+\mu_{32}}}^{\frac{u_2(\mu_{11}+\mu_{21})}{\mu_{32}}} \widehat{C}_1(u_1,u_2)dF_1(u_1)dF_2(u_2) \\
+ &\int_0^\infty \int_{\frac{u_2(\mu_{11}+\mu_{21})}{\mu_{32}}}^\infty \widehat{C}_2(u_1,u_2)dF_1(u_1)dF_2(u_2).
\end{aligned}
$$

We have shown in the proof of Lemma A.3.3 that, $\widehat{C}_1(u_1,u_2) \le \widehat{C}_9(u_1,u_2)$ for all $u_1, u_2 \in \mathbb{R}^+$, and if $u_1 > \frac{u_2\mu_{21}}{\mu_{12}+\mu_{32}}$, then $\widehat{C}_1(u_1,u_2) < \widehat{C}_{10}(u_1,u_2)$. We have shown in Lemma A.3.4 that $\widehat{C}_2(u_1,u_2) \le \widehat{C}_{10}(u_1,u_2)$ for all $u_1, u_2 \in \mathbb{R}^+$. Hence, we can conclude that $E_{112} \le E_{212}$, and action a_{212} cannot be optimal.

Similarly,

$$E_{122} = \int_0^\infty \int_0^{\frac{u_2\mu_{11}}{\mu_{22}+\mu_{32}}} \widehat{C}_3(u_1,u_2)dF_1(u_1)dF_2(u_2) + \int_0^\infty \int_{\frac{u_2\mu_{11}}{\mu_{22}+\mu_{32}}}^{\frac{u_2(\mu_{11}+\mu_{31})}{\mu_{22}}} \widehat{C}_4(u_1,u_2)dF_1(u_1)dF_2(u_2)$$
$$+ \quad \int_0^\infty \int_{\frac{u_2(\mu_{11}+\mu_{31})}{\mu_{22}}}^\infty \widehat{C}_4(u_1,u_2)dF_1(u_1)dF_2(u_2).$$

We know that $\widehat{C}_3(u_1,u_2) \le \widehat{C}_5(u_1,u_2)$ for all u_1, $u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_3(u_1,u_2) \le \widehat{C}_5(u_1,u_2) \Leftrightarrow \frac{\mu_{12}}{\mu_{11}} \le \frac{\mu_{12}+\mu_{32}}{\mu_{11}+\mu_{31}} \Leftrightarrow \mu_{12}\mu_{31} \le \mu_{11}\mu_{32} \Leftrightarrow \frac{\mu_{11}}{\mu_{12}} \ge \frac{\mu_{31}}{\mu_{32}}.$$

Furthermore, we have shown in the proof of Lemma A.3.2 that $\widehat{C}_4(u_1,u_2) \le \widehat{C}_6(u_1,u_2)$ for all $u_1, u_2 \in \mathbb{R}^+$, and if $u_1 \le \frac{u_2(\mu_{11}+\mu_{31})}{\mu_{22}}$, then $\widehat{C}_4(u_1,u_2) \le \widehat{C}_5(u_1,u_2)$. Hence, we can conclude that $E_{122} \le E_{121}$, and action a_{121} cannot be optimal.

The above discussion shows that either action a_{112} or a_{122} is optimal in state 1. If $E_{112} \le E_{122}$, then action a_{112} is no worse than action a_{122}, and hence action a_{112} is optimal. To the contrary, if $E_{122} < E_{112}$, then action a_{122} is better than action a_{112}, and we can conclude that action a_{122} is optimal. □

Theorems 3.4.1, 3.4.2, and 3.4.3 show the robustness of the form of the optimal policy to the service time distribution (i.e., threshold policies are optimal in both the Markovian and non-Markovian settings). However, the optimal value of the threshold for server 2 is sensitive to the service time distributions. Note that when $B > 0$, the threshold is arbitrary for the deterministic system for some values of u_1, u_2, and the service rates. However, this is not correct in the same generality for the Markovian system, see Andradóttir and Ayhan [6]. To further illustrate this point, define

$$r_M = \frac{(\mu_{11}+\mu_{21}+\mu_{31})(\mu_{11}\mu_{22}-\mu_{12}\mu_{21})}{(\mu_{12}+\mu_{22}+\mu_{32})(\mu_{21}\mu_{32}-\mu_{22}\mu_{31})}.$$

Assume that $B = 0$ and $m(1) = m(2) = 1$, and consider the optimal action in state 1. If the service times are deterministic, then Theorem 3.4.2 shows that server 2 works at station 1 if $r < 1$, or at station 2 if $r \geq 1$. In the Markovian system, server 2 works at station 1 if $r_M < 1$, or at station 2 if $r_M \geq 1$ (see Theorem 3.1 of Andradóttir and Ayhan [6]). Hence the threshold for server 2 (the state where server 2 starts working at station 2), is either 1 or 2 in both cases, but the value of the threshold may be different in Markovian and deterministic systems (e.g., for the system with $\mu_{11} = \mu_{32} = 2, \mu_{12} = \mu_{22} = 1, \mu_{21} = 0.5$, and $\mu_{31} = 0.1$, we have $r \simeq 0.94$ and $r_M \simeq 1.08$). This is consistent with the results of Andradóttir, Ayhan, and Down [11] for Markovian systems with server failures, where the optimal policy was found to be of threshold type (as for systems without server failures), with the value of the threshold depending on the server failures. Together these results suggest that the optimality of the threshold policy of Andradóttir and Ayhan [6] is quite robust to the assumptions it is derived under, but the value of the threshold must be determined using the circumstances of the problem at hand.

3.5 Numerical Results

In this section, we present numerical results that support the conjecture that the optimal server assignment policy is robust to the service time distributions. For this purpose, we provide simulation results for systems with two stations, two flexible servers, and non-exponential service times in Section 3.5.1 that show the effectiveness of the optimal policy for Markovian systems in non-Markovian settings. We then describe several heuristic server assignment policies for larger systems in Section 3.5.2, and provide numerical results that suggest that our heuristics yield good throughput performance and that their relative performance does not depend much on the service time distribution.

3.5.1 Optimal Policy for Small Systems

The results in Section 3.3 support the robustness of the optimal server assignment policy to the service time distribution. In particular, we showed that the optimal policy for Markovian systems with two servers and two stations is also optimal for systems with deterministic service time distributions and arbitrary buffer size and for systems with arbitrary service time distributions and zero buffer size. Here, we provide numerical results for systems with two servers, two stations, and an intermediate buffer of size one, two, or three that support the conjecture of robustness of the optimal server assignment policy to the service time distribution. (Because of the prohibitive amount of required computation time, systems with buffer sizes bigger than three were not considered.) Our results complement the numerical results of Andradóttir and Ayhan [6] who show that the optimal policy for Markovian systems with two stations and three servers also significantly outperforms the expedite policy of Van Oyen, Gel, and Hopp [67] (in which all servers work as a team that moves with each job through the line) for certain non-Markovian systems.

More specifically, we assume that the service times at stations 1 and 2 are either independent Erlang(2) random variables with CDF $F(x) = 1-e^{-2x}-2xe^{-2x}$ for $x \geq 0$ and squared coefficient of variation $c^2 = 0.5$, or independent hyperexponential random variables with CDF $F(x) = 2/3(1-e^{-2x}) + 1/3(1-e^{-x/2})$ for $x \geq 0$ and squared coefficient of variation $c^2 = 2$. These same distributions were used to study non-Markovian systems (with respect to the state space S) by Andradóttir and Ayhan [6]. The service rates are randomly generated from a uniform distribution in the interval $(0, 100)$. In particular, for each service time distribution and each value of $B \in \{1, 2, 3\}$, we create 5,000 sets of service rates $\{\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}\}$ and estimate the long-run average throughput under all non-idling server assignment policies in Π (idling policies cannot be optimal when $M = 2$, see Lemmas 3.2.1 and 3.2.2). Note that these systems can be modeled as Markov chains with state space larger than S because their service times come from the exponential distribution family. Hence the policy iteration algorithm could be used to determine the optimal policy. However, the set of Markovian stationary deterministic policies corresponding to their Markovian state space is larger than Π, and not all decision epochs are observable in this case. Hence simulation was instead used to compare the systems under the non-idling policies in Π. We estimate the long-run average throughput by simulating each system for 1,050,000 time units and truncating the first 50,000 time units. It is seen that the policy proven to be optimal for Markovian systems in Andradóttir, Ayhan, and Down [7] gives the best average throughput in all of the 5,000 random systems generated for the Erlang(2) and hyperexponential distributions and each choice of B.

3.5.2 Heuristics for Larger Systems

For larger systems (with three or more stations and three or more servers), it can be difficult to identify the optimal policy, even for Markovian systems. Furthermore, the optimal policy may be undesirable for use in practice (e.g., because it is complicated or difficult to implement). Thus, it is worthwhile to identify heuristic server assignment policies for larger systems that are easy to implement and yield good throughput performance. In this section, without loss of generality, we assume that $m(j) = 1$ for $j \in \{1, \ldots, N\}$ to facilitate the definitions of the heuristics (otherwise one can replace μ_{ij} by $\mu_{ij}/m(j)$ throughout). This implies that the service rates at each station are to be interpreted as the task completion rates for the servers at the stations, instead of the actual processing rates of the servers at the stations. We concentrate on situations where $M \geq N$, so that we have enough workforce to operate all stations simultaneously, but many of our ideas are applicable even when $M < N$. The only previous work we are aware of that studies heuristic server assignment policies for systems with more servers than stations is Andradóttir and Ayhan [6], but they consider the special case with $N = 2$.

Our heuristics have two parts, namely a primary assignment and a contingency plan, as in Andradóttir, Ayhan, and Down [7]. The primary assignment for each server indicates the station where the server is assigned as long as that station is not blocked or starved. The contingency plan shows where the server moves if the station (s)he is primarily assigned to is blocked or starved. It should be noted that these heuristics belong to Π.

More specifically, we present three heuristic primary assignments and six heuristic contingency plans, and compare the performance of different combinations of these. Let SR_j denote the sum of the service rates of servers with primary assignment at station $j \in \{1, \ldots, N\}$.

We consider the following primary assignments:

PA1: Maximize $\Pi_{j=1}^{N} SR_j$.

PA2: Maximize $\min_{j \in \{1,\ldots,N\}} SR_j$.

PA3: Assign server i to station $j^* = \arg\max_{k \in \{1,\ldots,N\}} \mu_{ik}$.

Note that PA2 has the objective of balancing the line, PA3 uses each server's capability as much as possible without taking into account other servers, and PA1 attempts to balance the line and take advantage of relative capabilities of the servers. PA1 was proposed by Andradóttir, Ayhan, and Down [7] for Markovian systems with $M = N$ but has not been considered for systems with $M > N$. PA2 and PA3 are new to this work.

We consider the following contingency plans:

CP1: When a station is starved but not blocked, servers having primary assignment at that station move to the closest upstream station that is operating (neither blocked or starved); when it is blocked, servers having primary assignment at that station move to the closest downstream station that is operating.

CP2: This is the local heuristic with the modification that all servers unable to work at their assigned station are working at the first station if the number of jobs in the system is less than the number of stations.

CP3: Whenever a server is unable to work at the station (s)he is primarily assigned to, (s)he works at the station that is operating (not blocked or starved) where (s)he has the highest rate compared to the other stations that are operating; i.e., server i works at station $j^* = \arg\max_{k \in I} \mu_{ik}$, where I is the set of stations that are operating.

CP4: Whenever a server is unable to work at the station (s)he is primarily assigned to, (s)he works at the station that is operating where (s)he has the highest relative rate with respect to the cumulative rate of all servers with primary assignment at that station (compared to the other stations that are operating); i.e., server i works at station $j^* = \arg\max_{k \in I} \mu_{ik}/SR_k$, where I is the set of stations that are operating.

CP5: This is a combination of CP1 and CP3. When a server is only starved, (s)he moves to the upstream station that is operating where (s)he has the highest rate; when a server is blocked, (s)he moves to the downstream station that is operating where (s)he has the highest rate.

CP6: This is a combination of CP1 and CP4 with a logic similar to that used in CP5.

Note that CP1 is the contingency plan of the local heuristic defined in Andradóttir, Ayhan, and Down [7]. By contrast, CP2 through CP6 are first proposed in this work. In CP1 servers create work for themselves, while in CP2 servers also try to push more jobs into the system when the total number of jobs is low. CP3 and CP4, like PA3, try to use each server's capabilities to the extent possible. CP4 also considers servers' relative rates compared to other servers, which implies that even though a server has a high service rate at one station, if the other servers at that station already have sufficiently high service rates, then this server is assigned to a station where (s)he is needed more. CP5 and CP6 attempt to use each server's capabilities to the extent possible, and at the same time they try to create job for the servers at the stations where they are primarily assigned to.

We consider all combinations of the three primary assignment policies PA1, PA2, and PA3 and the six contingency plans CP1 through CP6, with the exception that PA3 is not used with CP4 or CP6. This is because when PA3 is used, it is possible not to assign any servers to some stations primarily. Consequently, the initial service rate is zero in these stations, leading to frequent ties when deciding where the blocked or idle servers should move, and the CP4 and CP6 contingency plans lose their intended benefits.

We performed simulations to compare the different combinations of primary assignments and contingency plans. The service requirements at the stations were all i.i.d. with either the exponential, hyperexponential, or Erlang(2) distributions. The parameters of the hyperexponential and Erlang(2) distributions were selected as in Section 3.5.1, and the exponential distribution had rate 1. Hence, all the distributions had mean 1. We study systems with $M = N = 3$ and $M = 6, N = 4$. The smaller system has the same number of servers as stations, and the bigger system is a longer line with more servers than stations. Hence, we can observe the performance of our heuristics in both balanced and over-staffed tandem lines. For the system with $M = N = 3$, the service rates were drawn independently from a uniform distribution with range [0.5,2.5]. For the system with $M = 6, N = 4$, the service rates were drawn independently from a uniform distribution with range (0,100). In other words, we randomly generated sets of service rates $\{\mu_{ij}$, for $1 \leq i \leq M$ and $1 \leq j \leq N\}$, and each experiment consists of estimating the long-run average throughput of such a random system. The bigger system is already expected to have higher variability than the smaller system, and we increase its variability further by choosing its service rates from a larger range (corresponding to a more diverse workforce). In both cases, the same sets of service rates were used for systems with different service time distributions. Constant buffer sizes of one or four were used in each setting to understand the effects of the buffer size on the effectiveness of the heuristics. Tables 1, 2, 3, and 4 display the mean and half length of 95% confidence intervals for the long-run average throughput obtained by each heuristic under different system configurations. Each confidence interval in Tables 1, 2, 3, and 4 was obtained from 50,000, 10,000, 10,000, and 5,000 experiments, respectively. The number of simulation experiments was decreased when the buffer size and number of stations got bigger because of the long required computational time.

For each set of generated service rates, we also determine the optimal policy when the service times are exponentially distributed, and then employ this policy in systems with other service time distributions as well. The last row in Tables 1 through 4 gives the throughput achieved by the optimal policy for the exponential distribution for the three distributions we consider. This policy provides the best possible performance for systems with exponential service times, and is also used to benchmark the performance of our heuristics for systems with Erlang(2) or hyperexponential service times. (Although, the policy iteration algorithm could be used to determine the optimal server assignment policy for systems with Erlang(2) and hyperexponential distributed service times, we do not do this because the decision epochs would not be observable in this case.) Furthermore, the first row in each table gives the average throughput achieved by the expedite policy (see Section 3.5.1), also for benchmarking purposes. (In this case, the long-run average throughput of each system can be estimated using Monte Carlo simulation only, because the expected time spent at station j given the service rates $\mu_{1j}, \dots, \mu_{Mj}$ is equal to $\frac{1}{\sum_{i=1}^{M} \mu_{ij}}$, for $1 \leq j \leq N$).

For systems with three stations and three flexible servers, Tables 1 and 2 show that PA1 performs better than PA2 and PA3 for all three distributions, both buffer sizes, and all six contingency plans. Among the contingency plans, we see that CP3, CP4, CP5, and CP6 perform well, however in general CP4 and CP6 outperform the others, and CP4 seems to be the best contingency plan. Finally, we note that CP1 yields worse performance than the other five contingency plans for PA1 (and

Table 1: Performance of Heuristics for Systems with Three Stations, Three Servers, and Common Buffer Size One

Policy	*exponential*	*Erlang*	*hyperexponential*
expedite	1.449 ± 0.003	1.449 ± 0.003	1.449 ± 0.003
PA1-CP1	1.675 ± 0.002	1.713 ± 0.002	1.583 ± 0.002
PA1-CP2	1.676 ± 0.002	1.714 ± 0.002	1.584 ± 0.002
PA1-CP3	1.692 ± 0.002	1.724 ± 0.002	1.599 ± 0.002
PA1-CP4	1.698 ± 0.002	1.731 ± 0.002	1.605 ± 0.002
PA1-CP5	1.691 ± 0.002	1.726 ± 0.002	1.601 ± 0.002
PA1-CP6	1.693 ± 0.002	1.729 ± 0.002	1.602 ± 0.002
PA2-CP1	1.658 ± 0.002	1.691 ± 0.002	1.560 ± 0.002
PA2-CP2	1.656 ± 0.002	1.690 ± 0.002	1.559 ± 0.002
PA2-CP3	1.676 ± 0.002	1.705 ± 0.002	1.574 ± 0.002
PA2-CP4	1.679 ± 0.002	1.709 ± 0.002	1.582 ± 0.002
PA2-CP5	1.672 ± 0.002	1.704 ± 0.002	1.576 ± 0.002
PA2-CP6	1.673 ± 0.002	1.705 ± 0.002	1.578 ± 0.002
PA3-CP1	1.614 ± 0.002	1.623 ± 0.002	1.526 ± 0.002
PA3-CP2	1.606 ± 0.002	1.628 ± 0.002	1.520 ± 0.002
PA3-CP3	1.641 ± 0.002	1.660 ± 0.002	1.550 ± 0.002
PA3-CP5	1.635 ± 0.002	1.656 ± 0.002	1.544 ± 0.002
exp opt	1.713 ± 0.002	1.747 ± 0.002	1.615 ± 0.002

usually for also PA2 and PA3) even though $M = N$ in this example. Thus, the five heuristics composed of PA1 and one of CP2 through CP6 outperform the local heuristic of Andradóttir, Ayhan, and Down [7] in this example, even though $M = N$ (the local heuristic is designed for such systems). Nevertheless, CP1 is easy to implement compared to some better performing contingency plans, and it may also increase server motivation because every server concentrates on his/her own station, by either working at that station or creating work for that station.

Similar conclusions follow for the system with four stations and six flexible servers, see Tables 3 and 4. Even though there are now more servers than stations, and the variability in the service rates is larger (reflecting a more diverse set of servers), our heuristics remain robust.

Tables 1 through 4 also show that in the Markovian setting, our best heuristic

Table 2: Performance of Heuristics for Systems with Three Stations, Three Servers, and Common Buffer Size Four

Policy	*exponential*	*Erlang*	*hyperexponential*
expedite	1.450 ± 0.004	1.450 ± 0.004	1.450 ± 0.004
PA1-CP1	1.725 ± 0.003	1.747 ± 0.003	1.663 ± 0.003
PA1-CP2	1.725 ± 0.003	1.753 ± 0.003	1.665 ± 0.003
PA1-CP3	1.732 ± 0.003	1.754 ± 0.003	1.682 ± 0.003
PA1-CP4	1.742 ± 0.003	1.763 ± 0.003	1.689 ± 0.003
PA1-CP5	1.730 ± 0.003	1.758 ± 0.003	1.683 ± 0.003
PA1-CP6	1.740 ± 0.003	1.762 ± 0.003	1.687 ± 0.003
PA2-CP1	1.699 ± 0.003	1.723 ± 0.003	1.640 ± 0.003
PA2-CP2	1.703 ± 0.003	1.727 ± 0.003	1.639 ± 0.003
PA2-CP3	1.711 ± 0.003	1.733 ± 0.003	1.660 ± 0.003
PA2-CP4	1.717 ± 0.003	1.739 ± 0.003	1.664 ± 0.003
PA2-CP5	1.707 ± 0.003	1.734 ± 0.003	1.658 ± 0.003
PA2-CP6	1.714 ± 0.003	1.736 ± 0.003	1.659 ± 0.003
PA3-CP1	1.627 ± 0.003	1.640 ± 0.003	1.592 ± 0.003
PA3-CP2	1.638 ± 0.003	1.656 ± 0.003	1.595 ± 0.003
PA3-CP3	1.660 ± 0.003	1.676 ± 0.003	1.621 ± 0.003
PA3-CP5	1.675 ± 0.003	1.675 ± 0.003	1.620 ± 0.003
exp opt	1.763 ± 0.003	1.777 ± 0.003	1.699 ± 0.003

server assignment policy (i.e., PA1 with CP4) results in near-optimal mean throughput; more specifically its performance is between 97% and 99% of the optimal throughput for the Markovian systems we consider. We observe that all of our heuristics outperform the expedite policy (by margins as high as 45%). Moreover, the optimal policy of the Markovian system also performs well for the other two distributions. Nevertheless, we do not consider it to be a good heuristic because it can be difficult to implement for actual systems. Finally, we observe that the average throughput values of our heuristics are affected by the variability in the service times. More specifically, for each heuristic, the systems with Erlang(2) service time distributions had the best throughput, and systems with hyperexponential service time distributions had the worst throughput. Such a result is expected, because Erlang(2) and hyperexponential distributions have the smallest and biggest coefficients of variations,

Table 3: Performance of Heuristics for Systems with Four Stations, Six Servers, and Common Buffer Size One

Policy	*exponential*	*Erlang*	*hyperexponential*
expedite	71.626 ± 0.188	71.626 ± 0.188	71.626 ± 0.188
PA1-CP1	97.123 ± 0.125	98.776 ± 0.122	93.666 ± 0.130
PA1-CP2	97.190 ± 0.129	98.897 ± 0.128	93.763 ± 0.135
PA1-CP3	98.129 ± 0.137	100.022 ± 0.137	95.101 ± 0.134
PA1-CP4	98.879 ± 0.138	101.484 ± 0.139	95.761 ± 0.143
PA1-CP5	98.191 ± 0.139	100.358 ± 0.137	95.210 ± 0.139
PA1-CP6	98.783 ± 0.134	101.386 ± 0.130	95.679 ± 0.138
PA2-CP1	95.343 ± 0.130	96.612 ± 0.129	91.414 ± 0.135
PA2-CP2	95.400 ± 0.129	96.810 ± 0.128	91.492 ± 0.137
PA2-CP3	96.013 ± 0.130	98.013 ± 0.130	92.641 ± 0.140
PA2-CP4	96.880 ± 0.129	98.561 ± 0.130	93.143 ± 0.138
PA2-CP5	96.213 ± 0.135	98.142 ± 0.132	92.732 ± 0.137
PA2-CP6	96.783 ± 0.133	98.490 ± 0.132	93.042 ± 0.135
PA3-CP1	89.425 ± 0.131	91.794 ± 0.129	84.985 ± 0.136
PA3-CP2	89.492 ± 0.139	91.951 ± 0.130	85.025 ± 0.139
PA3-CP3	91.051 ± 0.134	93.740 ± 0.133	86.783 ± 0.135
PA3-CP5	92.194 ± 0.137	94.840 ± 0.132	87.416 ± 0.135
exp opt	101.905 ± 0.121	103.734 ± 0.127	97.426 ± 0.138

respectively.

We conclude that the heuristic that starts with a good primary assignment and uses the relative efficiency of the servers in the contingency plan (i.e., the heuristic comprised of PA1 and CP4) has the best performance among the heuristics that we considered. The only problem with this heuristic involves ease of implementation, especially in longer lines. The heuristic that uses PA1 with CP6 has performance very close to the PA1-CP4 combination, and is easier to apply in actual systems. Hence, for shorter lines (where determining what stations in the line have work is not time consuming) we recommend the use of PA1 with CP4, but PA1 with CP6 may be an attractive option for longer lines (where identifying stations with work for the entire line is cumbersome).

Table 4: Performance of Heuristics for Systems with Four Stations, Six Servers, and Common Buffer Size Four

Policy	*exponential*	*Erlang*	*hyperexponential*
expedite	71.751 ± 0.267	71.751 ± 0.2670	71.751 ± 0.267
PA1-CP1	99.234 ± 0.153	101.881 ± 0.1584	95.313 ± 0.161
PA1-CP2	99.391 ± 0.154	102.084 ± 0.1633	95.439 ± 0.165
PA1-CP3	100.783 ± 0.163	103.453 ± 0.1732	96.431 ± 0.176
PA1-CP4	101.239 ± 0.168	104.064 ± 0.1719	97.319 ± 0.173
PA1-CP5	100.899 ± 0.166	103.148 ± 0.1692	96.642 ± 0.171
PA1-CP6	101.193 ± 0.170	104.055 ± 0.1704	97.215 ± 0.174
PA2-CP1	96.942 ± 0.160	99.643 ± 0.1583	92.611 ± 0.171
PA2-CP2	97.140 ± 0.169	99.742 ± 0.1683	92.790 ± 0.173
PA2-CP3	98.423 ± 0.165	100.703 ± 0.1690	93.940 ± 0.179
PA2-CP4	99.542 ± 0.175	101.841 ± 0.1714	95.444 ± 0.175
PA2-CP5	98.485 ± 0.165	100.627 ± 0.1703	94.001 ± 0.174
PA2-CP6	99.602 ± 0.170	101.873 ± 0.1701	95.464 ± 0.173
PA3-CP1	91.033 ± 0.167	93.493 ± 0.1674	86.356 ± 0.173
PA3-CP2	91.241 ± 0.164	93.664 ± 0.1701	87.564 ± 0.172
PA3-CP3	92.315 ± 0.170	94.948 ± 0.1722	88.654 ± 0.174
PA3-CP5	92.334 ± 0.171	95.130 ± 0.1689	88.678 ± 0.175
exp opt	104.425 ± 0.150	107.942 ± 0.1583	100.420 ± 0.164

3.6 Conclusion

We have studied non-Markovian tandem lines with finite buffers. For lines with two stations and two or three flexible servers, we identified the optimal server assignment policy for systems with deterministic service times and any finite buffer, and for systems with general service times and zero buffer. We also provided numerical results that strongly support the conjecture that the optimal server assignment policy is robust to the service time distribution for arbitrary buffer sizes. Our study supports the conjecture that the results of Andradóttir, Ayhan, and Down [7] and Andradóttir and Ayhan [6], obtained for Markovian systems, also hold for non-Markovian systems.

Finally, we proposed heuristics for larger systems that were effective for different service time distributions and system configurations (e.g., we could achieve up to 99%

of the optimal throughput in Markovian systems). In the process, we noted that the best performing heuristics take the relative efficiency of servers at stations into consideration when assigning them dynamically to tasks. Furthermore, the heuristics that perform well in the Markovian setting (including the optimal policy of the Markovian system) also perform well for systems with different service time distributions. These results support the conclusion that effective server assignment policies are robust to service time distributions, even in larger systems.

CHAPTER IV

FLEXIBLE SERVERS IN UNDERSTAFFED TANDEM LINES

4.1 Introduction

In this chapter, we consider a production line with $N > 2$ stations and $M \geq 2$ flexible servers. We assume that the line is understaffed (so that $N > M$) and, without loss of generality, that the expected service requirements at each station are equal to one (i.e., $m(j) = 1$ for all $j \in \{1, \ldots, N\}$). We further assume that the other assumptions described in Section 3.1 hold and we use the notation described there.

Understaffed tandem lines with finite buffer spaces and more than one server are quite typical in the garment manufacturing industry, assembly plants, and warehouses (e.g., Bartholdi and Eisenstein [17, 18] and Lim and Yang [52]). In these settings, labor costs constitute a big proportion of the operating costs (see, e.g., Bartholdi and Hackman [20]), and hence it becomes an important task to effectively use the workforce.

Other researchers have addressed the dynamic server assignment problem when $N \geq M = 1$ (Andradóttir, Ayhan, and Down [7], Duenyas, Gupta, and Olsen [29], Iravani, Posner, and Buzacott [47]) and $2 = N \leq M$ (Andradóttir and Ayhan [6], Andradóttir, Ayhan, and Down [7, 11], Kırkızlar, Andradóttir, and Ayhan [50]). However, unlike the earlier work, we analyze understaffed lines in the presence of both finite buffers and multiple servers.

The benefits of partial flexibility in serial production lines also have been studied by some researchers (Andradóttir, Ayhan, and Down [10], Hopp, Tekin, and Van Oyen [45]). However, the works about partial flexibility in tandem systems only consider lines with equal number of servers and stations, or systems with more servers than stations. By contrast, in this work we consider an understaffed tandem line where stations do not have workers that are assigned to them initially. Hence, it is not a straightforward task to determine the bottleneck stations, and sometimes it is not possible to label any station as bottleneck. Furthermore, we consider a longer line than Andradóttir, Ayhan, and Down [10], and a different objective, release policy, and collaboration structure than Hopp, Tekin, and Van Oyen [45].

The remainder of this chapter is organized as follows. In Section 4.2, we characterize the optimal assignment policy for systems with deterministic service requirements, three stations and two flexible servers. In Section 4.3, we analyze the corresponding Markovian system with small buffer sizes and different flexibility structures. In Section 4.4, we propose heuristic server assignment policies, show that very simple server assignment rules can achieve near-optimal throughput in Markovian systems with larger buffer sizes, and provide some guidelines about how to select the best flexibility structure. In Section 4.5, we make some concluding remarks.

4.2 Deterministic Systems

In this section we determine the optimal server assignment policy for systems with deterministic service times. We also show that partial server flexibility can attain all the benefits of full flexibility and provide the conditions for each partial training strategy to be optimal.

Consider the following "allocation" linear program (LP) with decision variables λ

and $\{\delta_{ij}\}$:

$$
\begin{aligned}
\max \quad & \lambda \\
\text{s.t.} \quad & \delta_{11}\mu_{11} + \delta_{21}\mu_{21} \geq \lambda, \\
& \delta_{12}\mu_{12} + \delta_{22}\mu_{22} \geq \lambda, \\
& \delta_{13}\mu_{13} + \delta_{23}\mu_{23} \geq \lambda, \\
& \delta_{11} + \delta_{12} + \delta_{13} \leq 1, \\
& \delta_{21} + \delta_{22} + \delta_{23} \leq 1, \\
& \delta_{ij} \geq 0, \ \text{for all } i \in \{1,2\} \text{ and } j \in \{1,2,3\}.
\end{aligned}
$$

Let λ^* denote the optimal value of λ for this LP. Andradóttir, Ayhan, and Down [8] show that λ^* is the maximal capacity of an infinite-buffered tandem line with three stations, two flexible servers, and outside arrivals. Lemma 3.2.4 shows that λ^* is an upper bound on the throughput of our finite-buffered tandem line as well. Moreover, if $\{\delta^*_{ij}\}$ are optimal values of $\{\delta_{ij}\}$, then δ^*_{ij}, where $i \in \{1,2\}$ and $j \in \{1,2,3\}$, can be interpreted as the long-run proportion of time server i should be assigned to station j in order to achieve the maximal capacity λ^*.

The LP described in the previous paragraph has five constraints (in addition to the nonnegativity constraints) and seven variables. Since λ is always positive, we can conclude that at least two elements of the set $\{\delta_{ij}\}$ are zero (this also follows directly from Proposition 2 of Andradóttir, Ayhan, and Down [8]). This proves that four skills are sufficient to achieve the maximal capacity in systems with infinite buffers, and hence it is of interest to determine to what extent this is also true for systems with finite buffers. Consequently, we first analyze the system under the assumption that the two servers have a total of four skills, and then show the implications of this result for systems with fully cross-trained servers. We let (s_1, s_2) be the state of our system, where s_1 (s_2) is the number of jobs that have already been processed at station 1 (2) and are either waiting for service or being processed at station 2 (3).

Then,

$$S = \{(s_1, s_2) : \; s_1 \in \{0, 1, \dots, B_2 + 2\}, \; s_2 \in \{0, 1, \dots, B_3 + 2\}, \; s_1 + s_2 \le B_2 + B_3 + 3\}$$

is our state space. We say that a station is "operating" if that station is neither starved nor blocked.

In the following propositions, without loss of generality, we assume that the system initially starts in a state $s^0 = (s_1^0, s_2^0)$ where all the stations are operating and the jobs at each station have not started service yet, so that they all have the full service requirement. We let $S^0 \subseteq S$ be the set of such states. If the system does not start in a state in S^0, initially any policy that takes the process to such a state may be employed. Recall that we assume that there is at least one server with positive service rate at each station (since $\sum_{i=1}^{M} \mu_{ij} > 0$ for $j \in \{1, \dots, N\}$). For the system to reach a state in S^0, we can successively assign all servers to the station $j \in \{1, \dots, N\}$ that is closest to the end of the line among the stations that are either blocked or have jobs with remaining service requirements not equal to one (i.e., jobs that have already started their service). When there are no such stations left, we can assign all servers to the station $j \in \{1, \dots, N-1\}$ that is closest to the beginning of the line among the stations that are preceding a station that is starved. When this is no longer possible (because there are no such stations left), the system is in a state $s^0 \in S^0$ satisfying the conditions mentioned previously. This is achievable in finite time and will not affect the long-run average throughput.

We will consider systems with one dedicated and one fully flexible server in Section 4.2.1. Then, we will study systems with two partially flexible servers in Section 4.2.2. Finally, in Section 4.2.3, we will determine the critical skills needed to achieve the optimal performance of systems with fully flexible servers.

4.2.1 Systems with One Dedicated and One Fully Flexible Server

In this section, we provide three propositions identifying the optimal server assignment policies for systems with one dedicated and one fully flexible server. Without loss of generality, we assume that the first server is the dedicated server, since the other case is equivalent to this one by relabeling the servers.

Proposition 4.2.1 *Assume that $\mu_{12} = \mu_{13} = 0$ and that the system is initially in a state $s^0 \in S^0$. Then the following server assignment policy is optimal for a deterministic system with three stations and two servers:*

- *every time the system reaches state s^0, assign server 1 to station 1 until one job is completed at station 1, then (if this does not cause the system to return to state s^0) idle server 1 until the next time the process hits state s^0 ;*
- *every time the system reaches state s^0, assign server 2 to station 2 until one job is completed at station 2, then assign server 2 to station 3 until one job is completed at station 3, and finally (if the system is not already in state s^0) assign server 2 to station 1 until the next time the process hits state s^0.*

Moreover, this policy attains the maximal capacity of the system, regardless of the intermediate buffer sizes.

Proof: When $\mu_{12} = \mu_{13} = 0$, the allocation LP takes the simpler form:

$$\begin{aligned}
\max \quad & \lambda \\
\text{s.t.} \quad & \mu_{11} + \delta_{21}\mu_{21} \geq \lambda, && (2) \\
& \delta_{22}\mu_{22} \geq \lambda, && (3) \\
& \delta_{23}\mu_{23} \geq \lambda, && (4) \\
& \delta_{21} + \delta_{22} + \delta_{23} \leq 1, \\
& \delta_{2j} \geq 0, \text{ for all } j \in \{1, 2, 3\}.
\end{aligned}$$

Note that our assumptions on the service rates in Section 3.1 imply that $\mu_{11}, \mu_{22}, \mu_{23} > 0$, and our assumption that server 2 is fully flexible implies that $\mu_{21} > 0$. If $\mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$, we see that the left-hand side of the constraint (2) is always bigger than the left-hand sides of the constraints (3) and (4), and hence $\delta^*_{21} = 0$ in the optimal solution. Then, we find $\delta^*_{22} = \frac{\mu_{23}}{\mu_{22}+\mu_{23}}$ and $\delta^*_{23} = \frac{\mu_{22}}{\mu_{22}+\mu_{23}}$, by solving the equations $\delta^*_{22}\mu_{22} = \delta^*_{23}\mu_{23}$ and $\delta^*_{22} + \delta^*_{23} = 1$. On the other hand, if $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$, then we see that in the optimal solution all the constraints (2), (3), and (4) will be tight. Then, we find $\delta^*_{22} = \frac{\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}}$, $\delta^*_{23} = \frac{\mu_{22}(\mu_{11}+\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}}$, and $\delta^*_{21} = 1 - \delta^*_{22} - \delta^*_{23}$, by solving the equations $\mu_{11} + \delta^*_{21}\mu_{21} = \delta^*_{22}\mu_{22} = \delta^*_{23}\mu_{23}$ and $\delta^*_{21} + \delta^*_{22} + \delta^*_{23} = 1$. Consequently, the value of λ^* in the optimal solution is as follows:

$$\lambda^* = \begin{cases} \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}} & \text{if } \mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}, \\ \frac{\mu_{22}\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} & \text{if } \mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}. \end{cases} \tag{5}$$

Now, consider the policy described in the proposition and assume that the system is in state $s^0 = (s^0_1, s^0_2)$ at time T. When $\mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$ (i.e., $\frac{1}{\mu_{11}} < \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$), server 1 can complete the job at station 1 before server 2 finishes processing a job at stations 2 and 3; hence server 2 does not help server 1. The states of the system and the remaining service requirements for the jobs at each station will be as in Table 9 in Appendix B.1. When $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$ (i.e., $\frac{1}{\mu_{11}} \geq \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$), server 2 finishes processing a job at stations 2 and 3, and helps server 1 afterwards. The states of the system and the remaining service requirements for the jobs at each station will be as in Table 10 in Appendix B.1. We see that the system regenerates each time it hits the state s^0, that there is one departure from the system during each regenerative cycle, and that the length of the cycle is equal to the reciprocal of equation (5). Hence we can conclude that the policy given in the proposition is optimal. □

The optimal policy described in Proposition 4.2.1 also balances the line. In every regenerative cycle, the first server stops pushing new jobs into the system upon completion of one service at station 1. Moreover, the proof of the proposition shows that the first server is idle only if the service rate of this server is high enough so that utilizing him/her more would only cause blocking at the first station, rather than increasing the throughput. The second server helps the first server at station 1 if (s)he is fast enough to complete jobs at stations 2 and 3 before the first server completes one job at station 1. Finally, note that it is possible to attain the maximal capacity of the four-skilled system above even with three skills (i.e., with $\mu_{21} = 0$) when $\mu_{11} \geq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$.

Proposition 4.2.2 *Assume that $\mu_{11} = \mu_{13} = 0$ and that the system is initially in a state $s^0 \in S^0$. Then the following server assignment policy is optimal for a deterministic system with three stations and two servers:*

- *every time the system reaches state s^0, assign server 1 to station 2 until one job is completed at station 2, then (if this does not cause the system to return to state s^0) idle server 1 until the next time the process hits state s^0;*

- *every time the system reaches state s^0, do either of the following:*
  (a) assign server 2 to station 3 until one job is completed at station 3, then assign server 2 to station 1 until one job is completed at station 1, and finally (if the process is not already in state s^0) assign server 2 to station 2 until the next time the process hits state s^0; or
  (b) assign server 2 to station 1 until one job is completed at station 1, then assign server 2 to station 3 until one job is completed at station 3, and finally (if the system is not already in state s^0) assign server 2 to station 2 until the next time the process hits state s^0.

Moreover, this policy attains the maximal capacity of the system, regardless of the intermediate buffer sizes.

Proof: Proceeding as in the proof of Proposition 4.2.1, the value of λ in the optimal solution of the allocation LP can be found as follows:

$$\lambda^* = \begin{cases} \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}} & \text{if } \mu_{12} > \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}, \\ \frac{\mu_{21}\mu_{23}(\mu_{12}+\mu_{22})}{\mu_{21}\mu_{23}+\mu_{22}\mu_{23}+\mu_{21}\mu_{22}} & \text{if } \mu_{12} \leq \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}. \end{cases} \tag{6}$$

Consider the policy described in the proposition and assume that the system is in state $s^0 = (s_1^0, s_2^0)$ at time T. When $\mu_{12} > \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$, we obtain the states of the system and the remaining service requirements for the jobs at each station as in Table 11 or Table 13 in Appendix B.1, if we use assignment rule (a) or (b), respectively. When $\mu_{12} \leq \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$, we obtain Table 12 or Table 14 in Appendix B.1, if we use assignment rule (a) or (b), respectively. Optimality of the policy in the proposition can be shown using similar arguments as in the proof of Proposition 4.2.1. □

Similar to the previous proposition, we see that the dedicated server is idle only if his/her service rate is high enough so that (s)he would not increase the throughput by being utilized more. Moreover, by idling this server at certain times, we are able to keep both stations operating rather than causing starvation or blocking at the second station. Finally, note that the maximal capacity of the four-skilled system can be reached even with three skills (i.e., with $\mu_{22} = 0$) when $\mu_{12} \geq \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$.

Proposition 4.2.3 *Assume that $\mu_{11} = \mu_{12} = 0$ and that the system is initially in a state $s^0 \in S^0$. Then the following server assignment policy is optimal for a deterministic system with three stations and two servers:*

- *every time the system reaches state s^0, assign server 1 to station 3 until one job is completed at station 3, then (if this does not cause the system to return to state s^0) idle server 1 until the next time the process hits state s^0;*

- *every time the system reaches state s^0, assign server 2 to station 2 until one job is completed at station 2, then assign server 2 to station 1 until one job* *is completed at station 1, and finally (if the system is not already in state s^0) assign server 2 to station 3 until the next time the process hits state s^0.*

Moreover, this policy attains the maximal capacity of the system, regardless of the intermediate buffer sizes.

Proof: Proceeding as in the proof of Proposition 4.2.1, the value of λ in the optimal solution of the allocation LP can be found as follows:

$$\lambda^* = \begin{cases} \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}} & \text{if } \mu_{13} > \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}}, \\ \frac{\mu_{21}\mu_{22}(\mu_{13}+\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} & \text{if } \mu_{13} \leq \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}}. \end{cases} \tag{7}$$

Consider the policy described in the proposition and assume that the system is in state $s^0 = (s_1^0, s_2^0)$ at time T. When $\mu_{13} > \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}}$, we obtain the states of the system and the remaining service requirements for the jobs at each station as in Table 15 in Appendix B.1. When $\mu_{13} \leq \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}}$, we obtain Table 16 in Appendix B.1. Optimality of the policy in the proposition can be shown using similar arguments as in the proof of Proposition 4.2.1. □

We observe that when the dedicated server works at station 3, the optimal policy is "symmetrical" with respect to the case where the dedicated server works at station 1. Now, the flexible server starts working at station 2, moves to station 1 upon completion of the task at station 2, and finally moves to station 3. The decision of employing or idling the dedicated server is also a symmetrical one. Idling occurs only if the service rate of the dedicated server is high enough so that utilizing him/her more would only cause starvation at the third station, and not increase the throughput. Finally, note that the maximal capacity of the four-skilled system can be reached even with three skills (i.e., with $\mu_{23} = 0$) when $\mu_{13} \geq \frac{\mu_{21}\mu_{22}}{\mu_{22}+\mu_{22}}$.

We observe that whenever there is a dedicated and a fully flexible server, the optimal server assignment policy for the flexible server focuses on keeping the dedicated server's station operating at all times. Furthermore, when the dedicated server is at station 2, the policies that have the flexible server giving preference to prevent either blocking or starvation of station 2 are both optimal. Previous work (see the local heuristic in Andradóttir, Ayhan, and Down [7]) puts priority on preventing blocking. However, with deterministic service times, one can ensure that the dedicated server's station is always operating as long as the right assignment policy is used (in random systems it is generally not possible to avoid blocking or starvation). Finally, using similar arguments as in the proof of Proposition 4.2.2, one can show that, when the dedicated server is at station 1 (3), the policy that assigns the flexible server to station 3 (1) before station 2 in every regenerative cycle is also optimal. Hence, the flexible server can process the jobs at the stations where there is no dedicated server in arbitrary order, without any efficiency loss.

4.2.2 Systems with Two Partially Flexible Servers

In this section, we consider systems where each server is partially cross-trained; i.e., each server is capable of processing jobs at two stations. Again, we only consider three different cross-training strategies, and the optimal server assignment policy for the other three cases can be deduced from these by simply relabeling the servers.

Proposition 4.2.4 *Assume that $\mu_{13} = \mu_{22} = 0$ and that the system is initially in a state $s^0 \in S^0$. Then the following server assignment policy is optimal for a deterministic system with three stations and two servers:*

- *every time the system reaches state s^0, assign server 1 to station 2 until one job is completed at station 2, then (if the system is not already in state s^0) assign server 1 to station 1 until one job is completed at station 1, and finally (if the system is still not in state s^0) idle server 1 until the next time the process hits state s^0;*
- *every time the system reaches state s^0, assign server 2 to station 3 until one job is completed at station 3, then (if the system is not already in state s^0) assign*

server 2 to station 1 until one job is completed at station 1, and finally (if the system is still not in state s^0) idle server 2 until the next time the process hits state s^0.

Moreover, this policy attains the maximal capacity of the system, regardless of the intermediate buffer sizes.

Proof: When $\mu_{13} = \mu_{22} = 0$, the allocation LP takes the simpler form:

$$
\begin{aligned}
\max \quad & \lambda \\
\text{s.t.} \quad & \delta_{11}\mu_{11} + \delta_{21}\mu_{21} \geq \lambda, && (8) \\
& \delta_{12}\mu_{12} \geq \lambda, && (9) \\
& \delta_{23}\mu_{23} \geq \lambda, && (10) \\
& \delta_{11} + \delta_{12} \leq 1, \\
& \delta_{22} + \delta_{23} \leq 1, \\
& \delta_{11}, \delta_{12}, \delta_{22}, \delta_{23} \geq 0.
\end{aligned}
$$

Note that $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{23} > 0$ under our assumptions. If $\mu_{12} \leq \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$, we see that the left-hand side of the constraint (9) is always less than or equal to the left-hand sides of the constraints (8) and (10), and hence $\delta_{11}^* = 0$ in the optimal solution. Then, we find $\delta_{21}^* = \frac{\mu_{23}}{\mu_{21}+\mu_{23}}$ and $\delta_{23}^* = \frac{\mu_{21}}{\mu_{21}+\mu_{23}}$ by solving the equations $\delta_{21}^*\mu_{21} = \delta_{23}^*\mu_{23}$ and $\delta_{21}^* + \delta_{23}^* = 1$. If $\mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$, similar arguments show that $\delta_{11}^* = \frac{\mu_{12}}{\mu_{11}+\mu_{12}}$ and $\delta_{12}^* = \frac{\mu_{11}}{\mu_{11}+\mu_{12}}$. On the other hand, if $\mu_{12} > \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$ and $\mu_{23} > \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$, then we see that in the optimal solution all the constraints (8), (9), and (10) will be tight. Then, we find $\delta_{12}^* = \frac{\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}}$, $\delta_{23}^* = \frac{\mu_{12}(\mu_{11}+\mu_{21})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}}$, $\delta_{11}^* = 1 - \delta_{12}^*$ and $\delta_{22}^* = 1 - \delta_{23}^*$, by solving the equations $\delta_{11}^*\mu_{11} + \delta_{21}^*\mu_{21} = \delta_{12}^*\mu_{12} = \delta_{23}^*\mu_{23}$, $\delta_{11}^* + \delta_{12}^* = 1$, and $\delta^*_{21} + \delta^*_{23} = 1$. Consequently, the value of λ^* in the optimal solution is as follows:

$$\lambda^* = \begin{cases} \mu_{12} & \text{if } \mu_{12} \le \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}, \\ \mu_{23} & \text{if } \mu_{23} \le \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}, \\ \frac{\mu_{12}\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} & \text{if } \mu_{12} > \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}} \text{ and } \mu_{23} > \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}. \end{cases} \tag{11}$$

Note that $\mu_{12} \le \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$ (i.e., $\frac{1}{\mu_{12}} \ge \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$) and $\mu_{23} \le \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$ (i.e., $\frac{1}{\mu_{23}} \ge \frac{1}{\mu_{11}} + \frac{1}{\mu_{12}}$) cannot hold at the same time, since we assumed that all the service rates are finite.

Consider the policy described in the proposition and assume that the system is in state $s^0 = (s^0_1, s^0_2)$ at time T. When $\mu_{12} \le \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$, we obtain the states of the system and the remaining service requirements for the jobs at each station as in Table 17 in Appendix B.2. When $\mu_{23} \le \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$, we obtain Table 18 in Appendix B.2. When $\mu_{12} > \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$ and $\mu_{23} > \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$ (i.e., $\frac{1}{\mu_{12}} < \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$ and $\frac{1}{\mu_{23}} < \frac{1}{\mu_{11}} + \frac{1}{\mu_{12}}$), we obtain Table 19 in Appendix B.2. Optimality of the policy in the proposition can be shown using similar arguments as in the proof of Proposition 4.2.1. □

Note that in the system with $\mu_{13} = \mu_{22} = 0$, the maximal capacity of the four-skilled system can be reached even with three skills when either $\mu_{12} \le \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$ or $\mu_{23} \le \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$. In the former case (when $\mu_{12} \le \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$), the optimal throughput can be achieved with $\mu_{11} = 0$; in the latter case (when $\mu_{23} \le \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$), the optimal throughput can be achieved with $\mu_{21} = 0$.

Proposition 4.2.5 *Assume that $\mu_{13} = \mu_{21} = 0$ and that the system is initially in a state $s^0 \in S^0$. Then the following server assignment policy is optimal for a deterministic system with three stations and two servers:*

- *every time the system reaches state s^0, assign server 1 to station 1 until one job is completed at station 1, then (if the system is not already in state s^0) assign server 1 to station 2 until one job is completed at station 2, and finally (if the system is still not in state s^0) idle server 1 until the next time the process hits state s^0;*

- *every time the system reaches state s^0, assign server 2 to station 3 until one job is completed at station 3, then (if the system is not already in state s^0) assign server 2 to station 2 until one job is completed at station 2, and finally (if the system is still not in state s^0) idle server 2 until the next time the process hits state s^0.*

Moreover, this policy attains the maximal capacity of the system, regardless of the intermediate buffer sizes.

Proof: Proceeding as in the proof of Proposition 4.2.4, the value of λ in the optimal solution of the allocation LP can be found as follows:

$$\lambda^* = \begin{cases} \mu_{11} & \text{if } \mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}, \\ \mu_{23} & \text{if } \mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}, \\ \frac{\mu_{11}\mu_{23}(\mu_{12}+\mu_{22})}{\mu_{11}\mu_{22}+\mu_{12}\mu_{23}+\mu_{11}\mu_{23}} & \text{if } \mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}} \text{ and } \mu_{23} > \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}. \end{cases} \tag{12}$$

Note that $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$ (i.e., $\frac{1}{\mu_{11}} \geq \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$) and $\mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$ (i.e., $\frac{1}{\mu_{23}} \geq \frac{1}{\mu_{11}} + \frac{1}{\mu_{12}}$) cannot hold at the same time, since we assumed that all the service rates are finite.

Consider the policy described in the proposition and assume that the system is in state $s^0 = (s_1^0, s_2^0)$ at time T. When $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$, we obtain the states of the system and the remaining service requirements for the jobs at each station as in Table 20 in Appendix B.2. When $\mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$, we obtain Table 21 in Appendix B.2. When $\mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$ and $\mu_{23} > \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$, we obtain Table 22 in Appendix B.2. Optimality of the policy in the proposition can be shown using similar arguments as in the proof of Proposition 4.2.1. □

Note that in the system with $\mu_{13} = \mu_{21} = 0$, the maximal capacity of the four-skilled system can be reached even with three skills when either $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$ or $\mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$. In the former case (when $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$), the optimal throughput can be achieved with $\mu_{12} = 0$; in the latter case (when $\mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$), the optimal throughput can be achieved with $\mu_{22} = 0$.

Proposition 4.2.6 *Assume that $\mu_{12} = \mu_{21} = 0$ and that the system is initially in a state $s^0 \in S^0$. Then the following server assignment policy is optimal for a deterministic system with three stations and two servers:*

- *every time the system reaches state s^0, assign server 1 to station 1 until one job is completed at station 1, then (if the system is not already in state s^0) assign server 1 to station 3 until one job is completed at station 3, and finally (if the system is still not in state s^0) idle server 1 until the next time the process hits state s^0;*

- *every time the system reaches state s^0, assign server 2 to station 2 until one job is completed at station 2, then (if the system is not already in state s^0) assign server 2 to station 3 until one job is completed at station 3, and finally (if the system is still not in state s^0) idle server 2 until the next time the process hits state s^0.*

Moreover, this policy also attains the maximal capacity of the system, regardless of the intermediate buffer sizes.

Proof: Proceeding as in the proof of Proposition 4.2.4, the value of λ in the optimal solution of the allocation LP can be found as follows:

$$\lambda^* = \begin{cases} \mu_{11} & \text{if } \mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}, \\ \mu_{22} & \text{if } \mu_{22} \leq \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}, \\ \frac{\mu_{11}\mu_{22}(\mu_{13}+\mu_{23})}{\mu_{13}\mu_{22}+\mu_{11}\mu_{23}+\mu_{11}\mu_{22}} & \text{if } \mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}} \text{ and } \mu_{22} > \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}. \end{cases} \tag{13}$$

Note that $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$ (i.e., $\frac{1}{\mu_{11}} \geq \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$) and $\mu_{22} \leq \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}$ (i.e., $\frac{1}{\mu_{22}} \geq \frac{1}{\mu_{11}} + \frac{1}{\mu_{13}}$) cannot hold at the same time, since we assumed that all the service rates are finite.

Consider the policy described in the proposition and assume that the system is in state $s^0 = (s_1^0, s_2^0)$ at time T. When $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$, we obtain the states of the system and the remaining service requirements for the jobs at each station as in Table 23 in Appendix B.2. When $\mu_{22} \leq \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}$, we obtain Table 24 in Appendix B.2. When $\mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$ and $\mu_{22} > \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}$, we obtain Table 25 in Appendix B.2. Optimality of the policy in the proposition can be shown using similar arguments as in the proof of Proposition 4.2.1. $\square$

Note that in the system with $\mu_{12} = \mu_{21} = 0$, the maximal capacity of the four-skilled system can be reached even with three skills when either $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$ or $\mu_{22} \leq \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}$. In the former case (when $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$), the optimal throughput can be achieved with $\mu_{13} = 0$; in the latter case (when $\mu_{22} \leq \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}$), the optimal throughput can be achieved with $\mu_{23} = 0$.

The descriptions of the optimal policies in Propositions 4.2.4, 4.2.5, and 4.2.6 show that each server starts working at the station where (s)he is the only server trained to work. Then, after completing the job at the station they are primarily assigned to, the servers move to the other station where they are trained to work. Furthermore, idling occurs only when one server is so fast that (s)he can complete one job at two stations before the other server completes a job at one station. In this case, we idle the fast server in order to balance the line and keep all the stations operating, because utilizing the fast server more causes starvation or blocking in the system but does not increase the throughput. It is also observed that these policies utilize each server in order make sure that the other server is not blocked or starved. Perfect coordination of the servers in order to prevent any productivity loss is achievable since the service times are deterministic.

4.2.3 Identifying the Best Flexibility Structure

Propositions 4.2.1 through 4.2.6 show that when the servers have four skills and the service times are deterministic, it is possible to reach the maximal capacity of the corresponding four-skilled infinite-buffered systems in the finite-buffered settings. In this section, given the potential skill of each server at each task (if the server were trained to perform the task), we determine the four critical skills that are sufficient to attain the maximal capacity of the fully flexible system. In order to specify the best flexibility structure for a four-skilled system, we need the following conditions:

$$\{1\}\ \mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}; \qquad \{2\}\ \mu_{11}\mu_{22} < \mu_{12}\mu_{21}; \qquad \{3\}\ \mu_{11}\mu_{23} \geq \mu_{13}\mu_{21};$$

$$\{4\}\ \mu_{11}\mu_{23} < \mu_{13}\mu_{21}; \qquad \{5\}\ \mu_{12}\mu_{23} \geq \mu_{13}\mu_{22}; \qquad \{6\}\ \mu_{12}\mu_{23} < \mu_{13}\mu_{22};$$

$$\{7\}\ \mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}; \qquad \{8\}\ \mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}; \qquad \{9\}\ \mu_{12} \leq \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}};$$

$$\{10\}\ \mu_{12} > \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}; \qquad \{11\}\ \mu_{13} \leq \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}}; \qquad \{12\}\ \mu_{13} > \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}};$$

$$\{13\}\ \mu_{21} \leq \frac{\mu_{12}\mu_{13}}{\mu_{12}+\mu_{13}}; \qquad \{14\}\ \mu_{21} > \frac{\mu_{12}\mu_{13}}{\mu_{12}+\mu_{13}}; \qquad \{15\}\ \mu_{22} \leq \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}};$$

$$\{16\}\ \mu_{22} > \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}; \qquad \{17\}\ \mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}; \qquad \{18\}\ \mu_{23} > \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}.$$

Conditions $\{1\}$ through $\{6\}$ compare the relative speeds of servers at different stations. For example, condition $\{1\}$ implies that server 1 is relatively faster at station 1 than server 2 (at the same time server 2 is relatively faster at station 2 than server 1). Conditions $\{7\}$ through $\{18\}$ compare the service completion rate of servers in different zones. For example, condition $\{7\}$ implies that the service completion rate of server 2 in the zone consisting of stations 2 and 3 is higher than the service completion rate of server 1 at station 1, see the proof of Proposition 4.2.1.

The following theorem, whose proof is provided in Appendix B.3, specifies the best flexibility structure for a system with three stations and two servers.

Theorem 4.2.1 *For a tandem line with three stations, two flexible servers, arbitrary buffer sizes between the stations, and deterministic service times, the assignment (and hence cross-training) policy specified in Table 5 is optimal.*

Note that Theorem 4.2.1 employs the optimal solution of the allocation LP, and hence it also provides the optimal assignment policy for the corresponding infinite-buffered system. Next, we show that any set of service rates μ_{ij}, where $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$, has to satisfy exactly one of the cases mentioned in Theorem 4.2.1.

Table 5: Critical Skills and Optimal Server Assignment Policy for a Deterministic System with Three Stations and Two Servers

Case	Conditions Satisfied	Optimal Server Assignment Policy
a	{1}, {3}, {7}	Use Proposition 4.2.1
b	{2}, {4}, {13}	Relabel servers and use Proposition 4.2.1
c	{2}, {5}, {9}	Use Proposition 4.2.2
d	{1}, {6}, {15}	Relabel servers and use Proposition 4.2.2
e	{4}, {6}, {11}	Use Proposition 4.2.3
f	{3}, {5}, {17}	Relabel servers and use Proposition 4.2.3
g	{2}, {3}, {10}, {18}	Use Proposition 4.2.4
h	{1}, {4}, {12}, {16}	Relabel servers and use Proposition 4.2.4
i	{1}, {5}, {8}, {18}	Use Proposition 4.2.5
j	{2}, {6}, {12}, {14}	Relabel servers and use Proposition 4.2.5
k	{3}, {6}, {8}, {16}	Use Proposition 4.2.6
l	{4}, {5}, {10}, {14}	Relabel servers and use Proposition 4.2.6

Proposition 4.2.7 *The twelve cases* $\{a, \ldots, l\}$ *in Theorem 4.2.1 are mutually exclusive and collectively exhaustive.*

Proof: Consider a set of service rates $M = \{\mu_{ij} \mid i = 1, 2 \text{ and } j = 1, 2, 3\}$. The elements of M have to satisfy one of conditions {1} and {2}, one of conditions {3} and {4}, and one of conditions {5} and {6}. First assume that the elements of M satisfy the conditions {1}, {3}, and {5}. Then, none of the cases except a, f, and i can hold. The proof of Proposition 4.2.5 shows that conditions {7} and {17} are mutually exclusive. If in addition to {1}, {3}, and {5}, condition {7} is satisfied, then conditions {8} and {17} are not satisfied, and hence only case a holds. If in addition to {1}, {3}, and {5}, condition {17} is satisfied, then conditions {7} and {18} are not satisfied, and hence only case f holds. Finally, if conditions {1}, {3}, and {5} are satisfied and both of the conditions {7} and {17} are not satisfied, then conditions {8} and {18} are satisfied, and hence only case i holds.

Similar arguments show that if conditions {1}, {3}, and {6} are satisfied, then exactly one of the cases a, d, and k holds. If conditions {1}, {4}, and {6} are satisfied, then exactly one of the cases d, e, and h holds. If conditions $\{2\}$, $\{3\}$, and $\{5\}$ are satisfied, then exactly one of the cases c, f, and g holds. If conditions $\{2\}$, $\{4\}$, and $\{5\}$ are satisfied, then exactly one of the cases b, c, and l holds. If conditions $\{2\}$, $\{4\}$, and $\{6\}$ are satisfied, then exactly one of the cases b, e, and j holds.

Finally, note that conditions $\{1\}$, $\{4\}$, and $\{5\}$ cannot hold at the same time because conditions $\{1\}$ and $\{4\}$ together imply that condition $\{6\}$ is true. Similarly, the conditions $\{2\}$, $\{3\}$, and $\{6\}$ cannot hold at the same time because conditions $\{2\}$ and $\{3\}$ together imply that condition $\{5\}$ is true. This concludes the proof. $\square$

The criteria for deciding the best flexibility structure provided in Table 5 can be summarized as follows. If one server is relatively fast at one station with respect to the other stations (for example, conditions $\{1\}$ and $\{3\}$ imply that server 1 is relatively fast at station 1 in case a) and at that station (s)he cannot finish one job before the other server finishes service at both of the other stations (condition $\{7\}$ in case a), then this server should be dedicated to the station where (s)he is relatively fast.

On the other hand, if the two servers are relatively fast at different stations with respect to the same station (for example, conditions $\{2\}$ and $\{3\}$ imply that server 1 is relatively better than server 2 at station 2 compared to station 1 and server 2 is relatively better than server 1 at station 3 compared to station 1 in case g) and they can finish a job at the station they are relatively fast at before the other server can finish service at both of the other stations (conditions $\{10\}$ and $\{18\}$ in case g), then they should work at the station where they are relatively fast at, and also at the common station where they are both relatively slow.

The other cases (b through f and h through l) can be described similarly by simply changing the labeling of the servers and the stations they are relatively faster or slower at.

In summary, we have observed that the optimal cross-training strategy in a finite-buffered system with deterministic service times is the same as the one of the corresponding infinite-buffered system. Moreover, the maximal possible throughput (corresponding to full cross-training and infinite buffers) can be obtained with partial cross-training and finite buffers for deterministic systems, regardless of the size of the buffers.

4.3 Markovian Systems

In this section, we consider systems with three stations, two servers, and exponentially distributed service requirements at each station. In Section 4.3.1, we formulate the problem and provide our preliminary results. In Section 4.3.2, we present our observations about the form of the optimal server assignment policy using some numerical experiments. In Section 4.3.3, we show that four-skilled systems attain near-optimal throughput as compared to fully cross-trained systems. Finally, we identify the optimal server assignment policies for systems with one dedicated and one fully flexible server in Section 4.3.4, and for systems with two partially flexible servers in Section 4.3.5.

4.3.1 Problem Formulation

Let Π be the set of Markovian stationary deterministic policies corresponding to the state space S of the system, and let A_s denote the set of allowable actions in each state $s \in S$. For all $\pi \in \Pi$ and $t \geq 0$, let $D^\pi(t)$ be the number of departures under policy π by time t, and let

$$T^\pi = \lim_{t \to \infty} \frac{\mathbb{E}[D^\pi(t)]}{t}$$

be the long-run average throughput corresponding to the server assignment policy π. The existence of this limit follows from Proposition 8.1.1 of Puterman [58]. We are interested in finding a server assignment policy that maximizes the long-run average throughput. Theorem 9.1.8 of Puterman [58] shows the existence of an optimal deterministic stationary policy, because our state space S is finite and similarly A_s is finite for each $s \in S$. Hence, without loss of generality, we restrict ourselves to policies in Π (implying that the decision epochs correspond to the service completion times at the stations).

Specifically, for systems with three stations, we use the stochastic process $\{X(t) = (X_1(t), X_2(t)) : t \geq 0\}$ to keep track of how the state of the system evolves with time, where $X_1(t)$ ($X_2(t)$) is the number of jobs that have already been processed at station 1 (2) and are either waiting for service or being processed at station 2 (3) at time $t \geq 0$. Possible actions are idling a server or assigning the server to station 1, 2, or 3. When we use the term "idling," we refer to voluntary idling of a server. For example assigning a server to a station where (s)he is not cross-trained at is not considered as an idling action because in fact the server is assigned to a station (even if (s)he can not work there). We use the notation $a_{\sigma_1\sigma_2}$ for possible actions, where $\sigma_i \in \{0, 1, 2, 3\}$ for $i \in \{1, 2\}$, with $\sigma_i = 0$ when server i is idle and $\sigma_i = j \in \{1, 2, 3\}$ when server $i \in \{1, 2\}$ is assigned to station $j \in \{1, 2, 3\}$. In order to maximize the throughput in our system, we will identify the optimal action in each state.

The following lemma is a generalization of Lemmas 3.2.1 and 3.2.2 to the system with two servers and more than two stations.

Lemma 4.3.1 *For a tandem line with $N > 2$ and $M = 2$, there exists an optimal policy that is non-idling.*

Proof: If both of the servers are idle in some state s, then s is an absorbing state and the throughput is equal to zero. Hence, at least one of the servers should be assigned to at least one of the stations. First, assume that one server is assigned to station $j \in \{1, \ldots, N\}$ that is operating and the other server is left idle. Then, the only transition in the system will be to a state $s' \in S$ with probability one. The transition time to state s' is never longer if we assign both servers to station j; hence the throughput is never less for the policy that assigns both servers to the same station than for the policy that idles one of the servers. □

4.3.2 Fully Cross-Trained Servers

When both of the servers are cross-trained at all the stations (i.e., $\mu_{ij} > 0$ for $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$), the optimal server assignment policy is difficult to characterize. Even for systems with fewer skills and small buffer sizes, we show in Sections 4.3.4 and 4.3.5 that the optimal policy sometimes does not have an easy description, and the optimal assignment policy for fully flexible systems appears to be more complicated than these. Hence, we performed simulation experiments to determine the optimal assignment of fully cross-trained servers. We randomly generated 10,000 systems where the service rates were drawn independently from a uniform distribution with range [0.5,2.5]. Then, assuming that $B_2 = B_3 = B$, we used the policy iteration algorithm for communicating Markov chains to identify the optimal server assignment policy for each system and each $B \in \{0, 1, 2, 3, 4\}$. Here are our observations:

- At least one of the servers (sometimes both of them) appears to have a primary assignment. In other words, at least one server is assigned to a specific station as long as that station is neither starved or blocked.

- Primary assignment can change with the buffer size. In other words, sometimes a server that has a primary assignment for one buffer size does not have a primary assignment for another buffer size, or a server's primary assignment can be different for different buffer sizes.

- Primary assignment is not always where the server is fastest or according to a simple multiplicative rule (which is the case when there are two stations in tandem, as shown in Andradóttir, Ayhan, and Down [7]).

- If both of the servers have primary assignments, their primary assignment is not at the same station.

We observe in Sections 4.3.4 and 4.3.5 that the conclusions above sometimes hold even for the optimal assignment policy for partially flexible servers. More specifically, we will see that at least one server in lines with limited flexibility and small buffers has a primary assignment, and we will provide some guidelines about how to determine the primary assignment.

4.3.3 Partially Cross-Trained Servers

When the service requirements at each station are deterministic, it is shown in Section 4.2.3 that it is possible to attain the maximum throughput (corresponding to fully trained servers) when the servers are cross-trained to have four skills in total. In Markovian systems, it is not possible to reach such a conclusion because of the stochastic nature of the problem. Nevertheless, there is strong evidence that, especially for systems with medium to large buffer sizes, near-optimal throughput can be obtained with four skills only. We performed 50,000 experiments for the systems with the same parameters as in Section 4.3.2. We found the maximum throughput of all four-skilled systems and compared it to the maximum throughput of the system with six skills (i.e., both servers are fully cross-trained). The average performance of the best four-skilled systems compared to the six-skilled system is given below for $B_2 = B_3 \in \{0, 1, \ldots, 4\}$.

- 91.94% of the optimal throughput of the six-skilled system if $B_2 = B_3 = 0$;
- 96.67% of the optimal throughput of the six-skilled system if $B_2 = B_3 = 1$;
- 98.27% of the optimal throughput of the six-skilled system if $B_2 = B_3 = 2$;
- 99.01% of the optimal throughput of the six-skilled system if $B_2 = B_3 = 3$;

- 99.37% of the optimal throughput of the six-skilled system if $B_2 = B_3 = 4$.

We can conclude that even for small buffer sizes it is possible to achieve near-optimal throughput of the fully flexible system with only four skills. Hence, it is important to identify the optimal assignment policy for systems with four skills. We start by determining the optimal server assignment policy for systems with one dedicated and one fully flexible server in Section 4.3.4. Next, we consider systems with two partially flexible servers in Section 4.3.5. We limit ourselves to buffers of sizes zero or one in Section 4.3.4 and to buffers of size zero in Section 4.3.5 because the expressions become analytically intractable for larger buffer sizes.

4.3.4 Systems with One Dedicated and One Fully Flexible Server

In this section, we identify the optimal server assignment policy when one server is dedicated at stations 1, 2, or 3, respectively, and the other server is cross-trained at all stations. Without loss of generality, we assume that first server is the dedicated server because the otherwise we can relabel the servers. The proofs of the following propositions are provided in Appendix B.4.

Proposition 4.3.1 *Assume that* $\mu_{12} = \mu_{13} = 0$ *and* $B_2, B_3 \leq 1$*. Then the following server assignment policy is optimal for the Markovian system:*

- *assign server 1 to station 1;*
- *assign server 2 to station 2 if station 2 is operating, to station 3 if station 2 is not operating (i.e., blocked or starved) but station 3 is operating, and to station 1 otherwise.*

Proposition 4.3.2 *Assume that* $\mu_{11} = \mu_{13} = 0$ *and* $B_2, B_3 \leq 1$*. Then the following server assignment policy is optimal for the Markovian system with three stations and two servers:*

- *assign server 1 to station 2;*
- *assign server 2 to station 1 if station 1 is operating and station 2 is both blocked and starved, or not blocked and either will not become blocked or will become both blocked and starved if the next event were a service completion at station 2;*

  assign server 2 to station 3 if station 3 is operating and station 1 is not operating, or if stations 1 and 3 are operating and station 2 is either blocked but not starved or will become blocked but not starved if the next event were a service completion at station 2;

  assign server 2 to station 2 otherwise.

Proposition 4.3.3 *Assume that $\mu_{11} = \mu_{12} = 0$ and $B_2, B_3 \leq 1$. Then the following server assignment policy is optimal for the Markovian system with three stations and two servers:*

- *assign server 1 to station 3;*
- *assign server 2 to station 2 if station 2 is operating, to station 1 if station 2 is not operating but station 1 is operating, and to station 3 otherwise.*

We observe that if there is one dedicated and one flexible server, the flexible server does not work at the station where the dedicated server is working, as long as any of the other stations are operating. The assignment of the flexible server when both of the other stations are operating has the goal of keeping the dedicated server's station operating.

Propositions 4.3.1, and 4.3.3 are similar to Propositions 4.2.1, and 4.2.3, respectively. When the dedicated server is at station 1 (3), the main goal is to prevent blocking (starvation) at station 1 (3), and hence the flexible server gives priority to station 2, then to station 3 (1) if the dedicated server is at station 1 (3), and finally moves to the station where the dedicated server is working. This prioritization of the stations is the same as in the corresponding deterministic systems.

When the dedicated server is at station 2, two goals (to prevent starvation and blocking) conflict with each other. This explains the more complex structure of the optimal assignment of the flexible server in Proposition 4.3.2. In order to keep station 2 operating, the optimal policy gives priority to station 1 unless station 2 is blocked (but not starved) or about to be blocked (but not starved). After station 1, the flexible gives the second highest priority to station 3, and station 2 is the least preferred station. The optimal policy for the corresponding deterministic system also has a similar structure in that either of stations 1 or 3 can be given priority, but station 2 is the least preferred station, see Proposition 4.2.2. As mentioned earlier, the local heuristic in Andradóttir, Ayhan, and Down [7] gives preference to removing blocking rather than starving in longer lines. We see that the optimal policy in our system puts higher priority on removing starving than blocking, but it also considers the immediate blocking possibility in station 2 and tries to prevent blocking before it even happens.

We conclude this section by pointing out that the optimal policies provided in Propositions 4.3.1, 4.3.2, and 4.3.3 are not necessarily unique. For example, the proofs of Propositions 4.3.1 and 4.3.3 in Appendix B.4 for systems with a dedicated server at station 1 or 3 and $B_2 = B_3 = 0$ suggest that the actions a_{12} and a_{13} are both optimal whenever these actions are both in A_s. However, when $B_2 = B_3 = 1$, there are some states s with $a_{12}, a_{13} \in A_s$ where the action a_{12} is strictly better than the action a_{13}. Hence, the policy descriptions in the propositions were chosen so that the same policy would be optimal for systems with different buffer sizes.

4.3.5 Systems with Two Partially Flexible Servers

In this section, we consider four-skilled systems where each server is cross-trained to work at two stations. The following propositions provide the optimal server assignment policy under different cross-training strategies. We identify the optimal server assignment policy for three partially flexible systems out of six, because the other cases directly follow by relabeling the servers. We limit ourselves to the systems with zero buffer sizes, because for larger buffer sizes the expressions become analytically intractable and the optimal policy becomes difficult to characterize. The proofs of the following propositions are provided in Appendix B.5.

Proposition 4.3.4 *Assume that* $\mu_{13} = \mu_{22} = 0$ *and* $B_2 = B_3 = 0$. *Then the following server assignment policy is optimal for the Markovian system:*

- *assign server 1 to station 2 if station 2 is operating, and to station 1 otherwise;*
- *if* $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{23}$, *assign server 2 to station 3 if station 3 is operating, and to station 1 otherwise;*
- *if* $\mu_{11}\mu_{12} < \mu_{21}\mu_{23}$, *assign server 2 to station 3 if station 3 is operating and station 2 is either not starved or both starved and blocked, and to station 1 otherwise.*

When both servers are cross-trained at station 1, we observe that server 1 (the server that is cross-trained at stations 1 and 2) has a primary assignment at station 2. When $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{23}$ (which can be interpreted as server 1 having better overall performance), server 2 has a primary assignment at station 3. This is reasonable because server 1 is already performing well enough at stations 1 and 2, and the capacity of server 3 can be primarily given to station 3. When $\mu_{11}\mu_{12} < \mu_{21}\mu_{23}$, server 2 does not have a primary assignment at any station but gives priority to station 3 except when station 2 is starved but not blocked. In this case, since server 2 seems to perform well enough on stations 1 and 3, (s)he can shift some capacity to station 1 (whenever needed) without causing poor performance at station 3. This also allows the slower server (server 1 in this case) to spend more time on the task where the faster server cannot work. To summarize, in the state where station 3 is operating and station 2 is starved but not blocked, server 2 works at station 3 if $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{23}$, and at station 1 otherwise. Hence, we can conclude that the focus of server 2 changes depending on how the performance of server 1 compares to his/her own.

Proposition 4.3.5 *Assume that* $\mu_{13} = \mu_{21} = 0$ *and* $B_2 = B_3 = 0$*. Then the following server assignment policy is optimal for the Markovian system:*

- *assign server 2 to station 3 if station 3 is operating, and to station 2 otherwise;*
- *if* $\mu_{11}^2\mu_{12}^2 \leq \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$*, assign server 1 to station 1 if station 1 is operating, and to station 2 otherwise;*
- *if* $\mu_{11}^2\mu_{12}^2 > \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$*, assign server 1 to station 2 if station 2 is operating, and to station 1 otherwise.*

When both servers are cross-trained at station 2, both servers have primary assignments. Server 2 (the server that is cross-trained at stations 2 and 3) has a primary assignment at station 3 regardless of the service rates. However, server 1 can have a primary assignment at station 1 or 2. This shows a preference for clearing blocking in the system relative to starvation. If $\mu_{11}^2\mu_{12}^2 > \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$ holds, then server 1 has a primary assignment at station 2, and otherwise server 1 has a primary assignment at station 1. Unlike the corresponding condition in Proposition 4.3.4, each side of this inequality does not consist of simple multiplication of the rates of each server at the different stations. It seems to suggest which server has an overall better performance, but in a way that skews the selection of the better overall server towards server 2 because the right-hand side is always bigger than $\mu_{22}\mu_{23}\mu_{11}\mu_{12}$. In other words, it is more likely that server 1 is primarily assigned to station 1 (rather than station 2) with this inequality than under the condition $\mu_{11}\mu_{12} \leq \mu_{22}\mu_{23}$ (which is the criterion in Proposition 4.3.4 adapted to the current case).

Proposition 4.3.6 *Assume that* $\mu_{12} = \mu_{21} = 0$ *and* $B_2 = B_3 = 0$*. Then the following server assignment policy is optimal for the Markovian system:*

- *assign server 1 to station 1 if station 1 is operating, and to station 3 otherwise;*
- *assign server 2 to station 2 if station 2 is operating, and to station 3 otherwise.*

When the two servers are cross-trained at station 3, both servers have primary assignments at the stations where only one server is cross-trained to work. By contrast, Proposition 4.3.4 shows that when both servers are cross-trained at station 1, server 2 gives priority to station 3 but may move to station 1 even if station 3 (where only server 2 is trained to work) is operating when station 2 is starved but not blocked, depending on the service rates. By symmetry, when both servers are cross-trained at station 3, we would expect server 1 to give priority to station 1 and to move to station 3 when station 2 is blocked but not starved, for some service rates. Note, however, that when $B_2 = B_3 = 0$, $(1, 2)$ is the only state where station 2 is blocked but not starved. In fact, station 1 is also blocked in this state, and server 1 moves to station 3 even if the policy of the Proposition 4.3.6 is applied. Hence, even though the policies look different, a closer examination suggests that they are more symmetrical than it first appears.

The symmetry between the cases where both servers are trained at station 1 or 3, respectively, can be observed in systems with larger buffer sizes. Numerical experiments show that when both servers are cross-trained at station 3, the optimal policy sometimes appears to be of threshold type for the server that is cross-trained at stations 1 and 3. For example consider the case where the servers have the rates $\mu_{11} = 2$, $\mu_{12} = 0$, $\mu_{13} = 3$, $\mu_{21} = 0$, $\mu_{22} = 1$, and $\mu_{23} = 1$. Note that we chose these rates such that $\mu_{11}\mu_{13} \geq \mu_{22}\mu_{23}$ (which is the symmetrical to the inequality in Proposition 4.3.4). When $B_2 = 1$ and $B_3 = 0$, the optimal policy assigns server 2 to station 2 if station 2 is operating, and to station 3 otherwise; and assigns server 1 to station 1 if station 1 is operating and station 2 is not blocked or both blocked and starved, and to station 3 otherwise. In other words, the server that is cross-trained at stations 2 and 3 has a primary assignment at station 2 (symmetrical to the policy of Proposition 4.3.4), and the optimal assignment of the server that is cross-trained at stations 1 and 3 is of threshold type. The optimal policy results in a throughput of 0.8472, but the policy of Proposition 4.3.6 yields a throughput of 0.8100 (which corresponds to 95.61% of the optimal throughput). Similarly, the optimal policy appears to be of threshold type when $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{23}$ ($\mu_{11}\mu_{13} \geq \mu_{22}\mu_{23}$) and $B_2, B_3 > 0$ in cases where both servers are trained at station 1 (3).

Overall, we observe that in the four-skilled systems with two partially flexible servers, there is always one server with a primary assignment, and this result is consistent with what we observed for the fully-flexible system. For the small systems we considered, the primary assignment of one server does not depend on the service rates or buffer sizes. However, whether or not the other server has a primary assignment, and where (s)he is primarily assigned (see, e.g., Proposition 4.3.5), may depend on the service rates and the buffer sizes. Moreover, the policies in Propositions 4.3.4 and 4.3.6 are symmetrical versions of each other (even though the special structure of the system with $B_2 = B_3 = 0$ makes them seem different), while the policy in Proposition 4.3.5 is different from the others (as expected). When both servers are cross-trained at station 2, we observe that the optimal policy is more complex than in the other cases, which may result from the fact that station 2 can be starved or blocked, and the assignment policy has to prevent both of these events to the extent possible.

Propositions 4.3.4, 4.3.5, and 4.3.6 above also show that the optimal policy in the Markovian setting is slightly different from the optimal policy of the corresponding deterministic system. In the deterministic system, it is possible to coordinate the service completions so that no blocking or starvation occurs. Since this is not the case in the Markovian system, we see that the optimal policy is of a "threshold" type that also aims to keep the stations operating. Furthermore, we note that the form of the policy may be quite complicated, and that this special case with $B_2 = B_3 = 0$ does not generalize to systems with bigger buffer sizes. In fact, our numerical experiments in Section 4.4.1 suggest that the values of the thresholds, as well as the preferred assignment for each server, can not be determined by simply using the optimal policy for the system with $B_2 = B_3 = 0$. Note also that the policies specified in Propositions 4.3.4, 4.3.5, and 4.3.6 need not be unique. In fact the proofs of Propositions 4.3.4, 4.3.5, and 4.3.6 in Appendix B.5 show that there may be multiple actions that are optimal in some states.

The optimal throughput can be calculated for the cross-training strategies presented in Propositions 4.3.1 through 4.3.6, but the task of finding the best partially flexible system for a given set of (potential) service rates is not a simple task. When the expressions for the optimal throughputs are compared with each other, we obtain complex expressions that do not provide intuitive criteria to compare the flexibility structures. However, in the next section, we use the best flexibility structure for the corresponding deterministic system (see Theorem 4.2.1 and Table 5) and test the performance of the corresponding optimal policy (see Propositions 4.3.1 through 4.3.6) for small systems in larger Markovian systems. More specifically, we use the conditions in Table 5 to select a flexibility structure, and then we use Proposition $4.k$ instead of Proposition $3.k$, for $k \in \{1, \ldots, 6\}$, to specify a server assignment policy.

4.4 Numerical Results

In this section, we provide near-optimal heuristic server assignment policies and guidelines for selecting a good flexibility structure for understaffed Markovian lines. We first present and test our server assignment heuristics for tandem lines with three stations and two servers with four skills in Section 4.4.1. Then, in Section 4.4.2 we compare the performance of lines with limited flexibility with the performance of lines with full flexibility, and study guidelines for choosing a good partial flexibility structure, for four-skilled systems with three stations and two servers. Finally, in Section 4.4.3 we determine whether the solution of the allocation LP also provides a good flexibility structure in Markovian lines with more than three stations.

4.4.1 Heuristic Server Assignment Policies

In this section, we present and compare three heuristic server assignment policies for systems with three stations and two servers. Two of these policies use the results obtained for Markovian systems with small buffer sizes in Sections 4.3.4 and 4.3.5. More specifically, we consider the following heuristic policies for systems with four given server skills.

Policy 1: Assign priorities to stations for each flexible server. A flexible server works at a station with lower priority only if none of the stations with higher priority is operating.

Policy 2: The optimal assignment policy for Markovian systems with small buffer sizes (see Sections 4.3.4 and 4.3.5) is employed for systems with any buffer sizes.

Policy 3: A combination of the optimal assignment policy for Markovian systems with small buffer sizes and the optimal policy found numerically for various Markovian systems with larger buffer sizes. (This policy is only provided if the optimal policy has a different structure when the buffer sizes are larger, and it will be described in detail for each flexibility structure).

Note that for some flexibility structures (i.e., when there is a dedicated server at station 1 or 3 and a fully flexible server), some of the three policies above are identical.

In order to determine the priorities used in Policy 1, 50,000 systems were generated with service rates independently drawn from a uniform distribution with range [0.5,2.5] and the buffer sizes independently drawn from the discrete uniform distribution with range $\{0, 1, 2, 3, 4, 5\}$. Then, all possible assignments were compared and the one with the highest average throughput in 50,000 experiments was selected (in each case, the long-run average throughput was determined using the stationary distribution of the Markov chain $\{X(t)\}$). When there is a dedicated server at station 1 or 3, the best average priority policy coincides with the optimal policy of the corresponding system with smaller buffer size (see Propositions 4.3.1 and 4.3.3). When there is a dedicated server at station 2, the flexible server gives priority to station 1, then to station 3, and finally moves to station 2 (this is also consistent with Proposition 4.3.2 except for the states where station 2 is blocked but not starved or about to be blocked but not starved). When the servers have two skills, each server gives priority to the station where no other server is cross-trained. For example, when server 1 is cross-trained at stations 1 and 2 and server 2 is cross-trained at stations 1 and 3; server 1 gives priority to station 2 and server 2 gives priority to station 3. Policy 1 is the same as the optimal assignment policy given in Proposition 4.3.6 if both servers are trained at station 3. Policy 1 is also consistent with Proposition 4.3.4 except for the states where station 3 is operating and station 2 is either not starved or both blocked and starved, and when the service rates satisfy the condition $\mu_{11}\mu_{12} < \mu_{21}\mu_{23}$. Finally, Policy 1 is consistent with Proposition 4.3.5 except when the service rates satisfy the condition $\mu_{11}^2\mu_{12}^2 > \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$.

Policy 3 will be described only when both servers are cross-trained at two stations since no improvement over the optimal policy for small systems was found for the systems with one dedicated server and one fully flexible server (in this case we let Policy 3 be the same as Policy 2).

- When both servers are cross-trained at station 1, so that $\mu_{13} = \mu_{22} = 0$, Policy 3 is as follows. Server 1 has a primary assignment at station 2, but server 2 may or may not have a primary assignment. In particular, similar to the optimal policy for the system with $B_2 = B_3 = 0$, server 2 has a primary assignment at station 3 when $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{23}$. When $\mu_{11}\mu_{12} < \mu_{21}\mu_{23}$, Policy 3 differs from Policy 2 only in the states where station 2 would become starved but not blocked if the next event were a service completion at station 2. In these states, Policy 2 assigns server 2 to station 3 but Policy 3 assigns server 2 to station 1. Thus Policy 3 pushes more jobs into the system in order to keep the second station operating, and this idea is similar to that of Proposition 4.3.2 in that it tries to prevent starvation before it occurs.

- Policy 3 was defined using a similar idea when both servers are cross-trained at station 3, so that $\mu_{12} = \mu_{21} = 0$. In this case, Policy 3 differs from Policy 2 only in the states where station 2 is blocked but not starved, or will become blocked but not starved after the next service completion at station 2, and when the service rates satisfy the condition $\mu_{22}\mu_{23} < \mu_{11}\mu_{13}$. In this case, Policy 3 assigns server 1 to station 3, but Policy 2 assigns server 1 to station 1. This policy is different from the optimal policy of the system with $B_2 = B_3 = 0$ (where both servers have primary assignments) and is symmetric to Policy 3 for the case where both servers are cross-trained at station 1.

- Policy 3 for the system where both servers are cross-trained at station 2, so that $\mu_{13} = \mu_{21} = 0$, combines the ideas used in Policy 3 for the cases where both servers are cross-trained at station 1 or 3 with the optimal policy of the system with $B_2 = B_3 = 0$ when both servers are cross-trained at station 2.

In particular, when (a) $\mu_{11}^2\mu_{12}^2 \leq \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$ and (b) $\mu_{22}^2\mu_{23}^2 \leq \mu_{11}\mu_{12}(\mu_{22}\mu_{23} + \mu_{11}\mu_{23} + \mu_{11}\mu_{22} + \mu_{11}^2)$, server 1 has a primary assignment at station 1 and server 2 has a primary assignment at station 3. Otherwise, depending of which of these inequalities are satisfied or not satisfied, servers may move to station 2 in order to prevent blocking or starvation. More specifically, if inequality (a) is not satisfied, server 1 gives priority to station 1 but moves to station 3 if station 2 is blocked (but not starved) or will become blocked (but not starved) upon the next service completion at station 2. If inequality (b) is not satisfied, server 2 gives priority to station 3 but moves to station 1 if station 2 is starved (but not blocked) or will become starved (but not blocked) upon the next service completion at station 2.

In order to evaluate the performance of Policies 1, 2, and 3 for each of the twelve possible flexibility structures with four skills, we randomly generated 50,000 Markovian systems with service rates μ_{ij}, where $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$, independently drawn from a uniform distribution with range [0.5,2.5] and the buffer sizes B_2, B_3 independently drawn from the discrete uniform distribution with range $\{0, 1, 2, 3, 4, 5\}$. For each system and flexibility structure, we use the policy iteration algorithm to find the optimal throughput of the system. Table 6 shows the 95% confidence interval for the throughput of these policies and the throughput of the optimal policy. The first column shows the flexibility structure under consideration. More specifically, the first set of numbers shows the stations where server 1 is cross-trained at, and the second set of numbers (after the "-" sign) shows the stations where server 2 is cross-trained at. Note that, the throughput of Policy 2 is not provided for systems with a dedicated server at stations 1 or 3 and a fully flexible server, or with two partially flexible servers that are both cross-trained at station 3, because Policies 1 and 2 are identical in these cases. Similarly, for the systems with one dedicated and one fully flexible server, the throughput of Policy 3 is not provided, because Policy 3 is defined to be the same as Policy 2 in these cases.

Table 6: Performance of Heuristics for Partially Flexible Understaffed Systems

Flexibility Structure	Policy 1	Policy 2	Policy 3	Optimal Policy
1-123	0.6759 ± 0.0018	—	—	0.6759 ± 0.0018
123-1	0.6764 ± 0.0018	—	—	0.6764 ± 0.0018
2-123	0.6622 ± 0.0017	0.6714 ± 0.0018	—	0.6717 ± 0.0018
123-2	0.6611 ± 0.0017	0.6703 ± 0.0018	—	0.6706 ± 0.0018
3-123	0.6759 ± 0.0018	—	—	0.6759 ± 0.0018
123-3	0.6776 ± 0.0018	—	—	0.6776 ± 0.0018
12-13	0.8426 ± 0.0011	0.8518 ± 0.0011	0.8538 ± 0.0011	0.8566 ± 0.0011
13-12	0.8419 ± 0.0011	0.8484 ± 0.0011	0.8506 ± 0.0011	0.8533 ± 0.0011
12-23	0.8573 ± 0.0011	0.8588 ± 0.0011	0.8601 ± 0.0011	0.8656 ± 0.0011
23-12	0.8558 ± 0.0011	0.8567 ± 0.0011	0.8590 ± 0.0011	0.8643 ± 0.0011
13-23	0.8491 ± 0.0011	—	0.8529 ± 0.0011	0.8667 ± 0.0011
23-13	0.8500 ± 0.0011	—	0.8538 ± 0.0011	0.8577 ± 0.0011

From Table 6, we see that Policy 1 attained more that 99% of the optimal throughput in all the flexibility structures we considered. Moreover, when the dedicated server is at station 1 (3), Policy 2 described in Propositions 4.3.1 (4.3.3) attained the optimal throughput in all the experiments we performed. When there is a dedicated server at station 2, the policy described in Proposition 4.3.2 reached 99.95% of the optimal throughput. On the other hand, when both servers have two skills, we observe that the strict priority policy (Policy 1) performs well and the optimal policy for small systems (Policy 2) performs even better (it reaches more than 99% of the optimal throughput in all cases). Finally, a threshold policy (Policy 3) seems to close the optimality gap for Policy 2 by about 50% when there are two partially flexible servers.

We also observe that in the system with one dedicated and one flexible server, the flexibility structures with a dedicated server at the either end station seem to perform better compared to the one with a dedicated server at station 2. The reason may be that when the dedicated server is at one of the end stations, the flexible server can focus on making sure the dedicated server is not blocked (if (s)he is assigned to station 1) or not starved (if (s)he is assigned to station 3). By contrast, if the dedicated server is at station 2, the flexible server has to attempt to make sure the dedicated server is neither blocked nor starved. When there are two partially flexible servers, we see that the flexibility structure with both servers cross-trained at the middle station seems to perform better compared to the ones with both servers cross-trained at one of the end stations. This is reasonable because training both servers at the middle station allows each server to simultaneously be able to concentrate on one end of the line while being able to help with the operation of the middle station. We also observe that when Policy 1 is employed for systems with two partially flexible servers, the flexibility structure where both servers are trained at station 3 performs statistically better than the flexibility structure where both servers are trained at station 1. This is consistent with our results about the optimal policy for such systems in Section 4.3.5, where the optimal policy for the case with $\mu_{12} = \mu_{21} = 0$ and $B_2 = B_3 = 0$ was shown to be a strict priority policy (as in Policy 1) and the optimal policy for the case with $\mu_{13} = \mu_{22} = 0$ and $B_2 = B_3 = 0$ was shown to be a threshold policy for some service rates.

We conclude that server assignment policies that are of priority or threshold type are also effective in the systems with larger buffers sizes. The threshold policies described in Propositions 4.3.1 and 4.3.3 seem to be optimal for systems with larger buffer sizes as well. For the other cases, the form of the optimal server assignment policy seems complicated (as in Section 4.3.2), but it is still possible to attain near-optimal throughput with the simple heuristics described in this section.

4.4.2 Comparison with Full Flexibility and Selecting a Good Flexibility Structure

In this section, we compare the performance of partially flexible lines with four skills with the optimal performance of the corresponding fully flexible system. We perform

50,000 experiments, as described in Section 4.4.1. In each experiment, we first use the criteria in Theorem 4.2.1 to select a flexibility structure (that is known to be optimal for deterministic systems with finite buffers) and use either a heuristic or optimal server assignment policy to determine the throughput for this flexibility structure. The resulting 95% confidence intervals for the long-run average throughput are shown in the second column of Table 7. Then we determine the throughput of the best heuristic (Policy 3) for each of the twelve flexibility structures that are shown in the first column of Table 6, and the structure with the highest throughput is selected. The resulting 95% confidence intervals on the throughput are provided in the third column of Table 7. Finally, we compute the optimal long-run average throughput of the fully flexible system, see the last column of Table 7 for the 95% confidence interval on the optimal long-run average throughput.

Table 7: Comparison of the Throughput of Four-Skilled Systems with Six-Skilled Systems

Policy	Theorem 4.2.1	Best 4-skilled	6-skilled
Best Heuristic	1.0516 ± 0.0009	1.0552 ± 0.0009	—
Optimal	1.0567 ± 0.0009	1.0600 ± 0.0009	1.0822 ± 0.0010

We see from Table 7 that 97.51% of the benefits of full flexibility can be attained with only four skills and our heuristic assignment policies. When the optimal assignment policy is used with the best four-skilled flexibility structure, we see that the average throughput is 97.95% of that of the fully flexible system. Observe that the optimality gap is caused by the lack of two skills is larger than the optimality gap caused by the use of heuristic server assignment policies.

When the criteria in Theorem 4.2.1 are used to select the flexibility structure, Table 7 shows that the average throughput for the partially flexible systems is 97.17% of the optimal throughput of the fully flexible system when the best heuristic (Policy 3) is employed, and 97.64% of the optimal throughput of the fully flexible system when the optimal assignment policy is employed. This also corresponds to 99.66% and 99.67% of the corresponding results for the best four-skilled flexibility structure under the best heuristic and the optimal assignment policies, respectively. Hence, we can conclude that the criteria used for selecting the best system when the service times are deterministic also work well for the Markovian system.

Table 8 gives the frequency with which each flexibility structure is chosen in the 50,000 sets of service rates using the different criteria. The first column shows the flexibility structure. The second column shows the frequency of selecting each flexibility structure according to the selection rule of Theorem 4.2.1 (in other words it is the frequency with which each flexibility structure provides the optimal solution of the allocation LP). The third and four columns give the frequency for each flexibility structure when the system with highest throughput is selected if the best heuristic and the optimal assignment policies are employed, respectively.

Table 8: Frequency of Each Flexibility Structure Being the Best in Understaffed Systems

Flexibility Structure	Theorem 4.2.1	Best Heuristic	Optimal
1-123 or 123-1	2.90%	2.90%	2.80%
2-123 or 123-2	3.00%	2.51%	2.39%
3-123 or 123-3	2.93%	2.98%	2.83%
12-13 or 13-12	30.70%	30.85%	28.90%
12-23 or 23-13	30.26%	31.89%	33.33%
13-23 or 23-13	30.21%	28.87%	29.75%

We observe that the flexibility structures with two partially flexible servers are most of the time superior to the flexibility structures with one dedicated and one fully flexible server. More specifically, Table 8 shows that all three selection criteria select structures with two partially flexible servers almost ten times more often than the structures with one dedicated and one fully flexible server. However, when there is a dedicated and a fully flexible server in the system, we observe in Section 4.4.1

that there is no big difference in the performances of the systems where the dedicated server is at stations 1, 2, or 3, respectively. Similarly, when both servers have two skills each, we see that systems that have both servers trained at stations 1, 2, or 3 perform in a very similar manner. Theorem 4.2.1 chooses any of the three flexibility structures with one deterministic and one fully flexible server almost equally often, with the one with a dedicated server at station 2 being selected slightly more often than the other two. However, we observe that the cases with the dedicated server at stations 1 or 3 tend to be the best structures more often than the case with the dedicated server at station 2 when the best heuristic and optimal assignment policies are used, although the frequencies of all three flexibility structures are also very close to each other in this case. Among the flexibility structures where each server has two skills, the flexibility structure where both servers are cross-trained at station 2 is the best more often than the others when the best heuristic and optimal policies are employed, whereas Theorem 4.2.1 recommends it about as often as the other structures with two flexible servers (each cross-trained at two stations). We observed a similar result in Table 6 that showed that the flexibility structures that assign the dedicated server to the end stations are superior to the one with a dedicated server in the middle station, and the flexibility structure with both servers cross-trained at the middle station is superior to the ones with both servers cross-trained at the end stations.

We conclude that the solution of the allocation LP provides a good heuristic for finding a good flexibility structure for a tandem line with three stations and two servers. Even though this selection rule does not always find the best flexibility structure in each experiment, the performance of the flexibility structure it recommends is near-optimal as shown in this section. Furthermore, heuristic server assignment policies for systems with four-skills perform almost as well as optimal server assignment policies, and the frequency with which each flexibility structure is the best is very similar when the heuristic or optimal server assignment policies are employed. In the next section, we will study if these results generalize to longer Markovian lines.

4.4.3 Longer Markovian Lines

In Section 4.3.3 we observed that systems with the same number of skills as the optimal solution of the allocation LP attain near-optimal throughput in Markovian systems with three stations and two servers. In this section we test this conjecture for longer lines. More specifically, we consider tandem lines with two servers and four or five stations. We randomly generate the service rates of each server at each station with the same parameters as in Section 4.3.2. Then we find the optimal throughput of each flexibility structure under consideration and compare the highest average throughput among the all flexibility structures to the optimal throughput of the fully flexible system. We assume $B_j = B$ for all $j \in \{2, \ldots, N\}$, where $B \in \{0, 1, 2, 3\}$, and we repeat the experiment for each different value of B.

For systems with two flexible servers and four stations, there are eight possible skills and Proposition 2 of Andradóttir, Ayhan, and Down [8] shows that a five-skilled system would be the optimal solution of the allocation LP. There are $\binom{8}{5} = 56$ different choices for these five skills, but under our assumptions (e.g., that the service rate of both servers cannot be zero at the same station) it is sufficient to consider 32 different flexibility structures. We performed 50,000 experiments for each buffer size, found the optimal server assignment policy for each of the 32 flexibility structures, and identified best throughput among the optimal throughputs of the 32 flexibility structures. Then, we compare this best throughput to the optimal throughput of the corresponding fully flexible system, and on average the performance of the best five-skilled system is:

- 93.14% of the optimal throughput of the eight-skilled system if $B = 0$;
- 96.80% of the optimal throughput of the eight-skilled system if $B = 1$;

- 98.17% of the optimal throughput of the eight-skilled system if $B = 2$;
- 98.89% of the optimal throughput of the eight-skilled system if $B = 3$.

For the system with two flexible servers and five stations, there are ten possible skills and we know that there exists a six-skilled system that reaches the maximal capacity when the allocation LP is solved. Because of the high number of different possible combinations (80, to be more specific) for selecting these six skills out of ten while satisfying our assumptions, we only consider the 16 flexibility structures that consist of "zones" (because they are the easiest ones to apply in the real systems). In other words we allow each server to work only in consecutive stations. For example a server cannot work in stations 2 and 4 unless (s)he is also cross-trained at station 3. Because of the prohibitive amount of computational time, for each different value of $B = 0, 1, 2$ and 3, the number of experiments are 50,000, 10,000, 5,000 and 1,000, respectively. On average the best six-skilled system with zones has the following throughput:

- 86.02% of the optimal throughput of the ten-skilled system if $B = 0$;
- 87.92% of the optimal throughput of the ten-skilled system if $B = 1$;
- 88.22% of the optimal throughput of the ten-skilled system if $B = 2$;
- 88.45% of the optimal throughput of the ten-skilled system if $B = 3$.

For the system with four stations, even for the buffer sizes as small as one, the throughput of the best partially flexible system is near-optimal (i.e., attains more than 90% of the optimal throughput) compared to the fully flexible system. Especially when the buffer sizes reach three, the discrepancy between the performance of partial and fully flexible systems becomes very small. For the system with five stations, even though the number of structures considered here are 20% of all possible combinations, we still reach almost 90% of the optimal throughput of the fully flexible system. Hence, we can conclude that the solution of the allocation LP provides a good guideline for selecting a flexibility structure even for longer Markovian lines with finite buffers.

4.5 Conclusion

In this paper, we have studied understaffed tandem lines with finite buffers. More specifically, we determined the optimal server assignment policy for systems with three stations, two servers possessing four skills in total, and either deterministic service times and arbitrary buffer sizes, or exponential service times and small buffer sizes. Our results suggest that for deterministic systems, it is possible to attain the benefits of full flexibility with only partial flexibility, and we identified the optimal cross-training strategy for such systems. For Markovian systems, we observed that the optimal policy can be of threshold or priority type depending on the service rates and buffer sizes. Furthermore, we performed numerical experiments involving randomly generated Markovian systems that imply that even for small buffer sizes, partial flexibility together with a good server assignment policy can attain near-optimal throughput. Moreover, we observed that the optimal server assignment policies of small buffered systems also performed well in tandem lines with larger buffer sizes. Next, we determined the best flexibility structure for some random systems and showed that the solution of the allocation LP can be used to choose a good flexibility structure in systems with three stations and two flexible servers. Finally, we provided evidence that flexibility structures with the number of skills in the optimal solution of the allocation LP also perform well in Markovian tandem lines with finite buffers.

CHAPTER V

FLEXIBLE SERVERS IN TANDEM LINES WITH SETUPS

5.1 Introduction

In this chapter we study the queueing network described in Chapter 3 with the modification that a setup cost is incurred when servers move between the stations and the service requirements at each station are independent and exponentially distributed with mean one. For all $i \in \{1, \ldots, M\}$ and $j, k \in \{1, \ldots, N\}$, let $c_i(j, k)$ be the setup cost incurred when server i moves from station j to station k. We assume that $c_i(j, j) = 0$ and $0 \leq c_i(j, k) < \infty$ for $j \neq k$. We further assume that $c_i(j, k) \leq c_i(j, l) + c_i(l, k)$ for all $i \in \{1, \ldots, M\}$ and $j, k, l \in \{1, \ldots, N\}$, so that the least costly way of moving from one station to another does not include any intermediate stations. Every time there is a service completion at station N, a revenue of v is obtained. Without loss of generality, we assume that $v = 1$. Our goal is to find the dynamic server assignment policy that maximizes the long-run average profit.

Most existing works about systems with setups are on polling systems, where there is only one server in the system and the customers leave after being served at one station. Related work on polling systems includes Duenyas and Van Oyen [30], Gupta and Srinivasan [36], Hofri and Ross [44], Reiman and Wein [59], and references therein. We are only aware of a very limited number of works that study systems with setups apart from polling systems. In particular, Andradóttir, Ayhan, and Down [6, 9], Duenyas, Gupta, and Olsen [29], Iravani, Posner, and Buzacott [47], and Sennott, Van Oyen, and Iravani [61] consider tandem lines or general queueing networks with setups. However, all these papers assume that the storage spaces in the system have infinite capacity. By contrast, we study a system with setups and finite buffer spaces.

The remainder of this chapter is organized as follows. In Section 5.2 we formulate the problem. In Section 5.3 we provide preliminary results about tandem lines with two stations and setups. In Section 5.4 we consider systems with two stations and two generalist servers (i.e., servers that are equally skilled for all tasks) and identify the optimal server assignment policy for systems with small buffer sizes. In Section 5.5 we provide our observations about systems with larger buffer sizes and/or with specialist servers (i.e., servers who can be more skilled at some tasks than at others). Finally, in Section 5.6 we make some concluding remarks.

5.2 Problem Formulation

In this section, we formulate the dynamic server assignment problem in the presence of setups and illustrate our model for systems with two stations and two flexible servers operating under the policy known to be throughput optimal without setup costs.

For all all $t \geq 0$, let $Y_j(t) \in \{0, 1, \ldots, B_{j+1} + 2\}$ denote the number of jobs that have been served at station j and are either waiting for service or in service at station $j + 1$ at time t for $j \in \{1, \ldots, N - 1\}$. Similarly, for all $t \geq 0$ and $i \in \{1, \ldots, M\}$, let $Z_i(t)$ denote the station that server i was assigned to under the policy π at the time of the most recent service completion prior to time t in the queueing network (letting $Z_i(t)$ be the previous location of server i, rather than the current location of the server, will facilitate the translation of the optimization problem of interest into a Markov decision problem). We will use the stochastic process $\{X(t)\}$, where $X(t) = (Y(t), Z(t))$, $Y(t) = (Y_1(t), \ldots, Y_{N-1}(t))$, and $Z(t) = (Z_1(t), \ldots, Z_M(t))$ for all $t \geq 0$, to model the state of the system as a function of time.

We assume that the class Π of server assignment policies under consideration consists of all Markovian stationary deterministic policies corresponding to the state space $S \subset \mathbb{R}^{N+M-1}$ of the stochastic processes $\{X(t)\}$. In other words, the policies in Π specify whether each server is idle or not, and the station in the network that each non-idle server is assigned to as a function of the current state $x \in S$ of the stochastic process $\{X(t)\}$. Hence the server assignments may depend both on the status of the stations and buffers in the network and also on the previous location of the servers. Note that service may be preemptive when $M \geq 2$ (i.e., there is more than one server in the network) because a service completion at one station in the network may trigger the movement of servers that are currently working at other stations in the network. Without loss of generality, we do not consider actions that assign a server to another station and then keep the server idle. The reason is that by simply idling a server without any switchover, we obtain the same departure stream from the system and postpone or avoid the setup costs that could result from idling the server after a switchover (since $c_i(j,k) \leq c_i(j,l) + c_i(l,k)$ for all $i \in \{1,\ldots,M\}$ and $j,k,l \in \{1,\ldots,N\}$).

For all $x \in S$, let A_x denote the set of allowable actions in state x. We use the notation $a_{\sigma_1\sigma_2\ldots\sigma_M}$ to represent the actions, where σ_i is the assignment of server $i \in \{1,\ldots,M\}$ under this action. We use the convention that $\sigma_i = 0$ when server i is voluntarily idled at its current station, and this is treated differently from the case where server i is assigned to a station but is involuntarily idle since that station is not operating. Then, we have $A_x = \mathcal{A} = \bigcup_{\sigma \in \{0,\ldots,N\}^M}\{a_\sigma\}$ for all $x \in S$. However, in the proofs of Theorems 5.4.1, 5.4.2, 5.4.3, and 5.4.4, without loss of generality, we consider a smaller action set because some of the actions are known to be suboptimal in each state. We choose the decision rule d such that $d(x) \in A_x$ for all $x \in S$, and hence the policy $\pi \in \Pi$ corresponding to the decision rule d can be represented as $\pi = (d)^\infty$. Furthermore, we use the notation $d_i(x)$ to denote the assignment of server $i \in \{1,\ldots,M\}$ in state $x \in S$ under decision rule d. More specifically, $d_i(x) = \sigma_i$ for $i \in \{1,\ldots,M\}$ when $d(x) = a_{\sigma_1\sigma_2\ldots\sigma_M}$. Finally, we use the vector

$\delta_d(x) = (d_1(x), \ldots, d_M(x))$ to keep track of the assignments of all servers in state $x \in S$ under decision rule d.

For all $\pi \in \Pi$ and $t \geq 0$, let $D^\pi(t)$ be the number of departures from the network under the server assignment policy π by time t, and let

$$T^\pi = \lim_{t\to\infty} \mathbb{E}\left\{\frac{D^\pi(t)}{t}\right\} \tag{14}$$

be the long-run average throughput corresponding to the server assignment policy π. Moreover, for all $\pi \in \Pi$ and $t \geq 0$, let $C^\pi(t)$ be the (cumulative) setup cost incurred under the server assignment policy π in the period $[0, t]$, and let

$$C^\pi = \lim_{t\to\infty} \mathbb{E}\left\{\frac{C^\pi(t)}{t}\right\} \tag{15}$$

be the long-run average setup cost per unit time corresponding to the server assignment policy π. Note that Proposition 8.1.1 of Puterman [58] shows that the limits in equations (14) and (15) exist because we restrict ourselves to stationary policies, and the state space and immediate rewards are finite. Then, $T^\pi - C^\pi$ is the long-run average profit under policy $\pi \in \Pi$. We are interested in finding a policy in Π that maximizes the long-run average profit and we refer to this problem as the "original optimization problem."

We now explain how Andradóttir, Ayhan, and Kırkızlar [12] translate the original optimization problem into an equivalent (discrete time) Markov decision problem. Let $S_Y \subset \mathbb{R}^{N-1}$ and $S_Z = \{1, \ldots, N\}^M$ denote the state spaces of the stochastic processes $\{Y(t)\}$ and $\{Z(t)\}$, respectively. For the remainder of this paper, we use the decomposition $x = (y, z)$ and $x' = (y', z')$, where $x, x' \in S$, $y, y' \in S_Y$, and $z, z' \in S_Z$. For all $a \in \mathcal{A}$, let $\pi_a = (d_a)^\infty \in \Pi$ be the server assignment policy with $d_a(x) = a$ for all $x \in S$. Then it is clear that under our assumptions, the stochastic process $\{Y(t)\}$ is a continuous time Markov chain with state space S_Y for all $a \in \mathcal{A}$. For all $y, y' \in S_Y$ and all $a \in \mathcal{A}$, let $Q_a(y, y')$ be the rate at which the continuous time Markov chain $\{Y(t)\}$ goes from state y to state y' (under the server assignment policy

π_a). Then, it is not difficult to see that for all $\pi = (d)^\infty \in \Pi$, the stochastic process $\{X(t)\}$ is a continuous time Markov chain with state space S and with transition rates

$$q_d(x, x') = \begin{cases} Q_{d(x)}(y, y') & \text{if } z' = \delta_d(x) + I_z \ , \\ 0 & \text{otherwise,} \end{cases}$$

where I_z is an M-dimensional vector whose i^{th} element is equal to $\mathbf{1}_{(d_i(x)=0)} z_i$ and $\mathbf{1}$ is the indicator function. Hence, even if the decision rule voluntarily idles a server, we still keep track of this server's location in the state space.

It is also clear that for all $\pi = (d)^\infty \in \Pi$, there exists a scalar $q_\pi \leq \sum_{i=1}^{M} \max_{1 \leq j \leq N} \mu_{ij} < \infty$ such that the transition rates $\{q_d(x, x')\}$ of the continuous time Markov chain $\{X(t)\}$ satisfy $\sum_{x' \in S, x' \neq x} q_d(x, x') \leq q_\pi$ for all $x \in \mathcal{S}$. This shows that $\{X(t)\}$ is uniformizable for all $\pi \in \Pi$. We let $\{\tilde{X}(k)\}$ be the corresponding discrete time Markov chain, so that $\{\tilde{X}(k)\}$ has state space S and transition probabilities $p_d(x, x') = q_d(x, x')/q_\pi$ if $x' \neq x$ and $p_d(x, x) = 1 - \sum_{x' \in S, x' \neq x} q_d(x, x')/q_\pi$ for all $x \in S$. Andradóttir, Ayhan, and Kırkızlar [12] use the fact that $\{X(t)\}$ is uniformizable to translate the original optimization problem into an equivalent (discrete time) Markov decision problem (using uniformization to do this type of translation was proposed originally by Lippman [53]). In particular, it is well known that one can generate sample paths of the continuous time Markov chain $\{X(t)\}$, by generating a Poisson process $\{K(t)\}$ with rate q_π and at the times of the events of $\{K(t)\}$, the next state of the continuous time Markov chain $\{X(t)\}$ is generated using the transition probabilities of the discrete time Markov chain $\{\tilde{X}(k)\}$.

Let $\mathcal{D}_y = \{(y_1, \ldots, y_{N-2}, y_{N-1} - 1)\}$ for all $y \in S_Y$ with $y_{N-1} > 0$, and let $c_i(j, 0) = 0$ for all $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N\}$. Then Andradóttir, Ayhan, and Kırkızlar [12] show that if

$$R''_d(x) = \sum_{y' \in \mathcal{D}_y} Q_{d(x)}(y, y') - \left(\sum_{y' \in \mathcal{S}_Y \setminus \{y\}} Q_{d(x)}(y, y') \right) \times \left(\sum_{i=1}^{M} c_i(z_i, d_i(x)) \right)$$

for all $x \in S$ and $\pi = (d)^\infty \in \Pi$, then the original optimization problem (that maximizes the long-run average profit in our system) is equivalent to identifying the policy $\pi = (d)^\infty \in \Pi$ that maximizes the following quantity:

$$\lim_{K\to\infty} \mathbb{E}\left\{\frac{1}{K}\sum_{k=1}^{K} R''_d(\tilde{X}(k-1))\right\}. \tag{16}$$

In the remainder of this paper, we analyze the alternative formulation (16) of the original optimization problem.

In order to demonstrate the problem formulation more clearly, we provide an example that employs the server assignment policy that maximizes the throughput of the system with $M = N = 2$ and no setup costs if the servers are ordered such that $\mu_{11}\mu_{22} \geq \mu_{12}\mu_{21}$ (as shown by Andradóttir, Ayhan, and Down [7]). The description of the policy was modified in order to adapt it to our state space.

Example 5.2.1 Suppose that $M = N = 2$ and $B_2 = B < \infty$. Then

$$\begin{aligned} S \;=\; & \{(0,1,1),(1,1,1),\ldots,(B+2,1,1),(0,1,2),(1,1,2),\ldots,(B+2,1,2), \\ & (0,2,1),(1,2,1),\ldots,(B+2,2,1),(0,2,2),(1,2,2),\ldots,(B+2,2,2)\}, \end{aligned} \tag{17}$$

where in state $(l, k_1, k_2) \in S$, l refers to the number of jobs that have been processed at station 1 and are either in service or waiting for service at station 2, and k_m refers to the station that server m was previously assigned to (prior to the most recent service completion in the network) for $m = 1, 2$. Assume that for $i = 1, 2$, we have $c_i(j, j) = 0$ for $j = 1, 2$, $c_i(1, 2) = c_i^{\uparrow} \geq 0$, and $c_i(2, 1) = c_i^{\downarrow} \geq 0$.

Consider the policy $\pi_0 = (d_0)^\infty \in \Pi$, where

$$d_0(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,1,1),(0,1,2),(0,2,1),(0,2,2)\}, \\ a_{22} & \text{if } x \in \{(B+2,1,1),(B+2,1,2),(B+2,2,1),(B+2,2,2)\}, \\ a_{12} & \text{otherwise.} \end{cases}$$

Let $q = q_\pi = \mu_{11} + \mu_{12} + \mu_{21} + \mu_{22}$. Then

$$p_{d_0}(x, x') = \begin{cases} \frac{\mu_{12}+\mu_{22}}{q} & \text{if } y = 0,\ y' = 0, \text{ and } z' = z, \\ \frac{\mu_{11}+\mu_{21}}{q} & \text{if } y = 0,\ y' = 1, \text{ and } z' = (1,1), \\ \frac{\mu_{22}}{q} & \text{if } y = l,\ y' = l-1, \text{ and } z' = (1,2),\ \forall\, 0 < l < B+2, \\ \frac{\mu_{12}+\mu_{21}}{q} & \text{if } y = l,\ y' = l, \text{ and } z' = z,\ \forall\, 0 < l < B+2, \\ \frac{\mu_{11}}{q} & \text{if } y = l,\ y' = l+1, \text{ and } z' = (1,2),\ \forall\, 0 < l < B+2, \\ \frac{\mu_{12}+\mu_{22}}{q} & \text{if } y = B+2,\ y' = B+1, \text{ and } z' = (2,2), \\ \frac{\mu_{11}+\mu_{21}}{q} & \text{if } y = B+2,\ y' = B+2, \text{ and } z' = z, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$R''_{d_0}(x) = \begin{cases} 0 & \text{if } x = (0,1,1), \\ -(\mu_{11}+\mu_{21})c_2^{\downarrow} & \text{if } x = (0,1,2), \\ -(\mu_{11}+\mu_{21})c_1^{\downarrow} & \text{if } x = (0,2,1), \\ -(\mu_{11}+\mu_{21})(c_1^{\downarrow}+c_2^{\downarrow}) & \text{if } x = (0,2,2), \\ \mu_{22} - (\mu_{11}+\mu_{22})c_2^{\uparrow} & \text{if } x = (l,1,1),\ \forall\, 0 < l < B+2, \\ \mu_{22} & \text{if } x = (l,1,2),\ \forall\, 0 < l < B+2, \\ \mu_{22} - (\mu_{11}+\mu_{22})(c_1^{\downarrow}+c_2^{\uparrow}) & \text{if } x = (l,2,1),\ \forall\, 0 < l < B+2, \\ \mu_{22} - (\mu_{11}+\mu_{22})c_1^{\downarrow} & \text{if } x = (l,2,2),\ \forall\, 0 < l < B+2, \\ (\mu_{12}+\mu_{22})(1 - c_1^{\uparrow} - c_2^{\uparrow}) & \text{if } x = (B+2,1,1), \\ (\mu_{12}+\mu_{22})(1 - c_1^{\uparrow}) & \text{if } x = (B+2,1,2), \\ (\mu_{12}+\mu_{22})(1 - c_2^{\uparrow}) & \text{if } x = (B+2,2,1), \\ \mu_{12}+\mu_{22} & \text{if } x = (B+2,2,2). \end{cases}$$

Note that when the policy π_0 is used and $B > 0$, then there are only $B + 5$ positive recurrent states in S, namely $(1,1,1)$, $(B+1,2,2)$, and $(l,1,2)$, where $0 \le l \le B+2$. Similarly, when this policy π is used and $B = 0$, then there are only $B + 4$ positive recurrent states, namely $(0,1,2)$, $(1,1,1)$, $(1,2,2)$, and $(2,1,2)$.

5.3 Preliminary Results

In this section we provide some preliminary results about tandem lines with two stations and setup costs. We first present a result about the form of the optimal server assignment policy.

Lemma 5.3.1 *For a tandem line with $N = 2$, $M \geq 2$, and positive setup costs, there exists an optimal policy that does idle any server voluntarily when the first station is blocked or the second station is starved.*

Proof: When the first station is blocked, the system is in a state $s = (B+2, z_1, \ldots, z_M)$, where $(z_1, \ldots, z_M) \in S_Z$. Now compare two policies $\pi_1 = (d^1)^\infty$ and $\pi_2 = (d^2)^\infty$ that agree with each other apart from state s. Assume that $d_i^1(s) = z_i$ and $d_i^2(s) = 0$ for some $i \in \{1, \ldots, M\}$, and $d_j^1(s) = d_j^2(s)$ for $j \in \{1, \ldots, M\} \setminus \{i\}$. If $z_i = 1$, then the performance of π_1 and π_2 will be identical (since keeping a server at station 1 is equivalent to idling that server in terms of cost). If $z_i = 2$, then the next service completion under policy π_1 will never be later than the one under policy π_2, the system state will be the same after the next service completion, and no extra cost will have been incurred by keeping server i at the second station. Hence, $D^{\pi_1}(t) \geq D^{\pi_2}(t)$ for all $t \geq 0$, implying that $T^{\pi_1} \geq T^{\pi_2}$.

We now restrict ourselves to policies with nonzero long-run average throughput without loss of generality (this is possible because the optimal policy must have positive throughput when $M \geq 2$ since under our assumptions on the service rates provided in Section 3.1, there exists a policy with stationary servers and positive throughput). Define the setup cost per item produced up to time t under policy $\pi \in \Pi$ as $u^\pi(t) = C^\pi(t)/D^\pi(t)$ and let $u^\pi = \lim_{t\to\infty} \mathbb{E}\{u^\pi(t)\}$ be the long-run average setup cost per item produced for all $\pi \in \Pi$. The existence of this limit can be shown to follow from the strong law of large numbers for Markov Chains (see, e.g., Wolff [70], page 164) when the long-run average throughput is nonzero, because the setup cost incurred between two departures is finite (because we have a finite state space and finite setup costs) and we consider stationary policies. Under both policies (π_1 and π_2), the system goes through the same sequence of states, but at the time of each departure, the total setup cost incurred under policy π_1 is equal to the total setup cost incurred under policy π_2. Hence, we can conclude that $u^{\pi_1} = u^{\pi_2}$. Note that for all $\pi \in \Pi$, we have

$$\begin{aligned} T^\pi - C^\pi &= \lim_{t\to\infty} \mathbb{E}\left\{ \frac{D^\pi(t)}{t}\Big(1 - \frac{C^\pi(t)}{D^\pi(t)}\Big)\right\} \\ &= \lim_{t\to\infty} \mathbb{E}\left\{ \frac{D^\pi(t)}{t}\right\} \lim_{t\to\infty} \mathbb{E}\left\{1 - \frac{C^\pi(t)}{D^\pi(t)}\right\} \\ &= T^\pi(1 - u^\pi). \end{aligned}$$

Consequently, the long-run average profit under policy π_1 is never less than the long-run average profit under policy π_2. Hence, there exists an optimal policy that never idles the servers when the first station is blocked. A similar logic follows when the second station is starved. □

Now, for a system with two stations, consider the reversed line where station 1 is followed by station 2 and keep the labeling of the stations as in the original line (i.e., station 2 is the upstream station and station 1 is the downstream station in the reversed line). Let B denote the buffer size between the stations. Assume that the forward line operates under a policy $\pi = (d)^\infty$ and that the reversed line operates under a policy $\pi_R = (d_R)^\infty$, where $d_R(l, z) = d(B + 2 - l, z)$ for $0 \le l \le B + 2$ and $z \in S_Z$ (in both the forward and reversed lines, $z_i = j$ if the previous location of server i is station j). The following reversibility result will be used to simplify the proofs in the following sections.

Lemma 5.3.2 *When $N = 2$, the policy π is optimal for the forward line if and only if the policy π_R is optimal for the reversed line.*

Proof: Let $\kappa_{\pi,1}(x)$ and $\kappa_{\pi,2}(x)$ denote the sets of servers assigned to stations 1 and 2, respectively, under policy π when the original line is in state $x \in S$. Then we see that for $\{X(t)\}$, the transition rate from state $x = (l, z)$ to $(l+1, z')$ is $\sum_{i \in \kappa_{\pi,1}(x)} \mu_{i1}$ for $l \in \{0, \dots, B+1\}$ and the transition rate from state x to $(l-1, z')$ is $\sum_{i \in \kappa_{\pi,2}(x)} \mu_{i2}$ for $l \in \{1, \dots, B+2\}$ and $z, z' \in S_Z$ (where z' is determined by $\kappa_{\pi,1}(x)$ and $\kappa_{\pi,2}(x)$). Now, let $\{(Y_R(t), Z_R(t))\}$ be the Markov chain corresponding to the reversed line. It is easy to see that the stochastic process $\{(B + 2 - Y_R(t), Z_R(t))\}$ has the same transition rates as the stochastic process $\{(Y(t), Z(t))\}$. Hence these two processes are stochastically equivalent. Consequently, the long-run average profit of the forward line under policy π is equal to the long-run average profit of the reversed line under policy π_R (because the departures from one system correspond to departures from the first station of the other system, and the buffer size between the stations is finite), and the result follows. $\square$

5.4 Systems with Generalist Servers

In this section we consider a tandem line with two stations and two generalist servers. In other words, we assume that the service rate of a server at a station can be represented as the product of the server's speed and a constant related to the complexity level of the task at the station. Hence, we have $\mu_{ij} = \mu_i \gamma_j$ for $i, j \in \{1, 2\}$. Service rates of this form can be used to model situations where each server is equally skilled at all tasks. Furthermore, we assume that $c_i(1, 2) = c_i(2, 1) = c \geq 0$ for $i \in \{1, 2\}$. This is a reasonable assumption if the setup costs are due to the movement of the servers or if every machine requires similar setup procedures. Our state space S is as given in (17).

We first study the system where the service rates depend only on the station in Section 5.4.1. Then we consider systems where the service rates depend only on the server in Section 5.4.2. Finally we provide some results about systems where the services depend both on the server and the station in Section 5.4.3.

5.4.1 Service Rate Depends on the Station

In this section, we specify the optimal server assignment policy for small systems where the service rates depend only on the station (so that $\mu_i = 1$ for $i \in \{1,2\}$). We start with a system that has a buffer of size zero between the stations. The proof of the following theorem is provided in Appendix C. Note that the interval for c in part (ii) of Theorem 5.4.1 is non-empty when $\gamma_1 \geq \gamma_2$, and the interval in part (iii) of Theorem 5.4.1 is non-empty when $\gamma_1 < \gamma_2$.

Theorem 5.4.1 *For a Markovian tandem line with two stations, two flexible servers, and buffer of size zero between the stations, if $\mu_{ij} = \gamma_j$ for $i,j \in \{1,2\}$, then the optimal server assignment policy $\pi^* = (d^*)^\infty$ is as follows:*

(i) *If* $0 \leq c \leq \min\{\frac{\gamma_1}{2\gamma_1+4\gamma_2}, \frac{\gamma_2}{4\gamma_1+2\gamma_2}\}$, *then*

$$d^*(x) = \begin{cases} a_{11} & \text{if } x = (0,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z, \\ a_{12} & \text{if } x = (1,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z, \\ a_{22} & \text{if } x = (2,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z, \end{cases}$$

and the recurrent states are $(0,1,2),(1,1,1),(1,2,2),$ *and* $(2,1,2)$.

(ii) *If* $\gamma_1 \geq \gamma_2$ *and* $\frac{\gamma_2}{2\gamma_1+4\gamma_2} < c \leq \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$, *then*

$$d^*(x) = \begin{cases} a_{12} & \text{if } x = (0,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z \text{ or} \\ & \quad x = (1,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z \setminus \{(2,2)\}, \\ a_{22} & \text{if } x = (2,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z \text{ or } x = (1,2,2), \end{cases}$$

and the recurrent states are $(0,1,2),(0,2,2),(1,1,2),(1,2,2),$ *and* $(2,1,2)$.

(iii) *If* $\gamma_1 < \gamma_2$ *and* $\frac{\gamma_1}{4\gamma_1+2\gamma_2} < c \leq \frac{\gamma_2^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$, *then*

$$d^*(x) = \begin{cases} a_{11} & \text{if } x = (0,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z \text{ or } x = (1,1,1), \\ a_{12} & \text{if } x = (1,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z \setminus \{(1,1)\} \text{ or} \\ & \quad x = (2,z_1,z_2) \text{ for all } (z_1,z_2) \in S_Z, \end{cases}$$

and the recurrent states are $(0,1,2),(1,1,1),(1,1,2),(2,1,1),$ *and* $(2,1,2)$.

(iv) *If* $c > \max\{\frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}, \frac{\gamma_2^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}\}$, *then* $d^*(x) = a_{12}$ *for all* $x \in S$ *and the recurrent states are* $(0,1,2), (1,1,2)$, *and* $(2,1,2)$.

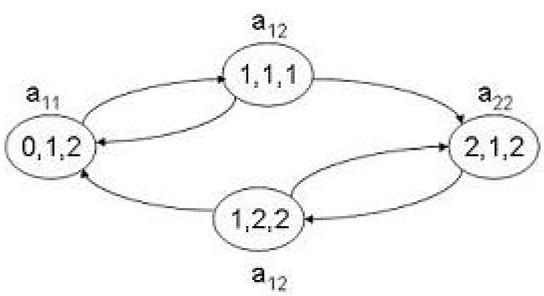
In the case without setups, we say that a server has a primary assignment at a station if (s)he works at this station as long as it is operating (i.e., neither blocked nor starved). However, in the presence of positive setup costs, servers may have a preferred station, but not a primary assignment, because they may work at a less preferred station even if their preferred station is operating, to avoid multiple switchovers. We see that in our case both servers have a preferred station, but not a primary assignment. More specifically, we say that server 1 has a preferred assignment at station 1 and server 2 has a preferred assignment at station 2 when the state $(l,1,2)$ is recurrent for some $l \in \{0,\dots,B+2\}$ and the state $(l,2,1)$ is transient for all $l \in \{0,\dots,B+2\}$. Note that since we have identical servers, the policy described in Theorem 5.4.1 is not unique. For every different type of policy, there is an alternative optimal policy where server 1 has a preferred assignment at station 2 and server 2 has a preferred assignment at station 1.

Theorem 5.4.1 also shows that the optimal policy is of one of the following three types :

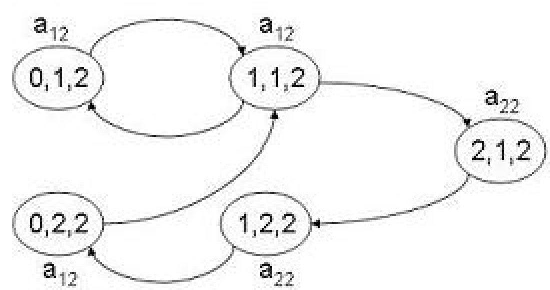
- Neither server switches (Type 0);
- Only one server switches (Type 1);
- Both servers switch (Type 2).

We observe that for small values of c, each server works at the station to which (s)he is primarily assigned as long as this station is operating, and works at the other station otherwise; hence the optimal policy is of Type 2. We also see that this policy is optimal for systems with $c = 0$. This is not surprising because any non-idling policy is known to be optimal for systems with $c = 0$ and generalist servers, see

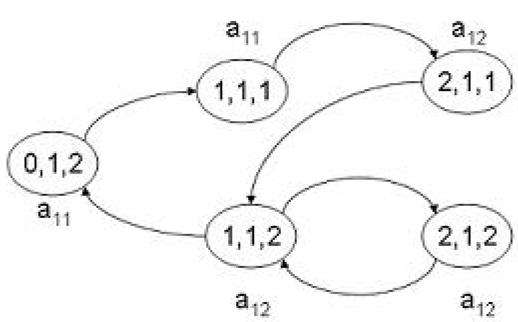
Andradóttir, Ayhan, and Down [10]. In our case the optimal assignment policy is more complicated because we also need to consider policies that involuntarily idle the servers. For intermediate values of c, we observe that one server stops switching to the other station (i.e., the optimal policy is of Type 1), and for large values of c, neither server switches to the other station (i.e., the optimal policy is of Type 0). Furthermore, if the policy is of Type 1, the switching server is the one that has a preferred assignment at the faster station. Note that idling occurs under both Type 1 and Type 0 policies. An examination of the bounds on c in Theorem 5.4.1 shows that the optimal policy is not of Type 2 for any value of $c > \frac{1}{6}$, and the optimal policy is of Type 0 for all values of $c > \frac{1}{2}$. The recurrent states together with the actions in these states under the optimal policy of Theorem 5.4.1 are depicted in Figures 1(a), 1(b), 1(c), and 1(d) (assuming that server 1 is the switching server in Figure 1(b) and server 2 is the switching server in Figure 1(c)).



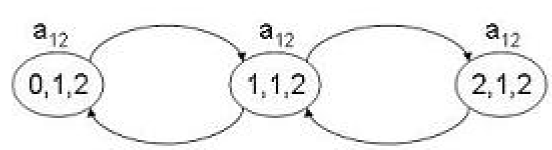
(a) Both Servers Switch (Type 2)



(b) Only Server 1 Switches (Type 1)



(c) Only Server 2 Switches (Type 1)



(d) Neither Server Switches (Type 0)

Figure 1: Recurrent States and Optimal Actions in Theorem 5.4.1

Theorem 5.4.1 also introduces the notion of "multiple threshold" policies. In other words, servers move between the stations when the number of jobs that are in service or waiting for service at station 2 reaches a threshold. Furthermore, the value of this threshold may depend on the locations of both servers prior to the most recent service completion in the network as well as on the station to which they are moving. We use the notation $t_i(z)$ to denote the threshold for server $i \in \{1, \ldots, M\}$ to switch from station z_i to the other station $3 - z_i$ when the previous locations of the servers are represented in the vector $z \in S_Z$. We use the convention that server i is assigned to station $3 - z_i$ when the system is in state $(t_i(z), z)$. In Figure 1(a), server 1 switches between stations in states $(1, 2), (2, 2)$ with $t_1(1, 2) = 2$, $t_1(2, 2) = 1$; and server 2 switches between stations in states $(1, 1), (1, 2)$ with $t_2(1, 1) = 1$, and $t_2(1, 2) = 0$. In Figure 1(b), server 1 switches between stations in states $(1, 2), (2, 2)$ with $t_1(1, 2) = 2$ and $t_1(2, 2) = 0$, but server 2 does not switch between stations at all in Figure 1(b).

Next, the following theorem provides the optimal server assignment policy for a system with a buffer of size one between the stations. Its proof is provided in Appendix C.

Theorem 5.4.2 *For a Markovian tandem line with two stations, two flexible servers, and buffer of size one between the stations, if* $\mu_{ij} = \gamma_j$ *for* $i, j \in \{1, 2\}$*, then the optimal server assignment policy* $\pi^* = (d^*)^\infty$ *is as follows:*

(i) *If* $0 \leq c \leq \min\{\frac{\gamma_1}{2\gamma_1 + 2\gamma_2}, \frac{\gamma_2}{2\gamma_1 + 2\gamma_2}\}$*, then*

$$d^*(x) = \begin{cases} a_{11} & \text{if } x = (0, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or } x = (1, 1, 1), \\ a_{12} & \text{if } x = (1, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \setminus \{(1, 1, 1)\} \text{ or} \\ & \quad x = (2, z_2, z_2) \text{ for all } (z_1, z_2) \in S_Z \setminus \{(2, 2, 2)\}, \\ a_{22} & \text{if } x = (3, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or } x = (2, 2, 2), \end{cases}$$

and the recurrent states are $(0, 1, 2)$*,* $(1, 1, 1)$*,* $(1, 1, 2)$*,* $(1, 2, 2)$*,* $(2, 1, 1)$*,* $(2, 1, 2)$*,* $(2, 2, 2)$*, and* $(3, 1, 2)$*.*

(ii) *If* $\gamma_1 \geq \gamma_2$ *and* $\frac{\gamma_2}{2\gamma_1+2\gamma_2} < c \leq \min\{\frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}, \frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2}\}$, *then*

$$d^*(x) = \begin{cases} a_{12} & \text{if } x = (y, z_1, z_2) \text{ for all } y \in \{0,1\} \text{ and } (z_1, z_2) \in S_Z \text{ or} \\ & x = (2, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \setminus \{(2,2)\}, \\ a_{22} & \text{if } x = (3, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or } x = (2,2,2), \end{cases}$$

and the recurrent states are $(0,1,2)$, $(1,1,2)$, $(1,2,2)$, $(2,1,2)$, $(2,2,2)$, *and* $(3,1,2)$.

(iii) *If* $\gamma_1 \geq \gamma_2$, $\gamma_1^2 \leq \gamma_1\gamma_2 + \gamma_2^2$, *and* $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}$, *then* $d^*(x) = a_{12}$ *for all* $x \in S$ *and the recurrent states are* $(0,1,2)$, $(1,1,2)$, $(2,1,2)$, *and* $(3,1,2)$.

(iv) *If* $\gamma_1 \geq \gamma_2$, $\gamma_1^2 > \gamma_1\gamma_2 + \gamma_2^2$, *and* $\frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2} < c \leq \frac{3\gamma_1^3+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$, *then*

$$d^*(x) = \begin{cases} a_{12} & \text{if } x = (0, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or} \\ & x = (y, z_1, z_2) \text{ for all } y \in \{1,2\} \text{ and } (z_1, z_2) \in S_Z \setminus \{(2,2)\}, \\ a_{22} & \text{if } x = (y, 2, 2) \text{ for all } y \in \{1,2\} \text{ or} \\ & x = (3, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z, \end{cases}$$

and the recurrent states are $(0,1,2)$, $(0,2,2)$, $(1,1,2)$, $(1,2,2)$, $(2,1,2)$, $(2,2,2)$, *and* $(3,1,2)$.

(v) *If* $\gamma_1 \geq \gamma_2$, $\gamma_1^2 > \gamma_1\gamma_2 + \gamma_2^2$ *and* $c > \frac{3\gamma_1^3+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$, *then* $d^*(x) = a_{12}$ *for all* $x \in S$ *and the recurrent states are* $(0,1,2)$, $(1,1,2)$, $(2,1,2)$, *and* $(3,1,2)$.

(vi) *If* $\gamma_1 < \gamma_2$ *and* $\frac{\gamma_1}{2\gamma_1+2\gamma_2} < c \leq \min\{\frac{\gamma_2^2}{2\gamma_1^2+2\gamma_2^2}, \frac{2\gamma_1\gamma_2+\gamma_1^2}{2\gamma_2^2+4\gamma_1\gamma_2}\}$, *then*

$$d^*(x) = \begin{cases} a_{11} & \text{if } x = (0, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or } x = (1,1,1), \\ a_{12} & \text{if } x = (1, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \setminus \{(1,1)\} \text{ or} \\ & x = (y, z_1, z_2) \text{ for all } y \in \{2,3\} \text{ and } (z_1, z_2) \in S_Z, \end{cases}$$

and the recurrent states are $(0,1,2)$, $(1,1,1)$, $(1,1,2)$, $(2,1,1)$, $(2,1,2)$, *and* $(3,1,2)$.

(vii) *If* $\gamma_1 < \gamma_2$, $\gamma_2^2 \leq \gamma_1\gamma_2 + \gamma_1^2$, *and* $c > \frac{\gamma_2^2}{2\gamma_1^2+2\gamma_2^2}$, *then* $d^*(x) = a_{12}$ *for all* $x \in S$ *and the recurrent states are* $(0,1,2)$, $(1,1,2)$, $(2,1,2)$, *and* $(3,1,2)$.

(viii) *If* $\gamma_1 < \gamma_2$, $\gamma_2^2 > \gamma_1\gamma_2 + \gamma_1^2$, *and* $\frac{2\gamma_1\gamma_2+\gamma_1^2}{2\gamma_2^2+4\gamma_1\gamma_2} < c \leq \frac{3\gamma_2^3+\gamma_1\gamma_2^2-\gamma_1^2\gamma_2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$, *then*

$$d^*(x) = \begin{cases} a_{11} & \text{if } x = (0, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or} \\ & \quad x = (y, 1, 1) \text{ for all } y \in \{1, 2\}, \\ a_{12} & \text{if } x = (y, z_1, z_2) \text{ for all } y \in \{1, 2\} \text{ and } (z_1, z_2) \in S_Z \setminus \{(1, 1)\} \text{ or} \\ & \quad x = (3, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z\ , \end{cases}$$

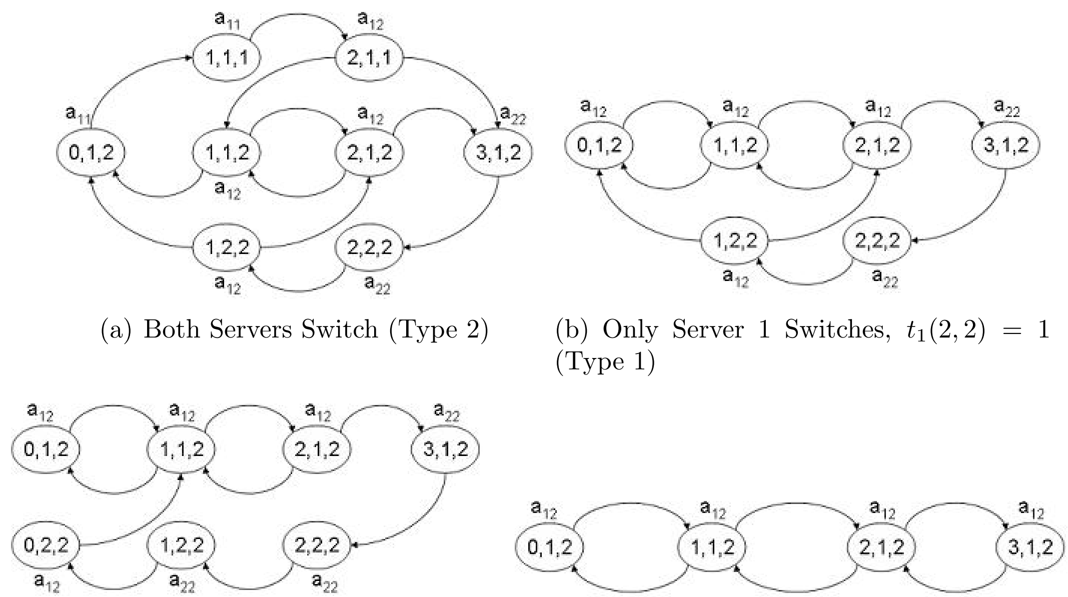
and the recurrent states are $(0,1,2)$, $(1,1,1)$, $(1,1,2)$, $(2,1,1)$, $(2,1,2)$, $(3,1,1)$, *and* $(3,1,2)$.

(ix) *If* $\gamma_1 < \gamma_2$, $\gamma_2^2 > \gamma_1\gamma_2 + \gamma_1^2$, *and* $c > \frac{3\gamma_2^3+\gamma_1\gamma_2^2-\gamma_1^2\gamma_2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$, *then* $d^*(x) = a_{12}$ *for all* $x \in S$ *and the recurrent states are* $(0,1,2)$, $(1,1,2)$, $(2,1,2)$, *and* $(3,1,2)$.

Note that the interval for c in part (ii) of Theorem 5.4.2 is non-empty when $\gamma_1 \geq \gamma_2$, and the interval in part (iv) of Theorem 5.4.2 is non-empty when $\gamma_1 \geq \gamma_2$ and $\gamma_1^2 > \gamma_1\gamma_2 + \gamma_2^2$. Similarly, the interval in part (vi) of Theorem 5.4.2 is non-empty when $\gamma_1 \geq \gamma_2$, and the interval in part (viii) of Theorem 5.4.2 is non-empty when $\gamma_1 \geq \gamma_2$ and $\gamma_1^2 > \gamma_1\gamma_2 + \gamma_2^2$.

We now depict the recurrent states and the optimal actions in Theorem 5.4.2. More specifically, Figure 2(a) shows the optimal policy of Type 2 corresponding to part (i) of Theorem 5.4.2. Figures 2(b) and 2(c) show the optimal policies of Type 1 (with different thresholds) where server 1 is the switching server corresponding to parts (ii) and (iv) of Theorem 5.4.2, respectively. Finally, Figure 2(c) shows the optimal policy of Type 0, corresponding to parts (iii), (v), (vii), and (ix) of Theorem 5.4.2.

We see that server 1 has a preferred assignment at station 1 and server 2 has a preferred assignment at station 2 for all values of c. However, note that this policy is



(a) Both Servers Switch (Type 2)

(b) Only Server 1 Switches, $t_1(2,2) = 1$ (Type 1)

(c) Only Server 1 Switches, $t_1(2,2) = 0$ (Type 1)

(d) Neither Server Switches (Type 0)

Figure 2: Recurrent States and Optimal Actions in Theorem 5.4.2

not the unique optimal policy. Since the servers are identical, we can relabel them and obtain alternative optimal policies where the preferred assignments of the servers are switched. Furthermore, we see that as c increases, all the systems go through the same set of optimal policies for $B = 0$ (although the cutoffs on the value of c depend on the service rates). However, this is no longer correct when $B = 1$. More specifically, when $B = 1$, we observe three or four different optimal policies for different values of the setup cost. If the first station is faster and the service rates satisfy the condition $\gamma_1^2 \leq \gamma_1\gamma_2 + \gamma_2^2$, then we observe three different optimal policies for different values of the setup cost. If the first station is faster and the service rates satisfy the condition $\gamma_1^2 > \gamma_1\gamma_2 + \gamma_2^2$, then we observe four different optimal policies, depending on the value of the setup cost (in particular, as the setup increases the first server completes more jobs at station 2 before switching back to station 1). Also note that the transition from one policy to another follows a similar pattern when $B = 0$ or $B = 1$. In both cases, for small values of c both servers switch and the optimal policy is of Type 2, for intermediate values of c only one server switches (server 1 is the switching server when $\gamma_1 \geq \gamma_2$ and server 2 is the switching server when $\gamma_1 < \gamma_2$) and the optimal policy is of Type 1, and for large values of c neither server switches and the optimal policy is of Type 0. Moreover, when the optimal policy is of Type 1, we observe that the switching server is the one that has a preferred assignment at the faster station. Finally, we see that the optimal policy is not of Type 2 when $c > \frac{1}{4}$ and is of Type 0 when $c > \frac{3}{4}$.

5.4.2 Service Rate Depends on the Server

In this section we study systems with small buffer sizes where the service rate depends only on the server (so that $\gamma_j = 1$ for $j \in \{1, 2\}$). Without loss of generality, assume that $\mu_1 \geq \mu_2$ because we can relabel the servers otherwise. We first identify the optimal server assignment policy for the system with buffer of size zero between the stations. We provide the proof of the following theorem in Appendix C. Note that the interval in part (ii) of Theorem 5.4.3 is non-empty when $\mu_1 \geq \mu_2$.

Theorem 5.4.3 *For a Markovian tandem line with two stations, two flexible servers, and buffer of size zero between the stations, if $\mu_{ij} = \mu_i$ for $i, j \in \{1, 2\}$, then the optimal server assignment policy $\pi^* = (d^*)^\infty$ is as follows:*

(i) *If $0 \leq c \leq \frac{\mu_2}{4\mu_1 + 2\mu_2}$, then*

$$d^*(x) = \begin{cases} a_{11} & \text{if } x = (0, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z, \\ a_{12} & \text{if } x = (1, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z, \\ a_{22} & \text{if } x = (2, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z, \end{cases}$$

and the recurrent states are $(0, 1, 2), (1, 1, 1), (1, 2, 2)$, and $(2, 1, 2)$.

(ii) *If* $\frac{\mu_2}{4\mu_1+2\mu_2} < c \leq \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$, *then*

$$d^*(x) = \begin{cases} a_{12} & \text{if } x = (y, z_1, z_2) \text{ for all } y \in \{0,1\} \text{ and } (z_1, z_2) \in S_Z \text{ or} \\ & \quad x = (2, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \setminus \{(2,2)\}, \\ a_{22} & \text{if } x = (3, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or } x = (2,2,2), \end{cases}$$

and the recurrent states are $(0,1,2), (0,2,2), (1,1,2), (1,2,2)$, *and* $(2,1,2)$.

(iii) *If* $c > \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$, *then* $d^*(x) = a_{12}$ *for all* $x \in S$ *and the recurrent states are* $(0,1,2), (1,1,2)$, *and* $(2,1,2)$.

We see that the optimal server assignment policy in Theorem 5.4.3 is similar to the optimal policy provided in Theorem 5.4.1 for the case where the service rate depends only on the station, and has one of forms shown in Figure 1. More specifically, both servers have preferred assignments. For small values of c they both switch to the other station when their own station is not operating. For intermediate values of c, the optimal policy becomes a multiple threshold policy with one switching server, and for large values of c, the optimal policy does not allow the servers to switch (in the recurrent states). However, we should note that the optimal policy is not unique in this case either. Lemma 5.3.2 shows that there is another optimal policy that assigns the faster server to the second station. In this case the preferred assignments of the servers will be switched and for intermediate values of c, there is an optimal Type 1 policy similar to the Type 1 policy of the system where the service rate depends on the station and $\gamma_1 < \gamma_2$.

We also see that the policy is not of Type 2 when $c > \frac{1}{6}$, and the policy is of Type 0 when $c > 1$. Note that the switching is possible for a larger range of setup costs compared to Theorem 5.4.1. For example when $\mu_1 = 10$ and $\mu_2 = 1$, switching policies are optimal when $c < 0.856$. However, we saw in Section 5.4.1 that when the service rates depend only on the server, no switching policy is optimal for $c > \frac{1}{2}$.

When a server is extremely fast compared to the other server, it may be advantageous to move this server, even for high values of the setup cost, to benefit from this high service rate. However, the same logic does not follow when a station is extremely fast compared to the other station because all the jobs have to be processed at both stations. In other words, we take advantage of the faster station not by assigning servers there, but by assigning servers disproportionally to the slower station.

Next, we provide the optimal assignment policy when the buffer size between the stations is equal to one. The proof of the following theorem is is provided in Appendix C.

Theorem 5.4.4 *For a Markovian tandem line with two stations, two flexible servers, and buffer of size one between the stations, if* $\mu_{ij} = \mu_j$ *for* $i, j \in \{1, 2\}$*, then the optimal server assignment policy* $\pi = (d)^\infty$ *is as follows:*

(i) *If* $0 \le c \le \frac{4\mu_1^2\mu_2+5\mu_1\mu_2^2+3\mu_2^3}{12\mu_1^3+20\mu_1^2\mu_2+12\mu_1\mu_2^2+4\mu_2^3}$, *then*

$$d^*(x) = \begin{cases} a_{11} & \text{if } x = (0, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or } x = (1, 1, 1), \\ a_{12} & \text{if } x \in \{(1, 1, 2), (1, 2, 2), (2, 1, 2)\}, \\ a_{21} & \text{if } x \in \{(1, 2, 1), (2, 1, 1), (2, 2, 1)\}, \\ a_{22} & \text{if } x = (3, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or } x = (2, 2, 2), \end{cases}$$

and the recurrent states are $(0, 1, 2)$, $(0, 2, 1)$, $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 1)$, $(1, 2, 2)$, $(2, 1, 1)$, $(2, 1, 2)$, $(2, 2, 1)$, $(2, 2, 2)$, $(3, 1, 2)$, *and* $(3, 2, 1)$.

(ii) *If* $\frac{4\mu_1^2\mu_2+5\mu_1\mu_2^2+3\mu_2^3}{12\mu_1^3+20\mu_1^2\mu_2+12\mu_1\mu_2^2+4\mu_2^3} \le c < \min\{\frac{\mu_2}{2\mu_1}, \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}\}$,

$$d^*(x) = \begin{cases} a_{12} & \text{if } x = (y, z_1, z_2) \text{ for all } y \in \{0, 1\} \text{ and } (z_1, z_2) \in S_Z \text{ or} \\ & \quad x = (2, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \setminus \{(2, 2)\}, \\ a_{22} & \text{if } x = (3, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or } x = (2, 2, 2), \end{cases}$$

and the recurrent states are $(0, 1, 2)$, $(1, 1, 2)$, $(1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 2)$, *and* $(3, 1, 2)$.

(iii) *If* $2\mu_1^5+\mu_1^4\mu_2 \geq \mu_1^3\mu_2^2+3\mu_1^2\mu_2^3+2\mu_1\mu_2^4+\mu_2^5$ *and* $\frac{\mu_2}{2\mu_1} < c \leq \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$,

$$d^*(x) = \begin{cases} a_{12} & \text{if } x = (0, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z \text{ or} \\ & \quad x = (y, z_1, z_2) \text{ for all } y \in \{1, 2\} \text{ and } (z_1, z_2) \in S_Z \setminus \{(2, 2)\}, \\ a_{22} & \text{if } x = (y, 2, 2) \text{ for all } y \in \{1, 2\} \text{ or} \\ & \quad x = (3, z_1, z_2) \text{ for all } (z_1, z_2) \in S_Z, \end{cases}$$

and the recurrent states are $(0, 1, 2)$, $(0, 2, 2)$, $(1, 1, 2)$, $(1, 2, 2)$, $(2, 1, 2)$, $(2, 2, 2)$, *and* $(3, 1, 2)$.

(iv) *If* $2\mu_1^5 + \mu_1^4\mu_2 \geq \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$ *and* $c > \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$, *then* $d^*(x) = a_{12}$ *for all* $x \in S$ *and the recurrent states are* $(0, 1, 2)$, $(1, 1, 2)$, $(2, 1, 2)$, *and* $(3, 1, 2)$.

(v) *If* $2\mu_1^5+\mu_1^4\mu_2 < \mu_1^3\mu_2^2+3\mu_1^2\mu_2^3+2\mu_1\mu_2^4+\mu_2^5$ *and* $c > \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$, *then* $d^*(x) = a_{12}$ *for all* $x \in S$ *and the recurrent states are* $(0, 1, 2)$, $(1, 1, 2)$, $(2, 1, 2)$, *and* $(3, 1, 2)$.

Note that the interval in part (ii) of Theorem 5.4.2 is non-empty when $\mu_1 \geq \mu_2$, and the interval in part (iii) of Theorem 5.4.2 is non-empty when $\mu_1 \geq \mu_2$ and $2\mu_1^5+\mu_1^4\mu_2 \geq \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$. Note further that $\frac{\mu_2}{2\mu_1} \leq \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$ when $2\mu_1^5 + \mu_1^4\mu_2 \geq \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$.

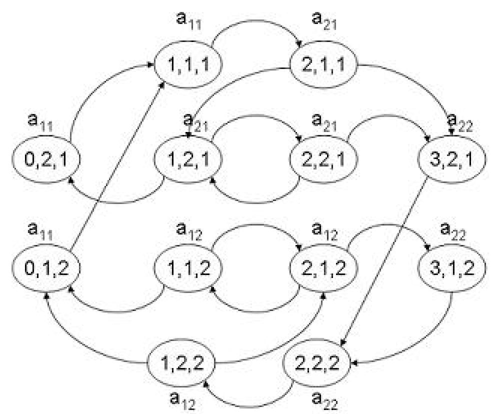
When $B = 1$, we see that for small values of c, there is no preferred assignment of the servers. More specifically, both servers are at the same station when the other station is not operating. When the number of jobs in the buffer reaches a certain threshold and both stations are operating, the faster server switches to the other station, and we thus cannot talk about a preferred station for each server. Furthermore, the slower server also switches between the stations for small values of c, because the increase in the throughput resulting from not idling this server dominates the setup cost associated with moving him/her. More specifically, we see that $t_1(1,1) = 2$, $t_1(1,2) = 3$, $t_1(2,1) = 0$, $t_1(2,2) = 1$, $t_2(1,2) = 3$, and $t_2(2,1) = 0$. We have four different thresholds for the first server that depend on the previous locations of both servers, the station from which (s)he is moving, and the station to which (s)he is moving.

Note that this optimal policy is not unique. The policy where the faster server has a preferred assignment at station 2 is optimal as well (as a result of Lemma 5.3.2). However, in this case the optimal server assignment policy of Type 1 would be similar to the Type 1 policy of Theorem 5.4.2 with $\gamma_1 < \gamma_2$.

As c increases, the optimal policy follows a similar pattern to the optimal policy given in Theorem 5.4.2 for the systems with $B = 1$ and station-dependent service rates. More specifically, for small values of c, the optimal policy is of Type 2 (however without any preferred assignments this time), for intermediate and high values of c, the optimal policy is of Type 1 and Type 0, respectively. The recurrent states and optimal actions for the optimal policy of Type 2 is depicted in Figure 3(a). The other cases are omitted because they are same as the ones shown in Figures 2(b), 2(c), and 2(d). We also see that the optimal policy is not of Type 2 when $c > \frac{1}{4}$ and is of Type 0 when $c > \frac{3}{2}$. When we let $\mu_1 = 10$ and $\mu_2 = 1$, then we see that the optimal Policy is of Type 0 when $c > 1.309$. Hence, the switching policies are optimal for a larger range of setup cost when $B = 1$ and the service rates depend only on the server, as compared to Theorem 5.4.2 where the service rates depend only on the station. This conclusion is similar to the one we made regarding the case with $B = 0$.

5.4.3 Service Rate Depends on Both the Server and the Station

We observed in Theorems 5.4.1 and 5.4.3 that the policy π_0 is optimal in systems with $B = 0$ when c is positive but small and the service rates depend either on the station or on the server. However, when $B > 0$, the optimal policy provided in Theorems 5.4.2 and 5.4.4 is different from π_0 even for small values of c. Hence, the servers



(a) Both Servers Switch (Type 2)

Figure 3: Recurrent States and Optimal Actions in Theorem 5.4.4

have a primary assignment when $B = 0$, but this is not correct when $B > 0$. In this section we consider systems with generalist servers whose service rates depend both on the server and the station. The following proposition shows that the policy π_0 is not optimal when $B > 0$ and $c > 0$.

Proposition 5.4.1 *In a tandem line with two stations, two generalist servers, and buffer of size $B > 0$ between the stations, π_0 is not optimal for the system with $c > 0$.*

Proof: First, assume that $\mu_1 \geq \mu_2$ and $\gamma_1 \geq \gamma_2$, where at least one of the inequalities is strict. Let $\pi_0 = (d_0)^\infty$ be as described in Example 5.2.1. It is not difficult to show that

$$T^{\pi_0} - C^{\pi_0} = \frac{(\mu_1+\mu_2)\gamma_1\gamma_2}{\gamma_1+\gamma_2} - \frac{2c(\mu_1+\mu_2)\gamma_1\gamma_2(\mu_1\gamma_1-\mu_2\gamma_2)\Big((\mu_1\gamma_1)^{B+1}+(\mu_2\gamma_2)^{B+1}\Big)}{(\gamma_1+\gamma_2)\Big((\mu_1\gamma_1)^{B+2}-(\mu_2\gamma_2)^{B+2}\Big)}.$$

Now define the policy $\pi_1 = (d^1)^\infty$ such that $d^1(1,1,1) = a_{11}$ and $d^1(x) = d_0(x)$ for $x \in S \setminus \{(1,1,1)\}$. In other words, π_1 is a multiple threshold policy that assigns both servers to station 1 if there are no jobs in the buffer and the servers are already at station 1. One can show that

$$\begin{aligned} T^{\pi_1} - C^{\pi_1} &= \frac{(\mu_1+\mu_2)\gamma_1\gamma_2}{\gamma_1+\gamma_2} \\ &\quad - \frac{2c(\mu_1+\mu_2)\gamma_1\gamma_2(\mu_1\gamma_1-\mu_2\gamma_2)\Big((\mu_1\gamma_1)^{B+1}+\mu_2\gamma_2\Big((\mu_1\gamma_1)^{B}+(\mu_2\gamma_2)^{B}\Big)\Big)}{(\gamma_1+\gamma_2)\Big((\mu_1\gamma_1)^{B+2}+\mu_2\gamma_2(\mu_1\gamma_1)^{B+1}-2(\mu_2\gamma_2)^{B+2}\Big)}. \end{aligned}$$

Some algebra shows that $(T^{\pi_1} - C^{\pi_1}) - (T^{\pi_0} - C^{\pi_0}) = \frac{\epsilon_1}{\epsilon_2}$ where

$$\begin{aligned}
\epsilon_1 &= 2c(\mu_1+\mu_2)\gamma_1\gamma_2(\mu_1\gamma_1 - \mu_2\gamma_2)(\mu_2\gamma_2)^{B+3}\Big((\mu_1\gamma_1)^B - (\mu_2\gamma_2)^B\Big),\\
\epsilon_2 &= (\gamma_1+\gamma_2)\Big((\mu_1\gamma_1)^{B+2} - (\mu_2\gamma_2)^{B+2}\Big)\Big[\Big((\mu_1\gamma_1)^{B+2} - (\mu_2\gamma_2)^{B+2}\Big)\\
&\quad +\mu_2\gamma_2\Big((\mu_1\gamma_1)^{B+1} - (\mu_2\gamma_2)^{B+1}\Big)\Big].
\end{aligned}$$

Then, it is easy to see that $\frac{\epsilon_1}{\epsilon_2} > 0$ and π_1 is a better policy than π_0. If $\mu_1 < \mu_2$, then we can relabel the servers. If $\gamma_1 < \gamma_2$, then define d^1 such that $d^1(B+1,2,2) = a_{22}$ and $d^1(x) = d(x)$ for $x \in S \setminus \{(B+1,2,2)\}$, and Lemma 5.3.2 implies that π_1 is a better policy than π_0. When $\mu_1 = \mu_2$ and $\gamma_1 = \gamma_2$, we can show that

$$T^{\pi_0} - C^{\pi_0} = \frac{\mu_1\gamma_1(2+B-4c)}{2+B}, T^{\pi_1} - C^{\pi_1} = \frac{\mu_1\gamma_1(3+2B-6c)}{3+2B}.$$

Then $(T^{\pi_1} - C^{\pi_1}) - (T^{\pi_0} - C^{\pi_0}) = \frac{2cB\mu_1\gamma_1}{6+7B+2B^2}$, and this quantity is strictly positive for $B > 0$. Consequently, when $c > 0$, policy π_0 is never optimal. □

Proposition 5.4.1 also shows that, unlike the case with $c = 0$, it is not true that all nonidling policies are optimal in the presence of small positive setup costs. Note that in the papers that study the case with $c = 0$ (e.g., Andradóttir, Ayhan, and Down [7]), Markovian stationary deterministic policies are employed, so that the same action is used each time the number of jobs $Y_1(t)$ processed by station 1 but not by station 2 reaches the same level. However, in our case it is possible to employ different actions in states with equal $Y_1(t)$ values, depending on the locations of the servers. More specifically, when there are jobs at each station and the buffer is empty, the policy π_1 considered in the proof of Proposition 5.4.1 sometimes assigns both servers to station 1, and sometimes assigns server 1 to station 1 and server 2 to station 2. Similarly, the optimal server assignment policies for small systems do not immediately move the servers back to their preferred stations for $c > 0$, as shown in Theorems 5.4.2 and 5.4.2 (when preferred assignments exist).

We conclude this section by pointing out that Proposition 5.4.1 does not necessarily hold for systems with specialist servers. In particular, we will show in Section

5.5.3 that the policy π_0 may be optimal if the servers are specialists.

5.5 Numerical Results

In this section, we perform numerical experiments and provide our observations about the form of the optimal policy for systems with generalist servers and larger buffers, or specialist servers. We consider systems with common setup costs at any station for both servers as in Section 5.4. In Section 5.5.1 we consider systems with an intermediate buffer of size $B > 1$ and servers whose service rates depend on either the station or the server. In Section 5.5.2 we consider systems with specialist servers and $B = 0$. Finally, in Section 5.5.3, we consider systems with specialist servers and an intermediate buffer of arbitrary size.

5.5.1 Systems with $B > 1$ and Service Rate Depending Either on the Station or the Server

In this section we provide our observations about the form of the optimal policy when the service rates depend on either the station or the server. Theorems 5.4.1 through 5.4.4 provide the optimal server assignment policy for these systems when the buffer size between the stations is zero or one. Consequently, in this section, we study systems with buffer sizes larger than one.

First we consider systems where the service rates depend only on the station. We randomly generate 50,000 systems with the service rate at each station independently drawn from a uniform distribution with range [0.5,2.5] and the setup cost drawn from a uniform distribution with range (0,0.5) (we have also tried a larger range for the setup cost and observed that most of the optimal policies ended up being of Type 0 with no switching). Furthermore, the buffer size B between the stations is drawn from a discrete uniform distribution with range $\{2, 3, 4, 5\}$. We determine the optimal server assignment policy using the policy iteration algorithm for communicating Markov chains.

Our numerical results for systems with service rates depending only on the station suggest that the optimal server assignment policy is similar to that of systems with an intermediate buffer of size one, see Theorem 5.4.2. Both servers have a preferred assignment and the optimal policy is a multiple threshold policy. Furthermore, some properties of the thresholds can also be determined. For example, consider a system where servers 1 and 2 have preferred assignments at stations 1 and 2, respectively and assume that the policy types 0, 1, 2 are as defined in Section 5.4.1. If the optimal policy is of Type 2, we observe that $t_1(1,2) = B+2$, $t_1(2,2) = l$, $t_2(1,1) = l+1$, and $t_2(1,2) = 0$ for some $l \in \{1, \ldots, B\}$ (note that $l = 1$ in the policy of Figure 2(a)). If the optimal policy is of Type 1 and server 1 is the switching server, then $t_1(1,2) = B+2$ and $t_1(2,2) = l$ for some $0 \in \{1, \ldots, B\}$ (note that $l = 1$ in the policy of Figure 2(b) and $l = 0$ in the policy of Figure 2(c)). Similarly, if the optimal policy is of Type 1 and server 2 is the switching server, then $t_2(1,1) = l$ and $t_2(1,2) = 0$ for some $l \in \{2, \ldots, B+2\}$.

Next, we study systems where the service rates depend only on the server. We randomly generate 50,000 systems with the service rate of each server independently drawn from a uniform distribution with range [0.5,2.5] and the parameters B and c chosen as before. We observe that the optimal policy is of multiple threshold type for any $c > 0$ and servers do not have preferred assignments for small values of c, as in Theorem 5.4.4. Furthermore, we are able to make some conclusions regarding the threshold values as well. For simplicity, we only provide our observations for the case where server 1 is the faster server. If the optimal policy is of Type 2, we observe that $t_1(1,2) = B+2$, $t_1(2,1) = 0$, $t_2(1,1) = l$, $t_2(1,2) = 0$, $t_2(2,1) = B+2$, and $t_2(2,2) \leq l$ for some $l \in \{2, \ldots, B+1\}$ (note that $l = 2$ and $t_2(2,2) = 1 \leq l$ in the policy of Figure 3(a)). In other words, the thresholds for switching to station 1 is never greater than the thresholds for switching to station 2 for the first server. If the optimal policy is of Type 1, we see that $t_1(1,2) = B+2$ and $t_1(2,2) = l$, for some

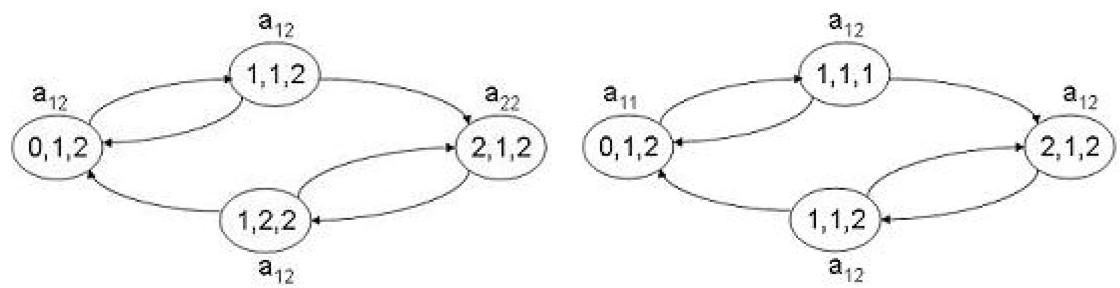
$l \in \{0, \ldots, B\}$.

We conclude that if the service rates depend only on either the station or the server, then the form of the optimal policy is robust to the buffer size, and the thresholds still have a certain dependency on each other for systems with $B > 1$. However, the thresholds can take values in a broader range and can not be calculated as easily as in Section 5.4.

5.5.2 Systems with $B = 0$ and Specialist Servers

Theorems 5.4.1 and 5.4.3 provide the optimal server assignment policy when $B = 0$ and the service rate depends on either the server or the station. More specifically, they show that the optimal policy is one of the multiple threshold type policies shown in Figure 1. In this section we provide our observations about the form of the optimal policy and the values of the thresholds if the servers are specialists; i.e., the service rates are not necessarily the products of two terms representing server skill and task difficulty. We randomly generate 100,000 systems with the service rates independently drawn from a uniform distribution with range $[0.5, 2.5]$ and the setup cost drawn from a uniform distribution with range $(0, 0.5)$.

We observe that both servers have preferred assignments in each experiment and the optimal policy is a multiple threshold policy. We now demonstrate these policies in more detail for a system where servers 1 and 2 have preferred assignments at stations 1 and 2, respectively. If the optimal policy is of Type 2, then we have $t_1(1, 2) = 2$, $t_1(2, 2) = 1$, $t_2(1, 1) = 1$, and $t_2(1, 2) = 0$, as in Figure 1(a). If the optimal policy is of Type 1 and server 1 is the switching server, then we have $t_1((1, 2), 1, 2) = 2$ and $t_1((2, 2), 1) \in \{0, 1\}$, as in Figures 1(b) and 4(a). If the optimal policy is of Type 1 and server 2 is the switching server, we have $t_2(1, 2) = 0$ and $t_2(1, 1) \in \{1, 2\}$, as in Figures 1(c) and 4(b). Finally, if the setup cost is big, the optimal policy is of Type 0, as in Figure 1(d). To summarize, the recurrent states under possible optimal policies for systems with $B = 0$ and specialist servers are as in Figures 1(a), 1(b), 1(c), 1(d), 4(a), and 4(b).



(a) Only Server 1 Switches (Type 1) (b) Only Server 2 Switches (Type 1)

Figure 4: Recurrent States and Optimal Actions in Section 5.5.2

Note that policies of Type 0, 1, or 2 were also observed for systems where the service rate depends on either the server or the station and $B = 0$, as shown in Section 5.4. However, for these systems, if the optimal policy is of Type 1 and the switching server is server $i \in \{1, 2\}$, then $t_i(2, 2)$ or $t_i(1, 1)$ is never equal to one; i.e., the policies shown in Figures 4(a) and 4(b) are never optimal. Hence, we conclude that the form of the policy is robust to the service rates, but the values of the thresholds can take values in a broader range in systems with specialist servers.

5.5.3 Systems with $B > 0$ and Specialist Servers

In this section, we study systems with specialist servers and $B > 0$. More specifically, we randomly generate 50,000 systems with the service rates and the setup cost chosen as in Section 5.5.2, and the buffer size drawn from a discrete uniform distribution with range $\{1, 2, 3, 4, 5\}$.

In all the experiments with specialist servers and $B > 0$ we observe that the optimal policy is of multiple threshold type. More specifically, the optimal policies are more general versions of the policies observed in Section 5.5.1. For example, consider the case where the optimal policy is of Type 2 with servers 1 and 2 having preferred assignments at stations 1 and 2, respectively. Then, we observe that $t_1(1, 2) = B + 2$,

$t_1(2,2) = k$, $t_2(1,1) = l$, and $t_2(1,2) = 0$, where $k, l \in \{1, \ldots, B+1\}$. Unlike in the case where the service rate depends only on the station or the server, we do not observe any simple relation between the thresholds k and l. Note that we observed some cases where $l = 1$ and $k = B+1$ in the optimal policy (i.e., π_0 was optimal). Hence, in the presence of specialist servers, the policy that has primary assignments (so that the servers switch back to their primary stations as soon as possible) can be optimal, unlike in the case with generalist servers considered in Proposition 5.4.1. Similar conclusions follow when the optimal policy is of Type 2 and the servers do not have preferred assignments (as in the case where service rate depends only on the servers). In this case, we observe that $t_1(1,2) = B+2$, $t_1(2,1) = 0$, $t_2(1,1) = k$, $t_2(1,2) = 0$, $t_2(2,1) = B+2$, and $t_2(2,2) = l$, where $k, l \in \{1, \ldots, B+1\}$. When the optimal policy is of Type 1, we observe similar patterns in the thresholds. For example, if server 1 is the switching server and (s)he has a preferred assignment at station 1, we see that $t_1(1,2) = B+2$ and $t_1(2,2) = k$, where $k \in \{1, \ldots, B+1\}$.

From these results, we conclude that the form of the optimal policy remains the same even when the buffer size is increased and the servers have different skills at different stations. However, the thresholds can take values in a bigger set, and we do not see simple dependencies between the thresholds as in Section 5.5.1.

5.6 Conclusion

In this work, we have studied the dynamic server assignment problem in the presence of setup costs. More specifically, we have determined the optimal server assignment policy for tandem systems with two stations, two servers, and small buffer sizes when the service rates depend only on either the station or the server. We have shown that the optimal policy is of “multiple threshold” type (i.e., servers move between stations when the number of jobs in the system reaches certain thresholds that may depend on the current locations of both servers). As the value of the setup cost increases, the optimal server assignment policy reduces the number of servers that move between the stations, and when there is only one switching server in the system, we have seen that the faster server or the server that is assigned to the faster station is the switching server. Moreover, the servers generally have preferred assignments (the only exception is the case when the service rates depend only on the server and the value of the setup cost is small). Finally, we have shown that server movements are more limited when the service rates depend only on the station than when the service rates depend only on the server.

For systems with larger buffer sizes and/or specialist servers (whose rates can not be described as products of two terms), we have performed numerical experiments that suggest that the form of the optimal policy also has a multiple threshold structure in this setting, and have provided our observations about the values of the thresholds. Consequently, the form of the optimal policy appears to be quite robust with respect to the service rates and the buffer sizes, but the thresholds can take more values in systems with larger buffer sizes and/or specialist servers.

CHAPTER VI

CONTRIBUTIONS AND FUTURE RESEARCH DIRECTIONS

6.1 Contributions

We have studied effective cross-training and dynamic server assignment strategies for tandem lines with finite buffers. First, we considered non-Markovian tandem lines. We provided analytical results for tandem systems with two stations and two or three flexible servers. More specifically, we have identified the optimal server assignment policy for systems with deterministic service times and an intermediate buffer of arbitrary size, and also for systems with general service time distributions and an intermediate buffer of zero size. We have observed that the form of the optimal policy is the same as for the corresponding Markovian system. For larger systems, we presented and compared several heuristic server assignment policies that performed well for various service time distributions. Systems with non exponential service times and finite buffers are very common in real life, however most of the time they are analytically intractable. Our research supports the conjecture that effective dynamic server assignment policies for flexible servers are robust to the service time distributions, and suggests that the analysis of Markovian lines can also provide insights for non-Markovian lines.

In this thesis, we have also studied understaffed tandem lines with fully or partially flexible servers. We have shown that, when the objective is to maximize the throughput, most of the benefits of full flexibility can be obtained even with partial flexibility. More specifically, for systems with three stations, two servers, and deterministic service times, we have determined the optimal server assignment policy when the intermediate buffers are of arbitrary size. Furthermore, we have identified the critical skills that are necessary to attain the benefits of full flexibility in systems with limited flexibility, and observed that these skills can be found by analyzing the corresponding system with infinite buffers. We have also studied Markovian systems with three stations and two servers under the partial flexibility structures that were optimal in deterministic systems. We have determined the optimal server assignment policies for such systems when the intermediate buffers are small and have developed near-optimal heuristic server assignment policies for systems with larger intermediate buffers. Finally, we have considered longer Markovian lines and provided numerical examples showing that the partial flexibility structures known to be optimal for systems with infinite buffers performed well in our system as well. Most of the time, tandem lines with more than two stations and finite buffers are very difficult to analyze exactly. However, our research suggests that analyzing systems with infinite buffers is effective with respect to identifying good flexibility structures for the corresponding finite-buffered systems.

Finally, we have incorporated setup costs into the dynamic server assignment problem. This problem had not been studied before in tandem systems with finite buffers. We have considered systems with two stations and two flexible servers. For systems with small buffer sizes and service times that depend on either the server or the station, we have shown that the form of the optimal policy is of multiple threshold type (i.e., there are different thresholds for the servers to switch to the other station depending on the locations of all servers in the system). Furthermore, we have determined the values of the thresholds and observed that server movements were more limited in systems with identical servers as compared to systems with identical stations. In the process, we observed that as the magnitude of the setup cost grows, the number of servers who switch between the stations becomes smaller in the optimal server assignment policy (i.e., first both servers switch, then only one, and finally none). For systems with larger buffer sizes and/or specialist servers, we have provided numerical evidence supporting the conjecture that the optimal server assignment policy is of multiple threshold type. However, the values of the thresholds become more unpredictable when the servers are specialists and/or the intermediate buffer size is large.

6.2 Future Research Directions

Our research in the area of performance improvements in tandem lines has provided us with valuable insights on how to analyze systems with finite buffers. The allocation LP (which is devised for infinite-buffered systems) has provided an attainable upper bound on the throughput of the deterministic systems we have considered. Hence, our first research direction is to generalize some of our results to systems with more servers and more stations. More specifically, the first problem of interest is studying non-Markovian tandem lines with two stations and more than two servers (this problem has been studied for Markovian systems in Andradóttir and Ayhan [6]). We believe that systems with general service time distributions are not analytically tractable for systems with positive buffer sizes, however our initial results suggest that the optimal server assignment policy can be determined for deterministic systems with an intermediate buffer of arbitrary size and also for systems with general service time distributions and an intermediate buffer of zero size. The second problem of interest is identifying an optimal server assignment policy for systems with two servers and more than three stations. The solution of the allocation LP is still analytically tractable in this case, and we believe that the maximal capacity found from the allocation LP is also attainable in deterministic systems with finite buffers.

Another future research direction is the analysis of finite parallel queueing systems. The dynamic server assignment problem has been studied in this setting as well, but mostly for systems with infinite buffers and under heavy traffic. These models have provided good insights for the analysis of certain call centers and computer systems. However, the analysis of more general service systems may require finite buffers (e.g., when an emergency room has reached its capacity, new patients may not be accepted unless their health condition does not allow them to be transported to another facility). The only work we are aware of that studies finite-buffered parallel systems is Ahn, Duenyas, and Zhang [2] and they study a clearing system with no arrivals. By contrast, we plan to study systems with multiple customer classes and outside arrivals. We also believe that other problems of interest, like admission control and dynamic pricing, can be considered in combination with the dynamic server assignment problem. Our first goal is to determine the optimal server assignment policy for simpler systems with finite buffers where every customer is allowed to enter the system as long as the buffers are not full. Then, we plan to add admission control or pricing mechanisms to our model.

APPENDIX A

SUPPLEMENTARY MATERIAL FOR CHAPTER 3

A.1 Lemmas Used in the Proofs of Theorems 3.3.1 and 3.3.2

Throughout this section, we will use the following notation.

$$\begin{aligned} C_1(u_1,u_2) &= \frac{u_1\mu_{12}}{\mu_{11}(\mu_{12}+\mu_{22})}+\frac{u_2}{\mu_{12}+\mu_{22}}, \quad C_2(u_1,u_2) = \frac{u_1\mu_{22}}{\mu_{21}(\mu_{12}+\mu_{22})}+\frac{u_2}{\mu_{12}+\mu_{22}}, \\ C_3(u_1,u_2) &= \frac{u_1}{\mu_{11}+\mu_{21}}+\frac{u_2\mu_{21}}{\mu_{22}(\mu_{11}+\mu_{21})}, \quad C_4(u_1,u_2) = \frac{u_1}{\mu_{11}+\mu_{21}}+\frac{u_2\mu_{11}}{\mu_{12}(\mu_{11}+\mu_{21})}, \\ C_5(u_1,u_2) &= \frac{u_1}{\mu_{11}+\mu_{21}}+\frac{u_2}{\mu_{12}+\mu_{22}}. \end{aligned}$$

Lemma A.1.1 *The maximal capacity of a tandem queueing network with two stations and two flexible servers is equal to*

$$\lambda^* = \begin{cases} \frac{1}{C_1(u_1,u_2)} & \text{if } \frac{u_1}{\mu_{11}} \le \frac{u_2}{\mu_{22}}, \\ \frac{1}{C_3(u_1,u_2)} & \text{if } \frac{u_1}{\mu_{11}} > \frac{u_2}{\mu_{22}}. \end{cases} \tag{19}$$

Proof: It is clear that it suffices to show that equation (19) provides the optimal value of λ in the allocation LP (see the proof of Theorem 3.3.1). We start by transforming this LP to the standard form as follows:

$$\begin{aligned} \min \quad & -\lambda \\ \text{s.t.} \quad & \lambda - \delta_{11}\frac{\mu_{11}}{u_1} - \delta_{21}\frac{\mu_{21}}{u_1} + s_1 = 0, \\ & \lambda - \delta_{12}\frac{\mu_{12}}{u_2} - \delta_{22}\frac{\mu_{22}}{u_2} + s_2 = 0, \\ & \delta_{11}+\delta_{12} = 1, && (20) \\ & \delta_{21}+\delta_{22} = 1, && (21) \\ & \delta_{ij} \ge 0, \text{ for all } i,j \in \{1,2\}, s_1, s_2 \ge 0. \end{aligned}$$

Note that there are no slack variables in equations (20) and (21), because it is always possible to satisfy these constraints as equalities without worsening the objective function value. Since there are four constraints (not including the nonnegativity constraints), every feasible basis will have four elements.

Let D be a basis for the above LP, c_B be the vector of coefficients of the elements of D in the objective function, $\mathbf{B}$ be the coefficients of the elements of D in the constraint matrix, and b be the right-hand side of the constraints. Also, let V denote the coefficients of the non-basic variables in the constraint matrix, and c_{NB} denote the vector of coefficients of the non-basic variables in the objective function. We let c_B and c_{NB} be row vectors, and b be a column vector. Then the following conditions guarantee that the basis D is optimal (see, e.g., Theorem 3.1 of Bertsimas and Tsitsiklis [25]):

$$\mathbf{B}^{-1}b \geq 0, \tag{22}$$

$$c_{NB} - c_B\mathbf{B}^{-1}V \geq 0. \tag{23}$$

First, consider the basis $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{22}\}$. Some algebra shows that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \mu_{11}(\mu_{12}+\mu_{22})/(u_1\mu_{12}+u_2\mu_{11}) \\ u_1(\mu_{12}+\mu_{22})/(u_1\mu_{12}+u_2\mu_{11}) \\ (u_2\mu_{11}-u_1\mu_{22})/(u_1\mu_{12}+u_2\mu_{11}) \\ 1 \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} (\mu_{11}\mu_{22}-\mu_{12}\mu_{21})/(u_1\mu_{12}+u_2\mu_{11}) \\ u_1\mu_{12}/(u_1\mu_{12}+u_2\mu_{11}) \\ u_2\mu_{11}/(u_1\mu_{12}+u_2\mu_{11}) \end{bmatrix}.$$

Since our assumptions on the service rates imply that $\mu_{11}\mu_{22} - \mu_{12}\mu_{21} \geq 0$, we can conclude that $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{22}\}$ is an optimal basis when $\frac{u_1}{\mu_{11}} \leq \frac{u_2}{\mu_{22}}$. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \frac{1}{C_1(u_1,u_2)}$ in this case.

Now consider the basis $D = \{\lambda, \delta_{11}, \delta_{21}, \delta_{22}\}$. Similar calculations show that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \mu_{22}(\mu_{11}+\mu_{21})/(u_1\mu_{22}+u_2\mu_{21}) \\ 1 \\ (u_1\mu_{22}-u_2\mu_{11})/(u_1\mu_{22}+u_2\mu_{21}) \\ u_2(\mu_{11}+\mu_{21})/(u_1\mu_{22}+u_2\mu_{21}) \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} (\mu_{11}\mu_{22}-\mu_{12}\mu_{21})/(u_1\mu_{22}+u_2\mu_{21}) \\ u_1\mu_{22}/(u_1\mu_{22}+u_2\mu_{21}) \\ u_2\mu_{11}/(u_1\mu_{22}+u_2\mu_{21}) \end{bmatrix}.$$

Hence, we can conclude that $D = \{\lambda, \delta_{11}, \delta_{21}, \delta_{22}\}$ is an optimal basis when $\frac{u_1}{\mu_{11}} > \frac{u_2}{\mu_{22}}$. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \frac{1}{C_3(u_1,u_2)}$ in this case, which provides the desired result. $\Box$

Lemma A.1.2 *Consider the policy* $\pi = (d)^\infty$, *where* $d(0) = a_{11}$, $d(B+2) = a_{22}$, *and* $d(s) = a_{12}$ *for* $1 \le s \le B+1$, *for a system with* $N = M = 2$. *If* $\frac{u_1}{\mu_{11}} \le \frac{u_2}{\mu_{22}}$, *then the long-run average throughput under the policy* π *is equal to* $\frac{1}{C_1(u_1,u_2)}$.

Proof: Suppose that $\frac{u_1}{\mu_{11}} < \frac{u_2}{\mu_{22}}$ and that the process $\{X(t)\}$ is in state s when the first service completion at station 1 takes place. At that time, the process $\{X(t)\}$ goes to state $s+1$. If $s+1 < B+2$, then a new job starts being processed at station 1, and the job being processed at station 2 has remaining service time $r_2 < u_2$. The process $\{X(t)\}$ will reach either state s or state $s+2$ next depending on whether $\frac{r_2}{\mu_{22}} < \frac{u_1}{\mu_{11}}$ or not. If the process $\{X(t)\}$ returns to state s, the remaining service time at station 1 is no longer than u_1. Consequently, the job at station 1 is completed first, and the process $\{X(t)\}$ goes next to state $s+1$, at which time the remaining service time at station 2 is

$$r_2^{(1)} = u_2 - \mu_{22}\Big(\frac{u_1 - \frac{r_2}{\mu_{22}}\mu_{11}}{\mu_{11}}\Big) = r_2 + \Big(u_2 - \frac{u_1}{\mu_{11}}\mu_{22}\Big) \in (r_2, u_2)$$

because $\frac{u_1}{\mu_{11}} < \frac{u_2}{\mu_{22}}$ and $\frac{r_2}{\mu_{22}} < \frac{u_1}{\mu_{11}}$. In this case, it is possible to have several $(s+1) \rightarrow s \rightarrow (s+1)$ cycles, but each time the process $\{X(t)\}$ reaches state $s+1$ from state s, a new job starts being processed at station 1 and the remaining service time at station 2 is longer than the previous time (by a constant margin). Hence, there will come a time when the job at station 1 will be finished before the job at station 2 in state $s+1$, and the process $\{X(t)\}$ will reach state $s+2$. Therefore, we can conclude that the process $\{X(t)\}$ never reaches state $s-1$ after the first service completion at station 1 in state s, and instead it eventually reaches state $B+2$.

Once the process $\{X(t)\}$ hits state $B+2$, it comes back to state $B+1$ after the service completion at station 2. New jobs start being processed at each station, the job at station 1 is finished first under the assumptions of the lemma, and the process $\{X(t)\}$ hits state $B+2$ again. Since one job leaves the system each time the process $\{X(t)\}$ reaches state $B+1$, the long-run average throughput is equal to $\frac{1}{C_1(u_1,u_2)}$, the reciprocal of the time between transitions to state $B+1$.

If $\frac{u_1}{\mu_{11}} = \frac{u_2}{\mu_{22}}$ and the policy π is employed, then the process is either confined to a single intermediate state, or to two adjacent intermediate states, and the long-run average throughput is the reciprocal of $\frac{u_1}{\mu_{11}}$ which is equal to $\frac{1}{C_1(u_1,u_2)}$. This completes the proof. □

Lemma A.1.3 *Consider the policy $\pi = (d)^\infty$, where $d(0) = a_{11}$, $d(B+2) = a_{22}$, and $d(s) = a_{12}$ for $1 \leq s \leq B+1$, for a system with $N = M = 2$. If $\frac{u_1}{\mu_{11}} > \frac{u_2}{\mu_{22}}$, then the long-run average throughput under the policy π is equal to $\frac{1}{C_3(u_1,u_2)}$.*

Proof: Using an analysis similar to that of Lemma A.1.2, it can be shown that the stochastic process $\{X(t)\}$ ends up being confined to states 0 and 1. Since one job leaves the system each time the process $\{X(t)\}$ reaches state 1, the long-run average throughput is equal to $\frac{1}{C_3(u_1,u_2)}$, the reciprocal of the time between transitions to state 1. This completes the proof. □

Lemma A.1.4 *For all* $u_1, u_2 \in \mathbb{R}^+$ *we have*

$$\max\{C_1(u_1,u_2), C_3(u_1,u_2)\} \leq \min\{C_2(u_1,u_2), C_4(u_1,u_2), C_5(u_1,u_2)\}.$$

Proof: It is easy to see that $C_1(u_1,u_2) \leq C_2(u_1,u_2)$, and $C_3(u_1,u_2) \leq C_4(u_1,u_2)$. Furthermore, for all $u_1, u_2 \in \mathbb{R}^+$ we have $C_1(u_1,u_2) \leq \min\{C_4(u_1,u_2), C_5(u_1,u_2)\}$ and $C_3(u_1,u_2) \leq \min\{C_2(u_1,u_2), C_5(u_1,u_2)\}$ because

$$\begin{aligned}
C_1(u_1,u_2) - C_4(u_1,u_2) &= \frac{(u_1\mu_{12} + u_2\mu_{11})(\mu_{12}\mu_{21} - \mu_{11}\mu_{22})}{\mu_{11}\mu_{12}(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22})} \leq 0,\\
C_1(u_1,u_2) - C_5(u_1,u_2) &= \frac{u_1(\mu_{12}\mu_{21} - \mu_{11}\mu_{22})}{\mu_{11}(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22})} \leq 0,\\
C_3(u_1,u_2) - C_2(u_1,u_2) &= \frac{(u_1\mu_{22} + u_2\mu_{21})(\mu_{12}\mu_{21} - \mu_{11}\mu_{22})}{\mu_{21}\mu_{22}(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22})} \leq 0,\\
C_3(u_1,u_2) - C_5(u_1,u_2) &= \frac{u_2(\mu_{12}\mu_{21} - \mu_{11}\mu_{22})}{\mu_{22}(\mu_{11}+\mu_{21})(\mu_{12}+\mu_{22})} \leq 0.
\end{aligned}$$

Hence the result follows. □

A.2 Lemmas Used in the Proof of Theorem 3.4.1

For simplicity, we will use the following notation:

$$\Sigma_1 = \mu_{11} + \mu_{21} + \mu_{31},\ \Sigma_2 = \mu_{12} + \mu_{22} + \mu_{32},$$

$$\Delta_{12} = \mu_{11}\mu_{22} - \mu_{12}\mu_{21},\ \Delta_{13} = \mu_{11}\mu_{32} - \mu_{12}\mu_{31},\ \Delta_{23} = \mu_{21}\mu_{32} - \mu_{22}\mu_{31}.$$

Note that we have assumed $\Delta_{12} \geq 0, \Delta_{13} \geq 0$, $\Delta_{23} \geq 0$, $\Sigma_1 > 0$, and $\Sigma_2 > 0$. Also, let $\gamma_j(u_1,u_2)$ for $j \in \{1,2,3\}$ be defined as follows.

$$\begin{aligned}
\gamma_1(u_1,u_2) &= \frac{\mu_{11}\Sigma_2}{u_1\mu_{12} + u_2\mu_{11}},\ \gamma_2(u_1,u_2) = \frac{\mu_{11}\mu_{22} + \mu_{21}\mu_{22} + \mu_{21}\mu_{32}}{u_1\mu_{22} + u_2\mu_{21}},\\
\gamma_3(u_1,u_2) &= \frac{\mu_{32}\Sigma_1}{u_1\mu_{32} + u_2\mu_{31}}.
\end{aligned}$$

Lemma A.2.1 *The maximal capacity of a tandem queueing network with two stations and three flexible servers is equal to*

$$\lambda^* = \begin{cases} \gamma_1(u_1,u_2) & \text{if } \frac{u_1}{u_2} \leq \frac{\mu_{11}}{\mu_{22}+\mu_{32}}, \\ \gamma_2(u_1,u_2) & \text{if } \frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}, \\ \gamma_3(u_1,u_2) & \text{if } \frac{u_1}{u_2} > \frac{\mu_{11}+\mu_{21}}{\mu_{32}}. \end{cases} \qquad (24)$$

Proof: It is clear that it suffices to show that equation (24) provides the optimal value of λ in the allocation LP (see the proof of Theorem 3.4.1). We start by transforming this LP to the standard form as follows:

$$\begin{aligned}
\min \quad & -\lambda \\
\text{s.t.} \quad & \lambda - \delta_{11}\frac{\mu_{11}}{u_1} - \delta_{21}\frac{\mu_{21}}{u_1} - \delta_{31}\frac{\mu_{31}}{u_1} + s_1 = 0, \\
& \lambda - \delta_{12}\frac{\mu_{12}}{u_2} - \delta_{22}\frac{\mu_{22}}{u_2} - \delta_{32}\frac{\mu_{32}}{u_2} + s_2 = 0, \\
& \delta_{11} + \delta_{12} = 1, && (25) \\
& \delta_{21} + \delta_{22} = 1, && (26) \\
& \delta_{31} + \delta_{32} = 1, && (27) \\
& \delta_{ij} \geq 0, \text{ for all } i \in \{1,2,3\}, j \in \{1,2\}, s_1, s_2 \geq 0.
\end{aligned}$$

Note that there are no slack variables in equations (25), (26), and (27) because it is always possible to satisfy these constraints as equalities without worsening the objective function value. Since there are five constraints (not including the nonnegativity constraints), every feasible basis will have five elements.

We will use the notation defined in the proof of Lemma A.1.1. Our approach will be to start with an initial basis and show that this basis is optimal by verifying the conditions (22) and (23).

First, consider the basis $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{22}, \delta_{32}\}$. Some algebra shows that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \gamma_1(u_1, u_2) \\ u_1\Sigma_2/(u_1\mu_{12} + u_2\mu_{11}) \\ (u_2\mu_{11} - u_1(\mu_{22} + \mu_{32}))/(u_1\mu_{12} + u_2\mu_{11}) \\ 1 \\ 1 \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \Delta_{12}/(u_1\mu_{12} + u_2\mu_{11}) \\ \Delta_{13}/(u_1\mu_{12} + u_2\mu_{11}) \\ u_1\mu_{12}/(u_1\mu_{12} + u_2\mu_{11}) \\ u_2\mu_{11}/(u_1\mu_{12} + u_2\mu_{11}) \end{bmatrix}.$$

Hence, we conclude that $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{22}, \delta_{32}\}$ is an optimal basis when $\frac{u_1}{u_2} \leq \frac{\mu_{11}}{\mu_{22}+\mu_{32}}$. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \gamma_1(u_1, u_2)$ in this case.

Next, consider the basis $D = \{\lambda, \delta_{11}, \delta_{21}, \delta_{22}, \delta_{32}\}$. Similar calculations show that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \gamma_2(u_1, u_2) \\ 1 \\ (u_1(\mu_{22} + \mu_{32}) - u_2\mu_{11})/(u_1\mu_{22} + u_2\mu_{21}) \\ (u_2(\mu_{11} + \mu_{21}) - u_1\mu_{32})/(u_1\mu_{22} + u_2\mu_{21}) \\ 1 \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \Delta_{12}/(u_1\mu_{22} + u_2\mu_{21}) \\ \Delta_{23}/(u_1\mu_{22} + u_2\mu_{21}) \\ u_1\mu_{22}/(u_1\mu_{22} + u_2\mu_{21}) \\ u_2\mu_{21}/(u_1\mu_{22} + u_2\mu_{21}) \end{bmatrix}.$$

Hence, we conclude that $D = \{\lambda, \delta_{11}, \delta_{21}, \delta_{22}, \delta_{32}\}$ is an optimal basis when $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} \leq \frac{u_1}{u_2} \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \gamma_2(u_1, u_2)$ in this case.

Finally, consider the basis $D = \{\lambda, \delta_{11}, \delta_{21}, \delta_{31}, \delta_{32}\}$. Similar calculations show that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \gamma_3(u_1, u_2) \\ 1 \\ 1 \\ (u_1\mu_{32} - u_2(\mu_{11} + \mu_{21}))/(u_1\mu_{32} + u_2\mu_{31}) \\ u_2\Sigma_1/(u_1\mu_{32} + u_2\mu_{31}) \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \Delta_{13}/(u_1\mu_{32} + u_2\mu_{31}) \\ \Delta_{23}/(u_1\mu_{32} + u_2\mu_{31}) \\ u_1\mu_{32}/(u_1\mu_{32} + u_2\mu_{31}) \\ u_2\mu_{31}/(u_1\mu_{32} + u_2\mu_{31}) \end{bmatrix}.$$

Hence, we can conclude that $D = \{\lambda, \delta_{11}, \delta_{21}, \delta_{31}, \delta_{32}\}$ is an optimal basis when $\frac{u_1}{u_2} \geq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \gamma_3(u_1, u_2)$ in this case, and this gives the desired result. □

Lemma A.2.2 *Consider the policy $\pi = (d)^\infty$, where $d(0) = a_{111}$, $d(B+2) = a_{222}$, and $d(s) = a_{122}$ for $1 \leq s \leq B+1$, for a system with $M = 3, N = 2$. If $\frac{u_1}{u_2} \leq \frac{\mu_{11}}{\mu_{22}+\mu_{32}}$, then the long-run average throughput under the policy π is equal to $\gamma_1(u_1, u_2)$ when $B \geq 0$.*

Proof: If $\frac{u_1}{\mu_{11}} < \frac{u_2}{\mu_{22}+\mu_{32}}$, using an analysis similar to that of Lemma A.1.2, it can be shown that the process $\{X(t)\}$ ends up being confined to states $B+1$ and $B+2$. Since one job leaves the system each time the process $\{X(t)\}$ reaches state $B+1$, the long-run average throughput is equal to the reciprocal of the time

$$\frac{u_1}{\mu_{11}} + \frac{u_2 - (\mu_{22}+\mu_{32})\frac{u_1}{\mu_{11}}}{\Sigma_2}$$

between transitions to state $B+1$, which equals $\gamma_1(u_1, u_2)$.

If $\frac{u_1}{u_2} = \frac{\mu_{11}}{\mu_{22}+\mu_{32}}$ and the policy π is employed, then the process is either confined to a single intermediate state, or to two adjacent intermediate states, and the long-run average throughput is the reciprocal of $\frac{u_1}{\mu_{11}}$ which is equal to $\gamma_1(u_1, u_2)$. This completes the proof. □

Lemma A.2.3 *Consider the policy $\pi = (d)^\infty$, where $d(0) = a_{111}$, $d(B+2) = a_{222}$, $d(s) = a_{112}$ for $1 \leq s < s^*$, and $d(s) = a_{122}$ for $s^* \leq s \leq B+1$, for a system with $M = 3, N = 2$. If $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$, then the long-run average throughput under the policy π is equal to $\gamma_2(u_1, u_2)$ when $B > 0$.*

Proof: First assume that $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} < \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$. Then, proceeding as in the proof of Lemma A.1.2, we can show that the process $\{X(t)\}$ eventually hits state s^*-1 regardless of whether the initial state s satisfies $s < s^*-1$ or $s > s^*-1$. We will show that the process $\{X(t)\}$ ends up being confined to the two intermediate states s^*-1 and s^* in the long-run, and compute the time between two consecutive visits to state s^*-1.

If the remaining service time at station 2 is $r_2^{(0)}$ upon hitting state s^*-1 from state s^*-2, then the process either goes next to state s^*-2 (if $\frac{r_2^{(0)}}{\mu_{32}} < \frac{u_1}{\mu_{11}+\mu_{21}}$) or to state s^* (if $\frac{r_2^{(0)}}{\mu_{32}} \geq \frac{u_1}{\mu_{11}+\mu_{21}}$). If the process next goes to state s^*-2, we can show (using ideas in the proof of Lemma A.1.2) that eventually it will come back to state s^*-1 with the remaining service times r_1 and r_2 at stations 1 and 2 satisfying $\frac{r_1}{\mu_{11}+\mu_{21}} < \frac{r_2}{\mu_{32}}$, and then the process will hit state s^* next. Similarly, when state s^*-1 is entered from state s^*, the job at station 1 will have a remaining service time smaller than u_1, a new job will start service at station 2, and the assumption $\frac{u_1}{\mu_{11}+\mu_{21}} < \frac{u_2}{\mu_{32}}$ guarantees that the process will go next to state s^*. Similarly, when state s^* is entered from state s^*-1, the remaining service time at station 2 will be smaller than u_2, and a new job will start service at station 1. The assumption that $\frac{u_2}{\mu_{22}+\mu_{32}} < \frac{u_1}{\mu_{11}}$ guarantees that the job at station 2 will be finished first, and the process will come back to state s^*-1. Hence, if T denotes the first time $\{X(t)\}$ hits state s^*-1 from state s^*, it is clear that $T < \infty$ and that $\{X(t)\}$ is confined to states s^*-1 and s^* for all $t \geq T$.

Suppose now that at time T, the remaining service time at station 1 equals $r_1^{(0)}$. Then, the remaining service times at station 1 the next n times the process enters state s^*-1 can be found using the following formula:

$$r_1^{(n)} = u_1 - \mu_{11}\Big(\frac{u_2 - \frac{r_1^{(n-1)}}{\mu_{11}+\mu_{21}}\mu_{32}}{\mu_{22}+\mu_{32}}\Big) \text{ for all } n \geq 1.$$

A simple induction now yields

$$r_1^{(n)} = u_1 \sum_{k=0}^{n-1} \Big(\frac{\mu_{11}\mu_{32}}{(\mu_{11}+\mu_{21})(\mu_{22}+\mu_{32})}\Big)^k - u_2 \frac{\mu_{11}}{\mu_{22}+\mu_{32}} \sum_{k=0}^{n-1} \Big(\frac{\mu_{11}\mu_{32}}{(\mu_{11}+\mu_{21})(\mu_{22}+\mu_{32})}\Big)^k + r_1^{(0)} \Big(\frac{\mu_{11}\mu_{32}}{(\mu_{11}+\mu_{21})(\mu_{22}+\mu_{32})}\Big)^n$$

for all $n \geq 1$. Hence, the time that $\{X(t)\}$ spends in state s^*-1 before going to state s^* for the n^{th} time is $c_1^{(n)} = \frac{r_1^{(n)}}{\mu_{11}+\mu_{21}}$.

Similarly, the first time $\{X(t)\}$ enters state s^* from state s^*-1 after time T, the remaining service time at station 2 equals $\bar{r}_2^{(0)} = u_2 - \frac{r_1^{(0)}}{\mu_{11}+\mu_{21}}\mu_{32}$. Then, the remaining service times at station 2 the next n times the process enters state s^* can be found using the following formula:

$$\bar{r}_2^{(n)} = u_2 - \mu_{32}\Big(\frac{u_1 - \frac{\bar{r}_2^{(n-1)}}{\mu_{22}+\mu_{32}}\mu_{11}}{\mu_{11}+\mu_{21}}\Big) \text{ for all } n \geq 1.$$

A simple induction now yields

$$\bar{r}_2^{(n)} = u_2 \sum_{k=0}^{n-1} \Big(\frac{\mu_{11}\mu_{32}}{(\mu_{11}+\mu_{21})(\mu_{22}+\mu_{32})}\Big)^k - u_1 \frac{\mu_{32}}{\mu_{11}+\mu_{21}} \sum_{k=0}^{n-1} \Big(\frac{\mu_{11}\mu_{32}}{(\mu_{11}+\mu_{21})(\mu_{22}+\mu_{32})}\Big)^k + \bar{r}_2^{(0)} \Big(\frac{\mu_{11}\mu_{32}}{(\mu_{11}+\mu_{21})(\mu_{22}+\mu_{32})}\Big)^n$$

for all $n \geq 1$. Hence, the time that $\{X(t)\}$ spends in state s^* before going to state s^*-1 for the n^{th} time is $c_2^{(n)} = \frac{\bar{r}_2^{(n)}}{\mu_{22}+\mu_{32}}$.

Consequently, in the limit, the time between two transitions to state s^*-1 is

$$\lim_{n\to\infty} \left(c_1^{(n)} + c_2^{(n)}\right) = \frac{u_1\mu_{22} + u_2\mu_{21}}{\mu_{11}\mu_{22} + \mu_{21}\mu_{22} + \mu_{21}\mu_{32}}. \tag{28}$$

Convergence in (28) follows since we have $\mu_{11} > 0$, $\mu_{32} > 0$, and $\mu_{21} + \mu_{22} > 0$, which in turn implies that $0 < \frac{\mu_{11}\mu_{32}}{(\mu_{11}+\mu_{21})(\mu_{22}+\mu_{32})} < 1$. Hence, the long-run average throughput is equal to the reciprocal of this expression, which is $\gamma_2(u_1, u_2)$.

If $\frac{u_1}{u_2} = \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$ and the policy π is employed, then the process is either confined to a single intermediate state s, or to two adjacent intermediate states s and $s+1$, where $s \in \{1, \ldots, s^*-1\}$ (and the same analysis as above will follow when $s = s^*-1$). In either case, the long-run average throughput is the reciprocal of $\frac{u_2}{\mu_{32}}$, which is equal to $\gamma_2(u_1, u_2)$. This completes the proof. □

Lemma A.2.4 *Consider the policy* $\pi = (d)^\infty$, *where* $d(0) = a_{111}$, $d(B+2) = a_{222}$, *and* $d(s) = a_{112}$ *for* $1 \leq s \leq B+1$, *for a system with* $M = 3, N = 2$. *If* $\frac{u_1}{u_2} > \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$, *then the long-run average throughput under the policy* π *is equal to* $\gamma_3(u_1, u_2)$ *when* $B \geq 0$.

Proof: Using an analysis similar to that of Lemma A.1.2, it can be shown that the process $\{X(t)\}$ ends up being confined to states 0 and 1. Since one job leaves the system each time the process $\{X(t)\}$ reaches state 1, long-run average throughput is equal to the reciprocal of the time

$$\frac{u_2}{\mu_{32}} + \frac{u_1 - (\mu_{11} + \mu_{21})\frac{u_2}{\mu_{32}}}{\Sigma_1}$$

between transitions to state 1, which equals $\gamma_3(u_1, u_2)$. Hence, the result follows. □

A.3 Lemmas Used in the Proofs of Theorems 3.4.2 and 3.4.3

According to the values of the service rates $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_{31}, \mu_{32}$, and the service times u_1 and u_2, we have twelve different cases depending on which server finishes his/her job first in state 1 under the actions $a_{112}, a_{122}, a_{121}, a_{211}, a_{212}$, and a_{221}, respectively:

Case 1: $\frac{u_1}{\mu_{11}+\mu_{21}} \leq \frac{u_2}{\mu_{32}}$,

Case 2: $\frac{u_1}{\mu_{11}+\mu_{21}} > \frac{u_2}{\mu_{32}}$,

Case 3: $\frac{u_1}{\mu_{11}} \leq \frac{u_2}{\mu_{22}+\mu_{32}}$,

Case 4: $\frac{u_1}{\mu_{11}} > \frac{u_2}{\mu_{22}+\mu_{32}}$,

Case 5: $\frac{u_1}{\mu_{11}+\mu_{31}} \leq \frac{u_2}{\mu_{22}}$,

Case 6: $\frac{u_1}{\mu_{11}+\mu_{31}} > \frac{u_2}{\mu_{22}}$,

Case 7: $\frac{u_1}{\mu_{21}+\mu_{31}} \leq \frac{u_2}{\mu_{12}}$,

Case 8: $\frac{u_1}{\mu_{21}+\mu_{31}} > \frac{u_2}{\mu_{12}}$,

Case 9: $\frac{u_1}{\mu_{21}} \leq \frac{u_2}{\mu_{12}+\mu_{32}}$,

Case 10: $\frac{u_1}{\mu_{21}} > \frac{u_2}{\mu_{12}+\mu_{32}}$,

Case 11: $\frac{u_1}{\mu_{31}} \leq \frac{u_2}{\mu_{12}+\mu_{22}}$,

Case 12: $\frac{u_1}{\mu_{31}} > \frac{u_2}{\mu_{12}+\mu_{22}}$.

Cases 1 and 2 occur under action a_{112}, Cases 3 and 4 occur under action a_{122}, Cases 5 and 6 occur under action a_{121}, Cases 7 and 8 occur under action a_{211}, Cases 9 and 10 occur under action a_{212}, and Cases 11 and 12 occur under action a_{221}.

The following are the times between two transitions to state 1 for these twelve cases:

$$\begin{aligned}
\widehat{C}_1(u_1,u_2) &= \frac{u_1(\mu_{12}+\mu_{22})}{(\mu_{11}+\mu_{21})\Sigma_2} + \frac{u_2}{\Sigma_2}, & \widehat{C}_2(u_1,u_2) &= \frac{u_1}{\Sigma_1} + \frac{u_2\mu_{31}}{\mu_{32}\Sigma_1},\\
\widehat{C}_3(u_1,u_2) &= \frac{u_1\mu_{12}}{\mu_{11}\Sigma_2} + \frac{u_2}{\Sigma_2}, & \widehat{C}_4(u_1,u_2) &= \frac{u_1}{\Sigma_1} + \frac{u_2(\mu_{21}+\mu_{31})}{(\mu_{22}+\mu_{32})\Sigma_1},\\
\widehat{C}_5(u_1,u_2) &= \frac{u_1(\mu_{12}+\mu_{32})}{(\mu_{11}+\mu_{31})\Sigma_2} + \frac{u_2}{\Sigma_2}, & \widehat{C}_6(u_1,u_2) &= \frac{u_1}{\Sigma_1} + \frac{u_2\mu_{21}}{\mu_{22}\Sigma_1},\\
\widehat{C}_7(u_1,u_2) &= \frac{u_1(\mu_{22}+\mu_{32})}{(\mu_{21}+\mu_{31})\Sigma_2} + \frac{u_2}{\Sigma_2}, & \widehat{C}_8(u_1,u_2) &= \frac{u_1}{\Sigma_1} + \frac{u_2\mu_{11}}{\mu_{12}\Sigma_1},\\
\widehat{C}_9(u_1,u_2) &= \frac{u_1\mu_{22}}{\mu_{21}\Sigma_2} + \frac{u_2}{\Sigma_2}, & \widehat{C}_{10}(u_1,u_2) &= \frac{u_1}{\Sigma_1} + \frac{u_2(\mu_{11}+\mu_{31})}{(\mu_{12}+\mu_{32})\Sigma_1},\\
\widehat{C}_{11}(u_1,u_2) &= \frac{u_1\mu_{32}}{\mu_{31}\Sigma_2} + \frac{u_2}{\Sigma_2}, & \widehat{C}_{12}(u_1,u_2) &= \frac{u_1}{\Sigma_1} + \frac{u_2(\mu_{11}+\mu_{21})}{(\mu_{12}+\mu_{22})\Sigma_1}.
\end{aligned}$$

Note that $\gamma_1(u_1,u_2) = \frac{1}{\widehat{C}_3(u_1,u_2)}$ and $\gamma_3(u_1,u_2) = \frac{1}{\widehat{C}_2(u_1,u_2)}$, where $\gamma_1(u_1,u_2)$ and $\gamma_3(u_1,u_2)$ are defined in Appendix A.2.

Proof of Lemma 3.4.1: Lemma 3.2.1 shows that it is never optimal to idle the servers in the end states. Hence it only remains to show that it is never optimal to idle them in state 1. Note that we will not use the inequalities $\frac{\mu_{11}}{\mu_{12}} \geq \frac{\mu_{21}}{\mu_{22}} \geq \frac{\mu_{31}}{\mu_{32}}$ in the proof. Hence, it suffices to show that the actions $a_{100}, a_{200}, a_{110}, a_{220}$, and a_{120} can never be optimal in state 1, because all other actions with idling can be converted to these actions by relabeling the servers. The proof of Lemma 3.2.2 shows that action a_{111} is better than actions a_{100} and a_{110}, and action a_{222} is better than actions a_{200} and a_{220}. Hence, it suffices to show that action a_{122} is better than action a_{120} in state 1 when actions a_{111} and a_{222} are used in states 0 and 2, respectively.

The expected time between two visits to state 1 under action a_{122} was defined as E_{122} in Section 3.4.2. The following is the expected time between two visits to state

1 under actions a_{120}:

$$E_{120} = \int_0^\infty \int_0^{\frac{u_2\mu_{11}}{\mu_{22}}} \widehat{C}_{13}(u_1,u_2)dF_1(u_1)dF_2(u_2) + \int_0^\infty \int_{\frac{u_2\mu_{11}}{\mu_{22}}}^\infty \widehat{C}_{14}(u_1,u_2)dF_1(u_1)dF_2(u_2),$$

where F_1, F_2 denote the CDF's of the service times at stations 1 and 2, respectively, and

$$\widehat{C}_{13}(u_1,u_2) = \frac{u_1(\mu_{12}+\mu_{32})}{\mu_{11}\Sigma_2} + \frac{u_2}{\Sigma_2}, \qquad \widehat{C}_{14}(u_1,u_2) = \frac{u_1}{\Sigma_1} + \frac{u_2(\mu_{21}+\mu_{31})}{\mu_{22}\Sigma_1}.$$

It is clear that $\widehat{C}_3(u_1,u_2) \le \widehat{C}_{13}(u_1,u_2)$ and $\widehat{C}_4(u_1,u_2) \le \widehat{C}_{14}(u_1,u_2)$ for all u_1, u_2 $\in \mathbb{R}^+$ since $\mu_{32} \ge 0$. Also,

$$\begin{aligned}\widehat{C}_4(u_1,u_2) \le \widehat{C}_{13}(u_1,u_2) \Leftrightarrow \Theta \;&:=\; u_1(\mu_{12}+\mu_{32})(\mu_{22}+\mu_{32})\Sigma_1 + u_2\mu_{11}(\mu_{22}+\mu_{32})\Sigma_1 \\ &\quad -u_1\mu_{11}(\mu_{22}+\mu_{32})\Sigma_2 - u_2\mu_{11}(\mu_{21}+\mu_{31})\Sigma_2 \ge 0.\end{aligned}$$

But the last inequality holds when $\frac{u_2}{\mu_{22}+\mu_{32}} \le \frac{u_1}{\mu_{11}} \le \frac{u_2}{\mu_{22}}$ because in this case

$$\begin{aligned}\Theta &= u_1(\mu_{12}+\mu_{32})(\mu_{22}+\mu_{32})(\mu_{21}+\mu_{31}) + u_2\mu_{11}^2(\mu_{22}+\mu_{32}) - u_1\mu_{11}\mu_{22}(\mu_{22}+\mu_{32}) \\ &\quad -u_2\mu_{11}\mu_{12}(\mu_{21}+\mu_{31}) \\ &\ge u_2\mu_{11}(\mu_{12}+\mu_{32})(\mu_{21}+\mu_{31}) + u_1\mu_{11}\mu_{22}(\mu_{22}+\mu_{32}) - u_1\mu_{11}\mu_{22}(\mu_{22}+\mu_{32}) \\ &\quad -u_2\mu_{11}\mu_{12}(\mu_{21}+\mu_{31}) \\ &= u_2\mu_{11}\mu_{32}(\mu_{21}+\mu_{31}) \ge 0.\end{aligned}$$

Hence,

$$\begin{aligned}E_{120} &= \int_0^\infty \int_0^{\frac{u_2\mu_{11}}{\mu_{22}+\mu_{32}}} \widehat{C}_{13}(u_1,u_2)dF_1(u_1)dF_2(u_2) \\ &\quad + \int_0^\infty \int_{\frac{u_2\mu_{11}}{\mu_{22}+\mu_{32}}}^{\frac{u_2\mu_{11}}{\mu_{22}}} \widehat{C}_{13}(u_1,u_2)dF_1(u_1)dF_2(u_2) + \int_0^\infty \int_{\frac{u_2\mu_{11}}{\mu_{22}}}^\infty \widehat{C}_{14}(u_1,u_2)dF_1(u_1)dF_2(u_2) \\ &\ge \int_0^\infty \int_0^{\frac{u_2\mu_{11}}{\mu_{22}+\mu_{32}}} \widehat{C}_3(u_1,u_2)dF_1(u_1)dF_2(u_2) + \int_0^\infty \int_{\frac{u_2\mu_{11}}{\mu_{22}+\mu_{32}}}^{\frac{u_2\mu_{11}}{\mu_{22}}} \widehat{C}_4(u_1,u_2)dF_1(u_1)dF_2(u_2) \\ &\quad + \int_0^\infty \int_{\frac{u_2\mu_{11}}{\mu_{22}}}^\infty \widehat{C}_4(u_1,u_2)dF_1(u_1)dF_2(u_2) \\ &= E_{122}.\end{aligned}$$

This implies that action a_{122} is better than action a_{120}, and using Lemma 3.2.3 we can conclude that the optimal policy should not allow servers to idle. □

Proof of Lemma 3.4.2: Lemma 3.2.1 shows that the optimal action in state 0 is a_{111}, and the optimal action in state 2 is a_{222}. When we use action a_{111} or a_{222} in state 1 and actions a_{111} and a_{222} in states 0 and 2, respectively, the expected time between two visits to state 1 can be found as

$$E_{111} = E_{222} = \int_0^\infty \int_0^\infty \widehat{C}_{15}(u_1, u_2) dF_1(u_1) dF_2(u_2),$$

where $\widehat{C}_{15}(u_1, u_2) = \frac{u_1}{\Sigma_1} + \frac{u_2}{\Sigma_2}$ for all u_1, $u_2 \in \mathbb{R}^+$. It is clear that $\widehat{C}_3(u_1, u_2) \leq \widehat{C}_{15}(u_1, u_2)$ for all u_1, $u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_3(u_1, u_2) \leq \widehat{C}_{15}(u_1, u_2) \Leftrightarrow \mu_{12}\Sigma_1 \leq \mu_{11}\Sigma_2 \Leftrightarrow 0 \leq \Delta_{12} + \Delta_{13}.$$

Similarly, it is clear that $\widehat{C}_4(u_1, u_2) \leq \widehat{C}_{15}(u_1, u_2)$ for all u_1, $u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_4(u_1, u_2) \leq \widehat{C}_{15}(u_1, u_2) \Leftrightarrow (\mu_{21} + \mu_{31})\Sigma_2 \leq (\mu_{22} + \mu_{32})\Sigma_1 \Leftrightarrow 0 \leq \Delta_{12} + \Delta_{13}.$$

Then, with E_{122} being as defined in Section 3.4.2, $E_{122} \leq E_{111} = E_{222}$. Lemma 3.2.3 now yields that action a_{122} is better than actions a_{111} and a_{222} in state 1, and hence actions a_{111} and a_{222} cannot be optimal in this state. □

Lemma A.3.1 *Let r be defined as in Section 3.4.1. Then,* $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} \leq r \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$.

Proof: It is seen that

$$\begin{aligned} r \leq \frac{\mu_{11} + \mu_{21}}{\mu_{32}} \quad &\Leftrightarrow \quad \mu_{32}\Delta_{12} \leq \mu_{22}(\Delta_{13} + \Delta_{23}) + \mu_{32}\Delta_{23} \\ &\Leftrightarrow \quad 0 \leq \Delta_{23}\Sigma_2. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\mu_{11}}{\mu_{22} + \mu_{32}} \leq r \quad &\Leftrightarrow \quad \mu_{11}\Delta_{23} \leq \mu_{11}\Delta_{12} + \mu_{21}(\Delta_{12} + \Delta_{13}) \\ &\Leftrightarrow \quad 0 \leq \Delta_{12}\Sigma_1. \end{aligned}$$

Hence, the result follows. □

Lemma A.3.2 *Consider the policy* $\pi = (d)^\infty$*, where* $d(0) = a_{111}$*,* $d(1) = a_{122}$*, and* $d(2) = a_{222}$*, for a system with* $M = 3, N = 2$*, and* $B = 0$*. If* $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \le r$*, then* π *is optimal.*

Proof: Note that the assumption of the lemma implies that Case 4 holds (corresponding to action a_{122}), and Cases 2 and 3 do not hold (see Lemma A.3.1). The other nine cases (corresponding to all other actions) may or may not hold depending on the values of u_1, u_2, and the service rates. By Lemmas 3.2.1 and 3.2.3, it suffices to show that $\widehat{C}_4(u_1, u_2)$ is no larger than the cycle times of the other cases that may hold for u_1 and u_2. We have:

- $\widehat{C}_4(u_1, u_2) \le \widehat{C}_1(u_1, u_2)$ when $\frac{u_1}{u_2} \le r$ because

$$\widehat{C}_4(u_1, u_2) \le \widehat{C}_1(u_1, u_2) \Leftrightarrow \frac{u_2(\Delta_{12} + \Delta_{13})}{\mu_{22} + \mu_{32}} \ge \frac{u_1(\Delta_{13} + \Delta_{23})}{\mu_{11} + \mu_{21}} \Leftrightarrow r \ge \frac{u_1}{u_2}. \tag{29}$$

- It is seen that

$$\widehat{C}_4(u_1, u_2) - \widehat{C}_5(u_1, u_2) = \frac{u_1(\Delta_{12} - \Delta_{23})}{(\mu_{11} + \mu_{31})\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{12} - \Delta_{13})}{(\mu_{22} + \mu_{32})\Sigma_1\Sigma_2}.$$

  Now we analyze the numerator of this expression.

$$\begin{aligned} & u_1(\mu_{22} + \mu_{32})\Delta_{12} - u_1(\mu_{22} + \mu_{32})\Delta_{23} - u_2(\mu_{11} + \mu_{31})(\Delta_{12} + \Delta_{13}) \\ \le\ & u_1(\mu_{22} + \mu_{32})\Delta_{12} - u_1(\mu_{22} + \mu_{32})\Delta_{23} - u_1\mu_{22}\Delta_{12} - u_1\mu_{22}\Delta_{13} \qquad (30) \\ =\ & u_1\mu_{32}\Delta_{12} - u_1(\mu_{22} + \mu_{32})\Delta_{23} - u_1\mu_{22}\Delta_{13} \\ =\ & -u_1(\mu_{22} + \mu_{32})\Delta_{23} + u_1(\mu_{11}\mu_{22}\mu_{32} - \mu_{12}\mu_{21}\mu_{32} - \mu_{11}\mu_{22}\mu_{32} + \mu_{12}\mu_{22}\mu_{31}) \\ =\ & -u_1(\mu_{22} + \mu_{32})\Delta_{23} - u_1\mu_{12}\Delta_{23} \le 0. \end{aligned}$$

  Here (30) follows since Case 5 holds. Thus, $\widehat{C}_4(u_1, u_2) \le \widehat{C}_5(u_1, u_2)$ when u_1 and u_2 are such that Case 5 holds.

- $\widehat{C}_4(u_1, u_2) \le \widehat{C}_6(u_1, u_2)$ for all u_1, $u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_4(u_1, u_2) \le \widehat{C}_6(u_1, u_2) \Leftrightarrow \mu_{21}\mu_{22} + \mu_{22}\mu_{31} \le \mu_{21}\mu_{22} + \mu_{21}\mu_{32} \Leftrightarrow 0 \le \Delta_{23}.$$

- $\widehat{C}_4(u_1,u_2) \le \widehat{C}_7(u_1,u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_4(u_1,u_2) - \widehat{C}_7(u_1,u_2) \quad = \quad \frac{u_1(-\Delta_{12}-\Delta_{13})}{(\mu_{21}+\mu_{31})\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{12}-\Delta_{13})}{(\mu_{22}+\mu_{32})\Sigma_1\Sigma_2} \le 0.$$

- $\widehat{C}_4(u_1,u_2) \le \widehat{C}_8(u_1,u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_4(u_1,u_2) \le \widehat{C}_8(u_1,u_2) \Leftrightarrow \mu_{12}\mu_{21} + \mu_{12}\mu_{31} \le \mu_{11}\mu_{22} + \mu_{11}\mu_{32} \Leftrightarrow 0 \le \Delta_{12} + \Delta_{13}.$$

- It is seen that

$$\widehat{C}_4(u_1,u_2) - \widehat{C}_9(u_1,u_2) = \frac{u_1(-\Delta_{12}+\Delta_{23})}{\mu_{21}\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{12}-\Delta_{13})}{(\mu_{22}+\mu_{32})\Sigma_1\Sigma_2}.$$

  Now we analyze the numerator of this expression.

$$\begin{aligned} & -u_1(\mu_{22}+\mu_{32})\Delta_{12} + u_1(\mu_{22}+\mu_{32})\Delta_{23} - u_2\mu_{21}(\Delta_{12}+\Delta_{13}) \\ \le \quad & -u_1(\mu_{22}+\mu_{32})(\Delta_{12}+\Delta_{13}) + u_2\mu_{11}(\Delta_{12}+\Delta_{13}) \qquad (31) \\ = \quad & (\Delta_{12}+\Delta_{13})\Big(-u_1(\mu_{22}+\mu_{32}) + u_2\mu_{11}\Big) \le 0. \end{aligned}$$

  Here (31) follows from $\frac{u_1}{u_2} \le r$ and (29). The last inequality implies that $\widehat{C}_4(u_1,u_2) \le \widehat{C}_9(u_1,u_2)$ when $\frac{\mu_{11}}{\mu_{22}+\mu_{32}} < \frac{u_1}{u_2} \le r$.

- It is seen that

$$\begin{aligned} & \widehat{C}_4(u_1,u_2) \le \widehat{C}_{10}(u_1,u_2) \\ & \quad \Leftrightarrow \mu_{12}\mu_{21} + \mu_{12}\mu_{31} + \mu_{21}\mu_{32} + \mu_{31}\mu_{32} \le \mu_{11}\mu_{22} + \mu_{11}\mu_{32} + \mu_{22}\mu_{31} + \mu_{31}\mu_{32} \\ & \quad \Leftrightarrow \Delta_{23} \le \Delta_{12} + \Delta_{13}. \qquad (32) \end{aligned}$$

  We will show that (32) holds whenever Cases 4 and 10 hold and $\frac{u_1}{u_2} \le r$. On the contrary, suppose that (32) does not hold, so that $\Delta_{23} > \Delta_{12} + \Delta_{13}$. Equation

(29) now gives:

$$
\begin{aligned}
& u_1(\mu_{22}+\mu_{32})(\Delta_{13}+\Delta_{23}) < u_2(\mu_{11}+\mu_{21})\Delta_{23} && (33)\\
\Rightarrow\quad & (\mu_{22}+\mu_{32})(\Delta_{13}+\Delta_{23}) < (\frac{\mu_{12}+\mu_{32}}{\mu_{21}})(\mu_{11}+\mu_{21})\Delta_{23} && (34)\\
\Rightarrow\quad & (\mu_{21}\mu_{22}+\mu_{21}\mu_{32})\Delta_{13} < (\mu_{11}\mu_{12}+\mu_{11}\mu_{32}+\mu_{12}\mu_{21}-\mu_{21}\mu_{22})\Delta_{23}\\
\Rightarrow\quad & \mu_{11}\mu_{21}\mu_{22}\mu_{32}-\mu_{12}\mu_{21}\mu_{22}\mu_{31}+\mu_{11}\mu_{21}\mu_{32}^2-\mu_{12}\mu_{21}\mu_{31}\mu_{32}\\
& < \mu_{11}\mu_{21}\mu_{32}^2-\mu_{11}\mu_{22}\mu_{31}\mu_{32}+\mu_{12}\mu_{21}^2\mu_{32}-\mu_{12}\mu_{21}\mu_{22}\mu_{31}\\
& +(\mu_{11}\mu_{12}-\mu_{21}\mu_{22})\Delta_{23}\\
\Rightarrow\quad & \mu_{11}\mu_{21}\mu_{22}\mu_{32}-\mu_{12}\mu_{21}^2\mu_{32}+\mu_{11}\mu_{22}\mu_{31}\mu_{32}-\mu_{12}\mu_{21}\mu_{31}\mu_{32}\\
& < (\mu_{11}\mu_{12}-\mu_{21}\mu_{22})\Delta_{23}\\
\Rightarrow\quad & (\mu_{21}\mu_{32}+\mu_{31}\mu_{32})\Delta_{12} < (\mu_{11}\mu_{12}-\mu_{21}\mu_{22})\Delta_{23}. && (35)
\end{aligned}
$$

Here (34) follows since Case 10 holds. If (35) is false, then we have a contradiction, and conclude that (32) is correct. If (35) holds, the term on the right-hand side of the inequality should be positive since the term on the left-hand side is nonnegative. Hence, we must have $\mu_{11}\mu_{12} > \mu_{21}\mu_{22}$.

Since Case 4 holds, we have $u_1(\mu_{22}+\mu_{32}) > u_2\mu_{11}$. Equation (33) now yields

$$
\begin{aligned}
& \mu_{11}(\Delta_{13}+\Delta_{23}) < (\mu_{11}+\mu_{21})\Delta_{23}\\
\Rightarrow\quad & \mu_{11}^2\mu_{32}-\mu_{11}\mu_{12}\mu_{31} < \mu_{21}^2\mu_{32}-\mu_{21}\mu_{22}\mu_{31}\\
\Rightarrow\quad & \mu_{32}(\mu_{11}^2-\mu_{21}^2) < \mu_{31}(\mu_{11}\mu_{12}-\mu_{21}\mu_{22})\\
\Rightarrow\quad & \frac{\mu_{11}^2-\mu_{21}^2}{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}} < \frac{\mu_{31}}{\mu_{32}} && (36)\\
\Rightarrow\quad & \frac{\mu_{11}^2-\mu_{21}^2}{\mu_{11}\mu_{12}-\mu_{21}\mu_{22}} < \frac{\mu_{21}}{\mu_{22}}\\
\Rightarrow\quad & \mu_{22}(\mu_{11}^2-\mu_{21}^2) < \mu_{21}(\mu_{11}\mu_{12}-\mu_{21}\mu_{22}) && (37)\\
\Rightarrow\quad & \mu_{11}\mu_{22} < \mu_{12}\mu_{21}. && (38)
\end{aligned}
$$

Here (36) and (37) follow from $\mu_{11}\mu_{12} > \mu_{21}\mu_{22}$. But (38) contradicts $\frac{\mu_{11}}{\mu_{12}} \geq \frac{\mu_{21}}{\mu_{22}}$. Therefore, we conclude that (32) holds. In other words $\widehat{C}_4(u_1,u_2) \leq \widehat{C}_{10}(u_1,u_2)$

when u_1 and u_2 are such that Cases 4 and 10 hold and $\frac{u_1}{u_2} \le r$.

- $\widehat{C}_4(u_1, u_2) \le \widehat{C}_{11}(u_1, u_2)$ for all u_1, $u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_4(u_1, u_2) - \widehat{C}_{11}(u_1, u_2) = \frac{u_1(-\Delta_{13} - \Delta_{23})}{\mu_{31}\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{12} - \Delta_{13})}{(\mu_{22} + \mu_{32})\Sigma_1\Sigma_2} \le 0.$$

- $\widehat{C}_4(u_1, u_2) \le \widehat{C}_{12}(u_1, u_2)$ for all u_1, $u_2 \in \mathbb{R}^+$ because

$$\begin{aligned}
&\widehat{C}_4(u_1, u_2) \le \widehat{C}_{12}(u_1, u_2) \\
&\quad \Leftrightarrow \mu_{12}\mu_{21} + \mu_{12}\mu_{31} + \mu_{21}\mu_{22} + \mu_{22}\mu_{31} \le \mu_{11}\mu_{22} + \mu_{11}\mu_{32} + \mu_{21}\mu_{22} + \mu_{21}\mu_{32} \\
&\quad \Leftrightarrow 0 \le \Delta_{12} + \Delta_{13} + \Delta_{23}.
\end{aligned}$$

The above arguments show that $\widehat{C}_4(u_1, u_2)$ is not longer than the cycle times corresponding to the other cases that may hold for the service times u_1 and u_2. This completes the proof. □

Lemma A.3.3 *Consider the policy* $\pi = (d)^\infty$*, where* $d(0) = a_{111}$*,* $d(1) = a_{112}$*, and* $d(2) = a_{222}$*, for a system with* $M = 3, N = 2$*, and* $B = 0$*. If* $r < \frac{u_1}{u_2} \le \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$*, then* π *is optimal.*

Proof: Note that the assumption of the lemma implies that Case 1 holds (corresponding to action a_{112}), and Cases 2 and 3 do not hold (see Lemma A.3.1). The other nine cases (corresponding to all other actions) may or may not hold depending on the values of u_1, u_2, and the service rates. By Lemmas 3.2.1 and 3.2.3, it suffices to show that $\widehat{C}_1(u_1, u_2)$ is no larger than the cycle times of the other cases that may hold for u_1 and u_2. We have:

- $\widehat{C}_1(u_1, u_2) < \widehat{C}_4(u_1, u_2)$ when $\frac{u_1}{u_2} > r$ because

$$\widehat{C}_1(u_1, u_2) < \widehat{C}_4(u_1, u_2) \Leftrightarrow \frac{u_2(\Delta_{12} + \Delta_{13})}{\mu_{22} + \mu_{32}} < \frac{u_1(\Delta_{13} + \Delta_{23})}{\mu_{11} + \mu_{21}} \Leftrightarrow r < \frac{u_1}{u_2}. \qquad (39)$$

- It is seen that

$$
\begin{aligned}
&\widehat{C}_1(u_1,u_2) \leq \widehat{C}_5(u_1,u_2) \\
&\quad\Leftrightarrow \mu_{11}\mu_{12}+\mu_{11}\mu_{22}+\mu_{12}\mu_{31}+\mu_{22}\mu_{31} \leq \mu_{11}\mu_{12}+\mu_{11}\mu_{32}+\mu_{12}\mu_{21}+\mu_{21}\mu_{32} \\
&\quad\Leftrightarrow \Delta_{12} \leq \Delta_{13}+\Delta_{23}. \qquad (40)
\end{aligned}
$$

We will show that (40) holds whenever Cases 1 and 5 hold and $\frac{u_1}{u_2} > r$. On the contrary, suppose that (40) does not hold, so that $\Delta_{12} > \Delta_{13}+\Delta_{23}$. Equation (39) now gives:

$$
\begin{aligned}
& u_2(\mu_{11}+\mu_{21})(\Delta_{12}+\Delta_{13}) < u_1(\mu_{22}+\mu_{32})\Delta_{12} \qquad (41)\\
&\quad\Rightarrow\quad (\mu_{11}+\mu_{21})(\Delta_{12}+\Delta_{13}) < (\mu_{22}+\mu_{32})\Delta_{12}\left(\frac{\mu_{11}+\mu_{31}}{\mu_{22}}\right) \qquad (42)\\
&\quad\Rightarrow\quad (\mu_{11}\mu_{22}+\mu_{21}\mu_{22})\Delta_{13} < (\mu_{11}\mu_{32}+\mu_{22}\mu_{31}+\mu_{31}\mu_{32}-\mu_{21}\mu_{22})\Delta_{12} \\
&\quad\Rightarrow\quad \mu_{11}^2\mu_{22}\mu_{32}-\mu_{11}\mu_{12}\mu_{22}\mu_{31}+\mu_{11}\mu_{21}\mu_{22}\mu_{32}-\mu_{12}\mu_{21}\mu_{22}\mu_{31} \\
&\qquad\quad < \mu_{11}^2\mu_{22}\mu_{32}+\mu_{11}\mu_{22}^2\mu_{31}-\mu_{11}\mu_{12}\mu_{21}\mu_{32}-\mu_{12}\mu_{21}\mu_{22}\mu_{31} \\
&\qquad\quad +(\mu_{31}\mu_{32}-\mu_{21}\mu_{22})\Delta_{12} \\
&\quad\Rightarrow\quad \mu_{11}\mu_{12}\mu_{21}\mu_{32}-\mu_{11}\mu_{12}\mu_{22}\mu_{31}+\mu_{11}\mu_{21}\mu_{22}\mu_{32}-\mu_{11}\mu_{22}^2\mu_{31} \\
&\qquad\quad < (\mu_{31}\mu_{32}-\mu_{21}\mu_{22})\Delta_{12} \\
&\quad\Rightarrow\quad (\mu_{11}\mu_{12}+\mu_{11}\mu_{22})\Delta_{23} < (\mu_{31}\mu_{32}-\mu_{21}\mu_{22})\Delta_{12}. \qquad (43)
\end{aligned}
$$

Here (42) follows since Case 5 holds. If (43) is false, then we have a contradiction, and conclude that (40) is correct. If (43) holds, the term on the right-hand side of the inequality should be positive since the term on the left-hand side is nonnegative. Hence, we must have $\mu_{31}\mu_{32} > \mu_{21}\mu_{22}$. Combining this together with $\frac{\mu_{21}}{\mu_{22}} \geq \frac{\mu_{31}}{\mu_{32}}$, we obtain $\mu_{32} > \mu_{22}$.

Since Case 1 holds, we have $u_2(\mu_{11}+\mu_{21}) \geq u_1\mu_{32}$. Equation (41) now yields

$$\begin{aligned}
&\mu_{32}(\Delta_{12}+\Delta_{13}) < (\mu_{22}+\mu_{32})\Delta_{12} \\
&\quad\Rightarrow\ \mu_{11}\mu_{32}^2 - \mu_{12}\mu_{31}\mu_{32} < \mu_{11}\mu_{22}^2 - \mu_{12}\mu_{21}\mu_{22} \\
&\quad\Rightarrow\ \mu_{11}(\mu_{32}^2 - \mu_{22}^2) < \mu_{12}(\mu_{31}\mu_{32} - \mu_{21}\mu_{22}) \\
&\quad\Rightarrow\ \frac{\mu_{11}}{\mu_{12}} < \frac{\mu_{31}\mu_{32} - \mu_{21}\mu_{22}}{\mu_{32}^2 - \mu_{22}^2} && (44) \\
&\quad\Rightarrow\ \frac{\mu_{21}}{\mu_{22}} < \frac{\mu_{31}\mu_{32} - \mu_{21}\mu_{22}}{\mu_{32}^2 - \mu_{22}^2} \\
&\quad\Rightarrow\ \mu_{21}(\mu_{32}^2 - \mu_{22}^2) < \mu_{22}(\mu_{31}\mu_{32} - \mu_{21}\mu_{22}) && (45) \\
&\quad\Rightarrow\ \mu_{21}\mu_{32} < \mu_{22}\mu_{31}. && (46)
\end{aligned}$$

Here (44) and (45) follow from $\mu_{32} > \mu_{22}$. But (46) contradicts $\frac{\mu_{21}}{\mu_{22}} \geq \frac{\mu_{31}}{\mu_{32}}$. Therefore, we conclude that (40) holds. In other words $\widehat{C}_1(u_1,u_2) \leq \widehat{C}_5(u_1,u_2)$ when u_1 and u_2 are such that Cases 1 and 5 hold and $\frac{u_1}{u_2} > r$.

- It is seen that

  $$\widehat{C}_1(u_1,u_2) - \widehat{C}_6(u_1,u_2) = \frac{u_1(-\Delta_{13}-\Delta_{23})}{(\mu_{11}+\mu_{21})\Sigma_1\Sigma_2} + \frac{u_2(\Delta_{12}-\Delta_{23})}{\mu_{22}\Sigma_1\Sigma_2}.$$

  Now we analyze the numerator of this expression.

  $$\begin{aligned}
  &-u_1\mu_{22}(\Delta_{13}+\Delta_{23}) + u_2(\mu_{11}+\mu_{21})(\Delta_{12}-\Delta_{23}) \\
  &\quad < \ -u_2(\mu_{11}+\mu_{21})(\Delta_{13}+\Delta_{23}) + u_1\mu_{32}(\Delta_{13}+\Delta_{23}) && (47) \\
  &\quad = \ (\Delta_{13}+\Delta_{23})\Big(-u_2(\mu_{11}+\mu_{21}) + u_1\mu_{32}\Big) \leq 0.
  \end{aligned}$$

  Here (47) follows from $r < \frac{u_1}{u_2}$ and (39). Thus, $\widehat{C}_1(u_1,u_2) < \widehat{C}_6(u_1,u_2)$ when $r < \frac{u_1}{u_2} \leq \frac{\mu_{11}+\mu_{21}}{\mu_{32}}$.

- $\widehat{C}_1(u_1,u_2) \leq \widehat{C}_7(u_1,u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

  $$\begin{aligned}
  &\widehat{C}_1(u_1,u_2) \leq \widehat{C}_7(u_1,u_2) \\
  &\quad\Leftrightarrow \mu_{12}\mu_{21} + \mu_{12}\mu_{31} + \mu_{21}\mu_{22} + \mu_{22}\mu_{31} \leq \mu_{11}\mu_{22} + \mu_{11}\mu_{32} + \mu_{21}\mu_{22} + \mu_{21}\mu_{32} \\
  &\quad\Leftrightarrow 0 \leq \Delta_{12} + \Delta_{13} + \Delta_{23}.
  \end{aligned}$$

- $\widehat{C}_1(u_1, u_2) \leq \widehat{C}_8(u_1, u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

  $$\widehat{C}_1(u_1, u_2) - \widehat{C}_8(u_1, u_2) = \frac{u_1(-\Delta_{13} - \Delta_{23})}{(\mu_{11} + \mu_{21})\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{12} - \Delta_{13})}{\mu_{12}\Sigma_1\Sigma_2} \leq 0.$$

- $\widehat{C}_1(u_1, u_2) \leq \widehat{C}_9(u_1, u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

  $$\widehat{C}_1(u_1, u_2) \leq \widehat{C}_9(u_1, u_2) \Leftrightarrow \mu_{12}\mu_{21} + \mu_{21}\mu_{22} \leq \mu_{11}\mu_{22} + \mu_{21}\mu_{22} \Leftrightarrow 0 \leq \Delta_{12}.$$

- It is seen that

  $$\widehat{C}_1(u_1, u_2) - \widehat{C}_{10}(u_1, u_2) = \frac{u_1(-\Delta_{13} - \Delta_{23})}{(\mu_{11} + \mu_{21})\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{12} + \Delta_{23})}{(\mu_{12} + \mu_{32})\Sigma_1\Sigma_2}.$$

  Now we analyze the numerator of this expression.

  $$\begin{aligned} & -u_1(\mu_{12} + \mu_{32})(\Delta_{13} + \Delta_{23}) - u_2(\mu_{11} + \mu_{21})(\Delta_{12} - \Delta_{23}) \qquad (48) \\ \leq\ & -u_2\mu_{21}(\Delta_{13} + \Delta_{23}) - u_2(\mu_{11} + \mu_{21})\Delta_{12} + u_2(\mu_{11} + \mu_{21})\Delta_{23} \\ =\ & -u_2\mu_{21}\Delta_{13} - u_2(\mu_{11} + \mu_{21})\Delta_{12} + u_2\mu_{11}\Delta_{23} \\ =\ & -u_2(\mu_{11}\mu_{21}\mu_{32} - \mu_{12}\mu_{21}\mu_{31} - \mu_{11}\mu_{21}\mu_{32} + \mu_{11}\mu_{22}\mu_{31}) - u_2(\mu_{11} + \mu_{21})\Delta_{12} \\ =\ & -u_2\Sigma_1\Delta_{12} \leq 0. \end{aligned}$$

  Here (48) follows since Case 10 holds. Thus, $\widehat{C}_1(u_1, u_2) \leq \widehat{C}_{10}(u_1, u_2)$ when u_1 and u_2 are such that Case 10 holds.

- $\widehat{C}_1(u_1, u_2) \leq \widehat{C}_{11}(u_1, u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

  $$\widehat{C}_1(u_1, u_2) \leq \widehat{C}_{11}(u_1, u_2) \Leftrightarrow \mu_{12}\mu_{31} + \mu_{22}\mu_{31} \leq \mu_{11}\mu_{32} + \mu_{21}\mu_{32} \Leftrightarrow 0 \leq \Delta_{13} + \Delta_{23}.$$

- $\widehat{C}_1(u_1, u_2) \leq \widehat{C}_{12}(u_1, u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

  $$\widehat{C}_1(u_1, u_2) - \widehat{C}_{12}(u_1, u_2) = \frac{u_1(-\Delta_{13} - \Delta_{23})}{(\mu_{11} + \mu_{21})\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{13} - \Delta_{23})}{(\mu_{12} + \mu_{22})\Sigma_1\Sigma_2} \leq 0.$$

The above arguments show that $\widehat{C}_1(u_1, u_2)$ is not longer than the cycle times corresponding to the other cases that may hold for the service times u_1 and u_2. This completes the proof. □

Lemma A.3.4 *For all* $u_1,\ u_2 \in \mathbb{R}^+$, *we have* $\widehat{C}_2(u_1,u_2) \leq \widehat{C}_j(u_1,u_2)$, *for* $j \in \{7,8,10,11,12\}$.

Proof: Here are the comparisons of $\widehat{C}_2(u_1,u_2)$ with $\widehat{C}_j(u_1,u_2)$ for $j \in \{7,8,10,11,12\}$.

- $\widehat{C}_2(u_1,u_2) \leq \widehat{C}_7(u_1,u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_2(u_1,u_2) - \widehat{C}_7(u_1,u_2) = \frac{u_1(-\Delta_{12}-\Delta_{13})}{(\mu_{21}+\mu_{31})\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{13}-\Delta_{23})}{\mu_{32}\Sigma_1\Sigma_2} \leq 0.$$

- $\widehat{C}_2(u_1,u_2) \leq \widehat{C}_8(u_1,u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_2(u_1,u_2) \leq \widehat{C}_8(u_1,u_2) \Leftrightarrow 0 \leq \Delta_{13}.$$

- $\widehat{C}_2(u_1,u_2) \leq \widehat{C}_{10}(u_1,u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_2(u_1,u_2) \leq \widehat{C}_{10}(u_1,u_2) \Leftrightarrow \mu_{12}\mu_{31} + \mu_{31}\mu_{32} \leq \mu_{11}\mu_{32} + \mu_{31}\mu_{32} \Leftrightarrow 0 \leq \Delta_{13}.$$

- $\widehat{C}_2(u_1,u_2) \leq \widehat{C}_{11}(u_1,u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

$$\widehat{C}_2(u_1,u_2) - \widehat{C}_{11}(u_1,u_2) = \frac{u_1(-\Delta_{13}-\Delta_{23})}{\mu_{31}\Sigma_1\Sigma_2} + \frac{u_2(-\Delta_{13}-\Delta_{23})}{\mu_{32}\Sigma_1\Sigma_2} \leq 0.$$

- $\widehat{C}_2(u_1,u_2) \leq \widehat{C}_{12}(u_1,u_2)$ for all $u_1,\ u_2 \in \mathbb{R}^+$ because

$$\begin{aligned}\widehat{C}_2(u_1,u_2) \leq \widehat{C}_{12}(u_1,u_2) \quad &\Leftrightarrow \quad \mu_{12}\mu_{31} + \mu_{22}\mu_{31} \leq \mu_{11}\mu_{32} + \mu_{21}\mu_{32} \\ &\Leftrightarrow \quad 0 \leq \Delta_{13} + \Delta_{23}. \ \square\end{aligned}$$

APPENDIX B

SUPPLEMENTARY MATERIAL FOR CHAPTER 4

B.1 Tables for Propositions 4.2.1, 4.2.2, and 4.2.3

In the last column of tables below, we use the convention that the service requirement for a station is equal to one if this station is starved.

Table 9: Sample Path for the Understaffed System with $\mu_{12} = \mu_{13} = 0$, and $\mu_{11} \geq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$

Case	Time	State	Remaining Service Requirement
$\frac{1}{\mu_{11}} < \frac{1}{\mu_{22}}$	T	(s_1^0, s_2^0)	$(1, 1, 1)$
	$T + \frac{1}{\mu_{11}}$	$(s_1^0 + 1, s_2^0)$	$(1, 1 - \frac{1}{\mu_{11}}\mu_{22}, 1)$
	$T + \frac{1}{\mu_{22}}$	$(s_1^0, s_2^0 + 1)$	$(1, 1, 1)$
	$T + \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1, 1, 1)$
$\frac{1}{\mu_{11}} > \frac{1}{\mu_{22}}$	T	(s_1^0, s_2^0)	$(1, 1, 1)$
	$T + \frac{1}{\mu_{22}}$	$(s_1^0 - 1, s_2^0 + 1)$	$(1 - \frac{1}{\mu_{22}}\mu_{11}, 1, 1)$
	$T + \frac{1}{\mu_{11}}$	$(s_1^0, s_2^0 + 1)$	$(1, 1, 1 - \left(\frac{1}{\mu_{11}} - \frac{1}{\mu_{22}}\right)\mu_{23})$
	$T + \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1, 1, 1)$
$\frac{1}{\mu_{11}} = \frac{1}{\mu_{22}}$	T	(s_1^0, s_2^0)	$(1, 1, 1)$
	$T + \frac{1}{\mu_{22}}$	$(s_1^0, s_2^0 + 1)$	$(1, 1, 1)$
	$T + \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1, 1, 1)$

Table 10: Sample Path for the Understaffed System with $\mu_{12} = \mu_{13} = 0$, and $\mu_{11} < \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{22}}$	$(s_1^0 - 1, s_2^0 + 1)$	$(1 - \frac{1}{\mu_{22}}\mu_{11}, 1, 1)$
$T + \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$	$(s_1^0 - 1, s_2^0)$	$(1 - \left(\frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}\right)\mu_{11}, 1, 1)$
$T + \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}} + \left(\frac{1}{\mu_{11}+\mu_{21}}\right)\left(1 - \mu_{11}\left(\frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}\right)\right)$	(s_1^0, s_2^0)	$(1,1,1)$

Table 11: Sample Path for the Understaffed System with $\mu_{11} = \mu_{13} = 0$, and $\mu_{12} \geq \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$ when assignment rule (a) is used

Case	Time	State	Remaining Service Requirement
$\frac{1}{\mu_{12}} < \frac{1}{\mu_{23}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{12}}$	$(s_1^0 - 1, s_2^0 + 1)$	$(1, 1, 1 - \frac{1}{\mu_{12}}\mu_{23})$
	$T + \frac{1}{\mu_{23}}$	$(s_1^0 - 1, s_2^0)$	$(1,1,1)$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{12}} > \frac{1}{\mu_{23}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{23}}$	$(s_1^0, s_2^0 - 1)$	$(1, 1 - \frac{1}{\mu_{23}}\mu_{12}, 1)$
	$T + \frac{1}{\mu_{12}}$	$(s_1^0 - 1, s_2^0)$	$(1 - \left(\frac{1}{\mu_{12}} - \frac{1}{\mu_{23}}\right)\mu_{21}, 1, 1)$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{12}} = \frac{1}{\mu_{23}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{23}}$	$(s_1^0 - 1, s_2^0)$	$(1,1,1)$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 12: Sample Path for the Understaffed System with $\mu_{11} = \mu_{13} = 0$, and $\mu_{12} < \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$ when assignment rule (a) is used

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{23}}$	$(s_1^0, s_2^0 - 1)$	$(1, 1 - \frac{1}{\mu_{23}}\mu_{12}, 1)$
$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	$(s_1^0 + 1, s_2^0 - 1)$	$(1, 1 - \left(\frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}\right)\mu_{12}, 1)$
$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}} + \left(\frac{1}{\mu_{12}+\mu_{22}}\right)\left(1 - \mu_{12}\left(\frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}\right)\right)$	(s_1^0, s_2^0)	$(1,1,1)$

Table 13: Sample Path for the Understaffed System with $\mu_{11} = \mu_{13} = 0$, and $\mu_{12} \geq \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$ when assignment rule (b) is used

Case	Time	State	Remaining Service Requirement
$\frac{1}{\mu_{12}} < \frac{1}{\mu_{21}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{12}}$	$(s_1^0 - 1, s_2^0 + 1)$	$(1 - \frac{1}{\mu_{12}}\mu_{21}, 1, 1)$
	$T + \frac{1}{\mu_{21}}$	$(s_1^0, s_2^0 + 1)$	$(1,1,1)$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{12}} > \frac{1}{\mu_{21}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{21}}$	$(s_1^0 + 1, s_2^0)$	$(1, 1 - \frac{1}{\mu_{21}}\mu_{12}, 1)$
	$T + \frac{1}{\mu_{12}}$	$(s_1^0, s_2^0 + 1)$	$(1, 1, 1 - \left(\frac{1}{\mu_{12}} - \frac{1}{\mu_{21}}\right)\mu_{23})$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{12}} = \frac{1}{\mu_{21}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{21}}$	$(s_1^0, s_2^0 + 1)$	$(1,1,1)$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 14: Sample Path for the Understaffed System with $\mu_{11} = \mu_{13} = 0$, and $\mu_{12} < \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$ when assignment rule (b) is used

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{21}}$	$(s_1^0 + 1, s_2^0)$	$(1, 1 - \frac{1}{\mu_{21}}\mu_{12}, 1)$
$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	$(s_1^0 + 1, s_2^0 - 1)$	$(1, 1 - \left(\frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}\right)\mu_{12}, 1)$
$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$ $+\left(\frac{1}{\mu_{12}+\mu_{22}}\right)\left(1 - \mu_{12}\left(\frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}\right)\right)$	(s_1^0, s_2^0)	$(1,1,1)$

Table 15: Sample Path for the Understaffed System with $\mu_{11} = \mu_{12} = 0$, and $\mu_{13} \geq \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}}$

Case	Time	State	Remaining Service Requirement
$\frac{1}{\mu_{13}} < \frac{1}{\mu_{22}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{13}}$	$(s_1^0, s_2^0 - 1)$	$(1, 1 - \frac{1}{\mu_{13}}\mu_{22}, 1)$
	$T + \frac{1}{\mu_{22}}$	$(s_1^0 - 1, s_2^0)$	$(1,1,1)$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{22}}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{13}} > \frac{1}{\mu_{22}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{22}}$	$(s_1^0 - 1, s_2^0 + 1)$	$(1, 1, 1 - \frac{1}{\mu_{22}}\mu_{13})$
	$T + \frac{1}{\mu_{13}}$	$(s_1^0 - 1, s_2^0)$	$(1 - \left(\frac{1}{\mu_{13}} - \frac{1}{\mu_{22}}\right)\mu_{21}, 1, 1)$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{22}}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{13}} = \frac{1}{\mu_{22}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{22}}$	$(s_1^0 - 1, s_2^0)$	$(1,1,1)$
	$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{22}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 16: Sample Path for the Understaffed System with $\mu_{11} = \mu_{12} = 0$, and $\mu_{13} < \frac{\mu_{21}\mu_{22}}{\mu_{21}+\mu_{22}}$

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{22}}$	$(s_1^0 - 1, s_2^0 + 1)$	$(1,1,1-\frac{1}{\mu_{22}}\mu_{13})$
$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{22}}$	$(s_1^0, s_2^0 + 1)$	$(1,1,1-\left(\frac{1}{\mu_{21}} + \frac{1}{\mu_{22}}\right)\mu_{13})$
$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{22}} + \left(\frac{1}{\mu_{13}+\mu_{23}}\right)\left(1 - \mu_{13}\left(\frac{1}{\mu_{21}} + \frac{1}{\mu_{22}}\right)\right)$	(s_1^0, s_2^0)	$(1,1,1)$

B.2 Tables for Propositions 4.2.4, 4.2.5, and 4.2.6

In the last column of tables below, we use the convention that the service requirement for a station is equal to one if this station is starved.

Table 17: Sample Path for the Understaffed System with $\mu_{13} = \mu_{22} = 0$, and $\mu_{12} \leq \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{23}}$	$(s_1^0, s_2^0 - 1)$	$(1, 1 - \frac{1}{\mu_{23}}\mu_{12}, 1)$
$T + \frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}$	$(s_1^0 + 1, s_2^0 - 1)$	$(1, 1 - \left(\frac{1}{\mu_{21}} + \frac{1}{\mu_{23}}\right)\mu_{12}, 1)$
$T + \frac{1}{\mu_{12}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 18: Sample Path for the Understaffed System with $\mu_{13} = \mu_{22} = 0$, and $\mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{12}}$	$(s_1^0 - 1, s_2^0 + 1)$	$(1, 1, 1 - \frac{1}{\mu_{12}}\mu_{23})$
$T + \frac{1}{\mu_{11}} + \frac{1}{\mu_{12}}$	$(s_1^0, s_2^0 + 1)$	$(1, 1, 1 - \left(\frac{1}{\mu_{11}} + \frac{1}{\mu_{21}}\right)\mu_{23})$
$T + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 19: Sample Path for the Understaffed System with $\mu_{13} = \mu_{22} = 0$, $\mu_{12} > \frac{\mu_{21}\mu_{23}}{\mu_{21}+\mu_{23}}$, and $\mu_{23} > \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$

Case	Time	State	Remaining Service Requirement
$\frac{1}{\mu_{12}} < \frac{1}{\mu_{23}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{12}}$	$(s_1^0 - 1, s_2^0 + 1)$	$(1,1,1 - \frac{1}{\mu_{12}}\mu_{23})$
	$T + \frac{1}{\mu_{23}}$	$(s_1^0 - 1, s_2^0)$	$(1 - \left(\frac{1}{\mu_{23}} - \frac{1}{\mu_{12}}\right)\mu_{11}, 1, 1)$
	$T + \frac{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}}{\mu_{12}\mu_{23}(\mu_{11}+\mu_{21})}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{12}} > \frac{1}{\mu_{23}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{23}}$	$(s_1^0, s_2^0 - 1)$	$(1, 1 - \frac{1}{\mu_{23}}\mu_{12}, 1)$
	$T + \frac{1}{\mu_{12}}$	$(s_1^0 - 1, s_2^0)$	$(1 - \left(\frac{1}{\mu_{12}} - \frac{1}{\mu_{23}}\right)\mu_{21}, 1, 1)$
	$T + \frac{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}}{\mu_{12}\mu_{23}(\mu_{11}+\mu_{21})}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{12}} = \frac{1}{\mu_{23}}$	$T + \frac{1}{\mu_{12}}$	$(s_1^0 - 1, s_2^0)$	$(1,1,1)$
	$T + \frac{1}{\mu_{12}} + \frac{1}{\mu_{11}+\mu_{21}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 20: Sample Path for the Understaffed System with $\mu_{13} = \mu_{21} = 0$, and $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{23}}$	$(s_1^0, s_2^0 - 1)$	$(1 - \frac{1}{\mu_{23}}\mu_{11}, 1, 1)$
$T + \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$	$(s_1^0 - 1, s_2^0)$	$(1 - \left(\frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}\right)\mu_{11}, 1, 1)$
$T + \frac{1}{\mu_{11}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 21: Sample Path for the Understaffed System with $\mu_{13} = \mu_{21} = 0$, and $\mu_{23} \leq \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{11}}$	$(s_1^0 + 1, s_2^0)$	$(1,1,1 - \frac{1}{\mu_{11}}\mu_{23})$
$T + \frac{1}{\mu_{11}} + \frac{1}{\mu_{12}}$	$(s_1^0, s_2^0 + 1)$	$(1,1,1 - \left(\frac{1}{\mu_{11}} + \frac{1}{\mu_{12}}\right)\mu_{23})$
$T + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 22: Sample Path for the Understaffed System with $\mu_{13} = \mu_{21} = 0$, $\mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$, and $\mu_{23} > \frac{\mu_{11}\mu_{12}}{\mu_{11}+\mu_{12}}$

Case	Time	State	Remaining Service Requirement
$\frac{1}{\mu_{11}} < \frac{1}{\mu_{23}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{11}}$	(s_1^0+1, s_2^0)	$(1,1,1-\frac{1}{\mu_{11}}\mu_{23})$
	$T + \frac{1}{\mu_{23}}$	(s_1^0+1, s_2^0-1)	$\left(1,1-\left(\frac{1}{\mu_{23}}-\frac{1}{\mu_{11}}\right)\mu_{12},1\right)$
	$T + \frac{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}}{\mu_{11}\mu_{23}(\mu_{12}+\mu_{22})}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{11}} > \frac{1}{\mu_{23}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{23}}$	(s_1^0, s_2^0-1)	$(1-\frac{1}{\mu_{23}}\mu_{11},1,1)$
	$T + \frac{1}{\mu_{11}}$	(s_1^0+1, s_2^0-1)	$\left(1,1-\left(\frac{1}{\mu_{11}}-\frac{1}{\mu_{23}}\right)\mu_{12},1\right)$
	$T + \frac{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}}{\mu_{11}\mu_{23}(\mu_{12}+\mu_{22})}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{11}} = \frac{1}{\mu_{23}}$	$T + \frac{1}{\mu_{11}}$	(s_1^0+1, s_2^0-1)	$(1,1,1)$
	$T + \frac{1}{\mu_{11}} + \frac{1}{\mu_{12}+\mu_{22}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 23: Sample Path for the Understaffed System with $\mu_{12} = \mu_{21} = 0$, and $\mu_{11} \leq \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{22}}$	(s_1^0-1, s_2^0+1)	$(1-\frac{1}{\mu_{22}}\mu_{11},1,1)$
$T + \frac{1}{\mu_{22}} + \frac{1}{\mu_{23}}$	(s_1^0-1, s_2^0)	$\left(1-\left(\frac{1}{\mu_{22}}+\frac{1}{\mu_{23}}\right)\mu_{11},1,1\right)$
$T + \frac{1}{\mu_{11}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 24: Sample Path for the Understaffed System with $\mu_{12} = \mu_{21} = 0$, and $\mu_{22} \leq \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}$

Time	State	Remaining Service Requirement
T	(s_1^0, s_2^0)	$(1,1,1)$
$T + \frac{1}{\mu_{11}}$	(s_1^0+1, s_2^0)	$(1,1-\frac{1}{\mu_{11}}\mu_{22},1)$
$T + \frac{1}{\mu_{11}} + \frac{1}{\mu_{13}}$	(s_1^0+1, s_2^0-1)	$\left(1,1-\left(\frac{1}{\mu_{11}}+\frac{1}{\mu_{13}}\right)\mu_{22},1\right)$
$T + \frac{1}{\mu_{22}}$	(s_1^0, s_2^0)	$(1,1,1)$

Table 25: Sample Path for the Understaffed System with $\mu_{12} = \mu_{21} = 0$, $\mu_{11} > \frac{\mu_{22}\mu_{23}}{\mu_{22}+\mu_{23}}$, and $\mu_{22} > \frac{\mu_{11}\mu_{13}}{\mu_{11}+\mu_{13}}$

Case	Time	State	Remaining Service Requirement
$\frac{1}{\mu_{11}} < \frac{1}{\mu_{22}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{11}}$	(s_1^0+1, s_2^0)	$(1, 1-\frac{1}{\mu_{11}}\mu_{22}, 1)$
	$T + \frac{1}{\mu_{22}}$	(s_1^0, s_2^0+1)	$(1,1,1-\left(\frac{1}{\mu_{22}}-\frac{1}{\mu_{11}}\right)\mu_{13})$
	$T + \frac{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{13}\mu_{22}}{\mu_{11}\mu_{22}(\mu_{13}+\mu_{23})}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{11}} > \frac{1}{\mu_{22}}$	T	(s_1^0, s_2^0)	$(1,1,1)$
	$T + \frac{1}{\mu_{22}}$	(s_1^0-1, s_2^0+1)	$(1-\frac{1}{\mu_{22}}\mu_{11}, 1, 1)$
	$T + \frac{1}{\mu_{11}}$	(s_1^0, s_2^0+1)	$(1,1,1-\left(\frac{1}{\mu_{11}}-\frac{1}{\mu_{22}}\right)\mu_{23})$
	$T + \frac{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{13}\mu_{22}}{\mu_{11}\mu_{22}(\mu_{13}+\mu_{23})}$	(s_1^0, s_2^0)	$(1,1,1)$
$\frac{1}{\mu_{11}} = \frac{1}{\mu_{22}}$	$T + \frac{1}{\mu_{11}}$	(s_1^0, s_2^0+1)	$(1,1,1)$
	$T + \frac{1}{\mu_{11}} + \frac{1}{\mu_{13}+\mu_{23}}$	(s_1^0, s_2^0)	$(1,1,1)$

B.3 Proof of Theorem 4.2.1

It suffices to show that the optimal value of λ in the allocation LP in the presence of fully flexible servers is equal to the throughput of the system with partially flexible servers. First, we transform this LP to the standard form as follows:

$$\begin{aligned}
\min \quad & -\lambda \\
\text{s.t.} \quad & \lambda - \delta_{11}\mu_{11} - \delta_{21}\mu_{21} + s_1 = 0, \\
& \lambda - \delta_{12}\mu_{12} - \delta_{22}\mu_{22} + s_2 = 0, \\
& \lambda - \delta_{13}\mu_{13} - \delta_{23}\mu_{22} + s_3 = 0, \\
& \delta_{11} + \delta_{12} + \delta_{13} = 1, \qquad (49) \\
& \delta_{21} + \delta_{22} + \delta_{23} = 1, \qquad (50) \\
& \delta_{ij} \geq 0, \text{ for all } i \in \{1,2,3\}, j \in \{1,2\}, s_1, s_2, s_3 \geq 0.
\end{aligned}$$

Note that no slack variables are needed in equations (49) and (50), because these constraints can be satisfied as equalities without worsening the objective function value. Every feasible basis will have five elements because there are five constraints (not including the nonnegativity constraints) in the LP.

Let D be a basis for the above LP, c_B be the vector of coefficients of the elements of D in the objective function, $\mathbf{B}$ be the coefficients of the elements of D in the constraint matrix, and b be the right-hand side of the constraints. Also, let V denote the coefficients of the non-basic variables in the constraint matrix, and c_{NB} denote the vector of coefficients of the non-basic variables in the objective function. We let c_B and c_{NB} be row vectors, and b be a column vector. The following conditions guarantee that the basis D is optimal (see, e.g., Theorem 3.1 of Bertsimas and Tsitsiklis [25]):

$$\mathbf{B}^{-1}b \geq 0, \qquad (51)$$

$$c_{NB} - c_B\mathbf{B}^{-1}V \geq 0. \qquad (52)$$

Consider the basis $D = \{\lambda, \delta_{11}, \delta_{21}, \delta_{22}, \delta_{23}\}$. Some algebra shows that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \frac{\mu_{22}\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{22}\mu_{23}-\mu_{11}(\mu_{22}+\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{22}(\mu_{11}+\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ 1 \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \frac{\mu_{23}(\mu_{11}\mu_{22}-\mu_{12}\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{22}(\mu_{11}\mu_{23}-\mu_{13}\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{22}\mu_{23}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}\mu_{23}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}\mu_{22}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \end{bmatrix}.$$

Hence we can conclude that $D = \{\lambda, \delta_{11}, \delta_{21}, \delta_{22}, \delta_{23}\}$ is an optimal basis if conditions $\{1\}$, $\{3\}$, and $\{7\}$ hold. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \frac{\mu_{22}\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}}$ in this case. This result also implies that cross-training the server 2 at all stations and server 1 at only station 1 corresponds to the best flexibility structure when case a holds. Furthermore, the policy of Proposition 4.2.1 attains the maximal capacity in this case, hence it is the optimal server assignment policy.

Relabeling the servers, we also see that $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{13}, \delta_{21}\}$ is the optimal basis if conditions $\{2\}$, $\{4\}$, and $\{13\}$ hold. Hence, the best flexibility structure is the one where server 2 is dedicated to station 1 and server 1 is trained at all stations. Furthermore, the maximal capacity can be attained by relabeling the servers and employing the policy of Proposition 4.2.1.

Next, consider the basis $D = \{\lambda, \delta_{12}, \delta_{21}, \delta_{22}, \delta_{23}\}$. Some algebra shows that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \frac{\mu_{21}\mu_{23}(\mu_{12}+\mu_{22})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{23}(\mu_{12}+\mu_{22})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}\mu_{23}-\mu_{12}(\mu_{21}+\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}(\mu_{12}+\mu_{22})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ 1 \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \frac{\mu_{23}(\mu_{12}\mu_{21}-\mu_{11}\mu_{22})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}(\mu_{12}\mu_{23}-\mu_{13}\mu_{22})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{22}\mu_{23}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}\mu_{23}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}\mu_{22}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \end{bmatrix}.$$

Hence we can conclude that $D = \{\lambda, \delta_{12}, \delta_{21}, \delta_{22}, \delta_{23}\}$ is an optimal basis when conditions $\{2\}$, $\{5\}$, and $\{9\}$ hold. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \frac{\mu_{21}\mu_{23}(\mu_{12}+\mu_{22})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}}$ in this case. This result also implies that cross-training server 2 at all stations and server 1 at only station 2 corresponds to the best flexibility structure when case c holds. Furthermore, the policy of Proposition 4.2.2 attains the maximal capacity in this case, hence it is the optimal server assignment policy.

Relabeling the servers, it follows that $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{13}, \delta_{22}\}$ is the optimal basis if conditions $\{1\}$, $\{6\}$, and $\{15\}$ hold. Hence, the best flexibility structure is the one where server 2 is dedicated to station 2 and server 1 is trained at all stations. Furthermore, the maximal capacity can be attained by relabeling the servers and employing the policy of Proposition 4.2.2.

Now, consider the basis $D = \{\lambda, \delta_{13}, \delta_{21}, \delta_{22}, \delta_{23}\}$. Some algebra shows that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \frac{\mu_{21}\mu_{22}(\mu_{13}+\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{22}(\mu_{13}+\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}(\mu_{13}+\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}\mu_{22}-\mu_{13}(\mu_{21}+\mu_{22})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ 1 \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \frac{\mu_{22}(\mu_{13}\mu_{21}-\mu_{11}\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}(\mu_{13}\mu_{22}-\mu_{12}\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{22}\mu_{23}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}\mu_{23}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \\ \frac{\mu_{21}\mu_{22}}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}} \end{bmatrix}.$$

Hence we can conclude that $D = \{\lambda, \delta_{13}, \delta_{21}, \delta_{22}, \delta_{23}\}$ is an optimal basis if conditions $\{4\}$, $\{6\}$, and $\{11\}$ hold. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \frac{\mu_{21}\mu_{22}(\mu_{13}+\mu_{23})}{\mu_{21}\mu_{22}+\mu_{21}\mu_{23}+\mu_{22}\mu_{23}}$ in this case. This result also implies that cross-training server 2 at all stations and the server 1 at only station 3 corresponds to the best flexibility structure when case e holds. Furthermore, the policy of Proposition 4.2.3 attains the maximal capacity in this case, hence it is the optimal server assignment policy.

Relabeling the servers, it is clear that $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{13}, \delta_{23}\}$ is the optimal basis if conditions $\{3\}$, $\{5\}$, and $\{17\}$ hold. Hence, the best flexibility structure is the one where server 2 is dedicated to station 3 and server 1 is trained at all stations. Furthermore, the maximal capacity can be attained by relabeling the servers and employing the policy of Proposition 4.2.3.

Consider the basis $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{21}, \delta_{23}\}$. Some algebra shows that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \frac{\mu_{12}\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \\ \frac{\mu_{12}(\mu_{21}+\mu_{23})-\mu_{21}\mu_{23}}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \\ \frac{\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \\ \frac{\mu_{23}(\mu_{11}+\mu_{12})-\mu_{11}\mu_{12}}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \\ \frac{\mu_{12}(\mu_{11}+\mu_{21})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \frac{\mu_{12}(\mu_{11}\mu_{23}-\mu_{13}\mu_{21})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \\ \frac{\mu_{23}(\mu_{12}\mu_{21}-\mu_{11}\mu_{22})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \\ \frac{\mu_{12}\mu_{23}}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \\ \frac{\mu_{11}\mu_{23}}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \\ \frac{\mu_{12}\mu_{21}}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}} \end{bmatrix}.$$

Hence we can conclude that $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{21}, \delta_{23}\}$ is an optimal basis if conditions $\{2\}$, $\{3\}$, $\{10\}$, and $\{18\}$ hold. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \frac{\mu_{12}\mu_{23}(\mu_{11}+\mu_{21})}{\mu_{11}\mu_{23}+\mu_{12}\mu_{21}+\mu_{12}\mu_{23}}$ in this case. This result also implies that cross-training the first server at stations 1 and 2, and server 2 at stations 1 and 3 corresponds to the best flexibility structure when case g holds. Furthermore, the policy of Proposition 4.2.4 attains the maximal capacity in this case, hence it is the optimal server assignment policy.

Relabeling the servers, we also see that $D = \{\lambda, \delta_{11}, \delta_{13}, \delta_{21}, \delta_{22}\}$ is the optimal basis if conditions $\{1\}$, $\{4\}$, $\{12\}$ and $\{16\}$ hold. Hence, the best flexibility structure is the one where server 1 is trained at stations 1 and 3, and server 2 is trained at stations 1 and 2. Furthermore, the maximal capacity can be attained by relabeling the servers and employing the policy of Proposition 4.2.4.

Consider the basis $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{22}, \delta_{23}\}$. Some algebra shows that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \frac{\mu_{11}\mu_{23}(\mu_{12}+\mu_{22})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{23}(\mu_{12}+\mu_{22})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{11}(\mu_{22}+\mu_{23})-\mu_{22}\mu_{23}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{23}(\mu_{11}+\mu_{12})-\mu_{11}\mu_{12}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{11}(\mu_{12}+\mu_{22})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \frac{\mu_{11}(\mu_{12}\mu_{23}-\mu_{13}\mu_{22})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{23}(\mu_{11}\mu_{22}-\mu_{12}\mu_{21})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{12}\mu_{23}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{11}\mu_{23}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{11}\mu_{22}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \end{bmatrix}.$$

Hence we can conclude that $D = \{\lambda, \delta_{11}, \delta_{12}, \delta_{22}, \delta_{23}\}$ is an optimal basis if conditions $\{1\}$, $\{5\}$, $\{8\}$ and $\{18\}$ hold. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \frac{\mu_{11}\mu_{23}(\mu_{12}+\mu_{22})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}}$ in this case. This result also implies that cross-training the first server at stations 1 and 2, and server 2 at stations 2 and 3 corresponds to the best flexibility structure when case i holds. Furthermore, the policy of Proposition 4.2.5 attains the maximal capacity in this case, hence it is the optimal server assignment policy.

Relabeling the servers, we also see that $D = \{\lambda, \delta_{12}, \delta_{13}, \delta_{21}, \delta_{22}\}$ is the optimal basis if conditions $\{2\}$, $\{6\}$, $\{12\}$ and $\{14\}$ hold. Hence, the best flexibility structure is the one where server 1 is trained at stations 2 and 3, and server 2 is trained at stations 1 and 2. Furthermore, the maximal capacity can be attained by relabeling the servers and employing the policy of Proposition 4.2.5.

Finally, consider the basis $D = \{\lambda, \delta_{11}, \delta_{13}, \delta_{22}, \delta_{23}\}$. Some algebra shows that

$$\mathbf{B}^{-1}b = \begin{bmatrix} \frac{\mu_{11}\mu_{22}(\mu_{13}+\mu_{23})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{13}\mu_{22}} \\ \frac{\mu_{22}(\mu_{13}+\mu_{23})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{13}\mu_{22}} \\ \frac{\mu_{11}(\mu_{22}+\mu_{23})-\mu_{22}\mu_{23}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{13}\mu_{22}} \\ \frac{\mu_{11}(\mu_{13}+\mu_{23})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{13}\mu_{22}} \\ \frac{\mu_{22}(\mu_{11}+\mu_{13})-\mu_{11}\mu_{13}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{13}\mu_{22}} \end{bmatrix},$$

$$c_{NB} - c_B\mathbf{B}^{-1}V = \begin{bmatrix} \frac{\mu_{11}(\mu_{13}\mu_{22}-\mu_{12}\mu_{23})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{22}(\mu_{11}\mu_{23}-\mu_{13}\mu_{21})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{13}\mu_{22}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{11}\mu_{23}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \\ \frac{\mu_{11}\mu_{22}}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{12}\mu_{23}} \end{bmatrix}.$$

Hence we can conclude that $D = \{\lambda, \delta_{11}, \delta_{13}, \delta_{22}, \delta_{23}\}$ is an optimal basis if conditions $\{3\}$, $\{6\}$, $\{8\}$, and $\{16\}$ hold. The first element in the matrix $\mathbf{B}^{-1}b$ is the value of λ in the optimal basis, hence $\lambda^* = \frac{\mu_{11}\mu_{22}(\mu_{13}+\mu_{23})}{\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{13}\mu_{22}}$ in this case. This result also implies that cross-training the first server at stations 1 and 3, and server 2 at stations 2 and 3 corresponds to the best flexibility structure when case k holds. Furthermore, the policy of Proposition 4.2.6 attains the maximal capacity in this case, hence it is the optimal server assignment policy.

Relabeling the servers, we also see that $D = \{\lambda, \delta_{12}, \delta_{13}, \delta_{21}, \delta_{23}\}$ is the optimal basis when conditions $\{4\}$, $\{5\}$, $\{10\}$ and $\{14\}$ hold. Hence, the best flexibility structure is the one where server 1 is trained at stations 2 and 3, and server 2 is trained at stations 1 and 3. Furthermore, the maximal capacity can be attained by relabeling the servers and employing the policy of Proposition 4.2.6.

Moreover, we already showed that there is an optimal basis for this LP with at least two elements of the set $\{\delta_{ij}\}$ being equal to zero. There are $\binom{6}{2} = 15$ different ways of selecting two elements that will be equal to zero out of six. When $\delta_{1j} = \delta_{2j} = 0$ for some $j \in \{1, 2, 3\}$, the throughput is equal to zero. We can conclude that one of the twelve cases mentioned in the theorem will provide the optimal throughput since the basis of each case above is one of the remaining twelve that are candidates for the optimal basis (we also show this algebraically in Proposition 4.2.7). □

B.4 Proofs of Propositions 4.3.1, 4.3.2, and 4.3.3

Proof of Proposition 4.3.1: When the service times are exponentially distributed, it is easy to see that $\{X(t)\}$ is a continuous-time Markov chain (CTMC). As described in Section 4.3.1, the state space of this CTMC is $S = \{(s_1, s_2) : s_1 \in \{0, 1, \ldots, B_2 + 2\}, s_2 \in \{0, 1, \ldots, B_3 + 2\},$ and $s_1 + s_2 \leq B_2 + B_3 + 3\}$. Lemma 4.3.1 shows that it suffices to consider the policies that are non idling (even if a server is idle at a station (s)he is assigned to, for notational convenience we treat this differently than the case where that server is not assigned to any station). Then, the set of allowable actions in state $s \in S$ is

$$A_s = \begin{cases} a_{11} & \text{for } s = (0,0), \\ a_{12} & \text{for } s = (B_2+2, 0), \\ a_{13} & \text{for } s = (B_2+1, B_3+2), \\ \{a_{11}, a_{12}, a_{22}\} & \text{for } s = (i, 0), \text{ where } i \in \{1, \ldots, B_2+1\}, \\ \{a_{11}, a_{13}, a_{33}\} & \text{for } s = (0, j) \text{ or } s = (i, B_3+2), \text{ where} \\ & \quad i \in \{1, \ldots, B_2\} \text{ and } j \in \{1, \ldots, B_3+2\}, \\ \{a_{12}, a_{13}\} & \text{for } s = (B_2+2, j), \text{ where } j \in \{1, \ldots, B_3+1\}, \\ \{a_{11}, a_{12}, a_{13}, a_{22}, a_{33}\} & \text{for } s = (i, j), \text{ where } i \in \{1, \ldots, B_2+1\} \\ & \quad \text{and } j \in \{1, \ldots, B_3+1\}. \end{cases}$$

Note that we used the fact that assigning a server to a station that is blocked or starved is equivalent to idling this server. For example idling a server is equivalent to assigning that server to station 1 in state $(B_2+2, 0)$. Furthermore, in the states where more than one station is operating, it is necessary to consider the actions where both servers are assigned to the same station (even if one server is not cross-trained at that station).

Proof of Lemma 4.3.1 shows that assigning servers to the same station is better than idling one of the servers but it does not compare policies where they are assigned to the same station (even if this causes involuntary idling) with the policies that assign them to different stations (in which case both of them might be working). Under our assumptions on the service rates ($\sum_{i=1}^{M} \mu_{ij} > 0$ for $j \in \{1, \ldots, N\}$ and $\mu_{12} = \mu_{13} = 0$), it is clear that $\mu_{22} > 0$ and $\mu_{23} > 0$. Hence, we can conclude that the policy described in the theorem corresponds to an irreducible Markov chain. Furthermore, $\{X(t)\}$ is uniformizable with the uniformization constant $q = \mu_{11}+\mu_{21}+\mu_{22}+\mu_{23}$ (see, Lippman [53]). Consequently, we have a communicating Markov decision process. Thus, we can use the policy iteration algorithm for communicating models as described in Section 9.5.1 of Puterman [58].

Note that $\pi = (d)^{\infty}$ for every policy π in Π, if d is the corresponding decision rule with $d(s) \in A_s$ for all $s \in S$. Similarly, let P_d be the probability transition matrix corresponding to the policy π, and $r_d(s)$ denote the reward in state s when policy π is employed.

We start the policy iteration algorithm by choosing

$$d_0(s) = \begin{cases} a_{11} & \text{for } s = (0,0), \\ a_{12} & \text{for } s = (B_2+2, j), \text{ where } j \in \{0, \ldots, B_3+1\}, \\ a_{13} & \text{for } s = (B_2+1, B_3+2), \\ a_{12} & \text{for } s = (i,j), \text{ where } i \in \{1, \ldots, B_2+1\} \text{ and } j \in \{0, \ldots, B_3+1\}, \\ a_{13} & \text{for } s = (0,j) \text{ or } s = (i, B_3+2), \text{ where } i \in \{1, \ldots, B_2\} \\ & \qquad \text{and } j \in \{1, \ldots, B_3+2\}. \end{cases}$$

Then we obtain

$$r_{d_0}(s) = \begin{cases} 0 & \text{for } s = (0,0) \text{ or } s = (i,j), \text{ where } i \in \{1, \ldots, B_2+2\} \\ & \qquad \text{and } j \in \{0, \ldots, B_3+1\}, \\ \mu_{23} & \text{for } s = (0,j) \text{ or } s = (i, B_3+2), \text{ where } i \in \{1, \ldots, B_2+1\} \\ & \qquad \text{and } j \in \{1, \ldots, B_3+2\}, \end{cases}$$

and

$$
P_{d_0}(s,s') = \begin{cases}
\frac{\mu_{11}+\mu_{21}}{q} & \text{for } s=(0,0) \text{ and } s'=(1,0), \\
\frac{\mu_{22}+\mu_{23}}{q} & \text{for } s=s'=(0,0), \\
\frac{\mu_{11}}{q} & \text{for } s=(i,j),\ s'=(i+1,j), \text{ where } i\in\{1,\dots,B_2+1\} \\
 & \quad \text{and } j\in\{0,\dots,B_3+1\}, \\
\frac{\mu_{21}+\mu_{23}}{q} & \text{for } s=s'=(i,j), \text{ where } i\in\{1,\dots,B_2+1\} \\
 & \quad \text{and } j\in\{0,\dots,B_3+1\}, \\
\frac{\mu_{11}}{q} & \text{for } s=(i,j),\ s'=(i+1,j), \text{ where } i=0 \\
 & \quad \text{and } j\in\{1,\dots,B_3+1\}, \\
\frac{\mu_{21}+\mu_{22}}{q} & \text{for } s=s'=(i,j), \text{ where } i=0 \\
 & \quad \text{and } j\in\{1,\dots,B_3+1\}, \\
\frac{\mu_{11}}{q} & \text{for } s=(i,j),\ s'=(i+1,j), \text{ where } i\in\{0,\dots,B_2\} \\
 & \quad \text{and } j=B_3+2, \\
\frac{\mu_{21}+\mu_{22}}{q} & \text{for } s=s'=(i,j), \text{ where } i\in\{0,\dots,B_2\} \\
 & \quad \text{and } j=B_3+2, \\
\frac{\mu_{22}}{q} & \text{for } s=(i,j),\ s'=(i-1,j+1), \text{ where } i\in\{1,\dots,B_2+2\} \\
 & \quad \text{and } j\in\{0,\dots,B_3+1\}, \\
\frac{\mu_{11}+\mu_{21}+\mu_{23}}{q} & \text{for } s=(i,j),\ s'=(i-1,j+1) \text{ where } i=B_2+2 \\
 & \quad \text{and } j\in\{0,\dots,B_2+1\}, \\
\frac{\mu_{23}}{q} & \text{for } s=(i,j),\ s'=(i,j-1), \text{ where } i=0 \\
 & \quad \text{and } j\in\{1,\dots,B_3+1\}, \\
\frac{\mu_{21}+\mu_{22}}{q} & \text{for } s=(i,j),\ s'=(i,j-1), \text{ where } i=0 \\
 & \quad \text{and } j\in\{1,\dots,B_3+1\}, \\
\frac{\mu_{23}}{q} & \text{for } s=(i,j),\ s'=(i,j-1), \text{ where } i\in\{0,\dots,B_2+1\} \\
 & \quad \text{and } j=B_3+2, \\
\frac{\mu_{11}+\mu_{21}+\mu_{22}}{q} & \text{for } s=s'=(B_2+1,B_2+2).
\end{cases}
$$

For all $s,s'\in S$ and $a\in A_s$, we use $r(s,a)$ to denote the immediate reward in state s when action a is taken and $p(s'|s,a)$ to denote the one-step probability of going from state s to state s' when action a is chosen in state s. It is easy to see that $\{X(t)\}$ is an irreducible Markov chain, and hence we can solve the following set of equations to find a scalar g and a vector h, letting $h(0,0)=0$.

$$r_{d_0} - ge + (P_{d_0} - I)h = 0.$$

Then, we use the policy iteration algorithm to find $d(s)$, where

$$d(s) \in \arg\max_{a \in A_s} \Big\{ r(s,a) + \sum_{s' \in S} p(s'|s,a)h(s') \Big\}, \;\; \forall s \in S,$$

and set $d(s) = d_0(s)$ whenever possible. If one can show $d(s) = d_0(s)$ for all $s \in S$, then the policy π is optimal. In particular, for all $s \in A_s$ and $a \in A_s$, we want to show that the following inequality holds:

$$\Big(r(s, d_0(s)) + \sum_{s' \in S} p(s'|s, d_0(s))h(s') \Big) - r(s,a) - \sum_{s' \in S} p(s'|s,a)h(s') \geq 0.$$

In the calculations below, $\xi_k(s, B_2, B_3)$ for $k \in \{1, \ldots, 11\}$ and $\xi(B_2, B_3)$ are nonnegative constants that depend on the service rates, the state $s = (i,j) \in S$ under consideration, and the buffer sizes, and they are provided below. We assume that $B_2, B_3 \leq 1$ in the following calculations. First, consider the state $s = (i,0)$, where $i \in \{1, \ldots, B_2+1\}$, and recall that $d_0(s) = a_{12}$. With some algebra we have, for all $i \in \{1, \ldots, B_2+1\}$,

$$\begin{aligned}
\Big(r((i,0), a_{12}) + \sum_{s' \in S} p(s'|(i,0), a_{12})h(s') \Big) - r((i,0), a_{11}) - \sum_{s' \in S} p(s'|(i,0), a_{11})h(s') \\
= \frac{\xi_1((i,0), B_2, B_3)}{\xi(B_2, B_3)} \geq 0, \\
\Big(r((i,0), a_{12}) + \sum_{s' \in S} p(s'|(i,0), a_{12})h(s') \Big) - r((i,0), a_{22}) - \sum_{s' \in S} p(s'|(i,0), a_{22})h(s') \\
= \frac{\xi_2((i,0), B_2, B_3)}{\xi(B_2, B_3)} \geq 0.
\end{aligned}$$

Recall that $d_0(s) = a_{13}$ for $s = (0,j)$, where $j \in \{1, \ldots, B_3+2\}$. Then, we can show that, for all $j \in \{1, \ldots, B_3 + 2\}$,

$$\Big(r((0,j),a_{13}) + \sum_{s' \in S} p(s'|(0,j),a_{13})h(s')\Big) - r((0,j),a_{11}) - \sum_{s' \in S} p(s'|(0,j),a_{11})h(s')$$
$$= \frac{\xi_3((0,j),B_2,B_3)}{\xi(B_2,B_3)} \geq 0,$$
$$\Big(r((0,j),a_{13}) + \sum_{s' \in S} p(s'|(0,j),a_{13})h(s')\Big) - r((0,j),a_{33}) - \sum_{s' \in S} p(s'|(0,j),a_{33})h(s')$$
$$= \frac{\xi_4((0,j),B_2,B_3)}{\xi(B_2,B_3)} \geq 0.$$

Similarly, $d_0(s) = a_{13}$ for $s = (i, B_3 + 2)$, where $i \in \{1, \ldots, B_2\}$. We can show that, for all $i \in \{1, \ldots, B_2\}$,

$$\Big(r((i,B_3+2),a_{13}) + \sum_{s' \in S} p(s'|(i,B_3+2),a_{13})h(s')\Big) - r((i,B_3+2),a_{11})$$
$$- \sum_{s' \in S} p(s'|(i,B_3+2),a_{11})h(s') = \frac{\xi_5((i,B_3+2),B_2,B_3)}{\xi(B_2,B_3)} \geq 0,$$
$$\Big(r((i,B_3+2),a_{13}) + \sum_{s' \in S} p(s'|(i,B_3+2),a_{13})h(s')\Big) - r((i,B_3+2),a_{33})$$
$$- \sum_{s' \in S} p(s'|(i,B_3+2),a_{33})h(s') = \frac{\xi_6((i,B_3+2),B_2,B_3)}{\xi(B_2,B_3)} \geq 0.$$

For $s = (i,j)$, where $i \in \{1, \ldots, B_2+1\}$ and $j \in \{1, \ldots, B_3+1\}$, recall that $d_0(s) = a_{12}$. Some algebra shows that, for all $i \in \{1, \ldots, B_2 + 1\}$ and for all $j \in \{1, \ldots, B_3 + 1\}$,

$$\Big(r((i,j),a_{12}) + \sum_{s' \in S} p(s'|(i,j),a_{12})h(s')\Big) - r((i,j),a_{11}) - \sum_{s' \in S} p(s'|(i,j),a_{11})h(s')$$
$$= \frac{\xi_7((i,j),B_2,B_3)}{\xi(B_2,B_3)} \geq 0,$$
$$\Big(r((i,j),a_{12}) + \sum_{s' \in S} p(s'|(i,j),a_{12})h(s')\Big) - r((i,j),a_{22}) - \sum_{s' \in S} p(s'|(i,j),a_{22})h(s')$$
$$= \frac{\xi_8((i,j),B_2,B_3)}{\xi(B_2,B_3)} \geq 0,$$
$$\Big(r((i,j),a_{12}) + \sum_{s' \in S} p(s'|(i,j),a_{12})h(s')\Big) - r((i,j),a_{33}) - \sum_{s' \in S} p(s'|(i,j),a_{33})h(s')$$
$$= \frac{\xi_9((i,j),B_2,B_3)}{\xi(B_2,B_3)} \geq 0,$$

$$\Big(r((i,j),a_{12})+\sum_{s'\in S}p(s'|(i,j),a_{12})h(s')\Big)-r((i,j),a_{13})-\sum_{s'\in S}p(s'|(i,j),a_{13})h(s')$$
$$=\frac{\xi_{10}((i,j),B_2,B_3)}{\xi(B_2,B_3)}\geq 0.$$

Finally, $d_0(s)=a_{12}$ for $s=(B_2+2,j)$, where $j\in\{1,\ldots,B_3+1\}$. Some algebra shows that, for all $j\in\{1,\ldots,B_3+1\}$,

$$\Big(r((B_2+2,j),a_{12})+\sum_{s'\in S}p(s'|(B_2+2,j),a_{12})h(s')\Big)-r((B_2+2,j),a_{13})$$
$$-\sum_{s'\in S}p(s'|(B_2+2,j),a_{13})h(s')=\frac{\xi_{11}((B_2+2,j),B_2,B_3)}{\xi(B_2,B_3)}\geq 0.$$

When $B_2=B_3=0$, we obtain

$$\xi(0,0)=\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}+\mu_{11}\mu_{22}^2\mu_{23}^2+\mu_{11}^2\mu_{21}\mu_{22}^2+\mu_{11}^3\mu_{22}^2+\mu_{11}^2\mu_{22}^2\mu_{23}+\mu_{22}^2\mu_{23}^3$$
$$+\mu_{21}\mu_{22}^2\mu_{23}^2+2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2+3\mu_{11}^3\mu_{22}\mu_{23}+\mu_{11}^4\mu_{22}+\mu_{11}\mu_{22}\mu_{23}^3+\mu_{21}\mu_{22}\mu_{23}^3$$
$$+2\mu_{11}^2\mu_{22}\mu_{23}^2+\mu_{11}^3\mu_{21}\mu_{22}+3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}+\mu_{11}^3\mu_{21}\mu_{23}+\mu_{11}^4\mu_{23}+2\mu_{11}^3\mu_{23}^2$$
$$+\mu_{11}^2\mu_{23}^3+2\mu_{11}^2\mu_{21}\mu_{23}^2+\mu_{11}\mu_{21}\mu_{23}^3$$

$$\xi_1((1,0),0,0)=\mu_{11}\mu_{22}\mu_{23}(\mu_{11}+\mu_{21})(\mu_{11}^2+\mu_{11}\mu_{22}+2\mu_{11}\mu_{23}+\mu_{23}^2),$$

$$\xi_2((1,0),0,0)=\mu_{11}\mu_{22}^2\mu_{23}^2(\mu_{11}+\mu_{23}),$$

$$\xi_3((0,1),0,0)=\mu_{11}^2\mu_{22}\mu_{23}(\mu_{11}+\mu_{21})(\mu_{11}+\mu_{22}+\mu_{23}),$$

$$\xi_3((0,2),0,0)=\mu_{11}\mu_{22}\mu_{23}(\mu_{11}+\mu_{21})(\mu_{11}^2+\mu_{11}\mu_{22}+2\mu_{11}\mu_{23}+\mu_{22}\mu_{23}),$$

$$\xi_4((0,1),0,0)=\mu_{11}\mu_{22}\mu_{23}^2(\mu_{11}^2+\mu_{11}\mu_{22}+\mu_{11}\mu_{23}+\mu_{22}\mu_{23}),$$

$$\xi_4((0,2),0,0)=\mu_{11}\mu_{22}\mu_{23}^2(\mu_{11}+\mu_{22}),$$

$$\xi_7((1,1),0,0)=\mu_{11}\mu_{22}\mu_{23}(\mu_{11}+\mu_{21})(\mu_{11}+\mu_{23})(\mu_{11}+\mu_{22}+\mu_{23}),$$

$$\xi_8((1,1),0,0)=\xi_9((1,1),0,0)=\mu_{11}\mu_{22}^2\mu_{23}^3,$$

$$\xi_{10}((1,1),0,0)=\xi_{11}((2,1),0,0)=0.$$

When $B_2=1$ and $B_3=0$, we obtain

$$\xi(1,0)=\mu_{11}^5\mu_{21}\mu_{23}+3\mu_{11}^4\mu_{21}\mu_{23}^2+3\mu_{11}^3\mu_{22}\mu_{23}^3+\mu_{11}^5\mu_{21}\mu_{22}+3\mu_{11}^3\mu_{22}^2\mu_{23}^2$$
$$+\mu_{11}\mu_{22}^3\mu_{23}^3+6\mu_{11}^4\mu_{22}\mu_{23}^2+\mu_{11}^2\mu_{22}^3\mu_{23}^2+5\mu_{11}^5\mu_{22}\mu_{23}+\mu_{11}\mu_{22}^2\mu_{23}^4$$

$$+2\mu_{11}^4\mu_{21}\mu_{22}^2 + 2\mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{11}^2\mu_{21}\mu_{23}^4 + 3\mu_{11}^3\mu_{21}\mu_{23}^3 + \mu_{11}^3\mu_{22}^3\mu_{23}$$

$$+4\mu_{11}^4\mu_{22}^2\mu_{23} + 3\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{11}^2\mu_{22}\mu_{23}^4 + \mu_{11}^3\mu_{21}\mu_{22}^3 + \mu_{11}^6\mu_{22} + 2\mu_{11}^2\mu_{22}^2\mu_{23}^3$$

$$+\mu_{11}\mu_{22}^3\mu_{23}^3 + \mu_{11}^4\mu_{22}^3 + \mu_{22}^3\mu_{23}^4 + \mu_{11}^3\mu_{23}^4 + \mu_{11}^6\mu_{23} + 3\mu_{11}^5\mu_{23}^2 + 3\mu_{11}^4\mu_{23}^3$$

$$+\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23} + 2\mu_{11}^5\mu_{22}^2 + \mu_{11}\mu_{21}\mu_{22}\mu_{23}^4 + 5\mu_{11}^4\mu_{21}\mu_{22}\mu_{23} + 6\mu_{11}\mu_{21}\mu_{22}\mu_{23}^3$$

$$+4\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23} + 3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^3 + 2\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^3 + \mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^2,$$

$$\xi_1((1,0),1,0) = \mu_{11}^2\mu_{22}\mu_{23}(\mu_{11}^4 + \mu_{11}^3\mu_{21} + 2\mu_{11}^3\mu_{22} + 3\mu_{11}^3\mu_{23} + 2\mu_{11}^2\mu_{21}\mu_{22}$$

$$+3\mu_{11}^2\mu_{21}\mu_{23} + 2\mu_{11}^2\mu_{22}^2 + 2\mu_{11}^2\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{23}^3 + \mu_{11}\mu_{21}\mu_{22}^2 + 2\mu_{11}\mu_{21}\mu_{22}\mu_{23}$$

$$+\mu_{11}\mu_{23}^3 + 3\mu_{11}\mu_{21}\mu_{23}^2 + \mu_{21}\mu_{23}^3),$$

$$\xi_1((2,0),1,0) = \mu_{11}\mu_{22}\mu_{23}(\mu_{11}^5 + 3\mu_{11}^4\mu_{23} + 2\mu_{11}^4\mu_{22} + \mu_{11}^4\mu_{21} + 3\mu_{11}^3\mu_{23}^2 + \mu_{11}^3\mu_{22}^2$$

$$+2\mu_{11}^3\mu_{21}\mu_{22} + 3\mu_{11}^3\mu_{21}\mu_{23} + 2\mu_{11}^3\mu_{22}\mu_{23} + \mu_{11}^2\mu_{21}\mu_{22}^2 + \mu_{11}^2\mu_{23}^3 + 3\mu_{11}^2\mu_{21}\mu_{23}^2$$

$$+3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23} + 2\mu_{11}^2\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{22}^2\mu_{23} + \mu_{11}\mu_{21}\mu_{22}^2\mu_{23} + \mu_{11}\mu_{21}\mu_{23}^3$$

$$+\mu_{11}\mu_{22}\mu_{23}^3 + 2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2 + \mu_{21}\mu_{22}\mu_{23}^2),$$

$$\xi_2((1,0),1,0) = \mu_{11}\mu_{22}^2\mu_{23}^2(\mu_{11}^3 + \mu_{11}^2\mu_{22} + 2\mu_{11}^2\mu_{23} + \mu_{11}\mu_{22}\mu_{23} + \mu_{11}\mu_{23}^2 + \mu_{22}\mu_{23}^2),$$

$$\xi_2((2,0),1,0) = \mu_{11}\mu_{22}^3\mu_{23}^3(\mu_{11} + \mu_{23}),$$

$$\xi_3((0,1),1,0) = \mu_{11}^3\mu_{22}\mu_{23}(\mu_{11}^3 + \mu_{11}^2\mu_{21} + 2\mu_{11}^2\mu_{22} + 2\mu_{11}^2\mu_{23} + 2\mu_{11}\mu_{21}\mu_{23}$$

$$+2\mu_{11}\mu_{21}\mu_{22} + \mu_{11}\mu_{22}^2 + \mu_{11}\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23} + \mu_{21}\mu_{23}^2 + \mu_{21}\mu_{22}\mu_{23} + \mu_{21}\mu_{22}^2),$$

$$\xi_3((0,2),1,0) = \mu_{11}^2\mu_{22}\mu_{23}(\mu_{21}\mu_{22}^2\mu_{23} + \mu_{11}\mu_{21}\mu_{22}^2 + \mu_{11}\mu_{22}^2\mu_{23} + \mu_{11}^2\mu_{22}^2 + \mu_{11}^4$$

$$+3\mu_{11}\mu_{21}\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{22}\mu_{23} + 2\mu_{11}^2\mu_{21}\mu_{22} + 2\mu_{11}^3\mu_{22} + 2\mu_{11}\mu_{21}\mu_{23}^2 + 3\mu_{11}^3\mu_{23}$$

$$+2\mu_{11}^2\mu_{23}^2 + \mu_{11}^3\mu_{21} + 3\mu_{11}^2\mu_{21}\mu_{23}),$$

$$\xi_4((0,1),1,0) = \mu_{11}\mu_{22}\mu_{23}^2(2\mu_{11}^2\mu_{22}\mu_{23} + \mu_{22}^2\mu_{23}^2 + \mu_{11}^2\mu_{22}^2 + \mu_{11}^4 + 2\mu_{11}^3\mu_{22}$$

$$+2\mu_{11}^3\mu_{22} + \mu_{11}^2\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{22}^2\mu_{23}),$$

$$\xi_4((0,2),1,0) = \mu_{11}\mu_{22}\mu_{23}^2(\mu_{11}\mu_{22}\mu_{23} + 2\mu_{11}^2\mu_{22} + \mu_{11}^3 + \mu_{22}^2\mu_{22} + \mu_{11}^2\mu_{23} + \mu_{11}\mu_{22}^2),$$

$$\xi_5((1,2),1,0) = \mu_{11}\mu_{22}\mu_{23}(\mu_{11}^3\mu_{22}^2 + \mu_{11}\mu_{21}\mu_{22}^2\mu_{23} + \mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{11}^2\mu_{22}^2\mu_{23}$$

$$+\mu_{11}\mu_{22}^2\mu_{23}^2 + \mu_{11}^2\mu_{21}\mu_{22}^2 + 3\mu_{11}^3\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23} + 2\mu_{11}^4\mu_{22} + \mu_{11}^5$$

$$+2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2 + 2\mu_{11}^3\mu_{21}\mu_{22} + 2\mu_{11}^2\mu_{22}\mu_{23}^2 + 3\mu_{11}^4\mu_{23} + 3\mu_{11}^3\mu_{23}^2 + 3\mu_{11}^3\mu_{21}\mu_{23}$$

$$+\mu_{11}^4\mu_{21} + \mu_{11}^2\mu_{23}^3 + \mu_{11}\mu_{21}\mu_{23}^3 + 3\mu_{11}^2\mu_{21}\mu_{23}^2),$$

$$\xi_6((1,2),1,0) = \mu_{11}\mu_{22}^2\mu_{23}^4(\mu_{11} + \mu_{22}),$$

$$\xi_7((1,1),1,0) = \mu_{11}^2\mu_{22}\mu_{23}(\mu_{11}^4 + 2\mu_{11}^3\mu_{22} + \mu_{11}^3\mu_{21} + 3\mu_{11}^3\mu_{23} + 3\mu_{11}^2\mu_{21}\mu_{23}$$

$$+3\mu_{11}^2\mu_{23}^2 + \mu_{11}^2\mu_{22}^2 + 2\mu_{11}^2\mu_{21}\mu_{22} + 3\mu_{11}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{22}^2\mu_{23} + 3\mu_{11}\mu_{21}\mu_{23}^2$$

$$+\mu_{11}\mu_{23}^3 + 3\mu_{11}\mu_{21}\mu_{22}\mu_{23} + \mu_{11}\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{21}\mu_{22}^2 + \mu_{21}\mu_{23}^3 + \mu_{21}\mu_{22}\mu_{23}^2$$

$$+\mu_{21}\mu_{22}^2\mu_{23}),$$

$$\xi_7((1,1),1,0) = \mu_{11}\mu_{22}\mu_{23}(\mu_{11}^2\mu_{21}\mu_{22}^2 + \mu_{11}\mu_{21}\mu_{22}^2\mu_{23} + \mu_{11}^3\mu_{22}^2 + \mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{11}^5$$

$$+\mu_{11}\mu_{22}^2\mu_{23}^2 + \mu_{11}^2\mu_{22}^2\mu_{23} + \mu_{11}\mu_{22}\mu_{23}^3 + 3\mu_{11}^3\mu_{22}\mu_{23} + 2\mu_{11}^4\mu_{22} + \mu_{21}\mu_{22}\mu_{23}^3$$

$$+\mu_{11}^3\mu_{21}\mu_{22} + 2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2 + 2\mu_{11}^2\mu_{22}\mu_{23}^2 + 3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23} + \mu_{11}^3\mu_{21}\mu_{23}$$

$$+\mu_{11}\mu_{21}\mu_{23}^3 + 3\mu_{11}^2\mu_{21}\mu_{23}^2 + \mu_{11}^4\mu_{21} + 3\mu_{11}^3\mu_{23}^2 + \mu_{11}^2\mu_{23}^3 + 3\mu_{11}^4\mu_{23}),$$

$$\xi_8((1,1),1,0) = \mu_{11}\mu_{22}^2\mu_{23}^2(\mu_{11}^2 + \mu_{22}\mu_{23} + \mu_{11}\mu_{23} + \mu_{11}\mu_{22}),$$

$$\xi_8((2,1),1,0) = \mu_{11}\mu_{22}^3\mu_{23}^4,$$

$$\xi_9((1,1),1,0) = \mu_{11}\mu_{22}^2\mu_{23}^2(\mu_{11}^2 + \mu_{22}\mu_{23} + \mu_{11}\mu_{23} + \mu_{11}\mu_{22}),$$

$$\xi_9((2,1),1,0) = \mu_{11}\mu_{22}^3\mu_{23}^4,$$

$$\xi_{10}((1,1),1,0) = 0,$$

$$\xi_{10}((2,1),1,0) = 0,$$

$$\xi_{11}((3,1),1,0) = 0.$$

When $B_2 = 0$ and $B_3 = 1$, we obtain

$$\xi(0,1) = \mu_{21}\mu_{22}^3\mu_{23}^3 + 2\mu_{11}\mu_{22}^2\mu_{23}^4 + \mu_{11}^5\mu_{21}\mu_{23} + 3\mu_{11}^4\mu_{21}\mu_{23}^2 + 3\mu_{11}^3\mu_{21}\mu_{23}^3$$

$$+\mu_{11}^2\mu_{22}^3\mu_{23}^2 + \mu_{11}^3\mu_{22}^3\mu_{23} + 4\mu_{11}^3\mu_{22}^2\mu_{23}^2 + 5\mu_{11}^5\mu_{22}\mu_{23} + \mu_{11}^3\mu_{21}\mu_{22}^3 + \mu_{21}\mu_{22}^2\mu_{23}^4$$

$$+\mu_{11}^5\mu_{21}\mu_{22} + 2\mu_{11}^2\mu_{22}\mu_{23}^4 + 5\mu_{11}^4\mu_{22}^2\mu_{23} + \mu_{11}^2\mu_{21}\mu_{23}^4 + \mu_{11}\mu_{22}^3\mu_{23}^3 + 3\mu_{11}^2\mu_{22}^2\mu_{23}^2$$

$$+4\mu_{11}^3\mu_{22}\mu_{23}^3 + 7\mu_{11}^4\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23} + 2\mu_{11}^4\mu_{21}\mu_{22}^2 + \mu_{11}\mu_{21}\mu_{22}^3\mu_{23}$$

$$+2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^4 + 4\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^2 + 4\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^3 + 3\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^3 + \mu_{11}^6\mu_{23}$$

$+5\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23} + 7\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^2 + 5\mu_{11}^4\mu_{21}\mu_{22}\mu_{23} + \mu_{22}^3\mu_{23}^4 + \mu_{11}^6\mu_{22}$

$+\mu_{11}^4\mu_{22}^3 + \mu_{11}^3\mu_{23}^4 + 2\mu_{11}^5\mu_{22}^2 + 3\mu_{11}^5\mu_{23}^2 + 3\mu_{11}^4\mu_{23}^3,$

$\xi_1((1,0),0,1) = \mu_{11}^3\mu_{22}\mu_{23}(\mu_{11}^5 + 2\mu_{11}^4\mu_{22} + \mu_{11}^4\mu_{21} + 3\mu_{11}^4\mu_{23} + 2\mu_{11}^3\mu_{21}\mu_{22}$

$+3\mu_{11}^3\mu_{21}\mu_{23} + 3\mu_{11}^3\mu_{22}\mu_{23} + \mu_{11}^3\mu_{22}^2 + 3\mu_{11}^3\mu_{23}^2 + 3\mu_{11}^2\mu_{21}\mu_{23}^2 + \mu_{11}^2\mu_{22}\mu_{23}^2$

$+3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23} + \mu_{11}^2\mu_{21}\mu_{22}^2 + \mu_{11}^2\mu_{23}^3 + \mu_{11}\mu_{22}\mu_{23}^3 + \mu_{11}\mu_{21}\mu_{23}^3 + \mu_{21}\mu_{22}\mu_{23}^2$

$\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2)$

$\xi_2((1,0),0,1) = \mu_{11}\mu_{22}^2\mu_{23}^2(\mu_{11}^3 + \mu_{11}^2\mu_{22} + 2\mu_{11}^2\mu_{23} + \mu_{11}\mu_{22}\mu_{23} + \mu_{11}\mu_{23}^2 + \mu_{22}\mu_{23}^2),$

$\xi_3((0,1),0,1) = \mu_{11}^3\mu_{22}\mu_{23}(\mu_{11}^3 + \mu_{11}^2\mu_{21} + 2\mu_{11}^2\mu_{22} + 2\mu_{11}^2\mu_{23} + 2\mu_{11}\mu_{21}\mu_{23}$

$+2\mu_{11}\mu_{21}\mu_{22} + \mu_{11}\mu_{22}^2 + \mu_{11}\mu_{23}^2 + 2\mu_{11}\mu_{22}\mu_{23} + \mu_{21}\mu_{23}^2 + 2\mu_{21}\mu_{22}\mu_{23} + \mu_{21}\mu_{22}^2),$

$\xi_3((0,2),0,1) = \mu_{11}^2\mu_{22}\mu_{23}(\mu_{21}\mu_{22}^2\mu_{23} + 2\mu_{11}\mu_{21}\mu_{23}^2 + \mu_{21}\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{22}^2 + \mu_{11}^4$

$+4\mu_{11}\mu_{21}\mu_{22}\mu_{23} + \mu_{11}\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{22}^2\mu_{23} + 2\mu_{11}^3\mu_{22} + \mu_{11}\mu_{21}\mu_{22}^2 + \mu_{11}^3\mu_{21}$

$+2\mu_{11}^2\mu_{23}^2 + 3\mu_{11}^3\mu_{23} + 3\mu_{11}^2\mu_{21}\mu_{23}) + 2\mu_{11}^2\mu_{21}\mu_{22} + 4\mu_{11}^2\mu_{22}\mu_{23}),$

$\xi_3((0,3),0,1) = \mu_{11}\mu_{22}\mu_{23}(\mu_{11}^5 + 2\mu_{11}^4\mu_{22} + \mu_{11}^4\mu_{21} + 3\mu_{11}^4\mu_{23} + 3\mu_{11}^3\mu_{21}\mu_{23}$

$+2\mu_{11}^3\mu_{21}\mu_{22} + 4\mu_{11}^3\mu_{22}\mu_{23} + \mu_{11}^3\mu_{22}^2 + 3\mu_{11}^3\mu_{23}^2 + 3\mu_{11}^2\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{22}^2\mu_{23}$

$+4\mu_{11}^2\mu_{21}\mu_{22}\mu_{23} + \mu_{11}^2\mu_{21}\mu_{22}^2 + 3\mu_{11}^2\mu_{21}\mu_{23}^2 + \mu_{11}\mu_{21}\mu_{22}^2\mu_{23} + \mu_{11}\mu_{22}^2\mu_{23}^2$

$+3\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2 + \mu_{21}\mu_{22}^2\mu_{23}^2),$

$\xi_4((0,1),0,1) = \mu_{11}\mu_{22}\mu_{23}^2(\mu_{11}^4 + 2\mu_{11}^3\mu_{22} + 2\mu_{11}^3\mu_{23} + \mu_{11}^2\mu_{22}^2 + \mu_{11}\mu_{22}^2\mu_{23} + \mu_{11}^2\mu_{23}^2$

$+2\mu_{11}\mu_{22}\mu_{23}^2 + 3\mu_{11}^2\mu_{22}\mu_{23} + \mu_{22}^2\mu_{23}^2),$

$\xi_4((0,2),0,1) = \mu_{11}\mu_{22}\mu_{23}^3(\mu_{11}^3 + 2\mu_{11}^2\mu_{22} + \mu_{11}^2\mu_{23} + \mu_{11}\mu_{22}^2 + 2\mu_{11}\mu_{22}\mu_{23} + \mu_{22}^2\mu_{23}),$

$\xi_4((0,3),0,1) = \mu_{11}\mu_{22}\mu_{23}^4(\mu_{11}^2 + 2\mu_{11}\mu_{22} + \mu_{22}^2),$

$\xi_7((1,1),0,1) = \mu_{11}\mu_{22}\mu_{23}(\mu_{11}^5 + 2\mu_{11}^4\mu_{22} + \mu_{11}^4\mu_{21} + 3\mu_{11}^4\mu_{23} + 2\mu_{11}^3\mu_{21}\mu_{22}$

$+3\mu_{11}^3\mu_{23}^2 + 4\mu_{11}^3\mu_{22}\mu_{23} + \mu_{11}^3\mu_{22}^2 + 3\mu_{11}^3\mu_{21}\mu_{23} + \mu_{11}^2\mu_{22}^2\mu_{23} + 2\mu_{11}^2\mu_{22}\mu_{23}^2$

$+\mu_{11}^2\mu_{23}^3 + \mu_{11}^2\mu_{21}\mu_{22}^2 + 4\mu_{11}^2\mu_{21}\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{21}\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23}^3 + 2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2$

$+\mu_{11}\mu_{21}\mu_{22}^2\mu_{23} + \mu_{11}\mu_{21}\mu_{23}^3 + \mu_{21}\mu_{22}\mu_{23}^3),$

$$\xi_7((1,2),0,1) = \mu_{11}\mu_{22}\mu_{23}(\mu_{11}^5 + 2\mu_{11}^4\mu_{22} + \mu_{11}^4\mu_{21} + 3\mu_{11}^4\mu_{23} + 2\mu_{11}^3\mu_{21}\mu_{22}$$
$$+3\mu_{11}^3\mu_{23}^2 + 4\mu_{11}^3\mu_{22}\mu_{23} + \mu_{11}^3\mu_{22}^2 + 3\mu_{11}^3\mu_{21}\mu_{23} + \mu_{11}^2\mu_{22}^2\mu_{23} + 3\mu_{11}^2\mu_{22}\mu_{23}^2$$
$$+\mu_{11}^2\mu_{23}^3 + \mu_{11}^2\mu_{21}\mu_{22}^2 + 4\mu_{11}^2\mu_{21}\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{21}\mu_{23}^2 + \mu_{11}\mu_{22}^2\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23}^3$$
$$+\mu_{11}\mu_{21}\mu_{22}^2\mu_{23} + 3\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{21}\mu_{23}^3 + \mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{21}\mu_{22}\mu_{23}^3),$$
$$\xi_8((1,1),0,1) = \mu_{11}\mu_{22}^2\mu_{23}^2(\mu_{11}^2 + \mu_{11}\mu_{22} + \mu_{11}\mu_{23} + \mu_{22}\mu_{23}),$$
$$\xi_8((1,2),0,1) = \mu_{11}\mu_{22}^2\mu_{23}^4(\mu_{11} + \mu_{22}),$$
$$\xi_9((1,1),0,1) = \mu_{11}\mu_{22}^2\mu_{23}^2(\mu_{11}^2 + \mu_{11}\mu_{22} + 2\mu_{11}\mu_{23} + \mu_{22}\mu_{23}),$$
$$\xi_9((1,2),0,1) = \mu_{11}\mu_{22}^2\mu_{23}^4(\mu_{11} + \mu_{22}),$$
$$\xi_{10}((1,1),0,1) = \mu_{11}^2\mu_{22}^2\mu_{23}^4,$$
$$\xi_{10}((1,1),0,1) = 0,$$
$$\xi_{11}((2,1),0,1) = \mu_{11}\mu_{22}^2\mu_{23}^4(\mu_{11} + \mu_{22}),$$
$$\xi_{11}((1,2),0,1) = 0.$$

When $B_2 = B_3 = 1$, we obtain

$$\xi(1,1) = 4\mu_{11}^8\mu_{22}^2 + 4\mu_{11}^6\mu_{22}^4 + \mu_{22}^5\mu_{23}^5 + 3\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^5 + \mu_{11}^8\mu_{21}\mu_{23} + \mu_{21}\mu_{22}^5\mu_{23}^4$$
$$+7\mu_{11}^5\mu_{22}^4\mu_{23} + \mu_{11}\mu_{21}\mu_{22}^5\mu_{23}^3 + \mu_{11}^3\mu_{21}\mu_{22}^5\mu_{23} + 4\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^5 + \mu_{11}^5\mu_{23}^5$$
$$+4\mu_{11}^7\mu_{21}\mu_{23}^2 + 4\mu_{11}^6\mu_{23}^4 + 6\mu_{11}^7\mu_{23}^3 + 4\mu_{11}^2\mu_{22}^4\mu_{23}^4 + 6\mu_{11}^4\mu_{22}^4\mu_{23}^2 + 5\mu_{11}^3\mu_{22}^4\mu_{23}^3$$
$$+3\mu_{11}\mu_{22}^4\mu_{23}^5 + 4\mu_{11}^5\mu_{21}\mu_{22}^4 + 8\mu_{11}^3\mu_{22}^3\mu_{23}^4 + 4\mu_{11}^2\mu_{22}^3\mu_{23}^5 + 17\mu_{11}^6\mu_{22}^3\mu_{23}$$
$$+19\mu_{11}^5\mu_{22}^3\mu_{23}^2 + 13\mu_{11}^4\mu_{22}^3\mu_{23}^3 + 6\mu_{11}^6\mu_{21}\mu_{22}^3 + 4\mu_{11}^7\mu_{21}\mu_{22}^2 + 11\mu_{11}^4\mu_{22}^2\mu_{23}^4$$
$$+4\mu_{11}^3\mu_{22}^2\mu_{22}^5 + 18\mu_{11}^7\mu_{22}^2\mu_{23} + 28\mu_{11}^6\mu_{22}^2\mu_{23}^2 + 23\mu_{11}^5\mu_{22}^2\mu_{23}^3 + \mu_{11}^8\mu_{21}\mu_{22}$$
$$+19\mu_{11}^6\mu_{22}\mu_{23}^3 + 18\mu_{11}^7\mu_{22}\mu_{23}^2 + 8\mu_{11}^8\mu_{22}\mu_{23} + 11\mu_{11}^5\mu_{22}\mu_{23}^4 + 3\mu_{11}^4\mu_{22}\mu_{23}^5$$
$$+\mu_{11}\mu_{22}^5\mu_{23}^4 + 4\mu_{11}^8\mu_{23}^2 + \mu_{11}^9\mu_{23} + 4\mu_{11}^5\mu_{21}\mu_{23}^4 + \mu_{11}^4\mu_{21}\mu_{23}^5 + 6\mu_{11}^6\mu_{21}\mu_{23}^3$$
$$+6\mu_{11}^7\mu_{22}^3 + \mu_{11}^9\mu_{22} + \mu_{21}\mu_{22}^4\mu_{23}^5 + \mu_{11}^2\mu_{22}^5\mu_{23}^3 + \mu_{11}^4\mu_{21}\mu_{22}^5 + 3\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^5$$
$$+\mu_{11}^5\mu_{22}^5 + \mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^5 + 19\mu_{11}^4\mu_{21}\mu_{22}^3\mu_{23}^2 + 17\mu_{11}^5\mu_{21}\mu_{22}^3\mu_{23}^2 + 13\mu_{11}^3\mu_{21}\mu_{22}^3\mu_{23}^3$$
$$+23\mu_{11}^4\mu_{21}\mu_{22}^2\mu_{23}^3 + 6\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}^4 + 8\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23}^4 + 11\mu_{11}^4\mu_{21}\mu_{22}\mu_{23}^4$$

$$+19\mu_{11}^5\mu_{21}\mu_{22}\mu_{23}^3 + 18\mu_{11}^6\mu_{21}\mu_{22}\mu_{23}^2 + 8\mu_{11}^7\mu_{21}\mu_{22}\mu_{23} + 28\mu_{11}^5\mu_{21}\mu_{22}^2\mu_{23}$$

$$+7\mu_{11}^4\mu_{21}\mu_{22}^4\mu_{23} + 4\mu_{11}\mu_{21}\mu_{22}^4\mu_{23}^4 + 11\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}^4 + 18\mu_{11}^6\mu_{21}\mu_{22}^2\mu_{23}$$

$$+5\mu_{11}^2\mu_{21}\mu_{22}^4\mu_{23}^3 + \mu_{11}^3\mu_{22}^5\mu_{23}^2 + \mu_{11}^4\mu_{22}^5\mu_{23},$$

$$\xi_1((1,0),1,1) = \mu_{11}^2\mu_{22}\mu_{23}(3\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^2 + 6\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^3 + 10\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}$$

$$+12\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^2 + 11\mu_{11}^4\mu_{21}\mu_{22}\mu_{23} + 7\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^3 + 2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^4 + \mu_{11}^6\mu_{21}$$

$$+\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^3 + 4\mu_{11}^3\mu_{21}\mu_{23}^3 + 4\mu_{11}^5\mu_{21}\mu_{23} + \mu_{11}^2\mu_{21}\mu_{23}^4 + 6\mu_{11}^4\mu_{21}\mu_{23}^2 + 4\mu_{11}^6\mu_{23}$$

$$+4\mu_{11}^4\mu_{23}^3 + 6\mu_{11}^5\mu_{23}^2 + \mu_{11}^7 + \mu_{11}^3\mu_{23}^4 + 3\mu_{11}^3\mu_{22}^3\mu_{23} + \mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{11}\mu_{22}^2\mu_{23}^4$$

$$+6\mu_{11}^3\mu_{22}^2\mu_{23}^2 + 10\mu_{11}^4\mu_{22}^2\mu_{23} + \mu_{11}^2\mu_{22}^2\mu_{23}^3 + 11\mu_{11}^5\mu_{22}\mu_{23} + 12\mu_{11}^4\mu_{22}\mu_{23}^2 + \mu_{11}^3\mu_{22}^4$$

$$+2\mu_{11}^2\mu_{22}\mu_{23}^4 + 7\mu_{11}^3\mu_{22}\mu_{23}^3 + 4\mu_{11}^3\mu_{21}\mu_{22}^3 + 4\mu_{11}^4\mu_{22}^3 + 6\mu_{11}^4\mu_{21}\mu_{22}^2 + \mu_{11}^2\mu_{21}\mu_{22}^4$$

$$+4\mu_{11}^5\mu_{21}\mu_{22} + 4\mu_{11}^6\mu_{22} + 6\mu_{11}^5\mu_{22}^2),$$

$$\xi_1((2,0),1,1) = \mu_{11}\mu_{22}\mu_{23}(\mu_{11}^3\mu_{22}^4\mu_{23} + 4\mu_{11}^5\mu_{23}^3 + 4\mu_{11}^7\mu_{23} + 4\mu_{11}^4\mu_{22}^4 + \mu_{11}^7\mu_{21}$$

$$+3\mu_{11}^3\mu_{22}^3\mu_{23}^2 + 6\mu_{11}^4\mu_{22}^3\mu_{23} + 12\mu_{11}^4\mu_{22}^2\mu_{23}^2 + 13\mu_{11}^5\mu_{22}^2\mu_{23}^3 + 6\mu_{11}^3\mu_{22}^2\mu_{23}^3$$

$$+15\mu_{11}^5\mu_{22}\mu_{23}^2 + 12\mu_{11}^6\mu_{22}\mu_{23} + 10\mu_{11}^4\mu_{22}\mu_{23}^3 + \mu_{11}\mu_{22}^3\mu_{23}^4 + \mu_{11}^4\mu_{23}^3 + 6\mu_{11}^6\mu_{23}^2$$

$$+3\mu_{11}^3\mu_{22}\mu_{23}^4 + 3\mu_{11}^2\mu_{22}^2\mu_{23}^4 + \mu_{11}^8 + \mu_{11}^2\mu_{22}^3\mu_{23}^3 + 4\mu_{11}^5\mu_{23}^3 + 6\mu_{11}^6\mu_{22}^2 + 4\mu_{11}^7\mu_{22}$$

$$+3\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23}^2 + \mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^3 + 6\mu_{11}^3\mu_{21}\mu_{22}^3\mu_{23}^1 + 6\mu_{11}^5\mu_{21}\mu_{22}^2 + \mu_{21}\mu_{22}^3\mu_{23}^4$$

$$+3\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^4 + 12\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + 13\mu_{11}^4\mu_{21}\mu_{22}^2\mu_{23} + 6\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^3$$

$$+12\mu_{11}^5\mu_{21}\mu_{22}\mu_{23} + 15\mu_{11}^4\mu_{21}\mu_{22}\mu_{23}^2 + 3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^4 + 10\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^3$$

$$+\mu_{11}^2\mu_{21}\mu_{22}^4\mu_{23} + \mu_{11}^3\mu_{21}\mu_{22}^4 + 4\mu_{11}^4\mu_{21}\mu_{22}^3 + 6\mu_{11}^5\mu_{21}\mu_{23}^2 + \mu_{11}^3\mu_{21}\mu_{23}^4 + 4\mu_{11}^4\mu_{21}\mu_{23}^3$$

$$+4\mu_{11}^6\mu_{21}\mu_{22} + 4\mu_{11}^6\mu_{21}\mu_{23}),$$

$$\xi_2((1,0),1,1) = \mu_{11}\mu_{22}^2\mu_{23}^2(\mu_{22}^3\mu_{23}^3 + \mu_{11}^2\mu_{22}^3\mu_{23} + 4\mu_{11}^2\mu_{22}^2\mu_{23}^2 + 5\mu_{11}^3\mu_{22}^2\mu_{23} + \mu_{11}^3\mu_{22}^3$$

$$+7\mu_{11}^3\mu_{22}\mu_{23}^2 + 7\mu_{11}^4\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{22}\mu_{23}^3 + 3\mu_{11}\mu_{22}^2\mu_{23}^3 + \mu_{11}^3\mu_{23}^3 + 3\mu_{11}^5\mu_{23}$$

$$+3\mu_{11}^4\mu_{23}^2 + 3\mu_{11}^4\mu_{22}^2 + 3\mu_{11}^5\mu_{22} + \mu_{11}^6 + \mu_{11}\mu_{22}^2\mu_{23}^3),$$

$$\xi_2((2,0),1,1) = \mu_{11}\mu_{22}^3\mu_{23}^3(\mu_{11}^2\mu_{22}^2 + \mu_{11}\mu_{22}^2\mu_{23} + \mu_{22}^2\mu_{23}^2 + 3\mu_{11}^2\mu_{22}\mu_{23} + 2\mu_{11}^3\mu_{22}$$

$$+2\mu_{11}\mu_{22}\mu_{23}^2 + \mu_{11}^4 + \mu_{11}^2\mu_{23}^2 + 2\mu_{11}^3\mu_{23}),$$

$$
\begin{aligned}
\xi_3((0,1),1,1) &= \mu_{11}^4\mu_{22}\mu_{23}(\mu_{11}^5+\mu_{11}^3\mu_{23}^2+3\mu_{11}^4\mu_{23}+8\mu_{11}^3\mu_{22}\mu_{23}+6\mu_{11}^2\mu_{22}\mu_{23}^2\\
&+\mu_{11}^4\mu_{21}+\mu_{21}\mu_{22}^4+\mu_{11}^2\mu_{23}^3+4\mu_{11}^3\mu_{22}\mu_{23}+\mu_{11}\mu_{21}\mu_{23}^3+3\mu_{11}^3\mu_{21}\mu_{23}+4\mu_{11}^2\mu_{22}^3\\
&+3\mu_{11}^2\mu_{21}\mu_{23}^2+4\mu_{11}^4\mu_{22}+6\mu_{11}^2\mu_{21}\mu_{22}^2+4\mu_{11}\mu_{21}\mu_{22}^3+7\mu_{11}^2\mu_{22}^2\mu_{23}+\mu_{11}\mu_{22}^4\\
&+2\mu_{11}\mu_{22}^3\mu_{23}+7\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}+2\mu_{21}\mu_{22}^3\mu_{23}+\mu_{21}\mu_{21}^2\mu_{23}^2+2\mu_{21}\mu_{22}\mu_{23}^3\\
&+2\mu_{11}\mu_{22}\mu_{23}^3+6\mu_{11}\mu_{21}\mu_{22}\mu_{23}^2+8\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}+6\mu_{11}^3\mu_{22}^2+3\mu_{11}\mu_{22}^2\mu_{23}^2),\\
\xi_3((0,2),1,1) &= \mu_{11}^3\mu_{22}\mu_{23}(6\mu_{11}^4\mu_{22}^2+\mu_{11}^2\mu_{23}^4+11\mu_{11}^3\mu_{22}\mu_{23}^2+4\mu_{11}^5\mu_{22}+4\mu_{11}^2\mu_{22}\mu_{23}^3\\
&+\mu_{11}^5\mu_{21}+\mu_{21}\mu_{22}^2\mu_{23}^3+\mu_{11}\mu_{22}^2\mu_{23}^3+4\mu_{11}^3\mu_{22}^3+7\mu_{11}^2\mu_{22}^2\mu_{23}^2+\mu_{21}\mu_{22}^3\mu_{23}^2\\
&+\mu_{11}\mu_{22}^3\mu_{23}^2+\mu_{11}\mu_{22}^4\mu_{23}+12\mu_{11}^4\mu_{22}\mu_{23}+6\mu_{11}^2\mu_{22}^3\mu_{23}+\mu_{21}\mu_{22}^4\mu_{23}+13\mu_{11}^3\mu_{22}^2\mu_{23}\\
&+2\mu_{11}^3\mu_{23}^3+2\mu_{11}^2\mu_{21}\mu_{23}^3+5\mu_{11}^4\mu_{23}^2+5\mu_{11}^3\mu_{21}\mu_{23}^2+4\mu_{11}^4\mu_{21}\mu_{23}+\mu_{11}^6+4\mu_{11}^5\mu_{23}\\
&+4\mu_{11}\mu_{21}\mu_{22}\mu_{23}^3+7\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^2+11\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^2+6\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}+\mu_{11}\mu_{21}\mu_{22}^4\\
&+12\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}+12\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}+4\mu_{11}^4\mu_{21}\mu_{22}+4\mu_{11}^2\mu_{21}\mu_{22}^3+6\mu_{11}^3\mu_{21}\mu_{22}^2),\\
\xi_3((0,3),1,1) &= \mu_{11}^2\mu_{22}\mu_{23}(4\mu_{11}^6\mu_{23}+4\mu_{11}^5\mu_{21}\mu_{23}+3\mu_{11}^3\mu_{21}\mu_{23}^3+\mu_{11}^2\mu_{21}\mu_{22}^4+\mu_{11}^7\\
&+6\mu_{11}^5\mu_{23}^2+6\mu_{11}^4\mu_{21}\mu_{23}^2+5\mu_{11}^2\mu_{22}^3\mu_{23}^2+3\mu_{11}^4\mu_{23}^3+4\mu_{11}^3\mu_{21}\mu_{22}^3+6\mu_{11}^4\mu_{21}\mu_{22}^2\\
&+4\mu_{11}^5\mu_{21}\mu_{22}+4\mu_{11}^4\mu_{22}^3+6\mu_{11}^5\mu_{22}^2+\mu_{11}^3\mu_{22}^4+2\mu_{11}^2\mu_{22}^2\mu_{23}^3+6\mu_{11}^3\mu_{22}\mu_{23}^3\\
&+15\mu_{11}^4\mu_{22}\mu_{23}^2+\mu_{11}\mu_{22}^4\mu_{22}+13\mu_{11}^3\mu_{22}^2\mu_{23}^2+\mu_{21}\mu_{22}^4\mu_{23}^2+6\mu_{11}^3\mu_{22}^3\mu_{23}+\mu_{11}^2\mu_{22}^4\mu_{23} \qquad (53)\\
&+13\mu_{11}^4\mu_{22}^2\mu_{23}+12\mu_{11}^5\mu_{22}\mu_{23}+2\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^3+6\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^3+15\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^2\\
&+13\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^2+5\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^2+6\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23}+12\mu_{11}^4\mu_{21}\mu_{22}\mu_{23}+\mu_{11}^6\mu_{21}\\
&+13\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}+\mu_{11}\mu_{21}\mu_{22}^4\mu_{23}+4\mu_{11}^6\mu_{22}),\\
\xi_4((0,1),1,1) &= \mu_{11}\mu_{22}\mu_{23}^2(\mu_{11}^7+\mu_{11}^2\mu_{22}^4\mu_{23}+6\mu_{11}^5\mu_{22}^2+10\mu_{11}^4\mu_{22}^2\mu_{23}+8\mu_{11}^4\mu_{22}\mu_{23}^2\\
&+8\mu_{11}^3\mu_{22}^2\mu_{23}^2+4\mu_{11}^2\mu_{22}^2\mu_{23}^3+5\mu_{11}^3\mu_{22}^3\mu_{23}+9\mu_{11}^5\mu_{22}\mu_{23}+3\mu_{11}^6\mu_{23}+3\mu_{11}^5\mu_{23}^2\\
&+\mu_{11}^4\mu_{23}^3+4\mu_{11}^6\mu_{22}+3\mu_{11}^3\mu_{22}\mu_{23}^3+3\mu_{11}\mu_{22}^3\mu_{23}^3+4\mu_{11}^4\mu_{22}^3+\mu_{11}\mu_{22}^4\mu_{23}^2+\mu_{22}^4\mu_{23}^3\\
&+\mu_{11}^3\mu_{22}^4+4\mu_{11}^2\mu_{22}^3\mu_{23}^2),\\
\xi_4((0,2),1,1) &= \mu_{11}\mu_{22}\mu_{23}^3(\mu_{11}^6+2\mu_{11}^5\mu_{23}+3\mu_{11}\mu_{22}^3\mu_{23}^2+4\mu_{11}^3\mu_{22}^3+\mu_{11}^2\mu_{22}^4+\mu_{11}^5\mu_{22}\\
&+6\mu_{11}^4\mu_{22}^2+4\mu_{11}^2\mu_{22}^2\mu_{23}^2+\mu_{22}^4\mu_{23}^2+\mu_{11}\mu_{22}^4\mu_{23}+7\mu_{11}^3\mu_{22}^2\mu_{23}+3\mu_{11}^3\mu_{22}\mu_{23}^2
\end{aligned}
$$

$+6\mu_{11}^4\mu_{22}\mu_{23} + 4\mu_{11}^2\mu_{22}^3\mu_{23} + \mu_{11}^4\mu_{23}^3),$

$\xi_4((0,3),1,1) = \mu_{11}\mu_{22}\mu_{23}^4(\mu_{11}^5 + \mu_{11}\mu_{22}^4 + 4\mu_{11}^2\mu_{22}^3 + \mu_{11}^4\mu_{23} + 3\mu_{11}\mu_{22}^3\mu_{23} + 6\mu_{11}^3\mu_{22}^2$

$+3\mu_{11}^3\mu_{22}\mu_{23} + \mu_{22}^4\mu_{23} + 4\mu_{11}^4\mu_{22} + 4\mu_{11}^2\mu_{22}^2\mu_{23}),$

$\xi_5((1,3),1,1) = \mu_{11}\mu_{22}\mu_{23}(10\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^3 + 4\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^3 + 8\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^3 + \mu_{11}^8$

$+15\mu_{11}^4\mu_{21}\mu_{22}\mu_{23}^2 + 5\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23}^2 + 13\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{11}\mu_{21}\mu_{22}^4\mu_{23}^2 + \mu_{11}^3\mu_{21}\mu_{23}^4$

$+4\mu_{11}^4\mu_{21}\mu_{23}^3 + 6\mu_{11}^5\mu_{21}\mu_{23}^2 + \mu_{21}\mu_{22}^4\mu_{23}^3 + 4\mu_{11}^7\mu_{22} + 4\mu_{11}^5\mu_{22}^3 + \mu_{11}^7\mu_{23} + 6\mu_{11}^6\mu_{22}^2$

$+\mu_{11}^4\mu_{22}^4 + 4\mu_{11}^6\mu_{21}\mu_{22} + \mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{11}^4\mu_{23}^4 + 4\mu_{11}^5\mu_{23}^3 + 4\mu_{11}^6\mu_{21}\mu_{23}$

$+4\mu_{11}^7\mu_{23} + 2\mu_{11}^3\mu_{22}\mu_{23}^4 + \mu_{11}^2\mu_{22}^2\mu_{23}^4 + 4\mu_{11}^2\mu_{22}^3\mu_{23}^3 + 8\mu_{11}^3\mu_{22}^2\mu_{23}^3 + 10\mu_{11}^4\mu_{22}\mu_{23}^3$

$+4\mu_{11}^4\mu_{21}\mu_{23}^3 + 6\mu_{11}^5\mu_{21}\mu_{22}^2 + \mu_{11}^3\mu_{21}\mu_{22}^4 + \mu_{11}\mu_{22}^4\mu_{23}^3 + \mu_{11}^2\mu_{22}^4\mu_{23}^2 + 13\mu_{11}^4\mu_{22}^2\mu_{23}^2$

$+15\mu_{11}^5\mu_{22}\mu_{23}^2 + 5\mu_{11}^3\mu_{22}^3\mu_{23}^2 + 6\mu_{11}^4\mu_{22}^3\mu_{23} + 13\mu_{11}^5\mu_{22}^2\mu_{23} + \mu_{11}^3\mu_{22}^4\mu_{23} + \mu_{11}^6\mu_{23}^2$

$+12\mu_{11}^6\mu_{22}\mu_{23} + 12\mu_{11}^5\mu_{21}\mu_{22}\mu_{23} + 6\mu_{11}^3\mu_{21}\mu_{22}^3\mu_{23} + 13\mu_{11}^4\mu_{21}\mu_{22}^2\mu_{23}$

$+\mu_{11}^2\mu_{21}\mu_{22}^4\mu_{23}^2 + 2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^4),$

$\xi_6((1,3),1,1) = \mu_{11}\mu_{22}^2\mu_{23}^5(\mu_{11}^3 + 3\mu_{11}\mu_{22}^2 + 3\mu_{11}^2\mu_{22} + \mu_{23}^3),$

$\xi_7((1,1),1,1) = \mu_{11}^2\mu_{22}\mu_{23}(\mu_{11}^3\mu_{23}^4 + \mu_{11}^3\mu_{22}^4 + 4\mu_{11}^4\mu_{23}^3 + 4\mu_{11}^6\mu_{22} + 6\mu_{11}^5\mu_{22}^2 + \mu_{11}^7$

$+4\mu_{11}^5\mu_{21}\mu_{22} + 6\mu_{11}^4\mu_{21}\mu_{23}^3 + 6\mu_{11}^5\mu_{23}^2 + 4\mu_{11}^6\mu_{23} + 4\mu_{11}^4\mu_{22}^3 + 4\mu_{11}^5\mu_{21}\mu_{23}$

$+3\mu_{11}^3\mu_{21}\mu_{23}^3 + 4\mu_{11}^3\mu_{21}\mu_{22}^3 + 6\mu_{11}^4\mu_{21}\mu_{22}^2 + 14\mu_{11}^4\mu_{22}\mu_{23}^2 + 10\mu_{11}^3\mu_{22}^2\mu_{23}^2$

$+13\mu_{11}^4\mu_{22}^2\mu_{23} + \mu_{11}^2\mu_{22}^4\mu_{23} + 3\mu_{11}^2\mu_{22}^2\mu_{23}^3 + 12\mu_{11}^5\mu_{22}\mu_{23} + 6\mu_{11}^3\mu_{22}^3\mu_{23}$

$+8\mu_{11}^3\mu_{22}\mu_{23}^3 + 2\mu_{11}^2\mu_{22}^3\mu_{23}^2 + 3\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^3 + 10\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^2 + 13\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}$

$+12\mu_{11}^4\mu_{21}\mu_{22}\mu_{23} + 2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^4 + 14\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^2 + 8\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^3 + \mu_{11}^6\mu_{21}$

$+\mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{11}\mu_{22}^2\mu_{23}^4 + 2\mu_{11}^2\mu_{22}\mu_{23}^4 + \mu_{11}^2\mu_{21}\mu_{22}^4 + \mu_{11}\mu_{21}\mu_{22}^4\mu_{23} + \mu_{11}^2\mu_{21}\mu_{23}^4$

$+2\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^2 + 6\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23}),$

$\xi_7((2,1),1,1) = \mu_{11}\mu_{22}\mu_{23}(\mu_{11}^3\mu_{22}^4\mu_{23} + \mu_{11}^2\mu_{22}^4\mu_{23}^2 + 4\mu_{11}^5\mu_{22}^3 + 4\mu_{11}^7\mu_{23} + \mu_{11}^4\mu_{22}^4$

$+\mu_{11}^7\mu_{21} + 13\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + 3\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^4 + 12\mu_{11}^5\mu_{21}\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^4$

$+15\mu_{11}^4\mu_{21}\mu_{22}\mu_{23}^2 + 10\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^3 + 5\mu_{11}^3\mu_{22}^2\mu_{23}^3 + 6\mu_{11}^4\mu_{22}^3\mu_{23} + 13\mu_{11}^4\mu_{22}^2\mu_{23}^2$

$+13\mu_{11}^5\mu_{22}^2\mu_{23} + 7\mu_{11}^3\mu_{22}^2\mu_{23}^3 + 15\mu_{11}^5\mu_{22}\mu_{23}^2 + 12\mu_{11}^6\mu_{22}\mu_{23} + 10\mu_{11}^4\mu_{22}^1\mu_{23}^3$

$+\mu_{11}\mu_{22}^3\mu_{23}^4 + \mu_{11}^4\mu_{23}^4 + 6\mu_{11}^6\mu_{23}^2 + 7\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^3 + 13\mu_{11}^4\mu_{21}\mu_{22}^2\mu_{23} + \mu_{11}^8$

$+5\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23}^2 + 3\mu_{11}^3\mu_{22}\mu_{23}^4 + 3\mu_{11}^2\mu_{22}^2\mu_{23}^4 + 2\mu_{11}^2\mu_{22}^3\mu_{23}^3 + 2\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^3$

$+6\mu_{11}^3\mu_{21}\mu_{22}^3\mu_{23} + \mu_{11}^2\mu_{21}\mu_{22}^4\mu_{23} + \mu_{11}\mu_{21}\mu_{22}^4\mu_{23}^2 + 4\mu_{11}^5\mu_{23}^3 + 6\mu_{11}^6\mu_{22}^2 + 4\mu_{11}^7\mu_{22}$

$+\mu_{11}^3\mu_{21}\mu_{22}^4 + 4\mu_{11}^4\mu_{21}\mu_{22}^3 + 6\mu_{11}^5\mu_{21}\mu_{22}^2 + 4\mu_{11}^6\mu_{22}\mu_{23} + \mu_{21}\mu_{22}^3\mu_{23}^4 + \mu_{11}^3\mu_{21}\mu_{23}^4$

$+4\mu_{11}^6\mu_{21}\mu_{23} + 6\mu_{11}^5\mu_{21}\mu_{22}^2 + 4\mu_{11}^4\mu_{21}\mu_{22}^3),$

$\xi_7((1,2),1,1) = \mu_{11}^2\mu_{22}\mu_{23}(4\mu_{11}^4\mu_{23}^3 + 4\mu_{11}^5\mu_{21}\mu_{23} + 6\mu_{11}^5\mu_{23}^2 + \mu_{11}^3\mu_{23}^4 + 4\mu_{11}^3\mu_{21}\mu_{23}^3$

$+\mu_{11}^2\mu_{21}\mu_{23}^4 + 9\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^3 + 2\mu_{11}\mu_{21}\mu_{22}\mu_{23}^4 + 5\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^3 + 5\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^2$

$+4\mu_{11}^4\mu_{22}^3 + 4\mu_{11}^6\mu_{22} + \mu_{11}^3\mu_{22}^4 + 4\mu_{11}^3\mu_{21}\mu_{23}^3 + \mu_{11}^2\mu_{21}\mu_{22}^4 + 6\mu_{11}^5\mu_{22}^2 + 4\mu_{11}^5\mu_{21}\mu_{22}$

$+6\mu_{11}^4\mu_{21}\mu_{22}^2 + \mu_{11}^7 + \mu_{11}^6\mu_{21} + 13\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{21}\mu_{22}^3\mu_{23}^3$

$+\mu_{21}\mu_{22}^4\mu_{23}^2 + \mu_{11}\mu_{22}^2\mu_{23}^4 + \mu_{11}\mu_{21}\mu_{22}^4\mu_{23} + 13\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23} + 12\mu_{11}^5\mu_{22}\mu_{23}$

$+4\mu_{11}^6\mu_{23} + 13\mu_{11}^4\mu_{22}^2\mu_{23} + 15\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^2 + 15\mu_{11}^4\mu_{22}\mu_{23}^2 + 9\mu_{11}^3\mu_{22}\mu_{23}^3$

$+13\mu_{11}^3\mu_{22}^2\mu_{23}^2 + 6\mu_{11}^3\mu_{22}^3\mu_{23} + 5\mu_{11}^2\mu_{22}^3\mu_{23}^2 + 5\mu_{11}^2\mu_{22}^2\mu_{23}^3 + 2\mu_{11}^2\mu_{22}\mu_{23}^4$

$+\mu_{11}^2\mu_{22}^4\mu_{23} + \mu_{11}\mu_{22}^3\mu_{23}^3 + 4\mu_{11}\mu_{22}^4\mu_{23}^2 + 12\mu_{11}^4\mu_{21}\mu_{22}\mu_{23} + 6\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23}$

$+6\mu_{11}^4\mu_{21}\mu_{23}^2),$

$\xi_7((2,2),1,1) = \mu_{11}\mu_{22}\mu_{23}(6\mu_{11}^5\mu_{21}\mu_{23}^2 + 4\mu_{11}^4\mu_{21}\mu_{23}^3 + 4\mu_{11}^7\mu_{23} + 4\mu_{11}^4\mu_{23}^4$

$+4\mu_{11}^6\mu_{21}\mu_{23} + 6\mu_{11}^6\mu_{23}^2 + 6\mu_{11}^3\mu_{21}\mu_{22}^3\mu_{23} + \mu_{11}^3\mu_{21}\mu_{23}^4 + 4\mu_{11}^5\mu_{23}^3 + 6\mu_{11}^5\mu_{21}\mu_{22}^2$

$+\mu_{21}\mu_{22}^4\mu_{23}^3 + \mu_{11}^7\mu_{21} + 4\mu_{21}\mu_{22}^3\mu_{23}^4 + \mu_{11}^3\mu_{21}\mu_{22}^4 + 6\mu_{11}^6\mu_{22}^2 + 4\mu_{11}^7\mu_{22} + 4\mu_{11}^5\mu_{23}^3$

$+15\mu_{11}^4\mu_{21}\mu_{22}\mu_{23}^2 + \mu_{11}^4\mu_{22}^4 + 12\mu_{11}^5\mu_{21}\mu_{22}\mu_{23} + 12\mu_{11}^6\mu_{22}\mu_{23} + 15\mu_{11}^5\mu_{22}\mu_{23}^2$

$+13\mu_{11}^4\mu_{21}\mu_{22}^2\mu_{23} + 13\mu_{11}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + 10\mu_{11}^3\mu_{21}\mu_{22}\mu_{23}^3 + \mu_{11}^2\mu_{21}\mu_{22}^4\mu_{23}$

$+3\mu_{11}^2\mu_{21}\mu_{22}\mu_{23}^4 + 8\mu_{11}^2\mu_{21}\mu_{22}^2\mu_{23}^3 + 5\mu_{11}^2\mu_{21}\mu_{22}^3\mu_{23}^3 + 3\mu_{11}\mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{11}^8$

$+13\mu_{11}^5\mu_{22}^2\mu_{23} + 6\mu_{11}^4\mu_{22}^3\mu_{23} + 13\mu_{11}^4\mu_{22}^2\mu_{23}^2 + \mu_{11}^3\mu_{22}^4\mu_{23} + 3\mu_{11}^3\mu_{22}\mu_{23}^4$

$+8\mu_{11}^3\mu_{22}^2\mu_{23}^3 + 5\mu_{11}^3\mu_{22}^3\mu_{23}^2 + 4\mu_{11}^2\mu_{22}^3\mu_{23}^3 + 3\mu_{11}^2\mu_{22}^2\mu_{23}^4 + \mu_{11}^2\mu_{22}^4\mu_{23}^2 + \mu_{11}\mu_{22}^4\mu_{23}^3$

$+10\mu_{11}^4\mu_{22}\mu_{23}^3 + \mu_{11}^3\mu_{21}\mu_{22}^4 + 4\mu_{11}^6\mu_{21}\mu_{23} + \mu_{11}\mu_{22}^3\mu_{23}^4 + 4\mu_{11}\mu_{21}\mu_{22}^3\mu_{23}^3$

$\mu_{11}\mu_{21}\mu_{22}^4\mu_{23}^2),$

$$\xi_8((1,1),1,1) = \mu_{11}\mu_{22}^2\mu_{23}^3(5\mu_{11}^3\mu_{22}\mu_{23} + 3\mu_{11}^3\mu_{23}^2 + 3\mu_{11}^2\mu_{22}\mu_{23}^2 + 3\mu_{11}\mu_{22}^2\mu_{23}^2 + \mu_{11}^5$$

$$+4\mu_{11}^2\mu_{22}^2\mu_{23} + \mu_{11}^2\mu_{23}^3 + 3\mu_{11}^4\mu_{22} + \mu_{22}^3\mu_{23}^2 + \mu_{11}\mu_{22}^3\mu_{23} + 2\mu_{11}^4\mu_{23} + \mu_{11}^3\mu_{23}^2),$$

$$\xi_8((2,1),1,1) = \mu_{11}\mu_{22}^3\mu_{23}^4(\mu_{22}^2\mu_{23} + \mu_{11}\mu_{22}^2 + 2\mu_{11}\mu_{22}\mu_{23} + \mu_{11}^2\mu_{23} + \mu_{11}^3 + 2\mu_{11}^2\mu_{22}),$$

$$\xi_8((1,2),1,1) = \mu_{11}\mu_{22}^2\mu_{23}^4(\mu_{22}^3\mu_{23} + 3\mu_{11}^2\mu_{22}^2 + 3\mu_{11}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{23}^3 + \mu_{11}^4 + 3\mu_{11}^3\mu_{22}$$

$$\mu_{11}^3\mu_{23} + 3\mu_{11}\mu_{22}^2\mu_{23}),$$

$$\xi_8((2,2),1,1) = \mu_{11}\mu_{22}^3\mu_{23}^5(\mu_{22}^2 + 2\mu_{11}\mu_{22} + \mu_{11}^2),$$

$$\xi_9((1,1),1,1) = \mu_{11}\mu_{22}^2\mu_{23}^3(\mu_{22}^3\mu_{23}^2 + 3\mu_{11}\mu_{22}^2\mu_{23}^2 + 4\mu_{11}^2\mu_{22}^2\mu_{23} + \mu_{11}\mu_{22}^3\mu_{23} + \mu_{11}^5$$

$$+2\mu_{11}^4\mu_{23} + \mu_{11}^3\mu_{23}^2 + 5\mu_{11}^3\mu_{22}\mu_{23} + 4\mu_{11}^2\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{23}^3 + 3\mu_{11}^4\mu_{22} + 2\mu_{11}^3\mu_{22}^2),$$

$$\xi_9((2,1),1,1) = \mu_{11}\mu_{22}^3\mu_{23}^4(\mu_{22}^2\mu_{23} + \mu_{11}\mu_{22}^2 + 3\mu_{11}\mu_{22}\mu_{23} + 2\mu_{11}^2\mu_{23} + \mu_{11}^3$$

$$+2\mu_{11}^2\mu_{22}),$$

$$\xi_9((1,2),1,1) = \mu_{11}\mu_{22}^2\mu_{23}^4(\mu_{22}^3\mu_{23} + 3\mu_{11}^2\mu_{22}^2 + 3\mu_{11}^2\mu_{22}\mu_{23} + \mu_{11}^3\mu_{23} + \mu_{11}\mu_{22}^3$$

$$+3\mu_{11}^3\mu_{22} + \mu_{11}^4 + 3\mu_{11}\mu_{22}^2\mu_{23}),$$

$$\xi_9((2,2),1,1) = \mu_{11}\mu_{22}^3\mu_{23}^5(\mu_{22}^2 + 2\mu_{11}\mu_{22} + \mu_{11}^2),$$

$$\xi_{10}((1,1),1,1) = \mu_{11}^3\mu_{22}^3\mu_{23}^5,$$

$$\xi_{10}((2,1),1,1) = \mu_{11}^2\mu_{22}^3\mu_{23}^5(\mu_{11} + \mu_{22}),$$

$$\xi_{10}((1,2),1,1) = 0,$$

$$\xi_{10}((2,2),1,1) = 0,$$

$$\xi_{11}((3,1),1,1) = \mu_{11}\mu_{22}^3\mu_{23}^5(\mu_{11}^2 + 2\mu_{11}\mu_{22} + \mu_{22}^2).$$

These results together with Theorem 9.5.1 of Puterman [58] proves the optimality of the policy $\pi = (d_0)^\infty$. □

Proof of Proposition 4.3.2 : The set of allowable actions in state $s \in S$ is

$$A_s = \begin{cases} a_{21} & \text{for } s = (0,0), \\ a_{22} & \text{for } s = (B_2+2, 0), \\ a_{23} & \text{for } s = (B_2+1, B_3+2), \\ \{a_{11}, a_{21}, a_{22}\} & \text{for } s = (i, 0), \text{ where } i \in \{1, \ldots, B_2+1\}, \\ \{a_{21}, a_{23}\} & \text{for } s = (0, j) \text{ or } s = (i, B_3+2), \text{ where} \\ & \quad i \in \{1, \ldots, B_2\} \text{ and } j \in \{1, \ldots, B_3+2\}, \\ \{a_{22}, a_{23}, a_{33}\} & \text{for } s = (B_2+2, j), \text{ where } j \in \{1, \ldots, B_3+1\}, \\ \{a_{11}, a_{21}, a_{22}, a_{23}, a_{33}\} & \text{for } s = (i, j), \text{ where } i \in \{1, \ldots, B_2+1\} \\ & \quad \text{and } j \in \{1, \ldots, B_3+1\}. \end{cases}$$

Under our assumptions on the service rates ($\sum_{i=1}^{M} \mu_{ij} > 0$ for $j \in \{1, \ldots, N\}$ and $\mu_{11} = \mu_{13} = 0$), it is clear that $\mu_{21} > 0$ and $\mu_{23} > 0$. Hence, we can conclude that the policy described in the theorem corresponds to an irreducible Markov chain, and consequently we have a communicating Markov decision process. Thus, we can use the policy iteration algorithm for communicating models as described in Section 9.5.1 of Puterman [58]. We use the uniformization constant $q = \mu_{12} + \mu_{21} + \mu_{22} + \mu_{23}$.

We start the policy iteration algorithm by choosing

$$d_0(s) = \begin{cases} a_{21} & \text{for } s = (i, 0) \text{ where } i \in \{0, \ldots, B_2+1\}, \\ a_{22} & \text{for } s = (B_2+2, 0), \\ a_{23} & \text{for } s = (B_2+2, j) \text{ or } s = (i, B_3+2), \text{ where } i \in \{1, \ldots, B_2+1\}, \\ & \quad \text{and } j \in \{1, \ldots, B_3+1\}, \\ a_{21} & \text{for } s = (i, j), \text{ where } i \in \{0, \ldots, B_2+1\} \text{ and } j \in \{1, \ldots, B_3\}, \\ a_{21} & \text{for } s = (i, B_3+1) \text{ or } s = (0, B_3+2), \text{ where } i \in \{0, 1\}, \\ a_{23} & \text{for } s = (i, B_3+1), \text{ where } i \in \{2, \ldots, B_2+1\}. \end{cases}$$

Then, we proceed as in the proof of Proposition 4.3.1. In the calculations below, $\phi_k(s, B_2, B_3)$ for $k \in \{1, \ldots, 14\}$ and $\phi(B_2, B_3)$ are nonnegative constants that depend on the service rates, the state $s = (i, j) \in S$ under consideration, and the buffer sizes, and they are provided below. We assume that $B_2, B_3 \leq 1$ in the following calculations.

First, consider the state $s = (i, 0)$, where $i \in \{1, \ldots, B_2 + 1\}$, and recall that $d_0(s) = a_{21}$. Some algebra shows that, for all $i \in \{1, \ldots, B_2 + 1\}$,

$$\Big(r((i,0),a_{21}) + \sum_{s' \in S} p(s'|(i,0),a_{21})h(s')\Big) - r((i,0),a_{11}) - \sum_{s' \in S} p(s'|(i,0),a_{11})h(s') = \frac{\phi_1((i,0),B_2,B_3)}{\phi(B_2,B_3)} \geq 0,$$

$$\Big(r((i,0),a_{21}) + \sum_{s' \in S} p(s'|(i,0),a_{21})h(s')\Big) - r((i,0),a_{22}) - \sum_{s' \in S} p(s'|(i,0),a_{22})h(s') = \frac{\phi_2((i,0),B_2,B_3)}{\phi(B_2,B_3)} \geq 0.$$

Recall that $d_0(s) = a_{21}$ for $s = (0, j)$, where $j \in \{1, \ldots, B_3 + 2\}$. Then, we can show that, for all $j \in \{1, \ldots, B_3 + 2\}$,

$$\Big(r((0,j),a_{21}) + \sum_{s' \in S} p(s'|(0,j),a_{21})h(s')\Big) - r((0,j),a_{23}) - \sum_{s' \in S} p(s'|(0,j),a_{23})h(s') = \frac{\phi_3((0,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0.$$

Similarly, $d_0(s) = a_{23}$ for $s = (i, B_3 + 2)$, where $i \in \{1, \ldots, B_2\}$. We can show that, for all $i \in \{1, \ldots, B_2\}$,

$$\Big(r((i,B_3+2),a_{23}) + \sum_{s' \in S} p(s'|(i,B_3+2),a_{23})h(s')\Big) - r((i,B_3+2),a_{21}) - \sum_{s' \in S} p(s'|(i,B_3+2),a_{21})h(s') = \frac{\phi_4((i,B_3+2),B_2,B_3)}{\phi(B_2,B_3)} \geq 0.$$

For $s = (i, j)$, where $i \in \{1, \ldots, B_2 + 1\}$ and $j \in \{1, \ldots, B_3\}$ or $(i, j) = (1, B_3 + 1)$, recall that $d_0(s) = a_{21}$. Some algebra shows that, for all $i \in \{1, \ldots, B_2 + 1\}$ and $j \in \{1, \ldots, B_3\}$ or $(i, j) = (1, B_3 + 1)$,

$$\Big(r((i,j),a_{21}) + \sum_{s' \in S} p(s'|(i,j),a_{21})h(s')\Big) - r((i,j),a_{11}) - \sum_{s' \in S} p(s'|(i,j),a_{11})h(s') = \frac{\phi_5((i,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0,$$

$$\Big(r((i,j),a_{21}) + \sum_{s'\in S} p(s'|(i,j),a_{21})h(s')\Big) - r((i,j),a_{22}) - \sum_{s'\in S} p(s'|(i,j),a_{22})h(s')$$
$$= \frac{\phi_6((i,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0,$$
$$\Big(r((i,j),a_{21}) + \sum_{s'\in S} p(s'|(i,j),a_{21})h(s')\Big) - r((i,j),a_{33}) - \sum_{s'\in S} p(s'|(i,j),a_{33})h(s')$$
$$= \frac{\phi_7((i,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0,$$
$$\Big(r((i,j),a_{21}) + \sum_{s'\in S} p(s'|(i,j),a_{21})h(s')\Big) - r((i,j),a_{23}) - \sum_{s'\in S} p(s'|(i,j),a_{23})h(s')$$
$$= \frac{\phi_8((i,j),B_2,B_3)}{\phi} \geq 0.$$

For $s = (i, B_3 + 1)$, where $i \in \{2, \ldots, B_2 + 1\}$, recall that $d_0(s) = a_{23}$. Some algebra shows that, for all $i \in \{2, \ldots, B_2 + 1\}$,

$$\Big(r((i,j),a_{23}) + \sum_{s'\in S} p(s'|(i,j),a_{23})h(s')\Big) - r((i,j),a_{11}) - \sum_{s'\in S} p(s'|(i,j),a_{11})h(s')$$
$$= \frac{\phi_9((i,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0,$$
$$\Big(r((i,j),a_{23}) + \sum_{s'\in S} p(s'|(i,j),a_{23})h(s')\Big) - r((i,j),a_{22}) - \sum_{s'\in S} p(s'|(i,j),a_{22})h(s')$$
$$= \frac{\phi_{10}((i,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0,$$
$$\Big(r((i,j),a_{23}) + \sum_{s'\in S} p(s'|(i,j),a_{23})h(s')\Big) - r((i,j),a_{33}) - \sum_{s'\in S} p(s'|(i,j),a_{33})h(s')$$
$$= \frac{\phi_{11}((i,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0,$$
$$\Big(r((i,j),a_{23}) + \sum_{s'\in S} p(s'|(i,j),a_{23})h(s')\Big) - r((i,j),a_{21}) - \sum_{s'\in S} p(s'|(i,j),a_{21})h(s')$$
$$= \frac{\phi_{12}((i,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0.$$

Finally, $d_0(s) = a_{23}$ for $s = (B_2 + 2, j)$, where $j \in \{1, \ldots, B_3 + 1\}$. Some algebra shows that, for all $j \in \{1, \ldots, B_3 + 1\}$,

$$\Big(r((B_2+2,j),a_{23}) + \sum_{s'\in S} p(s'|(B_2+2,j),a_{23})h(s')\Big) - r((B_2+2,j),a_{22})$$
$$-\sum_{s'\in S} p(s'|(B_2+2,j),a_{22})h(s') = \frac{\phi_{13}((B_2+2,j),B_2,B_3)}{\phi(B_2,B_3)} \geq 0,$$

$$\Big(r((B_2+2,j),a_{23})+\sum_{s'\in S}p(s'|(B_2+2,j),a_{23})h(s')\Big)-r((B_2+2,j),a_{33})$$
$$-\sum_{s'\in S}p(s'|(B_2+2,j),a_{33})h(s')=\frac{\phi_{14}((B_2+2,j),B_2,B_3)}{\phi(B_2,B_3)}\geq 0.$$

When $B_2=B_3=0$, we obtain

$$\begin{aligned}
\phi(0,0) &= \mu_{12}\mu_{21}^2\mu_{23}+\mu_{21}\mu_{22}\mu_{23}^2+\mu_{21}^2\mu_{22}\mu_{23}+2\mu_{12}\mu_{21}\mu_{22}\mu_{23}+\mu_{12}^3\mu_{21}+\mu_{12}^2\mu_{21}\mu_{22}\\
&+2\mu_{12}^2\mu_{21}\mu_{23}+\mu_{12}\mu_{21}\mu_{23}^2+2\mu_{12}\mu_{21}^2\mu_{22}+\mu_{12}^2\mu_{21}^2+\mu_{21}^2\mu_{23}^2+\mu_{12}^2\mu_{22}\mu_{23}+\mu_{12}^3\mu_{23}\\
&+\mu_{12}\mu_{22}\mu_{23}^2+\mu_{12}^2\mu_{23}^2,\\
\phi_1((1,0),0,0) &= \mu_{12}\mu_{21}^2\mu_{23}^2,\\
\phi_2((1,0),0,0) &= \mu_{12}\mu_{21}\mu_{23}(\mu_{12}\mu_{22}+\mu_{12}^2+\mu_{22}\mu_{23}+\mu_{12}\mu_{23}),\\
\phi_3((0,1),0,0) &= \mu_{12}\mu_{21}^2\mu_{23}^2,\\
\phi_3((0,2),0,0) &= 0,\\
\phi_5((1,1),0,0) &= \mu_{12}\mu_{21}\mu_{23}(\mu_{12}+\mu_{22}),\\
\phi_6((1,1),0,0) &= \mu_{12}\mu_{21}\mu_{23}(\mu_{12}\mu_{22}+\mu_{12}^2+\mu_{22}\mu_{23}+\mu_{12}\mu_{23}+\mu_{12}\mu_{21}+\mu_{21}\mu_{22}),\\
\phi_7((1,1),0,0) &= \mu_{12}\mu_{21}\mu_{23}(\mu_{21}^2+\mu_{22}\mu_{23}+\mu_{21}\mu_{23}+\mu_{12}^2+\mu_{12}\mu_{21}),\\
\phi_8((1,1),0,0) &= 0,\\
\phi_{13}((2,1),0,0) &= \mu_{12}\mu_{21}\mu_{23}(\mu_{12}\mu_{22}+\mu_{12}^2+\mu_{21}\mu_{22}+\mu_{12}\mu_{21}),\\
\phi_{14}((2,1),0,0) &= \mu_{12}\mu_{21}\mu_{23}^3.
\end{aligned}$$

When $B_2=1$ and $B_3=0$, we obtain

$$\begin{aligned}
\phi(1,0) &= \mu_{12}^3\mu_{23}^3+2\mu_{12}^4\mu_{21}^2+\mu_{21}^3\mu_{23}^3+\mu_{12}^5\mu_{21}+2\mu_{12}^4\mu_{23}^2+\mu_{12}^5\mu_{23}+\mu_{12}\mu_{21}^3\mu_{23}^2\\
&+\mu_{12}^3\mu_{23}^3+\mu_{21}^3\mu_{22}\mu_{23}^2+\mu_{12}^2\mu_{21}^3\mu_{23}+3\mu_{12}^2\mu_{21}^2\mu_{23}^2+4\mu_{12}^4\mu_{21}\mu_{23}+4\mu_{12}^3\mu_{21}^2\mu_{23}\\
&+4\mu_{12}^3\mu_{21}\mu_{23}^2+2\mu_{12}^2\mu_{21}\mu_{23}^3+\mu_{21}^2\mu_{22}\mu_{23}^3+2\mu_{12}\mu_{21}^2\mu_{23}^3+\mu_{12}^2\mu_{21}^3\mu_{22}+\mu_{12}^4\mu_{21}\mu_{22}\\
&+\mu_{12}^4\mu_{22}\mu_{23}+2\mu_{12}^3\mu_{22}\mu_{23}^2+2\mu_{12}\mu_{21}^3\mu_{22}\mu_{23}+4\mu_{12}^2\mu_{21}^2\mu_{22}\mu_{23}+4\mu_{12}^2\mu_{21}\mu_{22}\mu_{23}^2\\
&+2\mu_{12}^3\mu_{21}^2\mu_{22}+\mu_{12}^2\mu_{22}\mu_{23}^3+3\mu_{12}\mu_{21}^2\mu_{22}\mu_{23}^2+4\mu_{12}^3\mu_{21}\mu_{22}\mu_{23}+2\mu_{12}\mu_{21}\mu_{22}\mu_{23}^3,\\
\phi_1((1,0),1,0) &= \mu_{12}^2\mu_{21}^2\mu_{23}^2(\mu_{12}\mu_{21}\mu_{23}+\mu_{12}^2\mu_{22}+\mu_{12}\mu_{22}\mu_{23}+\mu_{12}^2\mu_{23}+\mu_{12}^2\mu_{21}),
\end{aligned}$$

$$\phi_1((2,0),1,0) = \mu_{12}^2\mu_{21}^3\mu_{23}^3(\mu_{12}\mu_{21} + \mu_{12}^2 + \mu_{12}\mu_{22}),$$

$$\phi_2((1,0),1,0) = \mu_{12}\mu_{21}^2\mu_{23}^2(\mu_{12}\mu_{21}\mu_{23}^2 + \mu_{12}^2\mu_{21}\mu_{22} + \mu_{12}\mu_{21}\mu_{22}\mu_{23} + \mu_{12}^2\mu_{21}\mu_{23} + \mu_{12}^4$$

$$+\mu_{12}^3\mu_{21} + \mu_{21}\mu_{22}\mu_{23}^2 + \mu_{12}^3\mu_{22} + \mu_{12}\mu_{22}\mu_{23}^2 + \mu_{12}^2\mu_{23}^2 + 2\mu_{12}^2\mu_{22}\mu_{23} + 2\mu_{12}^3\mu_{23}),$$

$$\phi_2((2,0),1,0) = \mu_{12}^2\mu_{21}\mu_{23}^2(\mu_{12}\mu_{22} + \mu_{21}\mu_{22} + \mu_{22}\mu_{23} + \mu_{12}^2 + \mu_{12}\mu_{21} + \mu_{12}\mu_{23})$$

$$\phi_3((0,1),1,0) = \mu_{12}\mu_{21}^3\mu_{23}^2,$$

$$\phi_3((0,2),1,0) = 0,$$

$$\phi_4((1,2),1,0) = \mu_{12}\mu_{21}^3\mu_{23}^2,$$

$$\phi_5((1,1),1,0) = \mu_{12}\mu_{21}^2\mu_{23}(2\mu_{12}\mu_{23} + \mu_{12}\mu_{22} + \mu_{12}\mu_{21} + 2\mu_{22}\mu_{23} + \mu_{12}^2),$$

$$\phi_6((1,1),1,0) = \mu_{12}\mu_{21}\mu_{23}(\mu_{12}^3 + 2\mu_{12}^2\mu_{23} + \mu_{12}^2\mu_{22} + \mu_{12}^2\mu_{21} + 2\mu_{12}\mu_{22}\mu_{23} + \mu_{12}\mu_{23}^2$$

$$+\mu_{12}\mu_{21}\mu_{22} + \mu_{22}\mu_{23}^2),$$

$$\phi_7((1,1),1,0) = \mu_{12}\mu_{21}^2\mu_{23}^2(\mu_{12}^2 + 2\mu_{12}\mu_{22} + \mu_{22}\mu_{23} + \mu_{21}\mu_{22}),$$

$$\phi_8((1,1),1,0) = 0,$$

$$\phi_9((2,1),1,0) = \mu_{12}\mu_{21}^2\mu_{23}(\mu_{12}^2 + \mu_{12}\mu_{22} + 2\mu_{12}\mu_{21} + \mu_{21}\mu_{22}),$$

$$\phi_{10}((2,1),1,0) = \mu_{12}\mu_{21}\mu_{23}(\mu_{12}^3 + \mu_{12}\mu_{22}\mu_{23} + 2\mu_{12}\mu_{21}\mu_{22} + \mu_{12}^2\mu_{22} + \mu_{21}^2\mu_{22}$$

$$+\mu_{12}^2\mu_{23} + \mu_{12}\mu_{21}^2 + 2\mu_{12}^2\mu_{21}),$$

$$\phi_{11}((2,1),1,0) = \mu_{12}\mu_{21}\mu_{23}^2(\mu_{22}^2\mu_{23} + 2\mu_{12}\mu_{22}\mu_{23} + \mu_{22}\mu_{23}^2 + \mu_{12}^2\mu_{22} + \mu_{12}\mu_{22}^2$$

$$+\mu_{12}\mu_{23}^2)$$

$$\phi_{12}((2,1),1,0) = \mu_{12}^2\mu_{21}^3\mu_{23}^2,$$

$$\phi_{13}((3,1),1,0) = \mu_{12}\mu_{21}\mu_{23}(\mu_{21}^2\mu_{22}\mu_{23} + \mu_{12}\mu_{21}^2\mu_{22} + \mu_{12}\mu_{21}\mu_{22}\mu_{23} + 2\mu_{12}^2\mu_{21}\mu_{22}$$

$$+\mu_{12}^3\mu_{22} + \mu_{12}^2\mu_{22}\mu_{23} + \mu_{12}\mu_{21}^2\mu_{23} + \mu_{12}^2\mu_{21}^2 + \mu_{12}^2\mu_{21}\mu_{23} + 2\mu_{12}^2\mu_{21} + \mu_{12}^3\mu_{23}$$

$$+\mu_{12}^4),$$

$$\phi_{14}((3,1),1,0) = \mu_{12}\mu_{21}\mu_{23}^2(\mu_{12}\mu_{22}^2 + \mu_{12}\mu_{23}^2 + 2\mu_{21}\mu_{22}^2 + \mu_{21}^2\mu_{23} + \mu_{12}^3),$$

When $B_2 = 0$ and $B_3 = 1$, we obtain

$$\phi(0,1) = \mu_{21}^2\mu_{23}^3 + 2\mu_{12}^2\mu_{21}\mu_{22}\mu_{23} + \mu_{12}^3\mu_{21}\mu_{22} + \mu_{12}^3\mu_{22}\mu_{23} + \mu_{12}^4\mu_{21} + \mu_{12}^3\mu_{21}^2$$

$$+\mu_{12}^4\mu_{23} + \mu_{12}^2\mu_{22}\mu_{23}^2 + \mu_{12}^3\mu_{23}^2 + \mu_{12}\mu_{21}\mu_{23}^3 + \mu_{21}\mu_{22}\mu_{23}^3 + \mu_{12}\mu_{22}\mu_{23}^3 + \mu_{12}^2\mu_{23}^3$$

$$+2\mu_{12}^2\mu_{21}\mu_{23}^2 + \mu_{12}^2\mu_{21}^2\mu_{22} + 2\mu_{12}\mu_{21}\mu_{22}\mu_{23}^2 + 2\mu_{12}^3\mu_{21}\mu_{23} + \mu_{12}^2\mu_{21}^2\mu_{23} + \mu_{21}^2\mu_{22}\mu_{23}^2$$

$$+\mu_{12}\mu_{21}^2\mu_{22}\mu_{23} + \mu_{12}\mu_{21}^2\mu_{23}^2,$$

$$\phi_1((1,0),0,1) = \mu_{12}\mu_{21}^3\mu_{23}^2,$$

$$\phi_2((1,0),0,1) = \mu_{12}\mu_{21}\mu_{23}(\mu_{12}\mu_{23}^2 + \mu_{12}^3 + \mu_{22}\mu_{23}^2 + \mu_{12}^2\mu_{23} + \mu_{12}^2\mu_{22} + \mu_{12}\mu_{22}\mu_{23}),$$

$$\phi_3((0,1),0,1) = \mu_{12}\mu_{21}^2\mu_{23}^2(\mu_{12} + \mu_{23}),$$

$$\phi_3((0,2),0,1) = \mu_{12}\mu_{21}^2\mu_{23}^3,$$

$$\phi_3((0,3),0,1) = 0,$$

$$\phi_5((1,1),0,1) = \mu_{12}\mu_{21}^2\mu_{23}(\mu_{12}^2 + \mu_{12}\mu_{21} + \mu_{21}\mu_{22} + \mu_{22}^2),$$

$$\phi_5((1,2),0,1) = \mu_{12}^2\mu_{21}\mu_{23}(\mu_{12}\mu_{21} + \mu_{21}\mu_{23} + \mu_{21}\mu_{22} + \mu_{21}^2 + \mu_{22}\mu_{23} + \mu_{12}\mu_{23}$$

$$+\mu_{23}^2 + \mu_{12}\mu_{22}),$$

$$\phi_6((1,1),0,1) = \mu_{12}\mu_{21}\mu_{23}(\mu_{12}\mu_{21}\mu_{22} + \mu_{12}^2\mu_{21} + \mu_{12}\mu_{23}^2 + \mu_{22}\mu_{23}^2 + \mu_{12}^2\mu_{23} + \mu_{12}^3$$

$$+\mu_{12}\mu_{22}\mu_{23} + \mu_{12}^2\mu_{22}),$$

$$\phi_7((1,2),0,1) = \mu_{12}\mu_{21}\mu_{23}(\mu_{12}^2\mu_{21} + \mu_{12}\mu_{21}\mu_{23} + \mu_{12}\mu_{21}\mu_{22} + \mu_{21}\mu_{22}\mu_{23} + \mu_{12}^2\mu_{23}$$

$$+\mu_{12}^3 + \mu_{12}\mu_{22}\mu_{23} + \mu_{12}^2\mu_{22} + \mu_{12}\mu_{23}^2 + \mu_{22}\mu_{23}^2),$$

$$\phi_7((1,1),0,1) = \mu_{12}\mu_{21}^2\mu_{23}^2(\mu_{12} + \mu_{22}),$$

$$\phi_7((1,2),0,1) = \mu_{12}\mu_{21}\mu_{23}^2(\mu_{12}\mu_{23} + \mu_{23}^2 + \mu_{22}\mu_{23} + \mu_{12}\mu_{22}),$$

$$\phi_8((1,1),0,1) = \mu_{12}^2\mu_{21}^3\mu_{23}^2,$$

$$\phi_8((1,2),0,1) = \mu_{12}^2\mu_{21}^2\mu_{23}^3,$$

$$\phi_{13}((2,1),0,1) = \mu_{12}^2\mu_{21}\mu_{23}(\mu_{12}^2\mu_{21} + \mu_{12}\mu_{21}\mu_{23} + \mu_{12}\mu_{21}\mu_{22} + \mu_{21}\mu_{22}\mu_{23} + \mu_{12}^2\mu_{23}$$

$$+\mu_{12}^3 + \mu_{12}^2\mu_{22} + \mu_{12}\mu_{23}^2 + \mu_{12}\mu_{22}\mu_{23} + \mu_{22}\mu_{23}^2),$$

$$\phi_{13}((2,2),0,1) = \mu_{12}\mu_{21}\mu_{23}(\mu_{12}^4\mu_{21} + \mu_{12}^3\mu_{21}\mu_{22} + 3\mu_{12}^3\mu_{21}\mu_{23} + 3\mu_{12}^2\mu_{21}\mu_{23}^2 + 3\mu_{12}^4\mu_{23}$$

$$+\mu_{12}\mu_{21}\mu_{23}^3 + 3\mu_{12}\mu_{21}\mu_{22}\mu_{23}^2 + \mu_{21}\mu_{22}\mu_{23}^3 + \mu_{12}^5 + \mu_{22}\mu_{23}^2 + \mu_{12}^4\mu_{22} + 3\mu_{12}^2\mu_{21}\mu_{22}\mu_{23}$$

$$+3\mu_{12}^3\mu_{23}^2 + 3\mu_{12}^3\mu_{22}\mu_{23} + 2\mu_{12}^2\mu_{23}^3 + 3\mu_{12}^2\mu_{22}\mu_{23}^2 + 2\mu_{12}\mu_{22}\mu_{23}^3),$$

$$\phi_{14}((2,1),0,1) = \mu_{12}\mu_{21}\mu_{23}^2(\mu_{12}^2\mu_{22} + \mu_{12}\mu_{23}^2 + \mu_{21}\mu_{22}^2 + \mu_{12}\mu_{21}\mu_{23}),$$

$$\phi_{14}((2,2),0,1) = \mu_{12}\mu_{21}^2\mu_{23}^4.$$

When $B_2 = B_3 = 1$, we obtain

$$\begin{aligned}
\phi(1,1) = {} & 3\mu_{12}^3\mu_{22}^4\mu_{23} + \mu_{12}\mu_{21}\mu_{22}^5\mu_{23} + \mu_{12}\mu_{21}\mu_{22}^5\mu_{23} + 4\mu_{12}^2\mu_{21}\mu_{23}^5 + \mu_{12}^3\mu_{23}^5 \\
& +12\mu_{12}^5\mu_{22}\mu_{23}^2 + 10\mu_{12}^4\mu_{22}\mu_{23}^3 + 6\mu_{12}^4\mu_{21}\mu_{22}^3 + 4\mu_{12}^5\mu_{21}\mu_{22}^2 + 7\mu_{12}^4\mu_{22}^2\mu_{23}^2 \\
& +4\mu_{12}^3\mu_{22}^2\mu_{22}^3 + 9\mu_{12}^5\mu_{22}^2\mu_{23} + 4\mu_{12}^4\mu_{22}^2\mu_{23}^2 + 3\mu_{12}^5\mu_{23}^3 + \mu_{12}^6\mu_{21}\mu_{22} + 3\mu_{12}^6\mu_{22}^2 \\
& +4\mu_{12}^7\mu_{21} + 4\mu_{12}^4\mu_{23}^4 + 6\mu_{12}^6\mu_{23}^2 + 4\mu_{12}\mu_{22}^4\mu_{23}^3 + 2\mu_{12}^2\mu_{22}^4\mu_{23}^2 + 3\mu_{12}\mu_{22}^4\mu_{23}^3 \\
& +3\mu_{12}\mu_{22}^2\mu_{23}^5 + 4\mu_{12}^3\mu_{21}\mu_{22}^4 + 3\mu_{12}^3\mu_{22}\mu_{23}^4 + 4\mu_{12}^2\mu_{22}\mu_{23}^5 + 4\mu_{12}^5\mu_{22}^2\mu_{23} + \mu_{12}^4\mu_{21}^4 \\
& +4\mu_{12}^4\mu_{22}\mu_{23}^3 + 8\mu_{12}^6\mu_{22}\mu_{23} + 4\mu_{12}^7\mu_{22} + 16\mu_{12}^3\mu_{22}\mu_{23}^4 + 3\mu_{12}^3\mu_{22}\mu_{23}^4 + \mu_{12}^4\mu_{22}^4 \\
& +12\mu_{12}^3\mu_{22}^2\mu_{23}^3 + 6\mu_{12}\mu_{21}\mu_{22}^2\mu_{23}^4 + 4\mu_{12}^2\mu_{21}\mu_{22}^2\mu_{23}^3 + 11\mu_{12}^3\mu_{21}\mu_{22}\mu_{23}^3 + 13\mu_{12}^6\mu_{22}^2 \\
& +16\mu_{12}^5\mu_{21}\mu_{22}\mu_{23} + 13\mu_{12}^5\mu_{22}\mu_{23}^2 + 8\mu_{12}^4\mu_{21}^2\mu_{22}\mu_{23} + 25\mu_{12}^5\mu_{21}\mu_{22}\mu_{23} + \mu_{12}^7\mu_{22} \\
& +\mu_{12}\mu_{22}^3\mu_{23}^4 + 4\mu_{12}^6\mu_{23}^2 + \mu_{12}^7\mu_{23} + 2\mu_{12}^3\mu_{21}\mu_{23}^4 + \mu_{12}^4\mu_{21}\mu_{23}^3 + 4\mu_{12}^4\mu_{21}\mu_{23}^3 \\
& +9\mu_{12}^4\mu_{21}\mu_{22}^2\mu_{23} + 2\mu_{12}\mu_{21}\mu_{22}^3\mu_{23}^3 + 7\mu_{12}\mu_{21}\mu_{22}^2\mu_{23}^4 + 12\mu_{12}^4\mu_{21}\mu_{22}^2\mu_{23} + 2\mu_{12}^4\mu_{23}^4 \\
& +2\mu_{21}\mu_{22}^2\mu_{23}^5 + \mu_{12}^2\mu_{22}^3\mu_{23}^3 + \mu_{12}^2\mu_{21}\mu_{22}^5 + 3\mu_{12}\mu_{21}\mu_{22}\mu_{23}^5,
\end{aligned}$$

$$\begin{aligned}
\phi_1((1,0),1,1) = {} & \mu_{12}^2\mu_{21}\mu_{23}(\mu_{12}^2\mu_{21}^2\mu_{23} + 4\mu_{12}\mu_{21}\mu_{22}\mu_{23}^2 + 5\mu_{12}\mu_{21}\mu_{22}^2\mu_{23} + \mu_{12}^5 \\
& +6\mu_{12}^3\mu_{23}^2 + 6\mu_{12}^2\mu_{22}^2\mu_{23} + \mu_{12}\mu_{22}^2\mu_{23}^2 + 14\mu_{12}^3\mu_{22}\mu_{23} + 8\mu_{12}^2\mu_{22}\mu_{23}^2 + \mu_{12}\mu_{22}^4 \\
& +2\mu_{12}^2\mu_{22}^2\mu_{23} + 3\mu_{12}\mu_{22}^2\mu_{23}^2 + 4\mu_{12}^3\mu_{21}\mu_{22} + 4\mu_{12}^2\mu_{22}^3 + 4\mu_{12}^2\mu_{21}\mu_{22}^2 + \mu_{21}\mu_{22}^4 \\
& +8\mu_{12}^3\mu_{21}\mu_{22} + 7\mu_{12}^2\mu_{21}\mu_{22}\mu_{23} + 4\mu_{12}^2\mu_{22}\mu_{23}^2 + 2\mu_{12}\mu_{22}\mu_{23}^3 + \mu_{12}^5\mu_{21} + 2\mu_{12}^4\mu_{22} \\
& +4\mu_{12}^4\mu_{23} + 6\mu_{12}^3\mu_{23}^2 + \mu_{12}\mu_{23}^4 + 2\mu_{12}\mu_{22}^3\mu_{23} + \mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{12}\mu_{23}^4 \\
& +2\mu_{12}^3\mu_{21}\mu_{21} + 3\mu_{12}^3\mu_{22}^2),
\end{aligned}$$

$$\begin{aligned}
\phi_1((2,0),1,1) = {} & \mu_{12}\mu_{21}\mu_{23}(2\mu_{12}^2\mu_{22}^3\mu_{23} + 3\mu_{12}^3\mu_{23}^3 + 4\mu_{12}^5\mu_{23} + 2\mu_{12}^2\mu_{22}^4 + \mu_{12}^5\mu_{21} \\
& +2\mu_{12}^2\mu_{21}\mu_{22}\mu_{23}^2 + 3\mu_{12}\mu_{22}\mu_{23}^4 + \mu_{12}^6 + \mu_{12}\mu_{22}^2\mu_{23}^3 + 3\mu_{12}^5\mu_{23} + 5\mu_{12}^4\mu_{22}^2 + 4\mu_{12}^5\mu_{22} \\
& +3\mu_{12}\mu_{22}^3\mu_{23}^2 + 4\mu_{12}^2\mu_{21}\mu_{22}^3 + 10\mu_{12}^2\mu_{22}^2\mu_{23}^2 + 4\mu_{12}^3\mu_{22}^2\mu_{23} + 6\mu_{12}^2\mu_{22}\mu_{23}^3 \\
& +6\mu_{12}^4\mu_{22}\mu_{23} + 4\mu_{12}^2\mu_{22}\mu_{23}^3 + \mu_{12}\mu_{22}\mu_{23}^4 + \mu_{12}^3\mu_{23}^3 + 6\mu_{12}^4\mu_{23}^2 + 12\mu_{12}^3\mu_{22}\mu_{23}^2
\end{aligned}$$

$$+3\mu_{22}^2\mu_{23}^4 + 4\mu_{12}\mu_{21}\mu_{22}^2\mu_{23}^2 + 11\mu_{12}^3\mu_{22}^2\mu_{23} + 6\mu_{12}^2\mu_{21}\mu_{22}^2\mu_{23} + \mu_{21}\mu_{22}^4\mu_{23}$$

$$+\mu_{12}^2\mu_{22}^4 + 4\mu_{12}^3\mu_{21}^2\mu_{22} + 3\mu_{12}^3\mu_{21}\mu_{23}^2 + \mu_{12}^3\mu_{21}\mu_{23}^2 + 4\mu_{12}^5\mu_{21} + 2\mu_{12}^3\mu_{21}^3),$$

$$\phi_2((1,0),1,1) = \mu_{12}\mu_{21}^2\mu_{23}^2(\mu_{22}^3\mu_{23} + \mu_{12}^2\mu_{22}\mu_{23} + \mu_{22}^2\mu_{23}^2 + 3\mu_{12}\mu_{22}^2\mu_{23} + \mu_{12}\mu_{22}^3$$

$$+4\mu_{12}\mu_{22}\mu_{23}^2 + 9\mu_{12}^2\mu_{22}\mu_{23} + 3\mu_{12}^2\mu_{21}\mu_{23} + 2\mu_{12}\mu_{23}^3 + \mu_{12}^3\mu_{23} + 3\mu_{12}^3\mu_{23}),$$

$$\phi_2((2,0),1,1) = \mu_{12}\mu_{21}^3\mu_{23}^3(\mu_{12}\mu_{22}^2 + \mu_{12}\mu_{22}^2 + \mu_{22}^2\mu_{23} + 2\mu_{12}\mu_{22}\mu_{23} + 2\mu_{12}^2\mu_{22}$$

$$+\mu_{12}^3 + \mu_{12}^2\mu_{23} + 2\mu_{12}\mu_{23}^2),$$

$$\phi_3((0,1),1,1) = \mu_{12}^3\mu_{21}^2\mu_{23}(\mu_{12}^4 + \mu_{12}^2\mu_{23}^2 + 3\mu_{12}^3\mu_{23} + 3\mu_{12}^2\mu_{22}^2 + \mu_{12}^2\mu_{23}^2 + \mu_{12}\mu_{23}^3$$

$$+3\mu_{12}\mu_{21}\mu_{23}^2 + 2\mu_{12}^3\mu_{22} + 7\mu_{12}^2\mu_{21}\mu_{22} + 4\mu_{12}\mu_{21}^2\mu_{22} + 7\mu_{12}\mu_{22}^2\mu_{23} + \mu_{12}\mu_{22}^3$$

$$+3\mu_{12}\mu_{22}^2\mu_{23} + 2\mu_{21}^2\mu_{22}^2 + \mu_{21}\mu_{21}^2\mu_{23} + \mu_{12}^3\mu_{21}),$$

$$\phi_3((0,2),1,1) = \mu_{12}^2\mu_{21}\mu_{23}(\mu_{12}^3\mu_{22}^2 + \mu_{12}^2\mu_{23}^3 + 6\mu_{12}\mu_{22}\mu_{23}^3 + 4\mu_{12}^4\mu_{22} + 2\mu_{12}\mu_{22}^3\mu_{23}$$

$$+2\mu_{12}^3\mu_{23}^2 + 2\mu_{12}^2\mu_{21}\mu_{23}^2 + 5\mu_{12}^3\mu_{23}^2 + \mu_{12}^2\mu_{21}\mu_{23}^2 + 2\mu_{12}^3\mu_{21}\mu_{23} + \mu_{12}^5 + 4\mu_{12}^4\mu_{23}$$

$$+\mu_{12}^4\mu_{21} + \mu_{21}\mu_{22}^3\mu_{23} + \mu_{22}^2\mu_{23}^3 + 4\mu_{12}^2\mu_{22}^3 + 8\mu_{12}\mu_{22}^2\mu_{23}^2 + \mu_{21}\mu_{22}^3\mu_{23} + \mu_{12}\mu_{22}^2\mu_{23}^2$$

$$+4\mu_{12}^2\mu_{21}\mu_{23}^2 + \mu_{12}\mu_{22}^4 + 12\mu_{12}^3\mu_{22}\mu_{23} + 2\mu_{12}^2\mu_{22}^2\mu_{23} + 9\mu_{12}^2\mu_{22}^2\mu_{23} + 4\mu_{12}^2\mu_{21}\mu_{22}^2$$

$$+8\mu_{12}^2\mu_{21}\mu_{22}\mu_{23} + 8\mu_{12}\mu_{21}\mu_{22}^2\mu_{23} + 4\mu_{12}^3\mu_{21}\mu_{22}),$$

$$\phi_3((0,3),1,1) = \mu_{12}^2\mu_{21}\mu_{23}(\mu_{12}^4\mu_{23} + \mu_{12}^3\mu_{21}\mu_{23} + 3\mu_{12}\mu_{21}\mu_{23}^3 + \mu_{21}\mu_{22}^4 + \mu_{12}^5$$

$$+4\mu_{12}^3\mu_{21}\mu_{22} + 4\mu_{12}^4\mu_{22} + 6\mu_{12}^3\mu_{22}^2 + 2\mu_{12}\mu_{22}\mu_{23}^3 + 6\mu_{12}^3\mu_{22}\mu_{23}^3 + 3\mu_{12}^4\mu_{22}$$

$$+6\mu_{12}^3\mu_{23}^2 + 2\mu_{12}^2\mu_{21}\mu_{23}^2 + 3\mu_{12}^2\mu_{22}^2\mu_{23} + 3\mu_{12}^2\mu_{23}^3 + 6\mu_{12}\mu_{21}^2\mu_{22}^2 + 4\mu_{12}^2\mu_{21}\mu_{22}^2$$

$$+11\mu_{12}^4\mu_{23} + 6\mu_{12}^3\mu_{22}\mu_{23} + 2\mu_{12}\mu_{21}\mu_{23}^3 + 6\mu_{12}\mu_{21}\mu_{22}\mu_{23}^2 + 6\mu_{12}^2\mu_{21}\mu_{22}\mu_{23}$$

$$+12\mu_{12}\mu_{21}\mu_{22}^2\mu_{23} + \mu_{12}\mu_{21}^2\mu_{22}\mu_{23}),$$

$$\phi_4((1,3),1,1) = \mu_{12}\mu_{21}^2\mu_{23}(\mu_{12}^5 + \mu_{22}^4\mu_{23} + 4\mu_{12}^3\mu_{22}^2 + 8\mu_{12}^2\mu_{22}^2\mu_{23} + 4\mu_{12}^2\mu_{22}\mu_{23}^2$$

$$+3\mu_{12}\mu_{22}^2\mu_{23}^2 + 2\mu_{12}^2\mu_{22}^2\mu_{23} + 5\mu_{12}^3\mu_{22}\mu_{23} + 6\mu_{12}^4\mu_{23} + 3\mu_{12}^3\mu_{23}^2 + 4\mu_{12}^2\mu_{22}^2\mu_{23}$$

$$+7\mu_{12}^2\mu_{23}^3 + 4\mu_{12}^4\mu_{22} + \mu_{12}\mu_{22}\mu_{23}^3 + 3\mu_{12}\mu_{22}^2\mu_{23}^2 + \mu_{12}\mu_{22}^2\mu_{23}^2 + \mu_{21}^3\mu_{23}^2 + \mu_{12}^3\mu_{21}^2),$$

$$\phi_5((1,1),1,1) = \mu_{12}\mu_{21}^2\mu_{23}^3(2\mu_{12}^3\mu_{22}\mu_{23} + 3\mu_{12}^3\mu_{23}^2 + 4\mu_{12}^2\mu_{22}\mu_{23}^2 + \mu_{12}\mu_{22}^2\mu_{23}^2 + \mu_{12}^5$$

$$+2\mu_{12}^2\mu_{22}^2\mu_{23} + 2\mu_{12}^2\mu_{23}^3 + \mu_{12}^4\mu_{22} + \mu_{22}^3\mu_{23}^2 + \mu_{12}\mu_{22}^3\mu_{23} + \mu_{12}^4\mu_{23} + \mu_{12}^3\mu_{23}^2),$$

$$\phi_5((2,1),1,1) = \mu_{12}\mu_{21}^3\mu_{23}^4(\mu_{22}^2\mu_{23} + 2\mu_{12}\mu_{22}^2 + 2\mu_{12}\mu_{22}\mu_{23} + \mu_{12}^2\mu_{23} + \mu_{12}^3 + \mu_{12}^2\mu_{22}),$$

$$\phi_5((1,2),1,1) = \mu_{12}\mu_{21}^2\mu_{23}^4(\mu_{22}^3\mu_{23} + 2\mu_{12}^2\mu_{22}^2 + 2\mu_{12}^2\mu_{22}\mu_{23} + 3\mu_{12}\mu_{23}^3 + \mu_{12}^4 + \mu_{12}^3\mu_{22}$$
$$\mu_{12}^3\mu_{23} + 2\mu_{12}\mu_{22}^2\mu_{23}),$$

$$\phi_6((1,1),1,1) = \mu_{12}^2\mu_{21}\mu_{23}(\mu_{12}^3\mu_{23}^3 + \mu_{12}^2\mu_{22}^4 + 2\mu_{12}^4\mu_{23}^2 + 4\mu_{12}^5\mu_{22} + 3\mu_{12}^4\mu_{22}^2 + \mu_{12}^6$$
$$+13\mu_{12}^4\mu_{22}\mu_{23} + \mu_{12}\mu_{22}^4\mu_{23} + 2\mu_{12}\mu_{22}^2\mu_{23}^3 + 8\mu_{12}^4\mu_{22}\mu_{23} + 5\mu_{12}^2\mu_{22}^2\mu_{23} + \mu_{12}^5\mu_{21}$$
$$+4\mu_{12}^2\mu_{22}\mu_{23}^3 + 3\mu_{12}\mu_{22}^3\mu_{23}^2 + 2\mu_{12}\mu_{21}\mu_{22}^2\mu_{23}^2 + 7\mu_{12}\mu_{21}\mu_{22}^2\mu_{23} + 11\mu_{12}^2\mu_{21}\mu_{22}^2\mu_{23}$$
$$+3\mu_{12}^4\mu_{21}\mu_{22} + 6\mu_{12}^2\mu_{21}\mu_{23}^3 + 4\mu_{12}^4\mu_{23}^2 + 3\mu_{12}^5\mu_{23} + 4\mu_{12}^4\mu_{22}^2 + 6\mu_{12}^3\mu_{21}\mu_{22}\mu_{23}$$
$$+4\mu_{12}^5\mu_{21} + 2\mu_{12}\mu_{21}\mu_{22}\mu_{23}^3 + 11\mu_{12}^2\mu_{21}\mu_{22}\mu_{23}^2 + 7\mu_{12}\mu_{21}^2\mu_{22}\mu_{23}^2 + \mu_{22}^3\mu_{23}^2$$
$$+\mu_{21}\mu_{22}^2\mu_{23}^3 + \mu_{12}\mu_{22}\mu_{23}^4 + 3\mu_{12}\mu_{21}\mu_{23}^4 + \mu_{12}^2\mu_{21}\mu_{22}^3 + \mu_{21}\mu_{22}^4\mu_{23} + \mu_{12}^2\mu_{21}\mu_{23}^3$$
$$+4\mu_{12}\mu_{21}\mu_{22}^3\mu_{23}),$$

$$\phi_6((2,1),1,1) = \mu_{12}\mu_{21}\mu_{23}(\mu_{12}^2\mu_{22}^4\mu_{23} + \mu_{12}^2\mu_{22}^3\mu_{23}^2 + 2\mu_{12}^4\mu_{22}^3 + 3\mu_{12}^6\mu_{23} + \mu_{12}^3\mu_{22}^4$$
$$+13\mu_{12}^5\mu_{22}\mu_{23} + 7\mu_{12}^3\mu_{22}^2\mu_{23}^2 + 15\mu_{12}^4\mu_{22}\mu_{23}^2 + 12\mu_{12}^6\mu_{22} + 10\mu_{12}^4\mu_{23}^3$$
$$+\mu_{22}^3\mu_{23}^4 + \mu_{12}^4\mu_{23}^3 + 4\mu_{12}^5\mu_{23}^2 + 5\mu_{12}\mu_{21}\mu_{22}^2\mu_{23}^3 + 11\mu_{12}^3\mu_{21}\mu_{22}^2\mu_{23} + 4\mu_{12}^4\mu_{21}\mu_{22}^2$$
$$+\mu_{12}^6\mu_{21} + 2\mu_{12}^2\mu_{21}\mu_{22}^2\mu_{23}^2 + 2\mu_{12}\mu_{21}\mu_{22}\mu_{23}^4 + 11\mu_{12}^5\mu_{21}\mu_{23} + 2\mu_{12}^2\mu_{21}\mu_{22}\mu_{23}^3$$
$$+3\mu_{12}^2\mu_{21}\mu_{22}^2\mu_{23}^2 + 5\mu_{12}^3\mu_{22}\mu_{23}^3 + 5\mu_{12}^2\mu_{22}^2\mu_{23}^3 + 6\mu_{12}^4\mu_{22}^2\mu_{23} + 12\mu_{12}^3\mu_{22}^3\mu_{23} + \mu_{12}^7$$
$$+3\mu_{12}\mu_{21}\mu_{22}^3\mu_{23}^2 + 2\mu_{12}^2\mu_{22}\mu_{23}^4 + \mu_{12}^2\mu_{21}\mu_{23}^4 + 2\mu_{12}^2\mu_{22}^2\mu_{23}^3 + 2\mu_{12}\mu_{21}\mu_{22}^2\mu_{23}^3$$
$$+3\mu_{12}^5\mu_{21}\mu_{23} + 6\mu_{12}^4\mu_{21}\mu_{22}^2),$$

$$\phi_6((1,2),1,1) = \mu_{12}\mu_{21}\mu_{23}(2\mu_{12}^4\mu_{23}^3 + 5\mu_{12}^5\mu_{21}\mu_{23} + 2\mu_{12}^5\mu_{23}^2 + 2\mu_{12}^3\mu_{23}^4 + \mu_{12}^3\mu_{21}\mu_{23}^3$$
$$+3\mu_{12}^4\mu_{22}^3 + 2\mu_{12}^6\mu_{22} + \mu_{12}^3\mu_{22}^4 + 5\mu_{12}^3\mu_{21}\mu_{23}^3 + \mu_{12}^2\mu_{21}\mu_{22}^4 + 3\mu_{12}^5\mu_{22}^2 + \mu_{12}^5\mu_{21}\mu_{22}$$
$$+\mu_{21}\mu_{22}^4\mu_{23}^2 + \mu_{12}\mu_{22}^2\mu_{23}^4 + 3\mu_{12}\mu_{21}\mu_{22}^4\mu_{23} + 4\mu_{12}^3\mu_{21}\mu_{22}^2\mu_{23} + 10\mu_{12}^5\mu_{22}\mu_{23}$$
$$+7\mu_{12}^4\mu_{21}\mu_{22}^2 + \mu_{12}^6\mu_{21} + 9\mu_{12}^2\mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{21}\mu_{22}^2\mu_{23}^4 + 6\mu_{21}\mu_{22}^3\mu_{23}^3 + 3\mu_{12}^4\mu_{21}\mu_{23}^2$$
$$+2\mu_{12}^2\mu_{21}\mu_{23}^4 + 6\mu_{12}^2\mu_{21}\mu_{22}\mu_{23}^3 + 2\mu_{12}\mu_{21}\mu_{22}\mu_{23}^4 + 2\mu_{12}\mu_{21}\mu_{22}^2\mu_{23}^3 + 4\mu_{12}\mu_{21}\mu_{22}^3\mu_{23}^2$$
$$+11\mu_{12}^3\mu_{22}^2\mu_{23}^2 + 6\mu_{12}^3\mu_{22}^3\mu_{23} + 5\mu_{12}^2\mu_{22}^3\mu_{23}^2 + 4\mu_{12}^2\mu_{22}^2\mu_{23}^3 + 4\mu_{12}^2\mu_{22}\mu_{23}^4 + \mu_{12}^7),$$

$$\phi_7((1,1),1,1) = \mu_{12}\mu_{12}^2\mu_{23}^3(2\mu_{22}^3\mu_{23}^2 + \mu_{12}\mu_{22}^2\mu_{23}^2 + 3\mu_{12}^2\mu_{22}^2\mu_{23} + 2\mu_{12}\mu_{22}^3\mu_{23} + \mu_{12}^5$$

$+\mu_{12}^4\mu_{23} + \mu_{12}^3\mu_{23}^2 + 5\mu_{12}^3\mu_{22}\mu_{23} + 4\mu_{12}^2\mu_{22}\mu_{23}^2 + \mu_{12}^2\mu_{23}^3 + 2\mu_{12}^4\mu_{22}),$

$\phi_7((2,1),1,1) = \mu_{12}\mu_{21}^4\mu_{23}^3(2\mu_{22}^2\mu_{23} + 2\mu_{12}\mu_{22}^2 + \mu_{12}\mu_{22}\mu_{23} + \mu_{12}^2\mu_{23} + \mu_{12}^3$

$+2\mu_{12}^2\mu_{22}),$

$\phi_7((1,2),1,1) = \mu_{12}\mu_{21}^2\mu_{23}^4(\mu_{22}^3\mu_{23} + 2\mu_{12}^2\mu_{22}^2 + 4\mu_{12}^2\mu_{22}\mu_{23} + 2\mu_{12}^3\mu_{23} + \mu_{12}\mu_{22}^3$

$+2\mu_{12}^3\mu_{22} + \mu_{12}^4 + 3\mu_{12}\mu_{22}^2\mu_{23}),$

$\phi_8((1,1),1,1) = \mu_{12}^3\mu_{21}^3\mu_{23}^5,$

$\phi_8((2,1),1,1) = \mu_{12}^2\mu_{21}^3\mu_{23}^5(\mu_{12} + \mu_{22}),$

$\phi_8((1,2),1,1) = 0,$

$\phi_9((2,2),1,1) = \mu_{12}\mu_{21}^3\mu_{23}^5(\mu_{22}^2 + \mu_{12}\mu_{22} + \mu_{12}^2),$

$\phi_{10}((2,2),1,1) = \mu_{12}\mu_{21}\mu_{23}(2\mu_{12}^5\mu_{21}\mu_{23}^2 + 3\mu_{12}^2\mu_{21}\mu_{23}^5 + 4\mu_{12}^7\mu_{23} + 5\mu_{12}^4\mu_{23}^4$

$+3\mu_{21}\mu_{22}^4\mu_{23}^3 + 3\mu_{12}^7\mu_{21} + 2\mu_{21}\mu_{22}^3\mu_{23}^4 + \mu_{12}^2\mu_{21}\mu_{22}^4 + 5\mu_{12}^6\mu_{22}^2 + 2\mu_{12}^7\mu_{22} + 2\mu_{12}^5\mu_{23}^3$

$+11\mu_{12}^4\mu_{21}\mu_{22}\mu_{23}^2 + \mu_{12}^4\mu_{22}^4 + 8\mu_{12}^5\mu_{21}\mu_{22}\mu_{23} + 5\mu_{12}^6\mu_{22}\mu_{23} + 13\mu_{12}^5\mu_{22}\mu_{23}^2$

$+12\mu_{12}^4\mu_{21}\mu_{22}^2\mu_{23} + 13\mu_{12}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + 10\mu_{12}^3\mu_{21}\mu_{22}\mu_{23}^3 + \mu_{12}^2\mu_{21}\mu_{22}^4\mu_{23} + \mu_{12}\mu_{22}^3\mu_{23}^4$

$+8\mu_{12}^5\mu_{22}^2\mu_{23} + 6\mu_{12}^4\mu_{22}^3\mu_{23} + 11\mu_{12}^4\mu_{22}^2\mu_{23}^2 + \mu_{12}^3\mu_{22}^4\mu_{23} + 3\mu_{12}^3\mu_{22}\mu_{23}^4 + 6\mu_{12}^4\mu_{22}\mu_{23}^3$

$+2\mu_{12}^3\mu_{22}^2\mu_{23}^3 + 5\mu_{12}^3\mu_{22}^3\mu_{23}^2 + 4\mu_{12}^2\mu_{22}^3\mu_{23}^3 + 4\mu_{12}^2\mu_{22}^2\mu_{23}^4 + \mu_{12}^2\mu_{22}^4\mu_{23}^2 + 3\mu_{12}\mu_{22}^4\mu_{23}^3$

$+2\mu_{12}^3\mu_{21}\mu_{22}^4 + 4\mu_{12}^6\mu_{21}\mu_{23}^2 + 2\mu_{12}\mu_{21}\mu_{22}^3\mu_{23}^3),$

$\phi_{11}((2,2),1,1) = \mu_{12}\mu_{21}^3\mu_{23}^5(\mu_{22}^2 + \mu_{12}\mu_{22} + \mu_{12}^2),$

$\phi_{12}((2,2),1,1) = 0,$

$\phi_{13}((3,1),1,1) = \mu_{12}\mu_{21}\mu_{23}^3(\mu_{12}^6 + \mu_{12}^5\mu_{23} + 4\mu_{12}\mu_{22}^3\mu_{23}^2 + 2\mu_{12}^3\mu_{22}^3 + 3\mu_{12}^2\mu_{22}^4 + \mu_{12}^5\mu_{22}$

$+4\mu_{12}^4\mu_{22}\mu_{23} + 4\mu_{12}^2\mu_{22}^3\mu_{23} + \mu_{12}^4\mu_{23}^2),$

$\phi_{13}((3,2),1,1) = \mu_{12}\mu_{21}\mu_{23}(2\mu_{12}^3\mu_{21}\mu_{22}\mu_{23}^3 + 4\mu_{12}\mu_{21}\mu_{22}^3\mu_{23}^3 + 4\mu_{12}^2\mu_{21}\mu_{22}^2\mu_{23}^3 + \mu_{12}^8$

$+4\mu_{12}^4\mu_{21}\mu_{23}^3 + 6\mu_{12}^5\mu_{21}\mu_{23}^2 + \mu_{21}\mu_{22}^4\mu_{23}^3 + 4\mu_{12}^7\mu_{22} + 4\mu_{12}^5\mu_{22}^3 + \mu_{12}^7\mu_{23} + 6\mu_{12}^6\mu_{22}^2$

$+2\mu_{12}^4\mu_{21}\mu_{22}\mu_{23}^2 + 5\mu_{12}^2\mu_{21}\mu_{22}^3\mu_{23}^2 + 3\mu_{12}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{12}\mu_{21}\mu_{22}^4\mu_{23}^2 + \mu_{12}^3\mu_{21}\mu_{23}^4$

$+\mu_{12}^7\mu_{23} + 2\mu_{12}^3\mu_{22}\mu_{23}^4 + \mu_{12}^2\mu_{22}^2\mu_{23}^4 + 4\mu_{12}^2\mu_{22}^3\mu_{23}^3 + 8\mu_{12}^3\mu_{22}^2\mu_{23}^3 + 10\mu_{12}^4\mu_{22}\mu_{23}^3$

$$+\mu_{12}^4\mu_{22}^4 + 2\mu_{12}^6\mu_{21}\mu_{22} + \mu_{12}\mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{12}^4\mu_{23}^4 + 2\mu_{12}^5\mu_{23}^3 + 4\mu_{12}^6\mu_{21}\mu_{23}$$

$$+4\mu_{12}^4\mu_{21}\mu_{23}^3 + 6\mu_{12}^5\mu_{21}\mu_{22}^2 + \mu_{12}^3\mu_{21}\mu_{22}^4 + \mu_{12}\mu_{22}^4\mu_{23}^3 + \mu_{12}^2\mu_{22}^4\mu_{23}^2 + 13\mu_{12}^4\mu_{22}^2\mu_{23}^2$$

$$+12\mu_{12}^6\mu_{22}\mu_{23} + 12\mu_{12}^5\mu_{21}\mu_{22}\mu_{23} + 6\mu_{12}^3\mu_{21}\mu_{22}^3\mu_{23} + 13\mu_{12}^4\mu_{21}\mu_{22}^2\mu_{23}$$

$$+\mu_{12}^2\mu_{21}\mu_{22}^4\mu_{23}^2 + 4\mu_{12}\mu_{21}\mu_{22}\mu_{23}^4),$$

$$\phi_{14}((3,1),1,1) = \mu_{12}\mu_{21}^3\mu_{23}^5(\mu_{12}^2 + \mu_{12}\mu_{22} + \mu_{22}^2),$$

$$\phi_{14}((3,2),1,1) = \mu_{12}\mu_{21}^2\mu_{23}^5(\mu_{12}^3 + 4\mu_{12}\mu_{22}^2 + 2\mu_{12}^2\mu_{22} + \mu_{23}^3).$$

These results together with Theorem 9.5.1 of Puterman [58] proves the optimality of the policy $\pi = (d_0)^\infty$. $\square$

Proof of Proposition 4.3.3 : The set of allowable actions in state $s \in S$ is

$$A_s = \begin{cases} a_{31} & \text{for } s = (0,0), \\ a_{32} & \text{for } s = (B_2+2, 0), \\ a_{33} & \text{for } s = (B_2+1, B_3+2), \\ \{a_{31}, a_{32}\} & \text{for } s = (i,0), \text{ where } i \in \{1,\dots,B_2+1\}, \\ \{a_{11}, a_{13}, a_{33}\} & \text{for } s = (0,j) \text{ or } s = (i, B_3+2), \text{ where} \\ & \quad i \in \{1,\dots,B_2\} \text{ and } j \in \{1,\dots,B_3+2\}, \\ \{a_{22}, a_{32}, a_{33}\} & \text{for } s = (B_2+2, j), \text{ where } j \in \{1,\dots,B_3+1\}, \\ \{a_{11}, a_{22}, a_{31}, a_{32}, a_{33}\} & \text{for } s = (i,j), \text{ where } i \in \{1,\dots,B_2+1\} \\ & \quad \text{and } j \in \{1,\dots,B_3+1\}. \end{cases}$$

Under our assumptions on the service rates ($\sum_{i=1}^M \mu_{ij} > 0$ for $j \in \{1,\dots,N\}$ and $\mu_{11} = \mu_{12} = 0$), it is clear that $\mu_{21} > 0$ and $\mu_{22} > 0$. Hence, we can conclude that the policy described in the theorem corresponds to an irreducible Markov chain, and consequently we have a communicating Markov decision process. Thus, we can use the policy iteration algorithm for communicating models as described in Section 9.5.1 of Puterman [58]. We use the uniformization constant $q = \mu_{13} + \mu_{21} + \mu_{22} + \mu_{23}$.

We start the policy iteration algorithm by choosing

$$d_0(s) = \begin{cases} a_{31} & \text{for } s = (0,0), \\ a_{32} & \text{for } s = (B_2+2, j), \text{ where } j \in \{0,\ldots,B_3+1\}, \\ a_{33} & \text{for } s = (B_2+1, B_3+2), \\ a_{32} & \text{for } s = (i,j), \text{ where } i \in \{1,\ldots,B_2+1\} \text{ and } j \in \{0,\ldots,B_3+1\}, \\ a_{31} & \text{for } s = (0,j) \text{ or } s = (i, B_3+2), \text{ where } i \in \{1,\ldots,B_2\} \\ & \quad \text{and } j \in \{1,\ldots,B_3+2\}. \end{cases}$$

Then, we proceed as in the proof of Proposition 4.3.1. In the calculations below, $\psi_k(s, B_2, B_3)$ for $k \in \{1,\ldots,11\}$ and $\psi(B_2, B_3)$ are nonnegative constants that depend on the service rates, the state $s = (i,j) \in S$ under consideration, and the buffer sizes, and they are provided below. We assume that $B_2, B_3 \leq 1$ in the following calculations.

First, consider the state $s = (i, 0)$, where $i \in \{1,\ldots,B_2+1\}$, and recall that $d_0(s) = a_{32}$. Some algebra shows that, for all $i \in \{1,\ldots,B_2+1\}$,

$$\Big(r((i,0),a_{32}) + \sum_{s' \in S} p(s'|(i,0),a_{32})h(s')\Big) - r((i,0),a_{31}) - \sum_{s' \in S} p(s'|(i,0),a_{31})h(s') = \frac{\psi_1((i,0),B_2,B_3)}{\psi(B_2,B_3)} \geq 0.$$

Recall that $d_0(s) = a_{31}$ for $s = (0,j)$, where $j \in \{1,\ldots,B_3+2\}$. Then, we can show, for all $j \in \{1,\ldots,B_3+2\}$, that

$$\Big(r((0,j),a_{31}) + \sum_{s' \in S} p(s'|(0,j),a_{31})h(s')\Big) - r((0,j),a_{11}) - \sum_{s' \in S} p(s'|(0,j),a_{11})h(s') = \frac{\psi_2((0,j),B_2,B_3)}{\psi(B_2,B_3)} \geq 0,$$

$$\Big(r((0,j),a_{31}) + \sum_{s' \in S} p(s'|(0,j),a_{31})h(s')\Big) - r((0,j),a_{33}) - \sum_{s' \in S} p(s'|(0,j),a_{33})h(s') = \frac{\psi_3((0,j),B_2,B_3)}{\psi(B_2,B_3)} \geq 0.$$

Similarly, $d_0(s) = a_{31}$ for $s = (i, B_3+2)$, where $i \in \{1,\ldots,B_2\}$. We can show that, for all $i \in \{1, \ldots, B_2\}$,

$$\Big(r((i, B_3+2), a_{31}) + \sum_{s' \in S} p(s'|(i, B_3+2), a_{31})h(s')\Big) - r((i, B_3+2), a_{11}) - \sum_{s' \in S} p(s'|(i, B_3+2), a_{11})h(s') = \frac{\psi_4((i, B_3+2), B_2, B_3)}{\psi(B_2, B_3)} \geq 0,$$

$$\Big(r((i, B_3+2), a_{31}) + \sum_{s' \in S} p(s'|(i, B_3+2), a_{31})h(s')\Big) - r((i, B_3+2), a_{33}) - \sum_{s' \in S} p(s'|(i, B_3+2), a_{33})h(s') = \frac{\psi_5((i, B_3+2), B_2, B_3)}{\psi(B_2, B_3)} \geq 0.$$

For $s = (i, j)$, where $i \in \{1, \ldots, B_2+1\}$ and $j \in \{1, \ldots, B_3+1\}$, recall that $d_0(s) = a_{32}$. Some algebra shows that, for all $i \in \{1, \ldots, B_2+1\}$ and for all $j \in \{1, \ldots, B_3+1\}$,

$$\Big(r((i, j), a_{32}) + \sum_{s' \in S} p(s'|(i, j), a_{32})h(s')\Big) - r((i, j), a_{11}) - \sum_{s' \in S} p(s'|(i, j), a_{11})h(s') = \frac{\psi_6((i, j), B_2, B_3)}{\psi(B_2, B_3)} \geq 0,$$

$$\Big(r((i, j), a_{32}) + \sum_{s' \in S} p(s'|(i, j), a_{32})h(s')\Big) - r((i, j), a_{22}) - \sum_{s' \in S} p(s'|(i, j), a_{22})h(s') = \frac{\psi_7((i, j), B_2, B_3)}{\psi(B_2, B_3)} \geq 0,$$

$$\Big(r((i, j), a_{32}) + \sum_{s' \in S} p(s'|(i, j), a_{32})h(s')\Big) - r((i, j), a_{33}) - \sum_{s' \in S} p(s'|(i, j), a_{33})h(s') = \frac{\psi_8((i, j), B_2, B_3)}{\psi(B_2, B_3)} \geq 0,$$

$$\Big(r((i, j), a_{32}) + \sum_{s' \in S} p(s'|(i, j), a_{32})h(s')\Big) - r((i, j), a_{31}) - \sum_{s' \in S} p(s'|(i, j), a_{31})h(s') = \frac{\psi_9((i, j), B_2, B_3)}{\psi(B_2, B_3)} \geq 0.$$

Finally, $d_0(s) = a_{32}$ for $s = (B_2+2, j)$, where $j \in \{1, \ldots, B_3+1\}$. Some algebra shows that, for all $j \in \{1, \ldots, B_3+1\}$,

$$\Big(r((B_2+2, j), a_{32}) + \sum_{s' \in S} p(s'|(B_2+2, j), a_{32})h(s')\Big) - r((B_2+2, j), a_{22}) - \sum_{s' \in S} p(s'|(B_2+2, j), a_{22})h(s') = \frac{\psi_{10}((B_2+2, j), B_2, B_3)}{\psi(B_2, B_3)} \geq 0,$$

$$\left(r((B_2+2,j),a_{32})+\sum_{s'\in S}p(s'|(B_2+2,j),a_{32})h(s')\right)-r((B_2+2,j),a_{33})$$
$$-\sum_{s'\in S}p(s'|(B_2+2,j),a_{33})h(s')=\frac{\psi_{11}((B_2+2,j),B_2,B_3)}{\psi(B_2,B_3)}\geq 0.$$

When $B_2=B_3=0$, we obtain

$$\psi(0,0)=\mu_{13}^3\mu_{22}^2+\mu_{13}^4\mu_{21}+2\mu_{13}^2\mu_{21}^2\mu_{22}+\mu_{13}\mu_{21}^2\mu_{22}^2+\mu_{21}^3\mu_{22}^2+\mu_{13}\mu_{21}\mu_{23}^3$$
$$+\mu_{13}^2\mu_{21}\mu_{22}^2+\mu_{21}^3\mu_{22}\mu_{23}+2\mu_{13}^3\mu_{21}^2+\mu_{13}^4\mu_{22}+\mu_{13}^4\mu_{22}+3\mu_{13}^3\mu_{21}\mu_{22}+\mu_{13}^2\mu_{21}^3$$
$$+\mu_{13}^3\mu_{21}\mu_{23}+\mu_{13}^3\mu_{22}\mu_{23}+2\mu_{13}\mu_{21}^2\mu_{22}\mu_{23}+2\mu_{13}^2\mu_{13}^2\mu_{23}+\mu_{13}^2\mu_{22}^2\mu_{23}+\mu_{21}^2\mu_{22}^2\mu_{23}$$
$$+\mu_{13}\mu_{21}^3\mu_{23}+\mu_{13}\mu_{21}\mu_{22}\mu_{23}^2+3\mu_{13}^2\mu_{21}\mu_{22}\mu_{23}$$
$$\psi_1((1,0),0,0)=\mu_{13}\mu_{21}^3\mu_{22}^2,$$
$$\psi_2((0,1),0,0)=\mu_{13}\mu_{21}^3\mu_{22}^2,$$
$$\psi_2((0,2),0,0)=\mu_{13}\mu_{21}^2\mu_{22}(\mu_{13}\mu_{22}+\mu_{21}\mu_{22}+\mu_{13}^2+\mu_{13}\mu_{21}),$$
$$\psi_3((0,1),0,0)=\mu_{13}\mu_{21}\mu_{22}(\mu_{21}^2\mu_{23}+\mu_{13}^2\mu_{23}+2\mu_{13}\mu_{21}\mu_{23}+\mu_{21}\mu_{22}\mu_{23}+\mu_{13}\mu_{22}\mu_{23}$$
$$+\mu_{13}^3+\mu_{13}\mu_{21}^2+2\mu_{13}^2\mu_{21}+\mu_{13}\mu_{21}\mu_{22}+\mu_{13}^2\mu_{22}),$$
$$\psi_3((0,2),0,0)=\mu_{13}\mu_{21}\mu_{22}(\mu_{13}^2+\mu_{13}\mu_{22}+\mu_{13}\mu_{23}+\mu_{22}\mu_{23}+\mu_{21}\mu_{23}+\mu_{13}\mu_{21}),$$
$$\psi_6((1,1),0,0)=\mu_{13}\mu_{21}^2\mu_{22}^2(\mu_{13}^2+\mu_{13}\mu_{22}+2\mu_{13}\mu_{21}+\mu_{21}\mu_{22}),$$
$$\psi_7((1,1),0,0)=\mu_{13}\mu_{21}^2\mu_{22}^2(\mu_{13}+\mu_{21}),$$
$$\psi_8((1,1),0,0)=\mu_{13}\mu_{21}\mu_{22}(\mu_{21}^2\mu_{23}+\mu_{13}\mu_{22}\mu_{23}+2\mu_{13}\mu_{21}\mu_{23}+\mu_{13}^2\mu_{23}+\mu_{13}\mu_{21}^2$$
$$+\mu_{13}^3+\mu_{13}^2\mu_{22}+2\mu_{13}^2\mu_{21}),$$
$$\psi_9((1,1),0,0)=\mu_{13}^2\mu_{21}^3\mu_{22}^2,$$
$$\psi_{10}((2,1),0,0)=\mu_{13}\mu_{21}\mu_{22}^2(\mu_{21}^2\mu_{22}+\mu_{13}\mu_{21}\mu_{22}+\mu_{13}^2\mu_{22}+\mu_{13}^3+2\mu_{13}^2\mu_{21}+\mu_{13}\mu_{21}^2),$$
$$\psi_{11}((2,1),0,0)=\mu_{13}\mu_{21}\mu_{22}(\mu_{13}^2\mu_{22}\mu_{23}+\mu_{13}\mu_{21}\mu_{22}\mu_{23}+\mu_{13}^2\mu_{22}\mu_{23}+\mu_{13}\mu_{21}^2\mu_{23}$$
$$+\mu_{13}^4+2\mu_{13}^2\mu_{21}\mu_{23}+\mu_{13}^3\mu_{23}+\mu_{13}\mu_{21}^2\mu_{22}+\mu_{13}^2\mu_{21}\mu_{22}+\mu_{13}^3\mu_{22}+\mu_{13}^2\mu_{21}^2$$
$$+2\mu_{13}^3\mu_{21}),$$

When $B_2 = 1$ and $B_3 = 0$, we obtain

$$\psi(1,0) = \mu_{21}^4\mu_{22}^2\mu_{23} + 2\mu_{13}^4\mu_{22}\mu_{23}^2 + 2\mu_{13}\mu_{21}^4\mu_{22}^2 + \mu_{13}\mu_{21}^3\mu_{22}^3 + 3\mu_{13}^2\mu_{21}^3\mu_{22}^2$$
$$+\mu_{13}^2\mu_{21}^4\mu_{23} + 3\mu_{13}^3\mu_{21}^3\mu_{23} + 5\mu_{13}^4\mu_{21}\mu_{22}^2 + 2\mu_{13}^2\mu_{21}^4\mu_{22} + \mu_{13}^5\mu_{22}\mu_{23} + \mu_{13}^3\mu_{22}^3\mu_{23}$$
$$+4\mu_{13}^3\mu_{21}^3\mu_{22} + 5\mu_{13}^5\mu_{21}\mu_{22} + 7\mu_{13}^4\mu_{21}^2\mu_{22} + 4\mu_{13}^3\mu_{21}^2\mu_{22}^2 + \mu_{13}^3\mu_{21}\mu_{22}^3 + \mu_{13}^2\mu_{21}^2\mu_{22}^3$$
$$+\mu_{13}^5\mu_{21}\mu_{23} + \mu_{21}^3\mu_{22}^3\mu_{23} + \mu_{13}\mu_{21}^2\mu_{22}^3\mu_{23} + \mu_{13}^2\mu_{21}\mu_{22}^3\mu_{23} + 3\mu_{13}\mu_{21}^3\mu_{22}^2\mu_{23}$$
$$+4\mu_{13}^2\mu_{21}^3\mu_{22}\mu_{23} + 4\mu_{13}^2\mu_{21}^2\mu_{22}^2\mu_{23} + 5\mu_{13}^4\mu_{21}\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}^4\mu_{22}\mu_{23} + \mu_{13}^6\mu_{21}$$
$$+5\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23} + 7\mu_{13}^3\mu_{21}^2\mu_{22}\mu_{23} + \mu_{13}^3\mu_{21}^4 + \mu_{21}^4\mu_{22}^3 + \mu_{13}^4\mu_{22}^3 + \mu_{13}^6\mu_{22}$$
$$+2\mu_{13}^5\mu_{22}^2 + 3\mu_{13}^4\mu_{21}^3 + 3\mu_{13}^5\mu_{21}^2 + 3\mu_{13}^4\mu_{21}^2\mu_{22},$$

$$\psi_1((1,0),1,0) = \mu_{13}\mu_{21}^4\mu_{22}^2(\mu_{13} + \mu_{22}),$$

$$\psi_1((2,0),1,0) = \mu_{13}\mu_{21}^3\mu_{22}^2(\mu_{13}^2 + \mu_{13}\mu_{22} + 2\mu_{13}\mu_{21} + \mu_{21}\mu_{22}),$$

$$\psi_2((0,1),1,0) = \mu_{13}\mu_{21}^4\mu_{22}^2(\mu_{13} + \mu_{22}),$$

$$\psi_2((0,2),1,0) = \mu_{13}\mu_{21}^3\mu_{22}(\mu_{21}\mu_{22}^2 + \mu_{13}^2\mu_{21} + 2\mu_{13}\mu_{21}\mu_{22} + \mu_{13}^3 + 2\mu_{13}^2\mu_{22} + \mu_{13}\mu_{22}^2),$$

$$\psi_3((0,1),1,0) = \mu_{13}\mu_{21}\mu_{22}(\mu_{13}^5 + \mu_{13}^4\mu_{23} + 3\mu_{13}^4\mu_{21} + 2\mu_{13}^4\mu_{22} + 2\mu_{13}^3\mu_{22}\mu_{23} + \mu_{13}^3\mu_{22}^2$$
$$+3\mu_{13}^3\mu_{21}^2 + 4\mu_{13}^3\mu_{21}\mu_{23} + 3\mu_{13}^3\mu_{21}\mu_{23} + 3\mu_{13}^2\mu_{21}^2\mu_{22} + 4\mu_{13}^2\mu_{21}\mu_{22}\mu_{23} + \mu_{13}^2\mu_{22}^2\mu_{23}$$
$$+\mu_{13}^2\mu_{21}\mu_{22}^2 + \mu_{13}^2\mu_{21}^3 + 3\mu_{13}^2\mu_{21}^2\mu_{23} + \mu_{13}\mu_{21}^2\mu_{22}^2 + \mu_{13}\mu_{21}^3\mu_{22} + \mu_{13}\mu_{21}^3\mu_{23}$$
$$+\mu_{13}\mu_{21}\mu_{22}^2\mu_{23} + 3\mu_{13}\mu_{21}^2\mu_{22}\mu_{23} + \mu_{21}^2\mu_{22}^2\mu_{23} + \mu_{21}^3\mu_{22}\mu_{23}),$$

$$\psi_3((0,2),1,0) = \mu_{13}^2\mu_{21}\mu_{22}(3\mu_{13}^3\mu_{21} + \mu_{13}^3\mu_{23} + 2\mu_{13}^3\mu_{22} + 2\mu_{13}^2\mu_{21}\mu_{22} + 2\mu_{13}^2\mu_{21}^2$$
$$+3\mu_{13}^2\mu_{22}^2 + 3\mu_{13}^2\mu_{21}\mu_{23} + 2\mu_{13}^2\mu_{22}\mu_{23} + \mu_{13}\mu_{21}\mu_{22}^2 + 2\mu_{13}\mu_{21}^2\mu_{22} + \mu_{13}\mu_{22}^2\mu_{23}$$
$$+\mu_{13}^4 + 4\mu_{13}\mu_{21}\mu_{22}\mu_{23} + \mu_{21}^2\mu_{22}\mu_{23} + \mu_{21}\mu_{22}^2\mu_{23}),$$

$$\psi_4((1,2),1,0) = \mu_{13}\mu_{21}^2\mu_{22}(2\mu_{13}\mu_{21}^2\mu_{22} + \mu_{13}^2\mu_{21}^2 + \mu_{13}\mu_{21}\mu_{22}^2 + 2\mu_{13}^3\mu_{21} + \mu_{13}^2\mu_{22}^2$$
$$2\mu_{13}^4 + 2\mu_{13}^3\mu_{22} + \mu_{21}^2\mu_{22}^2 + 2\mu_{13}^2\mu_{21}\mu_{22}),$$

$$\psi_5((1,2),1,0) = \mu_{13}^3\mu_{21}\mu_{22}(\mu_{21}^2\mu_{23} + \mu_{13}^2\mu_{21} + 2\mu_{21}\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}\mu_{23} + \mu_{13}^3$$
$$+2\mu_{13}\mu_{21}\mu_{22} + 2\mu_{13}^2\mu_{21} + 2\mu_{22}^2\mu_{23} + 2\mu_{13}\mu_{22}\mu_{23} + \mu_{13}\mu_{22}^2 + 2\mu_{13}^2\mu_{22} + \mu_{13}^2\mu_{23}),$$

$$\psi_6((1,1),1,0) = \mu_{13}\mu_{21}^3\mu_{22}^2(\mu_{13}^2 + \mu_{13}\mu_{22} + \mu_{21}\mu_{22} + 2\mu_{13}\mu_{21}),$$

$$\psi_6((2,1),1,0) = \mu_{13}\mu_{21}^2\mu_{22}^2(\mu_{13}^4 + 3\mu_{13}^3\mu_{21} + 2\mu_{13}^3\mu_{22} + 3\mu_{13}^2\mu_{21}^2 + 4\mu_{13}^2\mu_{21}\mu_{22}$$
$$+\mu_{13}^2\mu_{22}^2 + 3\mu_{13}\mu_{21}^2\mu_{22} + \mu_{13}\mu_{21}\mu_{22}^2 + \mu_{21}^2\mu_{22}^2),$$
$$\psi_7((1,1),1,0) = \mu_{13}\mu_{21}^3\mu_{22}^2(\mu_{13}^2 + \mu_{13}\mu_{22} + \mu_{21}\mu_{22} + \mu_{13}\mu_{21}),$$
$$\psi_7((2,1),1,0) = \mu_{13}\mu_{21}^2\mu_{22}^2(4\mu_{13}\mu_{21}^2 + \mu_{21}^2\mu_{22} + \mu_{13}\mu_{21}\mu_{22} + 2\mu_{13}^2\mu_{21} + \mu_{13}^3 + \mu_{13}^2\mu_{22}),$$
$$\psi_8((1,1),1,0) = \mu_{13}\mu_{21}\mu_{22}(\mu_{13}^2\mu_{22}^2\mu_{23} + \mu_{21}^3\mu_{22}\mu_{23} + 2\mu_{13}^3\mu_{22}\mu_{23} + 3\mu_{13}^2\mu_{21}^2\mu_{23}$$
$$+\mu_{13}\mu_{21}\mu_{22}^2\mu_{23} + 3\mu_{13}^3\mu_{21}\mu_{23} + 2\mu_{13}\mu_{21}^2\mu_{22}\mu_{23} + 4\mu_{13}^2\mu_{21}\mu_{22}\mu_{23} + \mu_{13}\mu_{21}^3\mu_{23}$$
$$+\mu_{13}^4\mu_{23} + 3\mu_{13}^3\mu_{21}^2 + \mu_{13}^2\mu_{21}^3 + 3\mu_{13}^4\mu_{21} + 2\mu_{13}^4\mu_{22} + \mu_{13}^3\mu_{22}^2 + \mu_{13}^2\mu_{21}\mu_{22}^2 + \mu_{13}^5$$
$$+\mu_{13}\mu_{21}^3\mu_{22} + 2\mu_{13}^2\mu_{21}^2\mu_{22} + 4\mu_{13}^3\mu_{21}\mu_{22}),$$
$$\psi_8((1,2),1,0) = \mu_{13}\mu_{21}\mu_{22}(\mu_{13}^4\mu_{23} + 3\mu_{13}^4\mu_{21} + 2\mu_{13}^4\mu_{22} + 3\mu_{13}^3\mu_{21}\mu_{22} + 3\mu_{13}^3\mu_{21}^2$$
$$+\mu_{13}^5 + 2\mu_{13}^3\mu_{22}\mu_{23} + 3\mu_{13}^3\mu_{21}\mu_{23} + \mu_{13}^3\mu_{22}^2 + 3\mu_{13}^2\mu_{21}\mu_{22}\mu_{23} + \mu_{13}^2\mu_{21}^2\mu_{22} + \mu_{13}^2\mu_{21}^3$$
$$+\mu_{13}^2\mu_{22}^2\mu_{23} + 3\mu_{13}^2\mu_{21}^2\mu_{23} + \mu_{13}\mu_{21}^2\mu_{22}\mu_{23} + \mu_{13}\mu_{21}^3\mu_{22} + \mu_{13}\mu_{21}^3\mu_{23} + \mu_{13}^3\mu_{22}\mu_{23}),$$
$$\psi_9((1,1),1,0) = \mu_{13}^2\mu_{21}^4\mu_{22}^2,$$
$$\psi_9((1,2),1,0) = \mu_{13}^2\mu_{21}^3\mu_{22}^2(\mu_{13}^2 + \mu_{13}\mu_{22} + 2\mu_{13}\mu_{21} + \mu_{21}\mu_{22}),$$
$$\psi_{10}((3,1),1,0) = \mu_{13}\mu_{21}\mu_{22}(\mu_{13}^3\mu_{22}^2\mu_{23} + \mu_{13}\mu_{21}^3\mu_{22}^2 + 2\mu_{13}^2\mu_{21}^3\mu_{22} + 2\mu_{13}\mu_{21}^3\mu_{22}\mu_{23}$$
$$+\mu_{13}^2\mu_{21}^3\mu_{23} + \mu_{13}^3\mu_{21}^3 + \mu_{13}^2\mu_{21}^2\mu_{22}^2 + \mu_{13}\mu_{21}^2\mu_{22}^2\mu_{23} + 3\mu_{13}^2\mu_{21}^2\mu_{22}\mu_{23} + 3\mu_{13}^4\mu_{21}^2$$
$$+3\mu_{13}^3\mu_{21}^2\mu_{22} + 3\mu_{13}^3\mu_{21}^2\mu_{23} + \mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23} + \mu_{13}^3\mu_{21}\mu_{22}^2 + 4\mu_{13}^4\mu_{21}\mu_{22} + 3\mu_{13}^5\mu_{21}$$
$$+4\mu_{13}^3\mu_{21}\mu_{22}\mu_{23} + 3\mu_{13}^4\mu_{21}\mu_{23} + \mu_{13}^4\mu_{22}^2 + \mu_{13}^3\mu_{22}^2\mu_{23} + 2\mu_{13}^4\mu_{22}\mu_{23} + 2\mu_{13}^5\mu_{22}$$
$$\mu_{13}^6 + \mu_{13}^5\mu_{23},$$
$$\psi_{10}((3,1),1,0) = \mu_{13}\mu_{21}\mu_{22}^2(\mu_{13}^5 + 3\mu_{13}^4\mu_{21} + 2\mu_{13}^4\mu_{22} + 3\mu_{13}^3\mu_{21}^2 + 4\mu_{13}^3\mu_{21}\mu_{22}$$
$$+\mu_{13}^3\mu_{22}^2 + \mu_{13}^2\mu_{21}^3 + 3\mu_{13}^2\mu_{21}^2\mu_{22} + \mu_{13}^2\mu_{21}\mu_{22}^2 + 2\mu_{13}\mu_{21}^3\mu_{22} + \mu_{21}^3\mu_{22}^2 + \mu_{13}\mu_{21}^2\mu_{22}^2),$$

When $B_2 = 0$ and $B_3 = 1$, we obtain

$$\psi(0,1) = 6\mu_{13}^2\mu_{22}^2\mu_{23}^3 + \mu_{13}^4\mu_{22}^2\mu_{23} + 2\mu_{13}^5\mu_{21}\mu_{22} + \mu_{13}^4\mu_{22}^3 + 3\mu_{13}^4\mu_{21}^3 + 2\mu_{13}^5\mu_{22}^2$$
$$+\mu_{21}^3\mu_{22}^3\mu_{23} + \mu_{13}\mu_{21}^3\mu_{22}^3 + 3\mu_{13}^3\mu_{21}^3\mu_{23} + \mu_{13}^2\mu_{21}^4\mu_{23} + \mu_{13}^3\mu_{21}^4 + 2\mu_{13}^4\mu_{22}^2\mu_{23}$$

$$+\mu_{13}^4\mu_{22}^2\mu_{23} + \mu_{13}^5\mu_{22}\mu_{23} + \mu_{13}\mu_{21}^4\mu_{22}^2 + \mu_{21}^4\mu_{22}^3 + \mu_{13}^2\mu_{21}^4\mu_{22} + 3\mu_{13}^5\mu_{21}^2 + \mu_{13}^6\mu_{21}$$

$$+\mu_{13}^3\mu_{22}^3\mu_{23} + \mu_{13}\mu_{21}^4\mu_{22}\mu_{23} + \mu_{13}^6\mu_{22} + 5\mu_{13}^4\mu_{21}\mu_{22}\mu_{23} + 4\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23}$$

$$+4\mu_{13}^2\mu_{21}^3\mu_{22}\mu_{23} + 4\mu_{13}^2\mu_{21}^2\mu_{22}^2\mu_{23} + 5\mu_{13}^4\mu_{21}\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}^4\mu_{22}\mu_{23} + \mu_{13}^6\mu_{21}$$

$$+\mu_{13}^2\mu_{21}\mu_{22}^3\mu_{23} + 6\mu_{13}^3\mu_{21}^2\mu_{22}\mu_{23} + 3\mu_{13}^2\mu_{21}^2\mu_{22}^2\mu_{23} + 4\mu_{13}^4\mu_{21}\mu_{22}^2 + 3\mu_{13}^3\mu_{21}^2\mu_{22}^2$$

$$+\mu_{13}^3\mu_{21}\mu_{22}^3 + \mu_{13}^2\mu_{21}^2\mu_{22}^3 + 3\mu_{13}^4\mu_{21}^2\mu_{23} + \mu_{13}\mu_{21}^2\mu_{22}^3\mu_{23} + 3\mu_{13}^3\mu_{21}^3\mu_{22} + 2\mu_{13}^2\mu_{21}^3\mu_{22}^2$$

$$+3\mu_{13}^2\mu_{21}^3\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}^3\mu_{22}^2\mu_{23},$$

$$\psi_1((1,0),0,1) = \mu_{13}\mu_{21}^4\mu_{22}^3,$$

$$\psi_2((0,1),0,1) = \mu_{13}\mu_{21}^4\mu_{22}^3,$$

$$\psi_2((0,2),0,1) = \mu_{13}\mu_{21}^3\mu_{22}^2(\mu_{13}^2 + \mu_{13}\mu_{21} + \mu_{13}\mu_{22} + \mu_{21}\mu_{22}),$$

$$\psi_2((0,3),0,1) = \mu_{13}\mu_{21}^2\mu_{22}(\mu_{13}^2\mu_{22}^2 + \mu_{13}\mu_{21}\mu_{22}^2 + 2\mu_{13}^3\mu_{21} + \mu_{13}^2\mu_{21}^2 + 2\mu_{13}^3\mu_{22} + \mu_{13}^4$$

$$+\mu_{21}^2\mu_{22}^2 + \mu_{13}\mu_{21}^2\mu_{22} + 2\mu_{13}^2\mu_{21}\mu_{22}),$$

$$\psi_3((0,1),0,1) = \mu_{13}\mu_{21}\mu_{22}(\mu_{13}^2\mu_{22}^2\mu_{23} + \mu_{13}^2\mu_{21}\mu_{22}^2 + \mu_{13}^3\mu_{22}^2 + \mu_{21}^2\mu_{22}^2\mu_{23} + \mu_{13}\mu_{21}^2\mu_{22}^2$$

$$+\mu_{13}\mu_{21}\mu_{22}^2\mu_{23} + \mu_{13}\mu_{21}^3\mu_{22} + 3\mu_{13}^3\mu_{21}\mu_{23} + 2\mu_{13}^3\mu_{22}\mu_{23} + 2\mu_{13}^2\mu_{21}^2\mu_{22} + \mu_{13}^3\mu_{22}\mu_{23}$$

$$+3\mu_{13}^2\mu_{21}\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}^2\mu_{22}\mu_{23} + 2\mu_{13}^4\mu_{22} + 3\mu_{13}^3\mu_{21}^2 + 3\mu_{13}^4\mu_{21} + \mu_{13}^4\mu_{23}$$

$$+\mu_{13}^2\mu_{21}^3 + 3\mu_{13}^2\mu_{21}^2\mu_{23} + \mu_{13}\mu_{21}^3\mu_{23} + \mu_{13}^5 + 3\mu_{13}^3\mu_{21}\mu_{23}),$$

$$\psi_3((0,2),0,1) = \mu_{13}^2\mu_{21}\mu_{22}(\mu_{13}^3\mu_{23} + 2\mu_{13}^3\mu_{22} + 3\mu_{13}^3\mu_{21} + 2\mu_{13}^3\mu_{22} + 3\mu_{13}^2\mu_{21}\mu_{23}$$

$$+\mu_{13}^2\mu_{22}^2 + 2\mu_{13}^2\mu_{22}\mu_{23} + 3\mu_{13}^2\mu_{21}\mu_{22} + 3\mu_{13}^2\mu_{21}^2 + \mu_{13}\mu_{21}^2\mu_{22} + 3\mu_{13}\mu_{21}^2\mu_{23} + \mu_{13}^4$$

$$+\mu_{13}\mu_{21}^3 + 3\mu_{13}\mu_{21}\mu_{22}\mu_{23} + \mu_{13}\mu_{21}\mu_{22}^2 + \mu_{13}\mu_{22}^2\mu_{23} + \mu_{21}^3\mu_{23} + \mu_{21}^2\mu_{22}\mu_{23}$$

$$+\mu_{21}\mu_{22}^2\mu_{23}),$$

$$\psi_3((0,3),0,1) = \mu_{13}^3\mu_{21}\mu_{22}(\mu_{13}^3 + 2\mu_{13}^2\mu_{22} + 2\mu_{13}^2\mu_{21} + \mu_{13}^2\mu_{23} + 2\mu_{13}\mu_{21}\mu_{23}$$

$$+\mu_{13}\mu_{22}^2 + \mu_{13}\mu_{21}\mu_{22} + \mu_{13}\mu_{21}^2 + 2\mu_{13}\mu_{22}\mu_{23} + \mu_{22}^2\mu_{23} + \mu_{21}\mu_{22}\mu_{23} + \mu_{21}^2\mu_{23}),$$

$$\psi_6((1,1),0,1) = \mu_{13}\mu_{21}^3\mu_{22}^3(\mu_{13}^2 + 2\mu_{13}\mu_{21} + \mu_{13}\mu_{22} + \mu_{21}\mu_{22}),$$

$$\psi_6((1,2),0,1) = \mu_{13}\mu_{21}^2\mu_{22}^2(\mu_{13}\mu_{21}\mu_{22}^3 + 4\mu_{13}^2\mu_{21}\mu_{22}^2 + 5\mu_{13}^3\mu_{21}\mu_{22} + 4\mu_{13}^2\mu_{21}^2\mu_{22}$$

$$+3\mu_{13}\mu_{21}^2\mu_{22}^2 + \mu_{21}^2\mu_{22}^3 + \mu_{13}^3\mu_{21}^2 + 2\mu_{13}^4\mu_{21} + \mu_{13}^5 + 3\mu_{13}^4\mu_{22} + 3\mu_{13}^3\mu_{22}^2 + \mu_{13}^2\mu_{22}^3),$$

$\psi_7((1,1),0,1) = \mu_{13}\mu_{21}^3\mu_{22}^3(\mu_{13}+\mu_{21}),$

$\psi_7((1,2),0,1) = \mu_{13}\mu_{21}^2\mu_{22}^2(\mu_{13}\mu_{21}^2+\mu_{13}^2\mu_{21}+\mu_{13}^2\mu_{22}+\mu_{13}^3+\mu_{21}^2\mu_{22}+\mu_{13}\mu_{21}\mu_{22}),$

$\psi_8((1,1),0,1) = \mu_{13}\mu_{21}\mu_{22}(\mu_{13}^5+2\mu_{13}^4\mu_{22}+3\mu_{13}^4\mu_{21}+\mu_{13}^4\mu_{23}+2\mu_{13}^3\mu_{22}\mu_{23}+\mu_{13}^3\mu_{22}^2$
$+3\mu_{13}^3\mu_{21}\mu_{22}+3\mu_{13}^3\mu_{21}\mu_{23}+3\mu_{13}^3\mu_{21}^2+3\mu_{13}^2\mu_{21}\mu_{22}\mu_{23}+3\mu_{13}^2\mu_{21}^2\mu_{23}+\mu_{13}^2\mu_{21}\mu_{22}^2$
$+\mu_{13}^3\mu_{21}^2+2\mu_{13}^2\mu_{21}^2\mu_{22}+\mu_{13}^2\mu_{22}^2\mu_{23}+\mu_{13}\mu_{21}\mu_{22}^2\mu_{23}+\mu_{13}\mu_{21}^3\mu_{22}+2\mu_{13}\mu_{21}^2\mu_{22}\mu_{23}$
$+\mu_{13}\mu_{21}^3\mu_{23}+\mu_{21}^2\mu_{22}\mu_{23}^2),$

$\psi_8((1,2),0,1) = \mu_{13}^2\mu_{21}\mu_{22}(3\mu_{13}^3\mu_{21}+2\mu_{13}^3\mu_{22}+\mu_{13}^3\mu_{23}+2\mu_{13}^2\mu_{22}\mu_{23}+3\mu_{13}^2\mu_{21}^2$
$+3\mu_{13}^2\mu_{21}\mu_{23}+\mu_{13}^2\mu_{22}^2+2\mu_{13}\mu_{21}\mu_{22}\mu_{23}+\mu_{13}\mu_{22}^2\mu_{23}+3\mu_{13}\mu_{21}^2\mu_{23}+\mu_{13}^4+\mu_{13}\mu_{21}^3$
$+\mu_{21}^3\mu_{23}),$

$\psi_9((1,1),0,1) = \mu_{13}^2\mu_{21}^4\mu_{22}^3,$

$\psi_9((1,2),0,1) = \mu_{13}^3\mu_{21}^4\mu_{22}^3,$

$\psi_{10}((2,1),0,1) = \mu_{13}\mu_{21}\mu_{22}(\mu_{13}^3+\mu_{13}\mu_{21}^2+2\mu_{13}^2\mu_{21}+\mu_{13}^2\mu_{22}+\mu_{13}\mu_{21}\mu_{22}+\mu_{21}^2\mu_{22}),$

$\psi_{10}((2,2),0,1) = \mu_{13}\mu_{21}\mu_{22}^2(\mu_{21}^3\mu_{22}^3+3\mu_{13}\mu_{21}^3\mu_{22}^2+3\mu_{13}^2\mu_{21}^3\mu_{22}+\mu_{13}^3\mu_{21}^3+\mu_{13}\mu_{21}^2\mu_{22}^3$
$+4\mu_{13}^2\mu_{21}^2\mu_{22}^2+3\mu_{13}^3\mu_{21}^2\mu_{22}+3\mu_{13}^4\mu_{21}^2+\mu_{13}^2\mu_{21}\mu_{22}^3+5\mu_{13}^3\mu_{21}\mu_{22}^2+7\mu_{13}^4\mu_{21}\mu_{22}$
$+3\mu_{13}^5\mu_{21}+\mu_{13}^3\mu_{22}^3+3\mu_{13}^4\mu_{22}^2+3\mu_{13}^5\mu_{22}+\mu_{13}^6),$

$\psi_{11}((1,2),0,1) = \mu_{13}\mu_{21}\mu_{22}(3\mu_{13}^4\mu_{21}^2+\mu_{13}^2\mu_{21}^2\mu_{22}^2+\mu_{13}\mu_{21}^3\mu_{22}^2+3\mu_{13}^4\mu_{22}\mu_{23}+\mu_{13}^6$
$+\mu_{13}^2\mu_{22}^3\mu_{23}+3\mu_{13}^3\mu_{22}^2\mu_{23}+\mu_{13}^5\mu_{23}+3\mu_{13}^4\mu_{21}\mu_{23}+3\mu_{13}^3\mu_{21}^2\mu_{23}+\mu_{13}^2\mu_{21}^3\mu_{23}$
$+\mu_{13}\mu_{21}^2\mu_{22}^2\mu_{23}+3\mu_{13}^5\mu_{21}+6\mu_{13}^4\mu_{21}\mu_{22}+5\mu_{13}^3\mu_{21}^2\mu_{22}+2\mu_{13}^2\mu_{21}^3\mu_{22}+3\mu_{13}^3\mu_{21}\mu_{22}^2$
$+6\mu_{13}^3\mu_{21}\mu_{22}\mu_{23}+5\mu_{13}^2\mu_{21}^2\mu_{22}\mu_{23}+3\mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23}+3\mu_{13}^4\mu_{22}^2+3\mu_{13}^5\mu_{22}$
$+\mu_{21}^3\mu_{22}^2\mu_{23}+\mu_{13}^3\mu_{21}^3+2\mu_{13}\mu_{21}^3\mu_{22}\mu_{23}+\mu_{13}^3\mu_{22}^3),$

$\psi_{11}((2,2),0,1) = \mu_{13}^2\mu_{21}\mu_{22}(\mu_{21}^3\mu_{22}^2\mu_{23}+3\mu_{13}^4\mu_{21}\mu_{23}+\mu_{13}^2\mu_{21}^3\mu_{23}+3\mu_{13}^3\mu_{21}^2\mu_{23}+\mu_{13}^6$
$+3\mu_{13}^4\mu_{22}^2+\mu_{13}^3\mu_{22}^3+\mu_{13}^2\mu_{22}^3\mu_{23}+3\mu_{13}^3\mu_{22}^2\mu_{23}+\mu_{13}^5\mu_{23}+3\mu_{13}^5\mu_{22}+3\mu_{13}^4\mu_{22}\mu_{23}$
$+\mu_{13}\mu_{21}^2\mu_{22}^2\mu_{23}+5\mu_{13}^2\mu_{21}^2\mu_{22}\mu_{23}+2\mu_{13}\mu_{21}^3\mu_{22}\mu_{23}+6\mu_{13}^3\mu_{21}\mu_{22}\mu_{23}+5\mu_{13}^3\mu_{21}^2\mu_{22}$
$+3\mu_{13}^3\mu_{21}\mu_{22}^2+6\mu_{13}^4\mu_{21}\mu_{22}+2\mu_{13}^2\mu_{21}^3\mu_{22}+3\mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23}+\mu_{13}^2\mu_{21}^2\mu_{22}^2+\mu_{13}^3\mu_{21}^3$
$+\mu_{13}\mu_{21}^3\mu_{22}^2+3\mu_{13}^5\mu_{21}+3\mu_{13}^4\mu_{21}^2),$

When $B_2 = B_3 = 1$, we obtain

$$\psi(1,1) = \mu_{13}^7\mu_{22}^3 + 4\mu_{13}^6\mu_{22}^4 + \mu_{22}^5\mu_{21}^5 + 3\mu_{13}^3\mu_{21}^5\mu_{22}\mu_{23} + \mu_{13}^8\mu_{21}\mu_{23} + \mu_{21}\mu_{22}^5\mu_{23}^4$$
$$+3\mu_{13}\mu_{22}^4\mu_{23}^5 + 4\mu_{13}^5\mu_{21}\mu_{22}^4 + 4\mu_{13}^3\mu_{21}^4\mu_{22}^3 + 4\mu_{13}^2\mu_{22}^3\mu_{23}^5 + 17\mu_{13}^6\mu_{22}^3\mu_{23}$$
$$+19\mu_{13}^5\mu_{22}^3\mu_{23}^2 + 13\mu_{13}^4\mu_{21}^3\mu_{22}^3 + 6\mu_{13}^6\mu_{21}\mu_{22}^3 + 4\mu_{13}^7\mu_{21}\mu_{22}^2 + 11\mu_{13}^4\mu_{22}^2\mu_{23}^4$$
$$+4\mu_{13}^3\mu_{22}^2\mu_{23}^5 + 11\mu_{13}^7\mu_{22}^2\mu_{23} + 2\mu_{13}^6\mu_{22}^2\mu_{23}^2 + 7\mu_{13}^5\mu_{21}^3\mu_{22}^2 + \mu_{13}^8\mu_{21}\mu_{22}$$
$$+12\mu_{13}^5\mu_{22}^4\mu_{23} + 2\mu_{13}\mu_{21}\mu_{22}^5\mu_{23}^3 + \mu_{13}^3\mu_{21}\mu_{22}^5\mu_{23} + 4\mu_{13}^2\mu_{21}^5\mu_{22}^2\mu_{23} + \mu_{13}^5\mu_{21}^5$$
$$+3\mu_{13}^7\mu_{21}\mu_{23}^2 + 4\mu_{13}^6\mu_{23}^4 + 6\mu_{13}^7\mu_{23}^3 + 3\mu_{13}^2\mu_{22}^4\mu_{23}^4 + 6\mu_{13}^4\mu_{22}^4\mu_{23}^2 + 5\mu_{13}^3\mu_{22}^4\mu_{23}^3$$
$$+12\mu_{13}^6\mu_{22}\mu_{23}^3 + 4\mu_{13}^7\mu_{21}^2\mu_{22} + 2\mu_{13}^8\mu_{22}\mu_{23} + 5\mu_{13}^5\mu_{21}^4\mu_{22} + 3\mu_{13}^4\mu_{21}^5\mu_{22}$$
$$+\mu_{13}\mu_{22}^5\mu_{23}^4 + 4\mu_{13}^8\mu_{23}^2 + \mu_{13}^9\mu_{23} + 4\mu_{13}^5\mu_{21}\mu_{23}^4 + \mu_{13}^4\mu_{21}\mu_{23}^5 + 2\mu_{13}^6\mu_{21}\mu_{23}^3$$
$$+4\mu_{13}^7\mu_{22}^3 + \mu_{13}^9\mu_{22} + \mu_{21}\mu_{22}^4\mu_{23}^5 + \mu_{13}^2\mu_{21}^3\mu_{22}^5 + \mu_{13}^4\mu_{21}\mu_{22}^5 + 2\mu_{13}\mu_{21}\mu_{22}^3\mu_{23}^5$$
$$+\mu_{13}^5\mu_{22}^5 + \mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23}^5 + 16\mu_{13}^4\mu_{21}\mu_{22}^3\mu_{23}^2 + 17\mu_{13}^5\mu_{21}\mu_{22}^2\mu_{23}^2 + 7\mu_{13}^3\mu_{21}^4\mu_{22}^3$$
$$+2\mu_{13}^4\mu_{21}\mu_{22}^2\mu_{23}^3 + 4\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23}^4 + 8\mu_{13}^2\mu_{21}\mu_{22}^3\mu_{23}^4 + 11\mu_{13}^4\mu_{21}\mu_{22}\mu_{23}^4$$
$$+17\mu_{13}^5\mu_{21}\mu_{22}\mu_{23}^3 + 12\mu_{13}^6\mu_{21}\mu_{22}\mu_{23}^2 + 6\mu_{13}^7\mu_{21}\mu_{22}\mu_{23} + 7\mu_{13}^5\mu_{21}^2\mu_{22}^2\mu_{23},$$

$$\psi_1((1,0),1,1) = \mu_{13}^2\mu_{21}\mu_{22}(2\mu_{13}^2\mu_{21}\mu_{22}^3\mu_{23} + 6\mu_{13}^2\mu_{21}^2\mu_{22}^2\mu_{23} + 2\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23}$$
$$+\mu_{13}\mu_{21}\mu_{22}^2\mu_{23}^3 + 4\mu_{13}^3\mu_{21}^3\mu_{23} + 3\mu_{13}^5\mu_{21}\mu_{23} + \mu_{13}^2\mu_{21}\mu_{23}^4 + 2\mu_{13}^4\mu_{21}\mu_{23}^2 + 2\mu_{13}^6\mu_{23}$$
$$+2\mu_{13}^4\mu_{23}^3 + 6\mu_{13}^5\mu_{23}^2 + \mu_{13}^7 + \mu_{13}^3\mu_{23}^4 + 2\mu_{13}^3\mu_{22}^3\mu_{23} + \mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{13}\mu_{22}^2\mu_{23}^4$$
$$+5\mu_{13}^3\mu_{21}\mu_{22}\mu_{23}^2 + 5\mu_{13}^4\mu_{21}\mu_{22}\mu_{23} + 5\mu_{13}^2\mu_{21}^3\mu_{22}\mu_{23} + 3\mu_{13}\mu_{21}^4\mu_{22}\mu_{23} + \mu_{13}^6\mu_{21}$$
$$+6\mu_{13}^3\mu_{21}^2\mu_{22}^2 + 3\mu_{13}^4\mu_{22}^2\mu_{23} + \mu_{13}^2\mu_{21}^3\mu_{22}^2 + 9\mu_{13}^5\mu_{22}\mu_{23} + 12\mu_{13}^4\mu_{22}\mu_{23}^2 + \mu_{13}^3\mu_{22}^4$$
$$+4\mu_{13}^5\mu_{21}\mu_{23} + 2\mu_{13}^6\mu_{22} + 2\mu_{13}^5\mu_{22}^2),$$

$$\psi_1((2,0),1,1) = \mu_{13}\mu_{21}\mu_{22}(\mu_{13}^3\mu_{22}^4\mu_{23} + 2\mu_{13}^5\mu_{23}^3 + 4\mu_{13}^7\mu_{23} + 3\mu_{13}^4\mu_{22}^4 + \mu_{13}^7\mu_{21}$$
$$+11\mu_{13}^5\mu_{22}\mu_{23}^2 + 6\mu_{13}^6\mu_{22}\mu_{23} + 8\mu_{13}^4\mu_{22}\mu_{23}^3 + \mu_{13}\mu_{22}^3\mu_{23}^4 + \mu_{13}^4\mu_{23}^4 + 6\mu_{13}^6\mu_{23}^2$$
$$+2\mu_{13}^3\mu_{22}\mu_{23}^4 + 2\mu_{13}^2\mu_{21}^4\mu_{22}^2 + \mu_{13}^8 + \mu_{13}^2\mu_{22}^3\mu_{23}^3 + 3\mu_{13}^5\mu_{23}^3 + 3\mu_{13}^6\mu_{22}^2 + 4\mu_{13}^7\mu_{22}$$
$$+3\mu_{13}^3\mu_{22}^3\mu_{23}^2 + 3\mu_{13}^4\mu_{22}^3\mu_{23} + 6\mu_{13}^4\mu_{22}^2\mu_{23}^2 + 9\mu_{13}^5\mu_{22}^2\mu_{23} + 3\mu_{13}^3\mu_{22}^2\mu_{23}^3$$
$$+6\mu_{13}^5\mu_{21}\mu_{22}\mu_{23} + 15\mu_{13}^4\mu_{21}\mu_{22}\mu_{23}^2 + 2\mu_{13}^2\mu_{21}\mu_{22}\mu_{23}^4 + 10\mu_{13}^3\mu_{21}\mu_{22}\mu_{23}^3$$

$$+\mu_{13}^2\mu_{21}\mu_{22}^4\mu_{23} + \mu_{13}^3\mu_{21}\mu_{22}^4 + 4\mu_{13}^4\mu_{21}\mu_{22}^3 + 6\mu_{13}^5\mu_{21}\mu_{23}^2 + \mu_{13}^3\mu_{21}\mu_{23}^4 + 4\mu_{13}^4\mu_{21}\mu_{23}^3$$

$$+3\mu_{13}\mu_{21}\mu_{22}^2\mu_{23}^4 + 15\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + 4\mu_{13}^4\mu_{21}\mu_{22}^2\mu_{23} + 6\mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23}^3),$$

$$\psi_2((0,1),1,1) = \mu_{13}\mu_{21}^2\mu_{22}(\mu_{13}^7 + \mu_{13}^2\mu_{21}\mu_{22}^4 + 6\mu_{13}^5\mu_{22}^2 + 2\mu_{13}^4\mu_{22}^2\mu_{23} + 8\mu_{13}^4\mu_{21}^2\mu_{22}$$

$$+\mu_{13}^4\mu_{23}^3 + 4\mu_{13}^6\mu_{22} + 3\mu_{13}^3\mu_{22}\mu_{23}^3 + 3\mu_{13}\mu_{22}^3\mu_{23}^3 + 4\mu_{13}^4\mu_{22}^3 + \mu_{13}\mu_{22}^4\mu_{23}^2 + \mu_{22}^4\mu_{23}^3$$

$$+4\mu_{13}^3\mu_{22}^2\mu_{23}^2 + 2\mu_{13}^2\mu_{21}^3\mu_{22}^2 + 3\mu_{13}^3\mu_{22}^3\mu_{23} + 12\mu_{13}^5\mu_{22}\mu_{23} + 2\mu_{13}^6\mu_{23} + 3\mu_{13}^5\mu_{21}^2$$

$$+2\mu_{13}^2\mu_{22}^3\mu_{23}^2),$$

$$\psi_2((0,2),1,1) = \mu_{13}\mu_{21}^3\mu_{22}(\mu_{13}^6 + 2\mu_{13}^5\mu_{21} + 2\mu_{13}\mu_{21}^2\mu_{22}^3 + 4\mu_{13}^3\mu_{22}^3 + 2\mu_{13}^2\mu_{22}^4 + \mu_{13}^5\mu_{22}$$

$$+3\mu_{13}^4\mu_{22}^2 + 4\mu_{13}^2\mu_{22}^2\mu_{23}^2 + \mu_{22}^4\mu_{23}^2 + \mu_{13}\mu_{22}^4\mu_{23} + 7\mu_{13}^3\mu_{22}^2\mu_{23} + 3\mu_{13}^3\mu_{22}\mu_{23}^2$$

$$+6\mu_{13}^4\mu_{22}\mu_{23} + 4\mu_{13}^2\mu_{22}^3\mu_{23} + \mu_{13}^4\mu_{23}^2),$$

$$\psi_2((0,3),1,1) = \mu_{13}\mu_{21}^4\mu_{22}(\mu_{13}^5 + 2\mu_{13}\mu_{22}^4 + 4\mu_{13}^2\mu_{22}^3 + 3\mu_{13}^4\mu_{23} + 2\mu_{13}\mu_{22}^3\mu_{23} + \mu_{13}^3\mu_{22}^2$$

$$+3\mu_{13}^3\mu_{21}\mu_{22} + \mu_{22}^4\mu_{23} + 4\mu_{13}^4\mu_{22} + 2\mu_{13}^2\mu_{21}\mu_{22}^2),$$

$$\psi_3((0,1),1,1) = \mu_{13}^4\mu_{21}\mu_{22}(\mu_{13}^5 + \mu_{13}^3\mu_{23}^2 + 3\mu_{13}^4\mu_{23} + 4\mu_{13}^3\mu_{21}\mu_{22} + 3\mu_{13}^2\mu_{21}^2\mu_{22}$$

$$+3\mu_{13}^2\mu_{21}^2\mu_{23} + 4\mu_{13}^4\mu_{22} + 6\mu_{13}^2\mu_{21}\mu_{22}^2 + 4\mu_{13}\mu_{21}\mu_{22}^3 + 7\mu_{13}^2\mu_{22}^2\mu_{23} + \mu_{13}\mu_{22}^4$$

$$+\mu_{13}^4\mu_{21} + \mu_{21}\mu_{22}^4 + \mu_{13}^2\mu_{23}^3 + 2\mu_{13}^3\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}\mu_{23}^3 + 6\mu_{13}^3\mu_{21}\mu_{23} + 3\mu_{13}^2\mu_{22}^3$$

$$+2\mu_{13}\mu_{22}^3\mu_{23} + 5\mu_{13}\mu_{21}\mu_{22}^2\mu_{23} + 2\mu_{21}\mu_{22}^3\mu_{23} + \mu_{21}\mu_{21}^2\mu_{23}^2 + 2\mu_{21}\mu_{22}\mu_{23}^3),$$

$$\psi_3((0,2),1,1) = \mu_{13}^3\mu_{21}\mu_{22}(4\mu_{13}^4\mu_{22}^2 + 2\mu_{13}^2\mu_{23}^4 + 5\mu_{13}^3\mu_{22}\mu_{23}^2 + 2\mu_{13}^5\mu_{22} + 4\mu_{13}^2\mu_{22}\mu_{23}^3$$

$$+3\mu_{13}\mu_{22}^3\mu_{23}^2 + \mu_{13}\mu_{21}\mu_{22}^4 + 5\mu_{13}^4\mu_{21}\mu_{22} + 2\mu_{13}^2\mu_{22}^3\mu_{23} + 2\mu_{21}\mu_{22}^4\mu_{23} + 11\mu_{13}^3\mu_{21}\mu_{22}^2$$

$$+\mu_{13}^5\mu_{21} + \mu_{21}^3\mu_{22}^2\mu_{23} + \mu_{13}\mu_{21}^3\mu_{22}^2 + 4\mu_{13}^3\mu_{22}^3 + 7\mu_{13}^2\mu_{22}^2\mu_{23}^2 + \mu_{21}\mu_{22}^3\mu_{23}^2$$

$$+2\mu_{13}^3\mu_{23}^3 + 2\mu_{13}^2\mu_{21}\mu_{23}^3 + 5\mu_{13}^4\mu_{23}^2 + 5\mu_{13}^3\mu_{21}^2\mu_{23} + 4\mu_{13}^4\mu_{21}\mu_{23} + \mu_{13}^6 + 4\mu_{13}^5\mu_{23}$$

$$+2\mu_{13}\mu_{21}\mu_{22}\mu_{23}^3 + 2\mu_{13}\mu_{21}^2\mu_{22}^2\mu_{23} + 5\mu_{13}^2\mu_{21}^2\mu_{22}\mu_{23} + 3\mu_{13}\mu_{21}\mu_{22}^3\mu_{23} + \mu_{13}\mu_{21}\mu_{22}^4$$

$$+3\mu_{13}^3\mu_{21}\mu_{22}\mu_{23} + 5\mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23} + 2\mu_{13}^4\mu_{21}\mu_{22} + 4\mu_{13}^2\mu_{21}\mu_{22}^3 + 6\mu_{13}^3\mu_{21}\mu_{22}^2),$$

$$\psi_3((0,3),1,1) = \mu_{13}^2\mu_{21}\mu_{23}(2\mu_{13}^6\mu_{21} + 4\mu_{13}^5\mu_{21}\mu_{23} + 2\mu_{13}^3\mu_{21}^3\mu_{23} + \mu_{13}^2\mu_{21}\mu_{22}^4 + \mu_{13}^7$$

$$+4\mu_{13}^5\mu_{21}\mu_{22} + 4\mu_{13}^4\mu_{22}^3 + 6\mu_{13}^5\mu_{22}^2 + \mu_{13}^3\mu_{22}^4 + 2\mu_{13}^2\mu_{22}^2\mu_{23}^3 + 6\mu_{13}^3\mu_{22}\mu_{23}^3$$

$$+9\mu_{13}^4\mu_{22}\mu_{23}^2 + 2\mu_{13}\mu_{22}^4\mu_{22} + 13\mu_{13}^3\mu_{22}^2\mu_{23}^2 + \mu_{21}\mu_{22}^4\mu_{23}^2 + 3\mu_{13}^3\mu_{22}^3\mu_{23} + \mu_{13}^2\mu_{22}^4\mu_{23}$$

$+13\mu_{13}^4\mu_{22}^2\mu_{23} + 12\mu_{13}^5\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}\mu_{22}^2\mu_{23}^3 + 6\mu_{13}^2\mu_{21}\mu_{22}\mu_{23}^3 + 15\mu_{13}^3\mu_{21}^2\mu_{22}\mu_{23}$

$+2\mu_{13}^5\mu_{21}^2 + 6\mu_{13}^4\mu_{21}^2\mu_{23} + 2\mu_{13}^2\mu_{21}^3\mu_{22}^2 + 3\mu_{13}^4\mu_{21}^3 + 4\mu_{13}^3\mu_{21}\mu_{22}^3 + 3\mu_{13}^4\mu_{21}\mu_{22}^2$

$+4\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23} + 4\mu_{13}^6\mu_{22}),$

$\psi_4((1,3),1,1) = \mu_{13}\mu_{21}^3\mu_{23}^4(\mu_{13}^3 + 3\mu_{13}\mu_{22}^2 + 3\mu_{13}^2\mu_{22} + \mu_{21}^3),$

$\psi_5((1,3),1,1) = \mu_{13}\mu_{21}\mu_{23}(2\mu_{13}^3\mu_{21}^3\mu_{22}\mu_{23} + 4\mu_{13}\mu_{21}^3\mu_{22}^3\mu_{23} + 2\mu_{13}^2\mu_{21}^3\mu_{22}^2\mu_{23} + \mu_{13}^8$

$+2\mu_{13}^4\mu_{21}^3\mu_{23} + 2\mu_{13}^5\mu_{21}\mu_{23}^2 + \mu_{21}\mu_{22}^4\mu_{23}^3 + 4\mu_{13}^7\mu_{22} + 4\mu_{13}^3\mu_{22}^3\mu_{23}^2 + \mu_{13}^7\mu_{23} + 6\mu_{13}^6\mu_{22}^2$

$+2\mu_{13}^4\mu_{22}^4 + 4\mu_{13}^6\mu_{21}\mu_{22} + \mu_{13}\mu_{21}\mu_{22}^2\mu_{23}^4 + \mu_{13}^4\mu_{23}^4 + 4\mu_{13}^5\mu_{23}^3 + 4\mu_{13}^6\mu_{21}\mu_{23} + \mu_{13}^6\mu_{23}^2$

$+11\mu_{13}^4\mu_{21}\mu_{22}\mu_{23}^2 + 5\mu_{13}^2\mu_{21}\mu_{22}^3\mu_{23}^2 + 10\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{13}\mu_{21}\mu_{22}^4\mu_{23}^2 + \mu_{13}^3\mu_{21}\mu_{23}^4$

$+4\mu_{13}^7\mu_{23} + 2\mu_{13}^3\mu_{22}\mu_{23}^4 + \mu_{13}^2\mu_{22}^2\mu_{23}^4 + 4\mu_{13}^2\mu_{22}^3\mu_{23}^3 + 8\mu_{13}^3\mu_{22}^2\mu_{23}^3 + 10\mu_{13}^4\mu_{22}\mu_{23}^3$

$+4\mu_{13}^4\mu_{21}\mu_{23}^3 + 6\mu_{13}^5\mu_{21}\mu_{22}^2 + \mu_{13}^3\mu_{21}\mu_{22}^4 + \mu_{13}\mu_{22}^4\mu_{23}^3 + \mu_{13}^2\mu_{22}^4\mu_{23}^2 + 13\mu_{13}^4\mu_{22}^2\mu_{23}^2$

$+4\mu_{13}^5\mu_{21}^2\mu_{22} + 5\mu_{13}^3\mu_{21}^2\mu_{22}^3 + 6\mu_{13}^4\mu_{22}^3\mu_{23} + 2\mu_{13}^5\mu_{22}^2\mu_{23} + \mu_{13}^3\mu_{22}^4\mu_{23}),$

$\psi_6((1,1),1,1) = \mu_{13}\mu_{21}^3\mu_{22}^2(2\mu_{22}^3\mu_{21}^2 + \mu_{13}\mu_{21}^2\mu_{22}^2 + 2\mu_{13}^2\mu_{21}\mu_{22}^2 + \mu_{13}\mu_{21}\mu_{22}^3 + \mu_{13}^5$

$+4\mu_{13}^4\mu_{21} + \mu_{13}^3\mu_{21}^2 + 5\mu_{13}^3\mu_{23}\mu_{22} + 4\mu_{13}^2\mu_{21}^2\mu_{22} + \mu_{13}^2\mu_{21}^3 + \mu_{13}^4\mu_{22} + 3\mu_{13}^3\mu_{22}^2),$

$\psi_6((2,1),1,1) = \mu_{13}\mu_{21}^4\mu_{22}^3(\mu_{21}\mu_{22}^2 + \mu_{13}\mu_{22}^2 + 3\mu_{13}\mu_{21}\mu_{22} + \mu_{13}^2\mu_{21} + \mu_{13}^3$

$+2\mu_{13}^2\mu_{22}),$

$\psi_6((1,2),1,1) = \mu_{13}\mu_{21}^4\mu_{22}^2(\mu_{21}\mu_{22}^3 + 3\mu_{13}^2\mu_{22}^2 + 3\mu_{13}^2\mu_{23}\mu_{22} + \mu_{13}^3\mu_{21} + \mu_{13}\mu_{22}^3$

$+\mu_{13}^3\mu_{22} + \mu_{13}^4 + 2\mu_{13}\mu_{21}\mu_{22}^2),$

$\psi_6((2,2),1,1) = \mu_{13}\mu_{21}^5\mu_{22}^3(\mu_{22}^2 + 2\mu_{13}\mu_{22} + 2\mu_{13}^2),$

$\psi_7((1,1),1,1) = \mu_{13}\mu_{21}^3\mu_{22}^2(5\mu_{13}^3\mu_{21}\mu_{22} + 3\mu_{13}^3\mu_{21}^2 + 2\mu_{13}^2\mu_{21}^2\mu_{22} + 3\mu_{13}\mu_{21}^2\mu_{22}^2 + \mu_{13}^5$

$+2\mu_{13}^2\mu_{21}\mu_{22}^2 + \mu_{13}^2\mu_{21}^3 + 3\mu_{13}^4\mu_{22} + \mu_{21}^2\mu_{22}^3 + \mu_{13}\mu_{21}\mu_{22}^3 + \mu_{13}^3\mu_{21}^2),$

$\psi_7((2,1),1,1) = \mu_{13}\mu_{21}^4\mu_{22}^3(\mu_{22}^2\mu_{23} + \mu_{13}\mu_{22}^2 + 2\mu_{13}\mu_{22}\mu_{23} + \mu_{13}^2\mu_{23} + \mu_{13}^3 + 2\mu_{13}^2\mu_{22}),$

$\psi_7((1,2),1,1) = \mu_{13}\mu_{21}^4\mu_{22}^2(2\mu_{21}\mu_{22}^3 + 3\mu_{13}^2\mu_{22}^2 + 2\mu_{13}^2\mu_{21}\mu_{22} + \mu_{13}\mu_{21}^3 + \mu_{13}^4 + 3\mu_{13}^3\mu_{22}$

$\mu_{13}^3\mu_{21} + 4\mu_{13}\mu_{21}\mu_{22}^2),$

$\psi_7((2,2),1,1) = \mu_{13}\mu_{21}^5\mu_{22}^3(\mu_{22}^2 + 2\mu_{13}\mu_{22} + 2\mu_{13}^2),$

$$\psi_8((1,1),1,1) = \mu_{13}^2\mu_{21}\mu_{23}(\mu_{13}^3\mu_{21}^4 + \mu_{13}^3\mu_{22}^4 + 2\mu_{13}^4\mu_{21}^3 + 4\mu_{13}^6\mu_{22} + 2\mu_{13}^5\mu_{22}^2 + \mu_{13}^7$$
$$+3\mu_{13}^3\mu_{21}\mu_{23}^3 + 4\mu_{13}^3\mu_{21}\mu_{22}^3 + 6\mu_{13}^4\mu_{21}\mu_{22}^2 + 5\mu_{13}^4\mu_{21}^2\mu_{22} + 7\mu_{13}^3\mu_{21}^2\mu_{22}^2 + 6\mu_{13}^5\mu_{23}^2$$
$$+6\mu_{13}^4\mu_{21}\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}^4\mu_{22}\mu_{23} + 14\mu_{13}^3\mu_{21}^2\mu_{22}\mu_{23} + 2\mu_{13}^2\mu_{21}^3\mu_{22}\mu_{23} + \mu_{13}^6\mu_{21}$$
$$+7\mu_{13}^4\mu_{22}^2\mu_{23} + \mu_{13}^2\mu_{22}^4\mu_{23} + 3\mu_{13}^2\mu_{22}^2\mu_{23}^3 + 12\mu_{13}^5\mu_{22}\mu_{23} + 6\mu_{13}^3\mu_{22}^3\mu_{23} + 4\mu_{13}^4\mu_{22}^3$$
$$+4\mu_{13}^5\mu_{21}\mu_{22} + 6\mu_{13}^4\mu_{21}^2\mu_{23} + 4\mu_{13}^6\mu_{21} + 4\mu_{13}^5\mu_{21}\mu_{23} + 6\mu_{13}^2\mu_{21}\mu_{22}^3\mu_{23} + \mu_{21}^4\mu_{22}^2\mu_{23}$$
$$+8\mu_{13}^3\mu_{22}\mu_{23}^3 + 2\mu_{13}^2\mu_{22}^3\mu_{23}^2 + 3\mu_{13}\mu_{21}^3\mu_{22}^2\mu_{23} + 6\mu_{13}^2\mu_{21}^2\mu_{22}^2\mu_{23} + 7\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23}$$
$$+\mu_{13}\mu_{22}^2\mu_{23}^4 + 2\mu_{13}^2\mu_{22}\mu_{23}^4 + \mu_{13}^2\mu_{21}\mu_{22}^4 + \mu_{13}\mu_{21}\mu_{22}^4\mu_{23} + \mu_{13}^2\mu_{21}\mu_{23}^4),$$
$$\psi_8((2,1),1,1) = \mu_{13}\mu_{22}\mu_{23}(2\mu_{13}^3\mu_{21}\mu_{22}^4 + \mu_{13}^2\mu_{21}^2\mu_{22}^4 + 4\mu_{13}^5\mu_{22}^3 + 4\mu_{13}^7\mu_{21} + \mu_{13}^4\mu_{22}^4$$
$$+13\mu_{13}^4\mu_{21}^2\mu_{22}\mu_{23} + 5\mu_{13}^3\mu_{21}^3\mu_{22}\mu_{23} + 5\mu_{13}^3\mu_{21}^3\mu_{22}^2 + 6\mu_{13}^4\mu_{21}\mu_{22}^3 + 4\mu_{13}^4\mu_{21}^2\mu_{22}^2$$
$$+\mu_{13}^7\mu_{21} + 4\mu_{13}^3\mu_{21}^2\mu_{22}^2\mu_{23} + 3\mu_{13}\mu_{21}\mu_{22}^2\mu_{23}^4 + 12\mu_{13}^5\mu_{21}\mu_{22}\mu_{23} + 3\mu_{13}^2\mu_{21}\mu_{22}\mu_{23}^4$$
$$+\mu_{13}\mu_{22}^3\mu_{23}^4 + \mu_{13}^4\mu_{23}^4 + 6\mu_{13}^6\mu_{23}^2 + 7\mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23}^3 + 13\mu_{13}^4\mu_{21}\mu_{22}^2\mu_{23} + \mu_{13}^8$$
$$+2\mu_{13}^3\mu_{21}\mu_{22}^3\mu_{23} + \mu_{13}^2\mu_{21}\mu_{22}^4\mu_{23} + 3\mu_{13}\mu_{21}^2\mu_{22}^4\mu_{23} + 4\mu_{13}^5\mu_{21}^3 + 6\mu_{13}^6\mu_{22}^2 + 4\mu_{13}^7\mu_{22}$$
$$+\mu_{13}^3\mu_{21}\mu_{22}^4 + 4\mu_{13}^4\mu_{21}\mu_{22}^3 + 6\mu_{13}^5\mu_{21}\mu_{22}^2 + 2\mu_{13}^6\mu_{22}\mu_{23} + \mu_{21}\mu_{22}^3\mu_{23}^4 + \mu_{13}^3\mu_{21}\mu_{23}^4$$
$$+2\mu_{13}^6\mu_{21}\mu_{23} + 6\mu_{13}^5\mu_{21}\mu_{22}^2 + 4\mu_{13}^4\mu_{21}\mu_{22}^3),$$
$$\psi_8((1,2),1,1) = \mu_{13}^2\mu_{21}\mu_{23}(\mu_{13}^4\mu_{21}^3 + 4\mu_{13}^5\mu_{21}\mu_{23} + 6\mu_{13}^5\mu_{21}^2 + 3\mu_{13}^3\mu_{21}^4 + 3\mu_{13}^3\mu_{21}\mu_{21}^3$$
$$+4\mu_{13}^4\mu_{22}^3 + 4\mu_{13}^6\mu_{22} + \mu_{13}^3\mu_{22}^4 + 4\mu_{13}^3\mu_{21}\mu_{23}^3 + \mu_{13}^2\mu_{21}\mu_{22}^4 + 6\mu_{13}^5\mu_{22}^2 + 4\mu_{13}^5\mu_{21}\mu_{22}$$
$$+\mu_{13}^2\mu_{21}^4\mu_{23} + 6\mu_{13}^2\mu_{21}^3\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}^4\mu_{22}\mu_{23} + 2\mu_{13}\mu_{21}^3\mu_{22}^2\mu_{23} + 5\mu_{13}\mu_{21}\mu_{22}^3\mu_{23}^2$$
$$+2\mu_{13}^4\mu_{21}\mu_{22}^2 + \mu_{13}^7 + \mu_{13}^6\mu_{21} + 13\mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23}^2 + \mu_{21}\mu_{22}^2\mu_{23}^4 + 3\mu_{21}\mu_{21}^3\mu_{22}^3$$
$$+4\mu_{13}^6\mu_{23} + 13\mu_{13}^4\mu_{22}^2\mu_{23} + 15\mu_{13}^3\mu_{21}\mu_{22}\mu_{23}^2 + 15\mu_{13}^4\mu_{22}\mu_{23}^2 + 9\mu_{13}^3\mu_{22}\mu_{23}^3$$
$$+\mu_{13}^2\mu_{22}^4\mu_{23} + \mu_{13}\mu_{22}^3\mu_{23}^3 + 4\mu_{13}\mu_{22}^4\mu_{23}^2 + 12\mu_{13}^4\mu_{21}\mu_{22}\mu_{23} + 6\mu_{13}^2\mu_{21}\mu_{22}^3\mu_{23}$$
$$+6\mu_{13}^4\mu_{21}\mu_{23}^2),$$
$$\psi_8((2,2),1,1) = \mu_{13}\mu_{21}\mu_{23}(2\mu_{13}^5\mu_{21}^2\mu_{23} + 4\mu_{13}^4\mu_{21}^3\mu_{23} + 2\mu_{13}^7\mu_{21} + \mu_{13}\mu_{22}^4\mu_{23}^3 + \mu_{13}^8$$
$$+\mu_{21}\mu_{21}^3\mu_{22}^4 + 2\mu_{13}^7\mu_{21} + 3\mu_{21}^4\mu_{22}^3\mu_{23} + \mu_{13}^3\mu_{21}\mu_{22}^4 + 6\mu_{13}^6\mu_{22}^2 + 4\mu_{13}^7\mu_{22} + 4\mu_{13}^5\mu_{21}^3$$
$$+4\mu_{13}^6\mu_{21}\mu_{23} + 6\mu_{13}^6\mu_{23}^2 + 6\mu_{13}^3\mu_{21}\mu_{22}^3\mu_{23} + \mu_{13}^3\mu_{21}\mu_{23}^4 + 2\mu_{13}^5\mu_{23}^3 + 2\mu_{13}^5\mu_{21}\mu_{22}^2$$

$+2\mu_{13}^4\mu_{21}\mu_{22}^2\mu_{23} + 5\mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23}^2 + 10\mu_{13}^3\mu_{21}\mu_{22}\mu_{23}^3 + \mu_{13}^2\mu_{21}\mu_{22}^4\mu_{23} + \mu_{13}^2\mu_{22}^4\mu_{23}^2$

$+3\mu_{13}^2\mu_{21}\mu_{22}\mu_{23}^4 + 8\mu_{13}^2\mu_{21}\mu_{22}^2\mu_{23}^3 + 5\mu_{13}^2\mu_{22}^3\mu_{23}^3 + 3\mu_{13}\mu_{21}\mu_{22}^2\mu_{23}^4 + 5\mu_{13}^4\mu_{21}^4$

$+11\mu_{13}^5\mu_{22}^2\mu_{23} + 6\mu_{13}^4\mu_{22}^3\mu_{23} + 13\mu_{13}^4\mu_{22}^2\mu_{23}^2 + \mu_{13}^3\mu_{22}^4\mu_{23} + 3\mu_{13}^3\mu_{22}\mu_{23}^4$

$+4\mu_{13}^3\mu_{21}^3\mu_{22}^2 + 5\mu_{13}^3\mu_{22}^3\mu_{23}^2 + 4\mu_{13}^2\mu_{22}^3\mu_{23}^3 + 3\mu_{13}^2\mu_{22}^2\mu_{23}^4 + 6\mu_{13}^4\mu_{21}^3\mu_{22} + \mu_{13}^3\mu_{21}\mu_{22}^2\mu_{23}^2$

$+4\mu_{13}^5\mu_{21}\mu_{23}^2 + \mu_{13}\mu_{22}^3\mu_{23}^4 + 4\mu_{13}\mu_{21}\mu_{22}^3\mu_{23}^3 + \mu_{13}\mu_{21}^2\mu_{22}^4\mu_{23}),$

$\psi_9((1,1),1,1) = \mu_{13}^3\mu_{21}^5\mu_{22}^3,$

$\psi_9((2,1),1,1) = 0,$

$\psi_9((1,2),1,1) = \mu_{13}^2\mu_{21}^5\mu_{22}^3(\mu_{13} + \mu_{22}),$

$\psi_9((2,2),1,1) = 0,$

$\psi_{10}((3,1),1,1) = \mu_{13}\mu_{21}^5\mu_{22}^2(\mu_{13}^2 + 2\mu_{13}\mu_{22} + 2\mu_{22}^2).$

$\psi_{10}((3,1),1,1) = \mu_{13}\mu_{21}^5\mu_{22}^4,$

$\psi_{11}((3,1),1,1) = \mu_{13}\mu_{21}^2\mu_{22}^2(2\mu_{22}^3\mu_{21}^3 + \mu_{13}^2\mu_{21}\mu_{22}^3 + 3\mu_{13}^2\mu_{21}^2\mu_{22}^2 + \mu_{13}^3\mu_{21}\mu_{22}^2 + \mu_{13}^3\mu_{22}^3$

$+2\mu_{13}^3\mu_{21}^2\mu_{22} + 7\mu_{13}^4\mu_{21}\mu_{22} + 2\mu_{13}^2\mu_{21}^3\mu_{22} + 3\mu_{13}\mu_{22}^2\mu_{23}^3 + \mu_{13}^3\mu_{21}^3 + 3\mu_{13}^5\mu_{21} + 2\mu_{13}^6$

$+3\mu_{13}^4\mu_{22}^2 + 3\mu_{13}^5\mu_{22} + \mu_{13}\mu_{21}^3\mu_{22}^2 + 2\mu_{13}^4\mu_{21}^2),$

$\psi_{11}((3,2),1,1) = \mu_{13}\mu_{21}^3\mu_{22}^3(\mu_{13}^2\mu_{22}^2 + \mu_{13}\mu_{22}^2\mu_{23} + \mu_{22}^2\mu_{23}^2 + 3\mu_{13}^2\mu_{22}\mu_{23} + 2\mu_{13}^3\mu_{22}$

$+2\mu_{13}\mu_{22}\mu_{23}^2 + \mu_{13}^4 + \mu_{13}^2\mu_{23}^2 + 2\mu_{13}^3\mu_{23}).$

These results together with Theorem 9.5.1 of Puterman [58] proves the optimality of the policy $\pi = (d_0)^\infty$. □

B.5 Proofs of Propositions 4.3.4, 4.3.5, and 4.3.6

Proof of Proposition 4.3.4 : The set of allowable actions in state $s \in S$ is

$$A_s = \begin{cases} a_{11} & \text{for } s = (0,0), \\ a_{23} & \text{for } s = (2,0), \\ a_{13} & \text{for } s = (1,2), \\ \{a_{11}, a_{21}, a_{22}\} & \text{for } s = (1,0), \\ \{a_{11}, a_{13}, a_{33}\} & \text{for } s \in \{(0,1),(0,2)\}, \\ \{a_{22}, a_{23}, a_{33}\} & \text{for } s = (2,1), \\ \{a_{11}, a_{13}, a_{21}, a_{22}, a_{23}, a_{33}\} & \text{for } s = (1,1). \end{cases}$$

Under our assumptions on the service rates ($\sum_{i=1}^{M} \mu_{ij} > 0$ for $j \in \{1, \ldots, N\}$ and $\mu_{13} = \mu_{22} = 0$), it is clear that $\mu_{12} > 0$ and $\mu_{23} > 0$. Hence, we can conclude that the policy described in the theorem corresponds to an irreducible Markov chain, and consequently we have a communicating Markov decision process. Thus, we can use the policy iteration algorithm for communicating models as described in Section 9.5.1 of Puterman [58]. We use the uniformization constant $q = \mu_{11} + \mu_{12} + \mu_{21} + \mu_{23}$.

First assume that $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{23}$ and start the policy iteration algorithm by choosing

$$d_0(s) = \begin{cases} a_{11} & \text{for } s = (0,0), \\ a_{13} & \text{for } s \in \{(0,1),(0,2)\}, \\ a_{21} & \text{for } s = (1,0), \\ a_{31} & \text{for } s \in \{(1,1),(1,2),(2,0),(2,1)\}. \end{cases}$$

Then, we proceed as in the proof of Proposition 4.3.1. In the calculations below, α_k for $k \in \{1, \ldots, 13\}$ and α are nonnegative constants when $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{23}$. They depend on the service rates and the state under consideration, and they are provided below.

First, consider the state $s = (1,0)$, and recall that $d_0(s) = a_{21}$. Some algebra shows that,

$$\Big(r((1,0),a_{21})+\sum_{s'\in S}p(s'|(1,0),a_{21})h(s')\Big)-r((1,0),a_{11})-\sum_{s'\in S}p(s'|(1,0),a_{11})h(s')$$
$$=\frac{\alpha_1}{\alpha}\geq 0,$$
$$\Big(r((1,0),a_{21})+\sum_{s'\in S}p(s'|(1,0),a_{21})h(s')\Big)-r((1,0),a_{22})-\sum_{s'\in S}p(s'|(1,0),a_{22})h(s')$$
$$=\frac{\alpha_2}{\alpha}\geq 0.$$

Recall that $d_0(s)=a_{13}$ for $s\in\{(0,1),(0,2)\}$. Then, we can show that

$$\Big(r((0,1),a_{13})+\sum_{s'\in S}p(s'|(0,1),a_{13})h(s')\Big)-r((0,1),a_{11})-\sum_{s'\in S}p(s'|(0,1),a_{11})h(s')$$
$$=\frac{\alpha_3}{\alpha}\geq 0,$$
$$\Big(r((0,1),a_{13})+\sum_{s'\in S}p(s'|(0,1),a_{13})h(s')\Big)-r((0,1),a_{33})-\sum_{s'\in S}p(s'|(0,1),a_{33})h(s')$$
$$=\frac{\alpha_4}{\alpha}\geq 0,$$
$$\Big(r((0,2),a_{13})+\sum_{s'\in S}p(s'|(0,2),a_{13})h(s')\Big)-r((0,2),a_{11})-\sum_{s'\in S}p(s'|(0,2),a_{11})h(s')$$
$$=\frac{\alpha_5}{\alpha}\geq 0,$$
$$\Big(r((0,2),a_{13})+\sum_{s'\in S}p(s'|(0,2),a_{13})h(s')\Big)-r((0,2),a_{33})-\sum_{s'\in S}p(s'|(0,2),a_{33})h(s')$$
$$=\frac{\alpha_6}{\alpha}\geq 0.$$

For $s=(1,1)$, recall that $d_0(s)=a_{23}$. Some algebra shows that

$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{11})-\sum_{s'\in S}p(s'|(1,1),a_{11})h(s')$$
$$=\frac{\alpha_7}{\alpha}\geq 0,$$
$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{22})-\sum_{s'\in S}p(s'|(1,1),a_{22})h(s')$$
$$=\frac{\alpha_8}{\alpha}\geq 0,$$
$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{33})-\sum_{s'\in S}p(s'|(1,1),a_{33})h(s')$$
$$=\frac{\alpha_9}{\alpha}\geq 0,$$

$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{13})-\sum_{s'\in S}p(s'|(1,1),a_{13})h(s')$$
$$=\frac{\alpha_{10}}{\alpha}\geq 0,$$
$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{21})-\sum_{s'\in S}p(s'|(1,1),a_{21})h(s')$$
$$=\frac{\alpha_{11}}{\alpha}\geq 0.$$

Finally, $d_0(s)=a_{23}$ for $s=(2,1)$. Some algebra shows that

$$\Big(r((2,1),a_{23})+\sum_{s'\in S}p(s'|(2,1),a_{23})h(s')\Big)-r((2,1),a_{22})-\sum_{s'\in S}p(s'|(2,1),a_{22})h(s')$$
$$=\frac{\alpha_{12}}{\alpha}\geq 0,$$
$$\Big(r((2,1),a_{23})+\sum_{s'\in S}p(s'|(2,1),a_{23})h(s')\Big)-r((2,1),a_{33})-\sum_{s'\in S}p(s'|(2,1),a_{33})h(s')$$
$$=\frac{\alpha_{13}}{\alpha}\geq 0.$$

Next, assume that $\mu_{11}\mu_{12}<\mu_{21}\mu_{23}$ and start the policy iteration algorithm by choosing

$$d_0'(s)=\begin{cases}a_{11} & \text{for } s\in\{(0,0),(0,1)\},\\ a_{13} & \text{for } s=(0,2),\\ a_{21} & \text{for } s=(1,0),\\ a_{31} & \text{for } s\in\{(1,1),(1,2),(2,0),(2,1)\}.\end{cases}$$

Then, we proceed as in the proof of Proposition 4.3.1. In the calculations below, α_k' for $k\in\{1,\ldots,13\}$ and α' are nonnegative constants when $\mu_{11}\mu_{12}<\mu_{21}\mu_{23}$. They depend on the service rates and the state under consideration, and they are provided below.

First, consider the state $s=(1,0)$, and recall that $d_0'(s)=a_{21}$. Some algebra shows that

$$\Big(r((1,0),a_{21})+\sum_{s'\in S}p(s'|(1,0),a_{21})h(s')\Big)-r((1,0),a_{11})-\sum_{s'\in S}p(s'|(1,0),a_{11})h(s')$$
$$=\frac{\alpha_1'}{\alpha'}\geq 0,$$

$$\Big(r((1,0),a_{21}) + \sum_{s'\in S} p(s'|(1,0),a_{21})h(s')\Big) - r((1,0),a_{22}) - \sum_{s'\in S} p(s'|(1,0),a_{22})h(s')$$
$$= \frac{\alpha'_2}{\alpha'} \geq 0.$$

Recall that $d'_0(s) = a_{11}$ for $s = (0,1)$. Then, we can show that

$$\Big(r((0,1),a_{11}) + \sum_{s'\in S} p(s'|(0,1),a_{11})h(s')\Big) - r((0,1),a_{13}) - \sum_{s'\in S} p(s'|(0,1),a_{13})h(s')$$
$$= \frac{\alpha'_3}{\alpha'} \geq 0,$$
$$\Big(r((0,1),a_{11}) + \sum_{s'\in S} p(s'|(0,1),a_{11})h(s')\Big) - r((0,1),a_{33}) - \sum_{s'\in S} p(s'|(0,1),a_{33})h(s')$$
$$= \frac{\alpha'_4}{\alpha'} \geq 0.$$

Recall that $d'_0(s) = a_{13}$ for $s = (0,2)$. Then, we can show that

$$\Big(r((0,2),a_{13}) + \sum_{s'\in S} p(s'|(0,2),a_{13})h(s')\Big) - r((0,2),a_{11}) - \sum_{s'\in S} p(s'|(0,2),a_{11})h(s')$$
$$= \frac{\alpha'_5}{\alpha'} \geq 0,$$
$$\Big(r((0,2),a_{13}) + \sum_{s'\in S} p(s'|(0,2),a_{13})h(s')\Big) - r((0,2),a_{33}) - \sum_{s'\in S} p(s'|(0,2),a_{33})h(s')$$
$$= \frac{\alpha'_6}{\alpha'} \geq 0.$$

For $s = (1,1)$, recall that $d'_0(s) = a_{23}$. Some algebra shows that

$$\Big(r((1,1),a_{23}) + \sum_{s'\in S} p(s'|(1,1),a_{23})h(s')\Big) - r((1,1),a_{11}) - \sum_{s'\in S} p(s'|(1,1),a_{11})h(s')$$
$$= \frac{\alpha'_7}{\alpha'} \geq 0,$$
$$\Big(r((1,1),a_{23}) + \sum_{s'\in S} p(s'|(1,1),a_{23})h(s')\Big) - r((1,1),a_{22}) - \sum_{s'\in S} p(s'|(1,1),a_{22})h(s')$$
$$= \frac{\alpha'_8}{\alpha'} \geq 0,$$
$$\Big(r((1,1),a_{23}) + \sum_{s'\in S} p(s'|(1,1),a_{23})h(s')\Big) - r((1,1),a_{33}) - \sum_{s'\in S} p(s'|(1,1),a_{33})h(s')$$
$$= \frac{\alpha'_9}{\alpha'} \geq 0,$$

$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{13})-\sum_{s'\in S}p(s'|(1,1),a_{13})h(s')$$
$$=\frac{\alpha'_{10}}{\alpha'}\geq 0,$$
$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{21})-\sum_{s'\in S}p(s'|(1,1),a_{21})h(s')$$
$$=\frac{\alpha'_{11}}{\alpha'}\geq 0.$$

Finally, $d'_0(s)=a_{23}$ for $s=(2,1)$. Some algebra shows that

$$\Big(r((2,1),a_{23})+\sum_{s'\in S}p(s'|(2,1),a_{23})h(s')\Big)-r((2,1),a_{22})-\sum_{s'\in S}p(s'|(2,1),a_{22})h(s')$$
$$=\frac{\alpha'_{12}}{\alpha'}\geq 0,$$
$$\Big(r((2,1),a_{23})+\sum_{s'\in S}p(s'|(2,1),a_{23})h(s')\Big)-r((2,1),a_{33})-\sum_{s'\in S}p(s'|(2,1),a_{33})h(s')$$
$$=\frac{\alpha'_{13}}{\alpha'}\geq 0.$$

$$\begin{aligned}
\alpha &= 2\mu_{11}^2\mu_{21}\mu_{23}^3+\mu_{11}^3\mu_{12}^3+\mu_{11}^3\mu_{12}^2\mu_{23}+2\mu_{11}\mu_{12}^2\mu_{21}^2\mu_{23}+\mu_{11}\mu_{12}^3\mu_{21}\mu_{23}+\mu_{12}^2\mu_{23}^4\\
&\quad+\mu_{11}^2\mu_{12}^2\mu_{21}^2+\mu_{11}^2\mu_{12}^3\mu_{21}+\mu_{11}^3\mu_{12}^2\mu_{21}+\mu_{11}\mu_{12}^3\mu_{23}^2+\mu_{11}^2\mu_{12}^3\mu_{23}+2\mu_{12}^2\mu_{21}^2\mu_{23}^2\\
&\quad+4\mu_{11}\mu_{12}\mu_{21}\mu_{23}^3+4\mu_{11}\mu_{12}^2\mu_{21}\mu_{23}^2+3\mu_{11}^2\mu_{12}\mu_{21}\mu_{23}^2+2\mu_{11}\mu_{12}\mu_{21}^2\mu_{23}^2+\mu_{11}^2\mu_{21}^2\mu_{23}^2\\
&\quad+\mu_{11}^3\mu_{12}\mu_{21}\mu_{23}+2\mu_{11}^2\mu_{12}\mu_{23}^3+2\mu_{11}^2\mu_{12}^2\mu_{23}^2+\mu_{11}^3\mu_{12}\mu_{23}^2+\mu_{11}^3\mu_{21}\mu_{23}^3+\mu_{12}^3\mu_{23}^3\\
&\quad+2\mu_{11}\mu_{21}^2\mu_{23}^3+\mu_{11}\mu_{21}\mu_{23}^4+\mu_{12}\mu_{21}\mu_{23}^4+\mu_{11}\mu_{12}\mu_{21}^4+3\mu_{11}\mu_{12}^2\mu_{23}^3+3\mu_{12}^2\mu_{21}\mu_{23}^3\\
&\quad+2\mu_{12}\mu_{21}^2\mu_{23}^3+\mu_{21}^2\mu_{23}^4+\mu_{11}^2\mu_{12}\mu_{21}^2\mu_{23}+3\mu_{11}^2\mu_{12}^2\mu_{22}\mu_{23}+\mu_{12}^3\mu_{21}\mu_{23}^2,\\
\alpha_1 &= \mu_{12}\mu_{23}(\mu_{11}+\mu_{21})(\mu_{11}^2\mu_{21}\mu_{23}+\mu_{11}^2\mu_{12}^2+\mu_{11}^2\mu_{12}\mu_{21}+\mu_{11}\mu_{12}\mu_{21}\mu_{23}+\mu_{21}\mu_{23}^3\\
&\quad+2\mu_{11}\mu_{21}\mu_{23}^2+\mu_{12}\mu_{21}\mu_{23}^2),\\
\alpha_2 &= \mu_{12}^2\mu_{21}\mu_{23}^2(\mu_{11}^2+\mu_{11}\mu_{12}+2\mu_{11}\mu_{23}+\mu_{11}\mu_{21}+\mu_{12}\mu_{23}+\mu_{23}^2+\mu_{21}\mu_{23}),\\
\alpha_3 &= \mu_{12}\mu_{23}(\mu_{11}+\mu_{21})(\mu_{11}\mu_{12}-\mu_{21}\mu_{23})(\mu_{11}\mu_{12}+\mu_{11}\mu_{21}+\mu_{11}\mu_{23}),\\
\alpha_4 &= \mu_{11}\mu_{12}\mu_{23}^2(\mu_{11}^2\mu_{21}+\mu_{11}^2\mu_{21}+2\mu_{11}\mu_{21}\mu_{23}+\mu_{11}\mu_{21}^2+\mu_{11}\mu_{12}^2+2\mu_{11}\mu_{12}\mu_{21}\\
&\quad+2\mu_{11}\mu_{12}\mu_{23}+2\mu_{12}\mu_{21}\mu_{23}+\mu_{21}\mu_{23}^2+\mu_{21}^2\mu_{23}+\mu_{12}\mu_{23}^2+\mu_{12}^2\mu_{23}),
\end{aligned}$$

$$\alpha_5 = \mu_{12}\mu_{23}(\mu_{11}+\mu_{21})(\mu_{11}^2\mu_{12}\mu_{23} + \mu_{11}^2\mu_{21}\mu_{23} + \mu_{11}^2\mu_{12}^2 + \mu_{11}^2\mu_{12}\mu_{21} + \mu_{11}\mu_{12}^2\mu_{23}$$
$$+2\mu_{11}\mu_{12}\mu_{21}\mu_{23} + \mu_{11}\mu_{12}\mu_{23}^2 + \mu_{11}\mu_{21}\mu_{23}^2 - \mu_{21}^2\mu_{23}^2),$$
$$\alpha_6 = \mu_{11}\mu_{12}\mu_{23}^3(\mu_{11}\mu_{21} + \mu_{11}\mu_{12} + \mu_{12}^2 + \mu_{21}\mu_{23} + 2\mu_{12}\mu_{21} + \mu_{21}^2 + \mu_{12}\mu_{23}),$$
$$\alpha_7 = \frac{\mu_{12}\mu_{23}}{\mu_{11}+\mu_{23}}(\mu_{11}+\mu_{21})(\mu_{11}^2\mu_{21}^2\mu_{23} + \mu_{11}^2\mu_{12}^2\mu_{21} + 2\mu_{11}^2\mu_{12}^2\mu_{23} + \mu_{11}^2\mu_{12}^3 + \mu_{21}\mu_{23}^4$$
$$+2\mu_{11}^2\mu_{12}\mu_{21}\mu_{23} + 2\mu_{11}\mu_{21}\mu_{23}^3 + 2\mu_{11}\mu_{12}^2\mu_{21}\mu_{23} + 2\mu_{11}\mu_{12}\mu_{21}\mu_{23}^2 + \mu_{11}\mu_{12}^2\mu_{23}^2$$
$$+\mu_{11}\mu_{12}^3\mu_{23} - \mu_{12}\mu_{21}^2\mu_{23}^2 + \mu_{12}\mu_{21}\mu_{23}^3),$$
$$\alpha_8 = \mu_{12}^2\mu_{23}(\mu_{11}^3\mu_{21} + \mu_{11}^3\mu_{12} + \mu_{11}^2\mu_{12}\mu_{21} + 2\mu_{11}^2\mu_{21}\mu_{23} + \mu_{11}^2\mu_{12}\mu_{23} + \mu_{12}\mu_{21}\mu_{23}^2$$
$$+\mu_{11}^2\mu_{21}^2 + \mu_{11}\mu_{12}\mu_{21}\mu_{23} + 2\mu_{11}\mu_{21}^2\mu_{23} + 2\mu_{11}\mu_{21}\mu_{23}^2 + \mu_{21}\mu_{23}^3 + 2\mu_{21}^2\mu_{23}^2),$$
$$\alpha_9 = \mu_{12}\mu_{23}^2(\mu_{11}^3\mu_{21} + \mu_{11}^3\mu_{12} + 2\mu_{11}^2\mu_{12}\mu_{23} + \mu_{11}^2\mu_{12}\mu_{21} + \mu_{11}^2\mu_{21}^2 + 2\mu_{11}^2\mu_{21}\mu_{23}$$
$$+\mu_{11}\mu_{12}^2\mu_{23} + 2\mu_{11}\mu_{12}\mu_{21}\mu_{23} + \mu_{11}\mu_{12}\mu_{23}^2 + \mu_{11}\mu_{21}\mu_{23}^2 + 2\mu_{11}\mu_{21}^2\mu_{23} + \mu_{21}^2\mu_{23}^2),$$
$$\alpha_{10} = \mu_{12}\mu_{23}^2(\mu_{11}+\mu_{21})(\mu_{11}^2\mu_{21}\mu_{23} + \mu_{11}^2\mu_{12}^2 + \mu_{11}^2\mu_{12}\mu_{21} + \mu_{11}\mu_{12}\mu_{21}\mu_{23} + \mu_{21}\mu_{23}^3$$
$$+2\mu_{11}\mu_{21}\mu_{23}^2 + \mu_{12}\mu_{21}\mu_{23}^2),$$
$$\alpha_{11} = \frac{\mu_{12}^2\mu_{23}}{\mu_{11}+\mu_{23}}(\mu_{11}+\mu_{21})(\mu_{11}^2\mu_{12}\mu_{23} + \mu_{11}^2\mu_{21}\mu_{23} + \mu_{11}^2\mu_{12}^2 + \mu_{11}^2\mu_{12}\mu_{21} + \mu_{11}\mu_{12}^2\mu_{23}$$
$$+2\mu_{11}\mu_{12}\mu_{21}\mu_{23} + \mu_{11}\mu_{12}\mu_{23}^2 + \mu_{11}\mu_{21}\mu_{23}^2 - \mu_{21}^2\mu_{23}^2),$$
$$\alpha_{12} = \mu_{12}^2\mu_{23}(\mu_{11}^3\mu_{12}\mu_{21} + \mu_{11}^3\mu_{12}^2 + \mu_{11}^3\mu_{12}\mu_{23} + \mu_{11}^3\mu_{21}\mu_{23} + \mu_{11}^2\mu_{12}^2\mu_{21} + \mu_{11}^2\mu_{12}\mu_{21}$$
$$+3\mu_{11}^2\mu_{12}\mu_{21}\mu_{23} + 2\mu_{11}^2\mu_{21}\mu_{23}^2 + \mu_{11}^2\mu_{12}^2\mu_{23} + 2\mu_{11}^2\mu_{12}\mu_{23}^2 + \mu_{11}^2\mu_{21}^2\mu_{23} + \mu_{21}^2\mu_{23}^3$$
$$+\mu_{11}\mu_{12}^2\mu_{21}\mu_{23} + \mu_{11}\mu_{12}^2\mu_{23}^2 + \mu_{11}\mu_{12}\mu_{23}^3 + \mu_{11}\mu_{21}\mu_{23}^3 + 2\mu_{11}\mu_{12}\mu_{21}^2\mu_{23}$$
$$+2\mu_{11}\mu_{21}^2\mu_{23}^2 + 4\mu_{11}\mu_{12}\mu_{21}\mu_{23}^2 + \mu_{12}\mu_{21}\mu_{23}^3 + 2\mu_{12}\mu_{21}^2\mu_{23}^2 + \mu_{12}^2\mu_{21}\mu_{23}^2),$$
$$\alpha_{13} = \mu_{12}\mu_{23}^2(\mu_{11}^3\mu_{12}\mu_{21} + \mu_{11}^3\mu_{12}^2 + \mu_{11}^3\mu_{12}\mu_{23} + \mu_{11}^3\mu_{21}\mu_{23} + \mu_{11}^2\mu_{12}^2\mu_{21} + \mu_{11}^2\mu_{12}\mu_{21}^2$$
$$+3\mu_{11}^2\mu_{12}\mu_{21}\mu_{23} + 2\mu_{11}^2\mu_{21}\mu_{23}^2 + \mu_{11}^2\mu_{12}^2\mu_{23} + 2\mu_{11}^2\mu_{12}\mu_{23}^2 + \mu_{11}^2\mu_{21}^2\mu_{23} + \mu_{21}^2\mu_{23}^3$$
$$+\mu_{11}\mu_{12}^2\mu_{21}\mu_{23} + \mu_{11}\mu_{12}^2\mu_{23}^2 + \mu_{11}\mu_{12}\mu_{23}^3 + \mu_{11}\mu_{21}\mu_{23}^3 + 2\mu_{11}\mu_{12}\mu_{21}^2\mu_{23}$$
$$+2\mu_{11}\mu_{21}^2\mu_{23}^2 + 4\mu_{11}\mu_{12}\mu_{21}\mu_{23}^2 + \mu_{12}\mu_{21}\mu_{23}^3 + 2\mu_{12}\mu_{21}^2\mu_{23}^2 + \mu_{12}^2\mu_{21}\mu_{23}^2).$$
$$\alpha' = \mu_{11}^2\mu_{21}\mu_{23}^2 + \mu_{11}\mu_{12}^2\mu_{21}^2 + \mu_{12}^2\mu_{21}^2\mu_{23} + \mu_{21}^2\mu_{23}^3 + \mu_{11}\mu_{12}^3\mu_{21} + 2\mu_{11}\mu_{12}^2\mu_{23}^2$$
$$+\mu_{11}\mu_{21}\mu_{23}^3 + \mu_{11}^2\mu_{12}^2\mu_{21} + \mu_{11}\mu_{21}^2\mu_{23}^2 + 2\mu_{12}^2\mu_{12}\mu_{23}^2 + \mu_{12}\mu_{21}\mu_{23}^3 + \mu_{11}\mu_{12}\mu_{21}^2\mu_{23}$$

$$+\mu_{11}^2\mu_{12}\mu_{21}\mu_{23} + 2\mu_{11}\mu_{12}^2\mu_{21}\mu_{23} + 2\mu_{11}\mu_{12}\mu_{21}\mu_{23}^2 + \mu_{12}\mu_{21}^2\mu_{23}^2 + \mu_{11}^3\mu_{12}\mu_{23}$$

$$+\mu_{11}\mu_{12}^3\mu_{23} + \mu_{11}^2\mu_{12}^3 + \mu_{11}^2\mu_{12}\mu_{23}^2 + \mu_{11}\mu_{12}\mu_{23}^3 + \mu_{12}^3\mu_{21}\mu_{23} + \mu_{12}^3\mu_{23}^2 + \mu_{12}^2\mu_{23}^3),$$

$$\alpha_1' = \mu_{12}\mu_{21}\mu_{23}(\mu_{11} + \mu_{21})(\mu_{11} + \mu_{23})(\mu_{12} + \mu_{21}),$$

$$\alpha_2' = \mu_{12}^2\mu_{21}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2),$$

$$\alpha_3' = \mu_{12}\mu_{23}(\mu_{21}\mu_{23} - \mu_{11}\mu_{12})(\mu_{11}\mu_{12} + \mu_{11}\mu_{21} + \mu_{11}\mu_{23}),$$

$$\alpha_4' = \mu_{11}\mu_{12}\mu_{23}(\mu_{11}\mu_{12}\mu_{21} + \mu_{11}\mu_{12}^2 + \mu_{11}\mu_{21}\mu_{23} + \mu_{11}\mu_{12}\mu_{23} + \mu_{12}^2\mu_{23} + \mu_{21}\mu_{23}^2$$

$$+\mu_{12}\mu_{23}^2 + \mu_{12}\mu_{21}\mu_{23}),$$

$$\alpha_5' = \mu_{11}\mu_{12}\mu_{23}(\mu_{11} + \mu_{21})(\mu_{12} + \mu_{21})(\mu_{12} + \mu_{23}),$$

$$\alpha_6' = \mu_{11}\mu_{12}\mu_{23}^2(\mu_{12}\mu_{21} + \mu_{21}\mu_{23} + \mu_{12}^2 + \mu_{12}\mu_{23}),$$

$$\alpha_7' = \mu_{12}\mu_{23}(\mu_{11} + \mu_{21})(\mu_{11}\mu_{12}^2 + \mu_{11}\mu_{12}\mu_{21} + \mu_{11}\mu_{21}\mu_{23} + \mu_{21}\mu_{23}^2),$$

$$\alpha_8' = \mu_{12}^2\mu_{23}(\mu_{11}^2\mu_{12} + \mu_{11}^2\mu_{21} + \mu_{11}\mu_{12}\mu_{21} + \mu_{11}\mu_{21}\mu_{23} + \mu_{11}\mu_{21}^2 + \mu_{21}\mu_{23}^2 + \mu_{21}^2\mu_{23}$$

$$+\mu_{12}\mu_{21}\mu_{23}),$$

$$\alpha_9' = \mu_{12}\mu_{23}^2(\mu_{11}^2\mu_{21} + \mu_{11}^2\mu_{12} + \mu_{11}\mu_{12}^2 + \mu_{11}\mu_{12}\mu_{23} + \mu_{11}\mu_{12}\mu_{21} + \mu_{11}\mu_{21}^2 + \mu_{21}^2\mu_{23}$$

$$+\mu_{11}\mu_{21}\mu_{23}),$$

$$\alpha_{10}' = \mu_{12}\mu_{21}\mu_{23}^2(\mu_{11} + \mu_{21})(\mu_{11} + \mu_{23}),$$

$$\alpha_{11}' = \mu_{11}\mu_{12}^2\mu_{23}(\mu_{11} + \mu_{21})(\mu_{12} + \mu_{21}),$$

$$\alpha_{12}' = \mu_{12}^2\mu_{23}(\mu_{11}^2\mu_{21} + \mu_{11}^2\mu_{12} + \mu_{11}\mu_{21}^2 + \mu_{11}\mu_{12}\mu_{23} + \mu_{11}\mu_{12}\mu_{21} + \mu_{11}\mu_{21}\mu_{23}$$

$$+\mu_{21}^2\mu_{23} + \mu_{12}\mu_{21}\mu_{23}),$$

$$\alpha_{13}' = \mu_{12}\mu_{23}^2(\mu_{11}^2\mu_{21} + \mu_{11}^2\mu_{12} + \mu_{11}\mu_{21}^2 + \mu_{11}\mu_{12}\mu_{23} + \mu_{11}\mu_{12}\mu_{21} + \mu_{11}\mu_{21}\mu_{23}$$

$$+\mu_{21}^2\mu_{23} + \mu_{12}\mu_{21}\mu_{23}).$$

These results together with Theorem 9.5.1 of Puterman [58] proves the optimality of the policy $\pi = (d_0)^\infty$ when $\mu_{11}\mu_{12} \geq \mu_{21}\mu_{23}$ and the optimality of the policy $\pi = (d_0')^\infty$ when $\mu_{11}\mu_{12} < \mu_{21}\mu_{23}$. □

Proof of Proposition 4.3.5 : The set of allowable actions in state $s \in S$ is

$$A_s = \begin{cases} a_{11} & \text{for } s = (0,0), \\ a_{23} & \text{for } s = (2,0), \\ a_{13} & \text{for } s = (1,2), \\ \{a_{11}, a_{12}, a_{22}\} & \text{for } s = (1,0), \\ \{a_{11}, a_{13}, a_{33}\} & \text{for } s \in \{(0,1),(0,2)\}, \\ \{a_{22}, a_{23}, a_{33}\} & \text{for } s = (2,1), \\ \{a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}\} & \text{for } s = (1,1). \end{cases}$$

Under our assumptions on the service rates ($\sum_{i=1}^{M} \mu_{ij} > 0$ for $j \in \{1, \ldots, N\}$ and $\mu_{13} = \mu_{21} = 0$), it is clear that $\mu_{11} > 0$ and $\mu_{23} > 0$. Hence, we can conclude that the policy described in the theorem corresponds to an irreducible Markov chain, and consequently we have a communicating Markov decision process. Thus, we can use the policy iteration algorithm for communicating models as described in Section 9.5.1 of Puterman [58]. We use the uniformization constant $q = \mu_{11} + \mu_{12} + \mu_{22} + \mu_{23}$.

First assume that $\mu_{11}^2\mu_{12}^2 \leq \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$ and start the policy iteration algorithm by choosing

$$d_0(s) = \begin{cases} a_{12} & \text{for } s \in \{(0,0),(1,0)\}, \\ a_{13} & \text{for } s \in \{(0,1),(0,2),(1,1)\}, \\ a_{22} & \text{for } s = (2,0), \\ a_{23} & \text{for } s \in \{(1,2),(2,1)\}. \end{cases}$$

Then, we proceed as in the proof of Proposition 4.3.1. In the calculations below, β_k for $k \in \{1, \ldots, 13\}$ and β are nonnegative constants when $\mu_{11}^2\mu_{12}^2 \leq \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$. They depend on the service rates and the state under consideration, and they are provided below.

First, consider the state $s = (1,0)$, and recall that $d_0(s) = a_{21}$. Some algebra shows that

$$\Big(r((1,0),a_{12}) + \sum_{s'\in S} p(s'|(1,0),a_{12})h(s')\Big) - r((1,0),a_{11}) - \sum_{s'\in S} p(s'|(1,0),a_{11})h(s')$$
$$= \frac{\beta_1}{\beta} \geq 0,$$
$$\Big(r((1,0),a_{12}) + \sum_{s'\in S} p(s'|(1,0),a_{12})h(s')\Big) - r((1,0),a_{22}) - \sum_{s'\in S} p(s'|(1,0),a_{22})h(s')$$
$$= \frac{\beta_2}{\beta} \geq 0.$$

Recall that $d_0(s) = a_{13}$ for $s \in \{(0,1),(0,2)\}$. Then, we can show that

$$\Big(r((0,1),a_{13}) + \sum_{s'\in S} p(s'|(0,1),a_{13})h(s')\Big) - r((0,1),a_{11}) - \sum_{s'\in S} p(s'|(0,1),a_{11})h(s')$$
$$= \frac{\beta_3}{\beta} \geq 0,$$
$$\Big(r((0,1),a_{13}) + \sum_{s'\in S} p(s'|(0,1),a_{13})h(s')\Big) - r((0,1),a_{33}) - \sum_{s'\in S} p(s'|(0,1),a_{33})h(s')$$
$$= \frac{\beta_4}{\beta} \geq 0,$$
$$\Big(r((0,2),a_{13}) + \sum_{s'\in S} p(s'|(0,2),a_{13})h(s')\Big) - r((0,2),a_{11}) - \sum_{s'\in S} p(s'|(0,2),a_{11})h(s')$$
$$= \frac{\beta_5}{\beta} \geq 0,$$
$$\Big(r((0,2),a_{13}) + \sum_{s'\in S} p(s'|(0,2),a_{13})h(s')\Big) - r((0,2),a_{33}) - \sum_{s'\in S} p(s'|(0,2),a_{33})h(s')$$
$$= \frac{\beta_6}{\beta} \geq 0.$$

For $s = (1,1)$, recall that $d_0(s) = a_{13}$. Some algebra shows that

$$\Big(r((1,1),a_{13}) + \sum_{s'\in S} p(s'|(1,1),a_{13})h(s')\Big) - r((1,1),a_{11}) - \sum_{s'\in S} p(s'|(1,1),a_{11})h(s')$$
$$= \frac{\beta_7}{\beta} \geq 0,$$
$$\Big(r((1,1),a_{13}) + \sum_{s'\in S} p(s'|(1,1),a_{13})h(s')\Big) - r((1,1),a_{22}) - \sum_{s'\in S} p(s'|(1,1),a_{22})h(s')$$
$$= \frac{\beta_8}{\beta} \geq 0,$$

$$\Big(r((1,1),a_{13}) + \sum_{s'\in S} p(s'|(1,1),a_{13})h(s')\Big) - r((1,1),a_{33}) - \sum_{s'\in S} p(s'|(1,1),a_{33})h(s')$$
$$= \frac{\beta_9}{\beta} \geq 0,$$
$$\Big(r((1,1),a_{13}) + \sum_{s'\in S} p(s'|(1,1),a_{13})h(s')\Big) - r((1,1),a_{12}) - \sum_{s'\in S} p(s'|(1,1),a_{12})h(s')$$
$$= \frac{\beta_{10}}{\beta} \geq 0,$$
$$\Big(r((1,1),a_{13}) + \sum_{s'\in S} p(s'|(1,1),a_{13})h(s')\Big) - r((1,1),a_{23}) - \sum_{s'\in S} p(s'|(1,1),a_{23})h(s')$$
$$= \frac{\beta_{11}}{\beta} \geq 0.$$

Finally, $d_0(s) = a_{23}$ for $s = (2,1)$. Some algebra shows that

$$\Big(r((2,1),a_{23}) + \sum_{s'\in S} p(s'|(2,1),a_{23})h(s')\Big) - r((2,1),a_{22}) - \sum_{s'\in S} p(s'|(2,1),a_{22})h(s')$$
$$= \frac{\beta_{12}}{\beta} \geq 0,$$
$$\Big(r((2,1),a_{23}) + \sum_{s'\in S} p(s'|(2,1),a_{23})h(s')\Big) - r((2,1),a_{33}) - \sum_{s'\in S} p(s'|(2,1),a_{33})h(s')$$
$$= \frac{\beta_{13}}{\beta} \geq 0.$$

Next, assume that $\mu_{11}^2\mu_{12}^2 > \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$ and start the policy iteration algorithm by choosing

$$d_0'(s) = \begin{cases} a_{12} & \text{for } s = (0,0), \\ a_{13} & \text{for } s \in \{(0,1),(0,2)\}, \\ a_{22} & \text{for } s \in \{(1,0),(2,0)\}, \\ a_{23} & \text{for } s \in \{(1,1),(1,2),(2,1)\}. \end{cases}$$

Then, we proceed as in the proof of Proposition 4.3.1. In the calculations below, β_k' for $k \in \{1,\ldots,13\}$ and β' are nonnegative constants when $\mu_{11}^2\mu_{12}^2 > \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$. They depend on the service rates and the state under consideration, and they are provided below.

First, consider the state $s = (1,0)$, and recall that $d_0'(s) = a_{22}$. Some algebra shows that

$$\Big(r((1,0),a_{22})+\sum_{s'\in S}p(s'|(1,0),a_{22})h(s')\Big)-r((1,0),a_{11})-\sum_{s'\in S}p(s'|(1,0),a_{11})h(s')$$
$$=\frac{\beta_1'}{\beta'}\geq 0,$$
$$\Big(r((1,0),a_{22})+\sum_{s'\in S}p(s'|(1,0),a_{22})h(s')\Big)-r((1,0),a_{12})-\sum_{s'\in S}p(s'|(1,0),a_{12})h(s')$$
$$=\frac{\beta_2'}{\beta'}\geq 0.$$

Recall that $d_0(s)=a_{13}$ for $s\in\{(0,1),(0,2)\}$. Then, we can show that

$$\Big(r((0,1),a_{13})+\sum_{s'\in S}p(s'|(0,1),a_{13})h(s')\Big)-r((0,1),a_{11})-\sum_{s'\in S}p(s'|(0,1),a_{11})h(s')$$
$$=\frac{\beta_3'}{\beta'}\geq 0,$$
$$\Big(r((0,1),a_{13})+\sum_{s'\in S}p(s'|(0,1),a_{13})h(s')\Big)-r((0,1),a_{33})-\sum_{s'\in S}p(s'|(0,1),a_{33})h(s')$$
$$=\frac{\beta_4'}{\beta'}\geq 0,$$
$$\Big(r((0,2),a_{13})+\sum_{s'\in S}p(s'|(0,2),a_{13})h(s')\Big)-r((0,2),a_{11})-\sum_{s'\in S}p(s'|(0,2),a_{11})h(s')$$
$$=\frac{\beta_5'}{\beta'}\geq 0,$$
$$\Big(r((0,2),a_{13})+\sum_{s'\in S}p(s'|(0,2),a_{13})h(s')\Big)-r((0,2),a_{33})-\sum_{s'\in S}p(s'|(0,2),a_{33})h(s')$$
$$=\frac{\beta_6'}{\beta'}\geq 0.$$

For $s=(1,1)$, recall that $d_0'(s)=a_{23}$. Some algebra shows that

$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{11})-\sum_{s'\in S}p(s'|(1,1),a_{11})h(s')$$
$$=\frac{\beta_7'}{\beta'}\geq 0,$$
$$\Big(r((1,1),a_{23})+\sum_{s'\in S}p(s'|(1,1),a_{23})h(s')\Big)-r((1,1),a_{22})-\sum_{s'\in S}p(s'|(1,1),a_{22})h(s')$$
$$=\frac{\beta_8'}{\beta'}\geq 0,$$

$$\Big(r((1,1),a_{23}) + \sum_{s'\in S} p(s'|(1,1),a_{23})h(s')\Big) - r((1,1),a_{33}) - \sum_{s'\in S} p(s'|(1,1),a_{33})h(s')$$
$$= \frac{\beta'_9}{\beta'} \geq 0,$$
$$\Big(r((1,1),a_{23}) + \sum_{s'\in S} p(s'|(1,1),a_{23})h(s')\Big) - r((1,1),a_{12}) - \sum_{s'\in S} p(s'|(1,1),a_{12})h(s')$$
$$= \frac{\beta'_{10}}{\beta'} \geq 0,$$
$$\Big(r((1,1),a_{23}) + \sum_{s'\in S} p(s'|(1,1),a_{23})h(s')\Big) - r((1,1),a_{13}) - \sum_{s'\in S} p(s'|(1,1),a_{13})h(s')$$
$$= \frac{\beta'_{11}}{\beta'} \geq 0.$$

Finally, $d'_0(s) = a_{23}$ for $s = (2,1)$. Some algebra shows that

$$\Big(r((2,1),a_{23}) + \sum_{s'\in S} p(s'|(2,1),a_{23})h(s')\Big) - r((2,1),a_{22}) - \sum_{s'\in S} p(s'|(2,1),a_{22})h(s')$$
$$= \frac{\beta'_{12}}{\beta'} \geq 0,$$
$$\Big(r((2,1),a_{23}) + \sum_{s'\in S} p(s'|(2,1),a_{23})h(s')\Big) - r((2,1),a_{33}) - \sum_{s'\in S} p(s'|(2,1),a_{33})h(s')$$
$$= \frac{\beta'_{13}}{\beta'} \geq 0.$$

$$\begin{aligned}
\beta = {} & \mu_{11}^3\mu_{12}^2\mu_{22} + \mu_{11}^4\mu_{12}\mu_{23} + \mu_{11}^3\mu_{12}\mu_{23}^2 + \mu_{11}^3\mu_{22}^2\mu_{23} + 3\mu_{11}^3\mu_{12}\mu_{23}^2 + 2\mu_{11}^2\mu_{12}^2\mu_{23}^2 \\
& + \mu_{12}^2\mu_{22}\mu_{23}^3 + \mu_{12}\mu_{22}\mu_{23}^4 + \mu_{11}^4\mu_{22}\mu_{23} + \mu_{11}\mu_{22}^2\mu_{23}^3 + \mu_{11}\mu_{12}^2\mu_{23}^3 + \mu_{11}\mu_{22}\mu_{23}^4 \\
& + \mu_{11}\mu_{12}\mu_{23}^4 + \mu_{11}^2\mu_{22}^2\mu_{23}^2 + \mu_{12}\mu_{22}^2\mu_{23}^3 + \mu_{11}^3\mu_{12}^2\mu_{23} + \mu_{11}^4\mu_{12}\mu_{22} + 2\mu_{11}^2\mu_{22}\mu_{23}^3 \\
& + \mu_{11}^2\mu_{12}^2\mu_{22}\mu_{23} + \mu_{11}^2\mu_{12}\mu_{22}\mu_{23}^2 + 2\mu_{11}^3\mu_{23}^3 + \mu_{11}\mu_{12}^2\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{12}\mu_{22}^2\mu_{23}^2 \\
& + 2\mu_{11}^3\mu_{12}\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{12}\mu_{21}\mu_{23}^2 + 2\mu_{11}\mu_{12}\mu_{21}\mu_{23}^3 + 3\mu_{11}^2\mu_{12}\mu_{23}^3 + \mu_{11}^4\mu_{12}^2 \\
& + \mu_{11}^4\mu_{23}^2 + 3\mu_{11}^3\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{23}^4 + \mu_{22}^2\mu_{23}^4, \\
\beta_1 = {} & \mu_{11}^2\mu_{22}\mu_{23}(\mu_{11}\mu_{12}^2 + 2\mu_{11}\mu_{12}\mu_{23} + \mu_{11}^2\mu_{12} + \mu_{11}\mu_{12}\mu_{22} + \mu_{12}\mu_{23}^2 + \mu_{11}^2\mu_{23} \\
& + 2\mu_{11}\mu_{23}^2 + \mu_{23}^3 + \mu_{11}\mu_{22}\mu_{23}), \\
\beta_2 = {} & \mu_{11}\mu_{23}(\mu_{12} + \mu_{22})(\mu_{11}\mu_{12}\mu_{22}\mu_{23} + \mu_{12}\mu_{22}\mu_{23}^2 + \mu_{22}\mu_{23}^3 + \mu_{11}\mu_{22}\mu_{23}^2 - \mu_{11}^2\mu_{12}^2),
\end{aligned}$$

$$\beta_3 = \mu_{11}^3\mu_{23}(\mu_{11}\mu_{12}^2 + \mu_{12}^2\mu_{22} + \mu_{12}\mu_{22}\mu_{23} + \mu_{11}\mu_{12}\mu_{22} + \mu_{12}\mu_{22}^2 + \mu_{11}\mu_{22}\mu_{23}$$
$$+\mu_{22}\mu_{23}^2 + \mu_{22}^2\mu_{23}),$$
$$\beta_4 = \mu_{11}\mu_{23}^2(\mu_{11}\mu_{12}^2\mu_{22} + \mu_{11}\mu_{12}^2\mu_{23} + 2\mu_{11}^2\mu_{12}^2 + \mu_{12}^2\mu_{22}\mu_{23} + 2\mu_{11}^2\mu_{12}\mu_{23} + \mu_{11}^3\mu_{12}$$
$$+\mu_{12}\mu_{22}\mu_{23}^2 + 2\mu_{11}^2\mu_{12}\mu_{22} + \mu_{11}\mu_{12}\mu_{23}^2 + \mu_{12}\mu_{22}^2\mu_{23} + 2\mu_{11}\mu_{12}\mu_{22}\mu_{23} + \mu_{11}\mu_{12}\mu_{22}^2$$
$$+\mu_{22}^2\mu_{23}^2 + \mu_{11}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{22}^2\mu_{23}),$$
$$\beta_5 = \mu_{11}^2\mu_{23}(3\mu_{11}^2\mu_{12}^2\mu_{23} + \mu_{11}^3\mu_{12}^2 + 2\mu_{11}\mu_{12}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{12}^2\mu_{23}^2 + \mu_{11}^2\mu_{12}^2\mu_{22} + \mu_{22}^2\mu_{23}^3$$
$$+\mu_{12}^2\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{12}\mu_{22}^2 + \mu_{12}\mu_{22}\mu_{23}^3 + 2\mu_{11}\mu_{12}\mu_{22}^2\mu_{23} + 4\mu_{11}^2\mu_{12}\mu_{22}\mu_{23}$$
$$+\mu_{11}\mu_{12}\mu_{23}^3 + \mu_{11}^3\mu_{12}\mu_{22} + 3\mu_{11}\mu_{12}\mu_{22}\mu_{23}^2 + \mu_{11}^3\mu_{12}\mu_{23} + \mu_{12}\mu_{22}^2\mu_{23}^2 + 2\mu_{11}^2\mu_{12}\mu_{23}^2$$
$$+\mu_{11}^2\mu_{22}^2\mu_{23} + 2\mu_{11}\mu_{22}^2\mu_{23}^2 + 3\mu_{11}^2\mu_{22}\mu_{23}^2 + \mu_{11}^3\mu_{22}\mu_{23} + 2\mu_{11}\mu_{22}\mu_{23}^3),$$
$$\beta_6 = \mu_{11}\mu_{23}^3(2\mu_{11}^2\mu_{12}^2 + \mu_{11}\mu_{12}^2\mu_{22} + \mu_{11}\mu_{12}^2\mu_{23} + \mu_{12}^2\mu_{22}\mu_{23} + \mu_{11}^3\mu_{12} + 2\mu_{11}^2\mu_{12}\mu_{23}$$
$$+\mu_{12}\mu_{22}\mu_{23}^2 + 2\mu_{11}\mu_{12}\mu_{22}\mu_{23} + \mu_{11}\mu_{12}\mu_{22}^2 + 2\mu_{11}^2\mu_{12}\mu_{22} + \mu_{11}\mu_{12}\mu_{23}^2 + \mu_{12}\mu_{22}^2\mu_{23}$$
$$+\mu_{11}\mu_{22}^2\mu_{23} + \mu_{22}^2\mu_{23}^2 + \mu_{11}^2\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23}^2),$$
$$\beta_7 = \mu_{11}^2\mu_{23}(\mu_{12}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{12}^2\mu_{22} + \mu_{11}\mu_{12}^2\mu_{23} + \mu_{11}^2\mu_{12}^2 + \mu_{11}\mu_{12}\mu_{22}^2 + \mu_{11}^2\mu_{12}\mu_{22}$$
$$+2\mu_{11}\mu_{12}\mu_{22}\mu_{23} + \mu_{12}\mu_{22}^2\mu_{23} + 2\mu_{12}\mu_{22}\mu_{23}^2 + 2\mu_{11}\mu_{22}\mu_{23}^2 + \mu_{22}\mu_{23}^3 + \mu_{22}^2\mu_{23}^2$$
$$+\mu_{11}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{22}^2\mu_{23}),$$
$$\beta_8 = \frac{\mu_{11}\mu_{23}}{\mu_{11}+\mu_{23}}(\mu_{12}+\mu_{22})(-\mu_{11}^2\mu_{12}^2\mu_{23} + 2\mu_{11}^3\mu_{12}\mu_{23} + \mu_{11}^2\mu_{12}\mu_{23}^2 + \mu_{12}\mu_{22}\mu_{23}^3 + \mu_{22}\mu_{23}^4$$
$$+\mu_{11}^4\mu_{12} + 2\mu_{11}\mu_{12}\mu_{22}\mu_{23}^2 + \mu_{11}^3\mu_{12}\mu_{22} + \mu_{11}^2\mu_{12}\mu_{22}\mu_{23} + \mu_{11}\mu_{22}\mu_{23}^3),$$
$$\beta_9 = \mu_{12}\mu_{23}^2(\mu_{11}\mu_{12}^2\mu_{23} + \mu_{11}^2\mu_{12}^2 + \mu_{11}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{12}\mu_{23}^2 + \mu_{12}\mu_{22}\mu_{23}^2 + \mu_{22}^2\mu_{23}^2$$
$$+2\mu_{11}^2\mu_{12}\mu_{23} + \mu_{11}^2\mu_{12}\mu_{23} + \mu_{12}\mu_{22}^2\mu_{23} + \mu_{11}^3\mu_{12} + \mu_{11}\mu_{12}\mu_{22}\mu_{23}),$$
$$\beta_{10} = \mu_{11}^2\mu 12\mu_{23}(\mu_{12}+\mu_{22})(\mu_{11}^3 + 2\mu_{11}^2\mu_{23} + \mu_{11}^2\mu_{22} + \mu_{11}\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23} + \mu_{22}\mu_{23}^2),$$
$$\beta_{11} = \frac{\mu_{11}\mu_{23}^2}{\mu_{11}+\mu_{23}}(\mu_{12}+\mu_{22})(\mu_{11}\mu_{12}\mu_{22}\mu_{23} + \mu_{12}\mu_{22}\mu_{23}^2 + \mu_{22}\mu_{23}^3 + \mu_{11}\mu_{22}\mu_{23}^2 - \mu_{11}^2\mu_{12}^2),$$
$$\beta_{12} = \mu_{11}\mu 12\mu_{23}(\mu_{12}+\mu_{22})(\mu_{11}^3 + 2\mu_{11}^2\mu_{23} + \mu_{11}^2\mu_{22} + \mu_{11}\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23} + \mu_{22}\mu_{23}^2),$$
$$\beta_{13} = \mu_{11}\mu 12\mu_{23}^2(\mu_{11}^3 + 2\mu_{11}^2\mu_{23} + \mu_{11}^2\mu_{22} + \mu_{11}\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23} + \mu_{22}\mu_{23}^2),$$
$$\beta' = \mu_{11}\mu_{12}^2\mu_{23}^2 + \mu_{11}^3\mu_{23}^2 + \mu_{11}^3\mu_{12}\mu_{23} + 3\mu_{11}\mu_{12}\mu_{23}^3 + \mu_{11}^3\mu_{22}\mu_{23} + \mu_{11}^2\mu_{12}^2\mu_{23}$$

$$+2\mu_{11}^2\mu_{22}\mu_{23}^2 + \mu_{11}^3\mu_{12}\mu_{22} + 2\mu_{11}^3\mu_{12}\mu_{23} + 2\mu_{11}\mu_{22}\mu_{23}^3 + \mu_{11}\mu_{23}^4 + \mu_{22}\mu_{23}^4 + \mu_{12}^2\mu_{23}^3$$

$$+\mu_{12}\mu_{23}^4 + \mu_{11}^2\mu_{12}\mu_{22}\mu_{23} + \mu_{11}\mu_{12}\mu_{22}\mu_{23}^2 + \mu_{12}\mu_{22}\mu_{23}^3 + \mu_{11}^3\mu_{12}^2 + 2\mu_{11}^2\mu_{23}^3,$$

$$\beta'_1 = \mu_{11}^2\mu_{23}(\mu_{11}\mu_{12}^2 + \mu_{11}\mu_{12}\mu_{22} + \mu_{11}\mu_{22}\mu_{23} + \mu_{22}\mu_{23}^2),$$

$$\beta'_2 = \mu_{11}\mu_{23}(\mu_{11}^2\mu_{12}^2 - \mu_{11}\mu_{12}\mu_{22}\mu_{23} - \mu_{12}\mu_{22}\mu_{23}^2 - \mu_{22}\mu_{23}^3 - \mu_{11}\mu_{22}\mu_{23}^2),$$

$$\beta'_3 = \mu_{11}^2\mu_{23}(\mu_{11}\mu_{12}^2 + \mu_{11}\mu_{12}\mu_{22} + \mu_{22}\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23}),$$

$$\beta'_4 = \mu_{11}\mu_{23}^2(\mu_{11}\mu_{12}^2 + \mu_{12}^2\mu_{23} + \mu_{12}\mu_{22}\mu_{23} + \mu_{12}\mu_{23}^2 + 2\mu_{11}^2\mu_{12} + \mu_{11}\mu_{12}\mu_{22} + \mu_{22}\mu_{23}^2$$

$$+2\mu_{11}\mu_{12}\mu_{23} + \mu_{11}\mu_{22}\mu_{23}),$$

$$\beta'_5 = \mu_{11}^2\mu_{23}(\mu_{11}\mu_{12}^2 + \mu_{12}^2\mu_{23} + \mu_{12}^2\mu_{23} + \mu_{12}\mu_{22}\mu_{23} + \mu_{11}\mu_{12}\mu_{23} + \mu_{11}\mu_{12}\mu_{22}$$

$$+2\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23}),$$

$$\beta'_6 = \mu_{11}\mu_{23}^3(\mu_{12}^2 + \mu_{11}\mu_{12} + \mu_{12}\mu_{22} + \mu_{12}\mu_{23} + \mu_{22}\mu_{23}),$$

$$\beta'_7 = \mu_{11}^2\mu_{23}(\mu_{12}^3\mu_{23} + \mu_{11}\mu_{12}^3 + \mu_{12}^2\mu_{23}^2 + \mu_{12}^2\mu_{22}\mu_{23} + 2\mu_{11}\mu_{12}^2\mu_{23} + \mu_{11}\mu_{12}^2\mu_{22}$$

$$+\mu_{22}\mu_{23}^3 + 2\mu_{12}\mu_{22}\mu_{23}^2 + 2\mu_{11}\mu_{12}\mu_{22}\mu_{23} + \mu_{11}\mu_{22}\mu_{23}^2),$$

$$\beta'_8 = \frac{\mu_{11}\mu_{23}}{\mu_{12} + \mu_{23}}(\mu_{12} + \mu_{22})(\mu_{11}\mu_{12}\mu_{23} + \mu_{11}^2\mu_{23} - \mu_{22}\mu_{23}^2),$$

$$\beta'_9 = \mu_{11}\mu_{12}\mu_{23}^2(\mu_{12}\mu_{23} + 2\mu_{11}\mu_{23} + \mu_{22}\mu_{23} + \mu_{23}^2 + \mu_{11}^2),$$

$$\beta'_{10} = \mu_{11}\mu_{23}(\mu_{11}^2\mu_{12}^3 + \mu_{11}^2\mu_{12}^2\mu_{22} + 2\mu_{11}^2\mu_{12}^2\mu_{23} + \mu_{11}^2\mu_{12}\mu_{22}\mu_{23} + \mu_{11}\mu_{12}^3\mu_{23}$$

$$+\mu_{11}\mu_{12}^2\mu_{23}^2 + \mu_{11}\mu_{12}^2\mu_{22}\mu_{23} - \mu_{11}\mu_{22}\mu_{23}^3 - \mu_{12}^2\mu_{22}\mu_{23}^2 - \mu_{12}\mu_{22}^2\mu_{23}^2 - \mu_{22}^2\mu_{23}^3$$

$$-\mu_{12}\mu_{22}\mu_{23}^3 - \mu_{22}\mu_{23}^4),$$

$$\beta'_{11} = \frac{\mu_{11}\mu_{23}^2}{\mu_{12} + \mu_{23}}(\mu_{11}^2\mu_{12}^2 - \mu_{11}\mu_{12}\mu_{22}\mu_{23} - \mu_{12}\mu_{22}\mu_{23}^2 - \mu_{22}\mu_{23}^3 - \mu_{11}\mu_{22}\mu_{23}^2),$$

$$\beta'_{12} = \mu_{11}\mu 12\mu_{23}(\mu_{12} + \mu_{22})(\mu_{11}^2\mu_{23} + \mu_{11}^2\mu_{12} + \mu_{12}\mu_{23}^2 + \mu_{23}^3 + \mu_{11}\mu_{12}\mu_{23} + 2\mu_{11}\mu_{23}^2),$$

$$\beta'_{13} = \mu_{11}\mu 12\mu_{23}^2(\mu_{23}^3 + \mu_{12}\mu_{23}^2 + \mu_{11}^2\mu_{22} + \mu_{11}^2\mu_{23} + \mu_{11}\mu_{12}\mu_{23} + 2\mu_{11}\mu_{23}^2),$$

These results together with Theorem 9.5.1 of Puterman [58] proves the optimality of the policy $\pi = (d_0)^\infty$ when $\mu_{11}^2\mu_{12}^2 \leq \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$ and the optimality of the policy $\pi = (d'_0)^\infty$ when $\mu_{11}^2\mu_{12}^2 > \mu_{22}\mu_{23}(\mu_{11}\mu_{12} + \mu_{11}\mu_{23} + \mu_{12}\mu_{23} + \mu_{23}^2)$. □

Proof of Proposition 4.3.6 : The set of allowable actions in state $s \in S$ is

$$A_s = \begin{cases} a_{11} & \text{for } s = (0,0), \\ a_{12} & \text{for } s = (2,0), \\ a_{33} & \text{for } s = (1,2), \\ \{a_{11}, a_{12}, a_{22}\} & \text{for } s = (1,0), \\ \{a_{11}, a_{13}, a_{33}\} & \text{for } s \in \{(0,1),(0,2)\}, \\ \{a_{22}, a_{32}, a_{33}\} & \text{for } s = (2,1), \\ \{a_{11}, a_{12}, a_{13}, a_{22}, a_{32}, a_{33}\} & \text{for } s = (1,1). \end{cases}$$

Under our assumptions on the service rates ($\sum_{i=1}^{M} \mu_{ij} > 0$ for $j \in \{1, \dots, N\}$ and $\mu_{12} = \mu_{21} = 0$), it is clear that $\mu_{11} > 0$ and $\mu_{22} > 0$. Hence, we can conclude that the policy described in the theorem corresponds to an irreducible Markov chain, and consequently we have a communicating Markov decision process. Thus, we can use the policy iteration algorithm for communicating models as described in Section 9.5.1 of Puterman [58]. We use the uniformization constant $q = \mu_{11} + \mu_{13} + \mu_{22} + \mu_{23}$.

We start the policy iteration algorithm by choosing

$$d_0(s) = \begin{cases} a_{12} & \text{for } s \in \{(0,0),(1,0),(1,1),(2,0)\}, \\ a_{13} & \text{for } s \in \{(0,1),(0,2)\}, \\ a_{32} & \text{for } s = (2,1), \\ a_{33} & \text{for } s = (1,2). \end{cases}$$

Then, we proceed as in the proof of Proposition 4.3.1. In the calculations below, χ_k for $k \in \{1, \dots, 13\}$ and χ are nonnegative constants. They depend on the service rates and the state under consideration. They are provided below.

First, consider the state $s = (1,0)$, and recall that $d_0(s) = a_{21}$. Some algebra shows that

$$\Big(r((1,0),a_{12}) + \sum_{s' \in S} p(s'|(1,0),a_{12})h(s')\Big) - r((1,0),a_{11}) - \sum_{s' \in S} p(s'|(1,0),a_{11})h(s')$$
$$= \frac{\chi_1}{\chi} \geq 0,$$

$$\left(r((1,0),a_{12})+\sum_{s'\in S}p(s'|(1,0),a_{12})h(s')\right)-r((1,0),a_{22})-\sum_{s'\in S}p(s'|(1,0),a_{22})h(s')$$
$$=\frac{\chi_2}{\chi}\geq 0.$$

Recall that $d_0(s)=a_{13}$ for $s\in\{(0,1),(0,2)\}$. Then, we can show that

$$\left(r((0,1),a_{13})+\sum_{s'\in S}p(s'|(0,1),a_{13})h(s')\right)-r((0,1),a_{11})-\sum_{s'\in S}p(s'|(0,1),a_{11})h(s')$$
$$=\frac{\chi_3}{\chi}\geq 0,$$
$$\left(r((0,1),a_{13})+\sum_{s'\in S}p(s'|(0,1),a_{13})h(s')\right)-r((0,1),a_{33})-\sum_{s'\in S}p(s'|(0,1),a_{33})h(s')$$
$$=\frac{\chi_4}{\chi}\geq 0,$$
$$\left(r((0,2),a_{13})+\sum_{s'\in S}p(s'|(0,2),a_{13})h(s')\right)-r((0,2),a_{11})-\sum_{s'\in S}p(s'|(0,2),a_{11})h(s')$$
$$=\frac{\chi_5}{\chi}\geq 0,$$
$$\left(r((0,2),a_{13})+\sum_{s'\in S}p(s'|(0,2),a_{13})h(s')\right)-r((0,2),a_{33})-\sum_{s'\in S}p(s'|(0,2),a_{33})h(s')$$
$$=\frac{\chi_6}{\chi}\geq 0.$$

For $s=(1,1)$, recall that $d_0(s)=a_{12}$. Some algebra shows that

$$\left(r((1,1),a_{12})+\sum_{s'\in S}p(s'|(1,1),a_{12})h(s')\right)-r((1,1),a_{11})-\sum_{s'\in S}p(s'|(1,1),a_{11})h(s')$$
$$=\frac{\chi_7}{\chi}\geq 0,$$
$$\left(r((1,1),a_{12})+\sum_{s'\in S}p(s'|(1,1),a_{12})h(s')\right)-r((1,1),a_{22})-\sum_{s'\in S}p(s'|(1,1),a_{22})h(s')$$
$$=\frac{\chi_8}{\chi}\geq 0,$$
$$\left(r((1,1),a_{12})+\sum_{s'\in S}p(s'|(1,1),a_{12})h(s')\right)-r((1,1),a_{33})-\sum_{s'\in S}p(s'|(1,1),a_{33})h(s')$$
$$=\frac{\chi_9}{\chi}\geq 0,$$
$$\left(r((1,1),a_{12})+\sum_{s'\in S}p(s'|(1,1),a_{12})h(s')\right)-r((1,1),a_{13})-\sum_{s'\in S}p(s'|(1,1),a_{13})h(s')$$
$$=\frac{\chi_{10}}{\chi}\geq 0,$$

$$\left(r((1,1),a_{12})+\sum_{s'\in S}p(s'|(1,1),a_{12})h(s')\right)-r((1,1),a_{32})-\sum_{s'\in S}p(s'|(1,1),a_{32})h(s')$$
$$=\frac{\chi_{11}}{\chi}\geq 0.$$

Finally, $d_0(s)=a_{32}$ for $s=(2,1)$. Some algebra shows that

$$\left(r((2,1),a_{32})+\sum_{s'\in S}p(s'|(2,1),a_{32})h(s')\right)-r((2,1),a_{22})-\sum_{s'\in S}p(s'|(2,1),a_{22})h(s')$$
$$=\frac{\chi_{12}}{\chi}\geq 0,$$
$$\left(r((2,1),a_{32})+\sum_{s'\in S}p(s'|(2,1),a_{32})h(s')\right)-r((2,1),a_{33})-\sum_{s'\in S}p(s'|(2,1),a_{33})h(s')$$
$$=\frac{\chi_{13}}{\chi}\geq 0.$$

$$\begin{aligned}
\chi=\;&\mu_{11}^5\mu_{22}^2+\mu_{11}^5\mu_{13}\mu_{22}+\mu_{11}^3\mu_{13}^2\mu_{23}^2+\mu_{11}\mu_{13}^2\mu_{22}^3\mu_{23}+2\mu_{11}\mu_{22}^3\mu_{23}^3+4\mu_{11}^3\mu_{22}^3\mu_{23}\\
&+3\mu_{11}^2\mu_{22}^3\mu_{23}^2+\mu_{11}^5\mu_{13}\mu_{23}+\mu_{11}^5\mu_{22}\mu_{23}+2\mu_{11}^2\mu_{22}^2\mu_{23}^3+\mu_{11}^3\mu_{22}\mu_{23}^3+4\mu_{11}^4\mu_{22}^2\mu_{23}\\
&+4\mu_{11}^3\mu_{22}^2\mu_{23}^2+2\mu_{11}^4\mu_{22}\mu_{23}^2+4\mu_{11}^4\mu_{13}\mu_{22}\mu_{23}+4\mu_{11}^3\mu_{13}\mu_{22}\mu_{23}^2+6\mu_{11}^3\mu_{13}\mu_{22}^2\mu_{23}\\
&+5\mu_{11}^2\mu_{13}\mu_{22}^2\mu_{23}^2+2\mu_{11}^2\mu_{13}\mu_{22}\mu_{23}^3+3\mu_{11}^4\mu_{13}\mu_{22}^2+3\mu_{11}^3\mu_{13}\mu_{22}^3+2\mu_{11}^3\mu_{13}^2\mu_{22}^2\\
&+2\mu_{11}\mu_{13}\mu_{22}^2\mu_{23}^3+4\mu_{11}^2\mu_{13}\mu_{22}^3\mu_{23}+3\mu_{11}^2\mu_{13}^2\mu_{22}^2\mu_{23}+2\mu_{11}\mu_{13}^2\mu_{22}^2\mu_{23}^2+\mu_{22}^4\mu_{23}^3\\
&+3\mu_{11}\mu_{13}\mu_{22}^3\mu_{23}^2+\mu_{13}\mu_{22}^3\mu_{23}^3+\mu_{11}^3\mu_{13}\mu_{23}^3+2\mu_{11}^4\mu_{13}\mu_{23}^2+\mu_{11}^2\mu_{22}^4\mu_{23}+\mu_{11}^5\mu_{13}^2\\
&+\mu_{11}\mu_{22}^4\mu_{23}^2+2\mu_{11}^4\mu_{22}^3+\mu_{11}^3\mu_{23}^4+2\mu_{11}^4\mu_{13}^2\mu_{22}+3\mu_{11}^3\mu_{13}^2\mu_{22}\mu_{23}+2\mu_{11}^4\mu_{13}^2\mu_{23}\\
&+2\mu_{11}^4\mu_{13}^2\mu_{23}+2\mu_{11}^2\mu_{13}^2\mu_{22}\mu_{23}^2+\mu_{11}^2\mu_{13}\mu_{22}^4+\mu_{11}^2\mu_{13}^2\mu_{22}^3+\mu_{11}^3\mu_{13}^2\mu_{23}^2+\mu_{13}^2\mu_{22}^3\mu_{23}^2\\
&+\mu_{13}\mu_{22}^4\mu_{23}^2+\mu_{11}\mu_{13}\mu_{22}^4\mu_{23},\\
\chi_1=\;&\mu_{11}^2\mu_{22}(2\mu_{11}^2\mu_{13}^2\mu_{23}+4\mu_{11}\mu_{13}^2\mu_{22}\mu_{23}+3\mu_{13}^2\mu_{22}\mu_{23}^2+\mu_{11}\mu_{13}^2\mu_{23}^2+\mu_{11}^3\mu_{13}^2\\
&+\mu_{11}^2\mu_{13}^2\mu_{22}+3\mu_{11}^2\mu_{13}\mu_{22}\mu_{23}+\mu_{11}^3\mu_{13}\mu_{22}+2\mu_{11}\mu_{13}\mu_{22}\mu_{23}^2+\mu_{11}^3\mu_{13}\mu_{23}+\mu_{22}^2\mu_{23}^3\\
&+2\mu_{11}^2\mu_{13}\mu_{23}^2+\mu_{13}\mu_{22}\mu_{23}^3+\mu_{11}^2\mu_{13}\mu_{22}^2+\mu_{11}\mu_{13}\mu_{23}^3+2\mu_{11}\mu_{13}\mu_{22}^2\mu_{23}+\mu_{13}\mu_{22}^2\mu_{23}^2\\
&+\mu_{11}^3\mu_{22}\mu_{23}+2\mu_{11}^2\mu_{22}\mu_{23}^2+\mu_{11}\mu_{22}^3\mu_{23}+2\mu_{11}^2\mu_{22}^2\mu_{23}+2\mu_{11}\mu_{22}^2\mu_{23}^2+\mu_{11}\mu_{22}\mu_{23}^3),\\
\chi_2=\;&\mu_{11}\mu_{22}^2(\mu_{13}^2\mu_{22}\mu_{23}^2+\mu_{11}^2\mu_{13}^2\mu_{22}+\mu_{11}\mu_{13}^2\mu_{22}\mu_{23}+\mu_{11}\mu_{13}^2\mu_{23}^2+2\mu_{11}^2\mu_{13}^2\mu_{23}\\
&+\mu_{11}^3\mu_{13}^2+\mu_{11}\mu_{13}\mu_{23}^3+\mu_{13}\mu_{22}\mu_{23}^3+2\mu_{11}\mu_{13}\mu_{22}\mu_{23}^2+\mu_{11}^3\mu_{13}\mu_{22}+\mu_{11}^2\mu_{13}\mu_{22}^2
\end{aligned}$$

$$+2\mu_{11}^2\mu_{13}\mu_{22}\mu_{23} + \mu_{13}\mu_{22}^2\mu_{23}^2 + \mu_{11}^3\mu_{13}\mu_{23} + 2\mu_{11}^2\mu_{13}\mu_{23}^2 + \mu_{11}\mu_{13}\mu_{22}^2\mu_{23} + \mu_{22}^2\mu_{23}^3$$

$$+\mu_{11}^2\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{22}^2\mu_{23}^2 + \mu_{11}\mu_{22}\mu_{23}^3),$$

$$\chi_3 = \mu_{11}\mu_{22}(\mu_{13}\mu_{22}^2\mu_{23}^3 + \mu_{13}\mu_{22}^3\mu_{23}^2 + \mu_{13}^2\mu_{22}^2\mu_{23}^2 + 2\mu_{11}\mu_{13}\mu_{22}\mu_{23}^3 + 2\mu_{11}^3\mu_{13}\mu_{22}^2$$

$$+\mu_{11}^4\mu_{13}\mu_{22} + \mu_{11}^2\mu_{13}\mu_{22}^3 + \mu_{11}^2\mu_{13}^2\mu_{22}^2 + 2\mu_{11}^3\mu_{13}^2\mu_{22} + \mu_{22}^3\mu_{23}^3 + \mu_{11}^4\mu_{13}^2 + \mu_{11}^4\mu_{13}\mu_{23}$$

$$+2\mu_{11}\mu_{22}^2\mu_{23}^3 + 3\mu_{11}^2\mu_{13}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{13}^2\mu_{22}^2\mu_{23} + 2\mu_{11}\mu_{13}^2\mu_{22}\mu_{23}^2 + 4\mu_{11}^2\mu_{13}\mu_{22}\mu_{23}^2$$

$$+\mu_{11}\mu_{13}\mu_{22}^3\mu_{23} + 3\mu_{11}\mu_{13}\mu_{22}^2\mu_{23}^2 + 3\mu_{11}^3\mu_{13}\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{13}\mu_{22}^2\mu_{23} + \mu_{11}^2\mu_{22}\mu_{23}^3$$

$$+\mu_{11}^3\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{13}\mu_{23}^3 + 2\mu_{11}^3\mu_{13}\mu_{23}^2 + 2\mu_{11}^3\mu_{13}^2\mu_{23} + \mu_{11}^2\mu_{13}^2\mu_{23}^2 + 2\mu_{11}^2\mu_{22}^2\mu_{23}^2$$

$$\mu_{11}\mu_{22}^3\mu_{23}^2),$$

$$\chi_4 = \mu_{11}\mu_{22}(\mu_{11} + \mu_{22})(\mu_{11} + \mu_{23})(\mu_{13} + \mu_{23})(\mu_{11}^2\mu_{23} + \mu_{11}\mu_{22}\mu_{23} + \mu_{11}\mu_{13}\mu_{23}$$

$$+\mu_{22}^2\mu_{23} + \mu_{13}\mu_{22}\mu_{23}),$$

$$\chi_5 = \mu_{11}\mu_{22}(\mu_{13}\mu_{22}^2\mu_{23}^3 + \mu_{13}\mu_{22}^3\mu_{23}^2 + \mu_{13}^2\mu_{22}^2\mu_{23}^2 + 2\mu_{11}\mu_{13}\mu_{22}\mu_{23}^3 + 2\mu_{11}^3\mu_{13}\mu_{22}^2$$

$$+\mu_{11}^4\mu_{13}\mu_{22} + \mu_{11}^2\mu_{13}\mu_{22}^3 + \mu_{11}^2\mu_{13}^2\mu_{22}^2 + 2\mu_{11}^3\mu_{13}^2\mu_{22} + \mu_{22}^3\mu_{23}^3 + \mu_{11}^4\mu_{13}^2 + \mu_{11}^3\mu_{13}\mu_{23}^2$$

$$+2\mu_{11}\mu_{22}^2\mu_{23}^3 + 3\mu_{11}^2\mu_{13}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{13}^2\mu_{22}^2\mu_{23} + 2\mu_{11}\mu_{13}^2\mu_{22}\mu_{23}^2 + 2\mu_{11}^2\mu_{13}\mu_{22}\mu_{23}^2$$

$$+\mu_{11}\mu_{13}\mu_{22}^3\mu_{23} + 2\mu_{11}\mu_{13}\mu_{22}^2\mu_{23}^2 + 3\mu_{11}^3\mu_{13}\mu_{22}\mu_{23} + 3\mu_{11}^2\mu_{13}\mu_{22}^2\mu_{23} + \mu_{11}^2\mu_{22}\mu_{23}^3$$

$$+\mu_{11}^2\mu_{13}\mu_{23}^3 + 2\mu_{11}^3\mu_{13}^2\mu_{23} + \mu_{11}^2\mu_{13}^2\mu_{23}^2),$$

$$\chi_6 = \mu_{11}\mu_{22}\mu_{23}(\mu_{11} + \mu_{22})(\mu_{13} + \mu_{23})(\mu_{11}^2\mu_{13} + \mu_{11}\mu_{22}\mu_{23} + \mu_{11}\mu_{13}\mu_{23} + \mu_{22}^2\mu_{23}$$

$$+\mu_{13}\mu_{22}\mu_{23}),$$

$$\chi_7 = \mu_{11}\mu_{22}^2(\mu_{13}^2\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{13}^2\mu_{22}\mu_{23} + 2\mu_{11}^2\mu_{13}^2\mu_{23} + \mu_{11}^3\mu_{13}^2 + \mu_{11}\mu_{13}^2\mu_{23}^2 + \mu_{11}^4\mu_{13}$$

$$+\mu_{11}^2\mu_{13}^2\mu_{22} + \mu_{11}^2\mu_{13}\mu_{22}^2 + \mu_{13}\mu_{22}^2\mu_{23}^2 + 3\mu_{11}^2\mu_{13}\mu_{22}\mu_{23} + 2\mu_{11}\mu_{13}\mu_{22}\mu_{23}^2 + \mu_{22}^2\mu_{23}^3$$

$$+2\mu_{11}^3\mu_{13}\mu_{22} + \mu_{13}\mu_{22}\mu_{23}^3 + \mu_{11}\mu_{13}\mu_{22}^2\mu_{23} + \mu_{11}\mu_{13}\mu_{23}^3 + 2\mu_{11}^2\mu_{13}\mu_{23}^2 + 2\mu_{11}^3\mu_{13}\mu_{23}$$

$$+\mu_{11}\mu_{22}\mu_{23}^3),$$

$$\chi_8 = \mu_{13}^2\mu_{23}(\mu_{11}^3\mu_{22} + \mu_{11}^3\mu_{13} + \mu_{11}^2\mu_{13}\mu_{22} + 2\mu_{11}^2\mu_{22}\mu_{23} + \mu_{11}^2\mu_{13}\mu_{23} + \mu_{13}\mu_{22}\mu_{23}^2$$

$$+\mu_{11}^2\mu_{22}^2 + \mu_{11}\mu_{13}\mu_{22}\mu_{23} + 2\mu_{11}\mu_{22}^2\mu_{23} + 2\mu_{11}\mu_{22}\mu_{23}^2 + \mu_{22}\mu_{23}^3 + 2\mu_{22}^2\mu_{23}^2),$$

$$\chi_9 = \mu_{11}\mu_{22}(\mu_{11} + \mu_{22})(\mu_{13} + \mu_{23})(\mu_{11}^3\mu_{13} + 2\mu_{11}^2\mu_{13}\mu_{23} + \mu_{11}\mu_{13}\mu_{23}^2 + \mu_{13}\mu_{22}\mu_{23}^2$$

$$+\mu_{22}^2\mu_{23}^2),$$

$$\chi_{10} = \mu_{11}^2\mu_{13}\mu_{22}(\mu_{13} + \mu_{23})(\mu_{22}\mu_{23}^2 + \mu_{11}\mu_{23}^2 + 2\mu_{11}^2\mu_{23} + \mu_{11}\mu_{22}\mu_{23} + \mu_{11}^3 + \mu_{11}^2\mu_{22}),$$

$$\chi_{11} = \mu_{11}\mu_{22}^2\mu_{23}(\mu_{13} + \mu_{23})(\mu_{11}^2\mu_{13} + \mu_{11}\mu_{22}\mu_{23} + \mu_{11}\mu_{13}\mu_{23} + \mu_{22}^2\mu_{23} + \mu_{13}\mu_{22}\mu_{23}),$$

$$\chi_{12} = \mu_{11}\mu_{13}\mu_{22}^2(2\mu_{11}^3\mu_{23} + 2\mu_{11}\mu_{22}\mu_{23}^2 + \mu_{11}^2\mu_{23}^2 + 3\mu_{11}^2\mu_{22}\mu_{23} + \mu_{11}\mu_{22}^2\mu_{23} + \mu_{22}^2\mu_{23}^2$$
$$+\mu_{11}^2\mu_{22}^2 + 2\mu_{11}^3\mu_{22} + \mu_{11}^4),$$

$$\chi_{13} = \mu_{11}\mu_{13}\mu_{22}(\mu_{11} + \mu_{22})(\mu_{13} + \mu_{23})(\mu_{11}^2\mu_{22} + \mu_{22}\mu_{23}^2 + \mu_{11}\mu_{23}^2 + 2\mu_{11}^2\mu_{23} + \mu_{11}^3$$
$$+\mu_{11}\mu_{22}\mu_{23}).$$

These results together with Theorem 9.5.1 of Puterman [58] proves the optimality of the policy $\pi = (d_0)^\infty$. □

APPENDIX C

PROOFS FOR CHAPTER 5

Proof of Theorem 5.4.1: Lemma 5.3.1 shows that servers should not be voluntarily idle when station 1 is blocked or station 2 is starved (even though a server will not be working when (s)he is assigned to a station that is blocked or starved). Furthermore, when both stations are operating, if a server is at station $j \in \{1, 2\}$ before the previous server completion, any action that idles this server and assigns the other server to station j cannot be optimal. For example, actions a_{01} and a_{20} cannot be optimal at a state $(l, 1, 2)$ where $1 \leq l \leq B + 1$ since they are strictly dominated by actions a_{11} and a_{22}, respectively. The states $(0, 1, 1)$ and $(B + 2, 2, 2)$ are transient under any policy $\pi \in \Pi$ and the actions at these states do not affect the long-run average profit. Hence, they are omitted in the proof since any feasible action can be chosen in these states. Hence, we can use the following action space:

$$A_x = \begin{cases} \{a_{11}, a_{12}, a_{21}\} & \text{for } x \in \{(0,1,2), (0,2,1), (0,2,2)\}, \\ \{a_{02}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}\} & \text{for } x \in \{(l,1,1) \text{ where } 1 \leq l \leq B+1\}, \\ \{a_{02}, a_{10}, a_{11}, a_{12}, a_{21}, a_{22}\} & \text{for } x \in \{(l,1,2) \text{ where } 1 \leq l \leq B+1\}, \\ \{a_{01}, a_{11}, a_{12}, a_{20}, a_{21}, a_{22}\} & \text{for } x \in \{(l,2,1) \text{ where } 1 \leq l \leq B+1\}, \\ \{a_{01}, a_{10}, a_{11}, a_{12}, a_{21}, a_{22}\} & \text{for } x \in \{(l,2,2) \text{ where } 1 \leq l \leq B+1\}, \\ \{a_{12}, a_{21}, a_{22}\} & \text{for } x \in \{(B+2,1,1), (B+2,1,2), (B+2,2,1)\}. \end{cases}$$

Since the action space and the state space are finite, Theorem 9.1.8 of Puterman [58] shows the existence of an optimal Markovian stationary deterministic policy. Furthermore, under our assumptions on the service rates, we must have $\gamma_1, \gamma_2 > 0$. Hence, the policies described in the theorem correspond to an irreducible Markov chain and we can use the Linear Program (LP) approach for communicating Markov decision processes as in Sections 8.8.2 and 9.5.2 of Puterman [58].

Consider the following LP:

$$\left.\begin{array}{ll} \max & \sum_{x\in S}\sum_{a\in A_x} r(x,a)\omega(x,a) \\ \text{s.t.} & \sum_{a\in A_{x'}} \omega(x',a) - \sum_{x\in S}\sum_{a\in A_x} p(x'|x,a)\omega(x,a) = 0,\ \text{for all } x'\in S, \\ & \sum_{x\in S}\sum_{a\in A_x} \omega(x,a) = 1, \\ & \omega(x,a)\geq 0,\ \text{for all } x\in S, a\in A_x, \end{array}\right\} \quad (54)$$

where, for all $x \in S$ and $a \in A_x$, $r(x,a)$ is the immediate reward of choosing action a in state x and $p(x'|x,a)$ is the one-step transition probability from state x to x' if action a is chosen in state x. Then, in every basic feasible solution corresponding to a policy described in the theorem, we can conclude that for each $x \in S$ there exists at most a single action $a_x \in A_x$ such that $\omega(x, a_x) > 0$ as a result of Corollary 8.8.7 of Puterman [58] (which can be applied since the policies we consider in the description of the theorem result in a single recurrent class). Furthermore, for every basic feasible optimal solution x^* if we define $S_{w^*} = \{x \in S : \sum_{a\in A_x} w^*(x,a) > 0\}$, then the optimal decision rule is as follows:

$$d_{w^*}(x) = \begin{cases} a & \text{if } w^*(x,a) > 0 \text{ for } x \in S_{w^*}, \\ a' & \text{for some } a' \text{ such that } \sum_{y\in S_{w^*}} p(y|x,a') > 0 \text{ for } x \in S \setminus S_{w^*}. \end{cases}$$

We first prove the optimality of the policy for $0 \leq c \leq \min\{\frac{\gamma_2}{2\gamma_1+4\gamma_2}, \frac{\gamma_1}{4\gamma_1+2\gamma_2}\}$. Consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,1,2),(0,2,1),(0,2,2)\}, \\ a_{12} & \text{if } x \in \{(1,1,1),(1,1,2),(1,2,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1)\}, \\ a_{22} & \text{if } x \in \{(2,1,1),(2,1,2),(2,2,1)\}, \end{cases}$$

Now, let D be a basis for the LP (54), c_B be the vector of coefficients of the elements of D in the objective function, $\mathbf{B}$ be the coefficients of the elements of D in the constraint matrix, and b be the right-hand side of the constraints. More specifically, consider the basic solution ω corresponding to the policy $\pi = (d)^\infty$ with the basis

$$\begin{aligned} D &= \{\omega((0,1,2),(1,1)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,1)), \\ &\quad \omega((1,1,1),(1,2)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(1,2)), \\ &\quad \omega((2,1,1),(2,2)), \omega((2,1,2),(2,2)), \omega((2,2,1),(2,2))\}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} c_B &= \{-2c\gamma_1, -2c\gamma_1, -4c\gamma_1, \gamma_2 - c(\gamma_1+\gamma_2), \gamma_2, \gamma_2, \gamma_2 - c(\gamma_1+\gamma_2), \\ &\quad 2\gamma_2(1-2c), 2\gamma_2(1-c), 2\gamma_2(1-c)\}, \end{aligned}$$

$$B = \begin{bmatrix} 2\gamma_1/q & 0 & 0 & -\gamma_2/q & \dots & 0 & 0 \\ 0 & 2\gamma_1/q & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2\gamma_1/q & 0 & \dots & 0 & 0 \\ -2\gamma_1/q & -2\gamma_1/q & -2\gamma_1/q & (\gamma_1+\gamma_2)/q & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -2\gamma_2/q & -2\gamma_2/q \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 2\gamma_2/q & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix},$$

where q is the uniformization constant. Note that the equation corresponding to one of the states is redundant, and hence the equation corresponding to state $(2,2,1)$ is eliminated. Furthermore, it is easy to see that ω is also a stationary distribution for the Markov Chain X_π (since it has finite state space and one recurrent class, stationary distribution exists). Moreover, Corollary 8.8.7 of Puterman [58] implies that ω is a basic feasible solution. Then, in order to show the optimality of this basic feasible solution, we need only to show that

$$c_B \mathbf{B}^{-1} v_y - c_y \geq 0 \tag{55}$$

for each nonbasic variable y, where v_y is the column in the constraint matrix of LP (54), and c_y is the coefficient corresponding to y in the objective function. For states $(0,1,2),(0,2,1)$, and $(0,2,2)$, we have :

$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{12})} - c_{w((0,1,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})}$$
$$= c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{12})} - c_{w((0,2,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})}$$
$$= \frac{\gamma_1(\gamma_2 - 2c\gamma_1 - 4c\gamma_2)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})}$$
$$= \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2}.$$

It is clear that these quantities are nonnegative when $0 \le c \le \min\{\frac{\gamma_2}{2\gamma_1+4\gamma_2}, \frac{\gamma_1}{4\gamma_1+2\gamma_2}\}$. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})}$$
$$= \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = 0,$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = 4c\gamma_2,$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})}$$
$$= \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = 4c\gamma_1,$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = 2c(\gamma_1+\gamma_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = 4c\gamma_2,$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})}$$
$$= \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = 4c\gamma_1,$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = 2c(\gamma_1 + \gamma_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = 4c\gamma_2,$$

and for state $(1, 2, 2)$ we have :

$$\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} \\
&\quad = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2}, \\
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = 4c\gamma_1, \\
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{22})} - c_{w((1,2,2),a_{22})} = 0.
\end{aligned}$$

Finally, for states $(2, 1, 1), (2, 1, 2)$ and $(2, 2, 1)$ we have

$$\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{12})} - c_{w((2,1,2),a_{12})} \\
&\quad = c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{21})} - c_{w((2,2,1),a_{21})} \\
&\quad = \frac{\gamma_2(\gamma_1 - 4c\gamma_1 - 2c\gamma_2)}{\gamma_1+\gamma_2}, \\
&c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} \\
&\quad = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2}.
\end{aligned}$$

These quantities are also nonnegative when c, γ_1, and γ_2 satisfy the assumptions above. Hence we have shown that the inequality (55) is satisfied for all nonbasic variables. We can conclude that D is an optimal basis for LP (54), and consequently $\pi = (d)^\infty$ is an optimal policy when $0 \le c \le \min\{\frac{\gamma_2}{2\gamma_1+4\gamma_2}, \frac{\gamma_1}{4\gamma_1+2\gamma_2}\}$. We see that the recurrent states are $(0, 1, 2)$, $(1, 1, 1)$, $(1, 2, 2)$, and $(2, 1, 2)$ under this policy. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $0 \le c \le \min\{\frac{\gamma_2}{2\gamma_1+4\gamma_2}, \frac{\gamma_1}{4\gamma_1+2\gamma_2}\}$.

Next, let $\gamma_1 \geq \gamma_2$ and $\frac{\gamma_2}{2\gamma_1+4\gamma_2} < c \leq \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2)\}, \\ a_{21} & \text{if } x \in \{(0,2,1),(1,2,1)\}, \\ a_{22} & \text{if } x \in \{(1,2,2),(2,1,1),(2,1,2),(2,2,1)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D &= \{\omega((0,1,2),(1,2)), \omega((0,2,1),(2,1)), \omega((0,2,2),(1,2)), \\ &\quad \omega((1,1,1),(1,2)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(2,2)), \\ &\quad \omega((2,1,1),(2,2)), \omega((2,1,2),(2,2)), \omega((2,2,1),(2,2))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2)$, $(0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{11})} - c_{w((0,2,1),a_{11})} \\ & = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} = \frac{\gamma_1(2\gamma_1+\gamma_2)(4c\gamma_2+2c\gamma_1-\gamma_2)}{(\gamma_1+\gamma_2)^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = 2c\gamma_1, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} = 0. \end{aligned}$$

These quantities are nonnegative because $c > \frac{\gamma_2}{2\gamma_1+4\gamma_2}$. For state $(1,1,1)$ we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} \\ & \quad = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2}, \\ & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = \frac{\gamma_1\gamma_2(2c\gamma_1+4c\gamma_2-\gamma_2)}{(\gamma_1+\gamma_2)^2}, \\ & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = 0, \\ & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{\gamma_2(2c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1\gamma_2)}{(\gamma_1+\gamma_2)^2}, \end{aligned}$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\gamma_1\gamma_2(2c\gamma_2+\gamma_1)}{(\gamma_1+\gamma_2)^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = \frac{\gamma_1(4c\gamma_1^2+8c\gamma_2^2+10c\gamma_1\gamma_2-\gamma_2^2)}{(\gamma_1+\gamma_2)^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = 2c(\gamma_1+\gamma_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{\gamma_2(2c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1\gamma_2)}{(\gamma_1+\gamma_2)^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\gamma_1\gamma_2(\gamma_1+2c\gamma_2)}{(\gamma_1+\gamma_2)^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{\gamma_1(4c\gamma_1^2+8c\gamma_2^2+10c\gamma_1\gamma_2-\gamma_2^2)}{(\gamma_1+\gamma_2)^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = 2c(\gamma_1+\gamma_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{\gamma_2(2c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1\gamma_2^2)}{(\gamma_1+\gamma_2)^2},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})},$$
$$= \frac{\gamma_2(2c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1\gamma_2^2)}{(\gamma_1+\gamma_2)^2}$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{\gamma_1(6c\gamma_1+8c\gamma_2-\gamma_2)}{(\gamma_1+\gamma_2)^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})}$$
$$= \frac{\gamma_1(2c\gamma_1+4c\gamma_2-\gamma_2)}{2(\gamma_1+\gamma_2)}.$$

These quantities are nonnegative because $\frac{\gamma_2}{2\gamma_1+4\gamma_2} < c \leq \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$. Finally, for states $(2,1,1),(2,1,2)$ and $(2,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{12})} - c_{w((2,1,2),a_{12})}$$
$$= c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{21})} - c_{w((2,2,1),a_{21})}$$

$$= \frac{\gamma_2(\gamma_1^2 - 2c\gamma_1^2 - 2c\gamma_2^2 - 2c\gamma_1\gamma_2)}{(\gamma_1+\gamma_2)^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})}$$
$$= \frac{\gamma_1\gamma_2(2c\gamma_2+\gamma_1)}{(\gamma_1+\gamma_2)^2}.$$

These quantities are nonnegative because $c \leq \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(0,2,2)$, $(1,1,2)$, $(1,2,2)$, and $(2,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\gamma_1 \geq \gamma_2$ and $\frac{\gamma_2}{2\gamma_1+4\gamma_2} < c \leq \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$.

When $\gamma_1 < \gamma_2$ and $\frac{\gamma_1}{4\gamma_1+2\gamma_2} < c \leq \frac{\gamma_2^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,1,2),(0,2,1),(0,2,2),(1,1,1)\}, \\ a_{12} & \text{if } x \in \{(1,1,2),(1,2,2),(2,1,1),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,2,1)\}, \end{cases}$$

Lemma 5.3.2 and the previous result (for the case where $\gamma_1 \geq \gamma_2$ and $\frac{\gamma_2}{2\gamma_1+4\gamma_2} < c \leq \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$) show that the policy $\pi = (d)^\infty$ optimal. The recurrent states under this policy are $(0,1,2)$, $(1,1,1)$, $(1,1,2)$, $(2,1,1)$ and $(2,1,2)$; and in the transient states we can select the actions that take the process to one of the recurrent states. Hence, the policy π^* described in the theorem is optimal when $\gamma_1 < \gamma_2$ and $\frac{\gamma_1}{4\gamma_1+2\gamma_2} < c \leq \frac{\gamma_2^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$.

Next, let $\gamma_1 \geq \gamma_2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2),(2,1,1),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(0,2,1),(1,2,1)\}, \\ a_{22} & \text{if } x \in \{(1,2,2),(2,2,1)\}, \end{cases}$$

The basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D &= \{\omega((0,1,2),(1,2)), \omega((0,2,1),(2,1)), \omega((0,2,2),(1,2)), \\ &\quad \omega((1,1,1),(1,2)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(2,2)), \\ &\quad \omega((2,1,1),(1,2)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,2))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2), (0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\ &\quad = \frac{\gamma_1(4c\gamma_1^2 + 4c\gamma_2^2 + 4c\gamma_1\gamma_2 - \gamma_1\gamma_2 - \gamma_1^2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = \frac{\gamma_1(4c\gamma_1^2 + 4c\gamma_2^2 + 4c\gamma_1\gamma_2 - \gamma_1^2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{11})} - c_{w((0,2,1),a_{11})} = \frac{\gamma_1(\gamma_1 - \gamma_2)(2\gamma_1 + \gamma_2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{\gamma_1^3}{\gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} = \frac{\gamma_1(2c\gamma_1^2 + 2c\gamma_2^2 + 2c\gamma_1\gamma_2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}. \end{aligned}$$

These quantities are nonnegative because $\gamma_1 \geq \gamma_2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$. For state $(1,1,1)$ we have :

$$\begin{aligned} c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} &= \frac{\gamma_1\gamma_2^2}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}, \\ c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} &= \frac{\gamma_1(2c\gamma_1^2 + 2c\gamma_2^2 + 2c\gamma_1\gamma_2 + \gamma_1\gamma_2 - \gamma_1^2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}, \\ c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} &= \frac{\gamma_1\gamma_2(\gamma_1 - \gamma_2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}, \\ c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} &= \frac{(\gamma_1 + \gamma_2)(2c\gamma_1^2 + 2c\gamma_2^2 + 2c\gamma_1\gamma_2 - \gamma_1^2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}, \\ c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} &= \frac{\gamma_2(4c\gamma_1^2 + 4c\gamma_2^2 + 4c\gamma_1\gamma_2 + \gamma_1\gamma_2 - \gamma_1^2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}, \end{aligned}$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\gamma_1\gamma_2^2}{\gamma_1^2 + 2\gamma_1\gamma_2 + \gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\gamma_1(4c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1\gamma_2-2\gamma_1^2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = \frac{\gamma_1(4c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1\gamma_2-\gamma_2^2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = \frac{(\gamma_1+\gamma_2)(4c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2-\gamma_1^2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{\gamma_2(4c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1\gamma_2-\gamma_1^2)}{(\gamma_1+\gamma_2)^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\gamma_1^2\gamma_2}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\gamma_1\gamma_2^2}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{\gamma_1(\gamma_1+\gamma_2(2\gamma_1-\gamma_2))}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = \frac{\gamma_1^2(\gamma_1+\gamma_2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{\gamma_1\gamma_2(\gamma_1+\gamma_2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = \frac{\gamma_1(8c\gamma_1^2+8c\gamma_2^2+8c\gamma_1\gamma_2+\gamma_1\gamma_2-3\gamma_1^2)}{(\gamma_1+\gamma_2)^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} = \frac{\gamma_2(4c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1\gamma_2-\gamma_1^2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{\gamma_1(4c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2+\gamma_1^2-\gamma_2^2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = \frac{\gamma_1(\gamma_1-\gamma_2)(\gamma_1+\gamma_2)}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = \frac{(\gamma_1+\gamma_2)(4c\gamma_1^2+4c\gamma_2^2+4c\gamma_1\gamma_2-\gamma_1\gamma_2-\gamma_1^2)}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}.$$

These quantities are nonnegative because $\gamma_1 \geq \gamma_2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$. Finally, for states $(2,1,1), (2,1,2)$ and $(2,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = \frac{\gamma_2(2c\gamma_1^2+2c\gamma_2^2+2c\gamma_1\gamma_2-\gamma_1^2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})}$$
$$= \frac{2\gamma_2(2c\gamma_1^2 + 2c\gamma_2^2 + 2c\gamma_1\gamma_2 - \gamma_1^2)}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{\gamma_2(4c\gamma_1^2 + 4c\gamma_2^2 + 4c\gamma_1\gamma_2 - \gamma_1^2)}{2\gamma_1^2 + 2\gamma_1\gamma_2 + 2\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{\gamma_1^2\gamma_2}{\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2}$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{21})} - c_{w((2,2,1),a_{21})} = 0.$$

These quantities are nonnegative because $\gamma_1 \geq \gamma_2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, and $(2,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent classes and this shows that the policy π^* described in the theorem is optimal when $\gamma_1 \geq \gamma_2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$.

Finally, let $\gamma_1 < \gamma_2$ and $c > \frac{\gamma_2^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$, and consider the following decision rule:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1),(1,1,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,2),(1,2,2),(2,1,1),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,2,1)\}, \end{cases}$$

Lemma 5.3.2 and the previous result (for the case where $\gamma_1 \geq \gamma_2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$) show that this policy is optimal. Under the policy $\pi = (d)^\infty$, the recurrent states are $(0,1,2)$, $(1,1,1)$, $(1,1,2)$, $(2,1,1)$ and $(2,1,2)$; and in the transient states we can select the actions that take the process to one of the recurrent states. Hence, the policy π^* described in the theorem is optimal when $\gamma_1 < \gamma_2$ and $c > \frac{\gamma_2^2}{2\gamma_1^2+2\gamma_1\gamma_2+2\gamma_2^2}$. Hence the proof is complete. □

Proof of Theorem 5.4.2: We will use the LP approach for communicating Markov decision processes using the notation in the proof of Theorem 5.4.1.

Here, we prove the result when $\gamma_1 \geq \gamma_2$ because the result for the other case follows from the reversibility of two station tandem lines, as shown in Lemma 5.3.2. We first prove the optimality of the policy for $0 \leq c \leq \frac{\gamma_2}{2\gamma_1+2\gamma_2}$. Consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,1,2),(0,2,1),(0,2,2),(1,1,1)\}, \\ a_{12} & \text{if } x \in \{(1,1,2),(1,2,2),(2,1,1),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,2,1)\}, \\ a_{22} & \text{if } x \in \{(2,2,2),(3,1,1),(3,1,2),(3,2,1)\}, \end{cases}$$

More specifically, consider the basic solution ω corresponding to the policy $\pi = (d)^\infty$ with the basis

$$\begin{aligned} D \;=\; & \{\omega((0,1,2),(1,1)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,1)), \\ & \omega((1,1,1),(1,1)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(1,2)), \\ & \omega((2,1,1),(1,2)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\ & \omega((3,1,1),(2,2)), \omega((3,1,2),(2,2)), \omega((3,2,1),(2,2))\}. \end{aligned}$$

For states $(0,1,2), (0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B \mathbf{B}^{-1} v_{w((0,1,2),a_{12})} - c_{w((0,1,2),a_{12})} = c_B \mathbf{B}^{-1} v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} \\ & \quad = c_B \mathbf{B}^{-1} v_{w((0,2,2),a_{12})} - c_{w((0,2,2),a_{12})} = c_B \mathbf{B}^{-1} v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} \\ & \quad = \frac{\gamma_1(\gamma_2 - 2c\gamma_1 - 2c\gamma_2)}{\gamma_1+\gamma_2}, \\ & c_B \mathbf{B}^{-1} v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = c_B \mathbf{B}^{-1} v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} \\ & \quad = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}. \end{aligned}$$

It is clear that these quantities are nonnegative when $0 \leq c \leq \frac{\gamma_2}{2\gamma_1+2\gamma_2}$. For state

$(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})}$$
$$= \frac{\gamma_1\gamma_2(\gamma_1 + c\gamma_1)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{12})} - c_{w((1,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = c\gamma_2,$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = 6c\gamma_2,$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = \frac{2c\gamma_1(2\gamma_1+\gamma_2)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = 2c(\gamma_1+\gamma_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{2c\gamma_2(2\gamma_1+\gamma_2)}{\gamma_1+\gamma_2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{2c\gamma_1(2\gamma_1+\gamma_2)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = 2c(\gamma_1+\gamma_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{2c\gamma_2(2\gamma_1+\gamma_2)}{\gamma_1+\gamma_2},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})}$$
$$= \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{2c\gamma_1(2\gamma_1+\gamma_2)}{\gamma_1+\gamma_2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = 0,$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{22})} - c_{w((1,2,2),a_{22})} = \frac{2c\gamma_1\gamma_2}{\gamma_1+\gamma_2}.$$

For state $(2,1,1)$ we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})} \\ & \quad = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2}, \\ & c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{2c\gamma_1\gamma_2}{\gamma_1+\gamma_2}, \\ & c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = 0, \\ & c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = \frac{2c\gamma_2}{\gamma_1+2\gamma_2}, \end{aligned}$$

for state $(2,1,2)$ we obtain :

$$\begin{aligned} c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} &= \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2}, \\ c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} &= \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}, \\ c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})} &= \frac{2c\gamma_1(2\gamma_1+3\gamma_2)}{\gamma_1+\gamma_2}, \\ c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} &= 2c(\gamma_1+\gamma_2), \\ c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})} &= \frac{2c\gamma_2(2\gamma_1+2\gamma_2)}{\gamma_1+\gamma_2}, \end{aligned}$$

for state $(2,2,1)$ we obtain :

$$\begin{aligned} c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} &= \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}, \\ c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} &= \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2}, \\ c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} &= \frac{2c\gamma_1(2\gamma_1+3\gamma_2)}{\gamma_1+\gamma_2}, \\ c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} &= 2c(\gamma_1+\gamma_2), \\ c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} &= \frac{2c\gamma_2(2\gamma_1+2\gamma_2)}{\gamma_1+\gamma_2}, \end{aligned}$$

and for state $(2,2,2)$ we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})} \\ & \quad = \frac{\gamma_1(\gamma_2+c\gamma_1)}{\gamma_1+\gamma_2}, \\ & c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})} = 6c\gamma_1, \\ & c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((2,2,2),a_{21})} = 2c\gamma_1. \end{aligned}$$

Finally, for states $(3,1,1),(3,1,2)$ and $(3,2,1)$ we have

$$\begin{aligned}
& c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{12})} - c_{w((3,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{12})} - c_{w((3,1,2),a_{12})} \\
&\quad = c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})} \\
&\quad = \frac{\gamma_2(\gamma_1 - 2c\gamma_1 - 2c\gamma_2)}{\gamma_1+\gamma_2}, \\
& c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} \\
&\quad = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}.
\end{aligned}$$

These quantities are also nonnegative when c, γ_1, and γ_2 satisfy the assumptions above. Hence we have shown that the inequality (55) is satisfied for all nonbasic variables. We can conclude that D is an optimal basis for LP (54), and consequently $\pi = (d)^\infty$ is an optimal policy when $0 \le c \le \frac{\gamma_2}{2\gamma_1+2\gamma_2}$. We see that the recurrent states are $(0,1,2)$, $(1,1,1)$, $(1,1,2)$, $(1,2,2)$, $(2,1,1)$, $(2,1,2)$, $(2,2,2)$ and $(3,1,2)$ under this policy. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $0 \le c \le \frac{\gamma_2}{2\gamma_1+2\gamma_2}$.

Next, let $\frac{\gamma_2}{2\gamma_1+2\gamma_2} < c \le \min\{\frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}, \frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2}\}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(1,1,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,2),(1,2,2),(2,1,1),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(0,2,1),(1,2,1),(2,2,1)\}, \\ a_{22} & \text{if } x \in \{(2,2,2),(3,1,1),(3,1,2),(3,2,1)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned}
D \;=\; & \{\omega((0,1,2),(1,2)), \omega((0,2,1),(2,1)), \omega((0,2,2),(1,2)), \\
& \omega((1,1,1),(1,1)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(1,2)), \\
& \omega((2,1,1),(1,2)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\
& \omega((3,1,1),(2,2)), \omega((3,1,2),(2,2)), \omega((3,2,1),(2,2))\}.
\end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2),(0,2,1)$, and $(0,2,2)$, we have :

$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})}-c_{w((0,1,2),a_{11})}=c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{11})}-c_{w((0,2,1),a_{11})}$$
$$=c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})}-c_{w((0,2,2),a_{11})}=\frac{2\gamma_1(\gamma_1+\gamma_2)(2c\gamma_1+2c\gamma_2-\gamma_2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})}-c_{w((0,1,2),a_{21})}=c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})}-c_{w((0,2,1),a_{12})}=2c\gamma_1,$$
$$c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})}-c_{w((0,2,2),a_{21})}=0.$$

These quantities are nonnegative because $c>\frac{\gamma_2}{2\gamma_1+2\gamma_2}$. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})}-c_{w((1,1,1),a_{02})}=c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})}-c_{w((1,1,1),a_{20})}$$
$$=\frac{\gamma_2(2\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+\gamma_2^3-4c\gamma_1^3-6c\gamma_1^2\gamma_2-4c\gamma_1\gamma_2^2)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{12})}-c_{w((1,1,1),a_{12})}=c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})}-c_{w((1,1,1),a_{21})}$$
$$=\frac{\gamma_2(\gamma_2^2+2\gamma_1\gamma_2-2c\gamma_1^2-4c\gamma_1\gamma_2)}{2(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})}-c_{w((1,1,1),a_{22})}=\frac{\gamma_2(\gamma_2^2+2\gamma_1\gamma_2+2c\gamma_1^2+4c\gamma_1\gamma_2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})}-c_{w((1,1,2),a_{02})}=\frac{\gamma_1\gamma_2(\gamma_1+\gamma_2-2c\gamma_1)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})}-c_{w((1,1,2),a_{10})}=\frac{\gamma_1^2\gamma_2(\gamma_1^2+2c\gamma_2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})}-c_{w((1,1,2),a_{11})}=\frac{\gamma_1(2\gamma_1+\gamma_2)(2c\gamma_1^2+4c\gamma_1\gamma_2+4c\gamma_2^2-\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})}-c_{w((1,1,2),a_{21})}=2c(\gamma_1+\gamma_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})}-c_{w((1,1,2),a_{22})}$$
$$=\frac{\gamma_2(2\gamma_1^2\gamma_2+\gamma_1\gamma_2^2+2c\gamma_1^3+2c\gamma_1^2\gamma_2+8c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})}-c_{w((1,2,1),a_{01})}=\frac{\gamma_1^2\gamma_2(\gamma_1^2+2c\gamma_2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})}-c_{w((1,2,1),a_{20})}=\frac{\gamma_1\gamma_2(\gamma_1+\gamma_2-2c\gamma_1)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{\gamma_1(2\gamma_1+\gamma_2)(2c\gamma_1^2+4c\gamma_1\gamma_2+4c\gamma_2^2-\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = 2c(\gamma_1+\gamma_2),$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \\ = \frac{\gamma_2(2\gamma_1^2\gamma_2+\gamma_1\gamma_2^2+2c\gamma_1^3+2c\gamma_1^2\gamma_2+8c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})}, \\ = \frac{\gamma_1^2\gamma_2(\gamma_1+2c\gamma_2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{\gamma_1(2\gamma_1+\gamma_2)(2c\gamma_1^2+4c\gamma_1\gamma_2+4c\gamma_2^2-\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = 0,$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{22})} - c_{w((1,2,2),a_{22})} = \frac{\gamma_1\gamma_2(2\gamma_1\gamma_2+\gamma_2^2-2c\gamma_1^2-4c\gamma_1\gamma_2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)}.$$

For state $(2,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})} \\ = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{\gamma_1\gamma_2(2c\gamma_1^2+4c\gamma_1\gamma_2+4c\gamma_2^2-\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = 0,$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = \frac{\gamma_2(\gamma_1\gamma_2^2+2c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$

for state $(2,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\gamma_1\gamma_2(\gamma_1^2+2c\gamma_2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})} = \frac{\gamma_1(4c\gamma_1^3+10c\gamma_1^2\gamma_2+12c\gamma_1\gamma_2^2+8c\gamma_2^3-\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = 2c(\gamma_1+\gamma_2),$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})} = \frac{\gamma_2(\gamma_1\gamma_2^2+2c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},$$

for state $(2,2,1)$ we obtain :

$$\begin{aligned}
c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} &= \frac{\gamma_1\gamma_2(1-2c)}{\gamma_1+\gamma_2}, \\
c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} &= \frac{\gamma_1\gamma_2(\gamma_1^2+2c\gamma_2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2}, \\
c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} &= \frac{\gamma_1(4c\gamma_1^3+10c\gamma_1^2\gamma_2+12c\gamma_1\gamma_2^2+8c\gamma_2^3-\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)}, \\
c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} &= 2c(\gamma_1+\gamma_2), \\
c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} &= \frac{\gamma_2(\gamma_1\gamma_2^2+2c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)},
\end{aligned}$$

and for state $(2,2,2)$ we have :

$$\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})}, \\
&\quad = \frac{\gamma_1(2\gamma_1^2\gamma_2+\gamma_1\gamma_2^2+2c\gamma_1^3+2c\gamma_1^2\gamma_2+8c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)}, \\
&c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})} = \frac{\gamma_1(6c\gamma_1^2+8c\gamma_1\gamma_2+8c\gamma_2^2-\gamma_2^2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2}, \\
&c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})} \\
&\quad = \frac{\gamma_1(2c\gamma_1^2+4c\gamma_1\gamma_2+4c\gamma_2^2-\gamma_2^2)}{2(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)}.
\end{aligned}$$

These quantities are nonnegative because $\frac{\gamma_2}{2\gamma_1+2\gamma_2} < c \leq \min\{\frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}, \frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2}\}$. Finally, for states $(3,1,1), (3,1,2)$ and $(3,2,1)$ we have

$$\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{12})} - c_{w((3,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{12})} - c_{w((3,1,2),a_{12})} \\
&= c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})} \\
&\quad = \frac{\gamma_2(\gamma_1^2-2c\gamma_1^2-2c\gamma_2^2)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2}, \\
&c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} \\
&\quad = \frac{\gamma_1\gamma_2(2c\gamma_2+\gamma_1)}{\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2}.
\end{aligned}$$

These quantities are nonnegative because $c \leq \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, $(1,2,2)$, $(2,1,2)$, $(2,2,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\frac{\gamma_2}{2\gamma_1+2\gamma_2} < c \leq \min\{\frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}, \frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2}\}$.

Now, let $\gamma_1^2 \leq \gamma_1\gamma_2 + \gamma_2^2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(1,1,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,2),(1,2,2),(2,1,1),(2,1,2), \\ & \qquad (3,1,1),(3,1,2)\}, \\ a_{21} & \text{if } x \in \{(0,2,1),(1,2,1),(2,2,1)\}, \\ a_{22} & \text{if } x \in \{(2,2,2),(3,2,1)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D \;=\; & \{\omega((0,1,2),(1,2)), \omega((0,2,1),(2,1)), \omega((0,2,2),(1,2)), \\ & \omega((1,1,1),(1,1)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(1,2)), \\ & \omega((2,1,1),(1,2)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\ & \omega((3,1,1),(1,2)), \omega((3,1,2),(1,2)), \omega((3,2,1),(2,2))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2), (0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\ & \quad = \frac{2\gamma_1(2c\gamma_1^2 + 2c\gamma_2^2 - \gamma_2^2)}{\gamma_1^2+\gamma_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = \frac{\gamma_1(4c\gamma_1^2 + 4c\gamma_2^2 - \gamma_1^2)}{\gamma_1^2+\gamma_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{11})} - c_{w((0,2,1),a_{11})} = \frac{2\gamma_1(\gamma_1-\gamma_2)(\gamma_1+\gamma_2)}{\gamma_1^2+\gamma_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{\gamma_1^3}{\gamma_1^2+\gamma_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} = \frac{\gamma_1(2c\gamma_1^2 + 2c\gamma_2^2 - \gamma_1^2)}{\gamma_1^2+\gamma_2^2}. \end{aligned}$$

These quantities are nonnegative because $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}$. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = \frac{\gamma_2^2(\gamma_1^2+3\gamma_1\gamma_2+\gamma_2^2)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)}$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} = \frac{\gamma_2(4c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3-2\gamma_1^3-\gamma_1^2\gamma_2+3\gamma_1\gamma_2^2+\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{12})} - c_{w((1,1,1),a_{12})} = \frac{\gamma_2(\gamma_2^2+\gamma_1\gamma_2-\gamma_1^2)}{2(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} == \frac{4c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3-2\gamma_1^3-3\gamma_1^2\gamma_2+\gamma_1\gamma_2^2+\gamma_2^3}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{\gamma_2(\gamma_2^2+\gamma_1\gamma_2-\gamma_1^2+4c\gamma_1^3+4c\gamma_2^3)}{\gamma_1^2+\gamma_2^2},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\gamma_1\gamma_2^2}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\gamma_1^3\gamma_2}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = \frac{\gamma_1(\gamma_1^2\gamma_2-\gamma_1\gamma_2^2-\gamma_2^3+4c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = \frac{\gamma_1(4c\gamma_1^2+4c\gamma_2^2-\gamma_1^2)}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{\gamma_2(\gamma_1^2\gamma_2+\gamma_1\gamma_2^2-\gamma_1^3+4c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\gamma_1^3\gamma_2}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\gamma_1\gamma_2^2}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{\gamma_1(2\gamma_1+\gamma_2)(\gamma_1^2+\gamma_1\gamma_2-\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = \frac{\gamma_1^2(\gamma_1+\gamma_2)}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{\gamma_1\gamma_2(\gamma_1^2+3\gamma_1\gamma_2+\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = \frac{\gamma_1(2c\gamma_1^3 + 2c\gamma_1^2\gamma_2 + 2c\gamma_1\gamma_2^2 + 2c\gamma_2^3 - \gamma_1^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} = \frac{\gamma_1^3\gamma_2}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{\gamma_1(\gamma_1^2\gamma_2 - \gamma_1\gamma_2^2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = \frac{(\gamma_1+\gamma_2)(4c\gamma_1^2 + 4c\gamma_2^2 - \gamma_1^2)}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{22})} - c_{w((1,2,2),a_{22})} = \frac{\gamma_2(\gamma_1\gamma_2 - \gamma_1^2\gamma_2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)}.$$

For state $(2,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = \frac{\gamma_1\gamma_2^3}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})} = \frac{\gamma_1(2c\gamma_1^3 + 2c\gamma_1^2\gamma_2 + 2c\gamma_1\gamma_2^2 + 2c\gamma_2^3 + \gamma_1\gamma_2^2 - \gamma_1^2\gamma_2 - \gamma_1^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{\gamma_1\gamma_2(\gamma_1^2 + \gamma_1\gamma_2 - \gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = \frac{(\gamma_1+\gamma_2)(2c\gamma_1^2 + 2c\gamma_2^2 - \gamma_1^2)}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = \frac{\gamma_1(\gamma_1\gamma_2^2 - \gamma_1^2\gamma_2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

for state $(2,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\gamma_1\gamma_2^3}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\gamma_1^2\gamma_2}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})} = \frac{\gamma_1(\gamma_1\gamma_2^2 + \gamma_1^2\gamma_2 - \gamma_2^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{(\gamma_1+\gamma_2)(4c\gamma_1^2 + 4c\gamma_2^2 - \gamma_1^2)}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})}$$
$$= \frac{\gamma_1(\gamma_1\gamma_2^2 - \gamma_1^2\gamma_2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\gamma_1^2\gamma_2}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{\gamma_1\gamma_2^3}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} = \frac{\gamma_1(2\gamma_1^3 + 3\gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 - \gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{\gamma_1^2(\gamma_1+\gamma_2)}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} = \frac{\gamma_1\gamma_2(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

and for state $(2,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})}$$
$$= \frac{\gamma_1(\gamma_1\gamma_2^2 + \gamma_1^2\gamma_2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})} = \frac{\gamma_1^2 2(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})} = \frac{\gamma_1(\gamma_1^2+\gamma_1\gamma_2-\gamma_2^2+4c\gamma_1^2+4c\gamma_2^2)}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = \frac{\gamma_1(\gamma_1^2+\gamma_1\gamma_2-\gamma_2^2)}{2(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})}$$
$$= \frac{-\gamma_1\gamma_2^2 - \gamma_1^2\gamma_2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3}{2(\gamma_1^2+\gamma_2^2)},$$

These quantities are nonnegative because $\gamma_1^2 \leq \gamma_1\gamma_2 + \gamma_2^2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}$. Finally, for states $(3,1,1), (3,1,2)$ and $(3,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{12})} - c_{w((3,1,1),a_{21})} = \frac{\gamma_2(2c\gamma_1^2+2c\gamma_2^2-\gamma_1^2)}{\gamma_1^2+\gamma_2^2},$$

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{22})} - c_{w((3,1,1),a_{22})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{22})} - c_{w((3,1,2),a_{22})} \\
&\quad = \frac{2\gamma_2(2c\gamma_1^2 + 2c\gamma_2^2 - \gamma_1^2)}{\gamma_1^2 + \gamma_2^2}, \\
&c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})} = \frac{\gamma_2(4c\gamma_1^2 + 4c\gamma_2^2 - \gamma_1^2)}{\gamma_1^2 + \gamma_2^2}, \\
&c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})}\frac{\gamma_1^2\gamma_2}{\gamma_1^2 + \gamma_2^2} \\
&c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})} = 0.
\end{aligned}
$$

These quantities are nonnegative because $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, $(2,1,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\gamma_1^2 \leq \gamma_1\gamma_2 + \gamma_2^2$ and $c > \frac{\gamma_1^2}{2\gamma_1^2+2\gamma_2^2}$.

Next, let $\gamma_1^2 > \gamma_1\gamma_2 + \gamma_2^2$ and $\frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2} < c \leq \frac{3\gamma_1^2+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$
d^(x) = \begin{cases} a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2),(2,1,1),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(0,2,1),(1,2,1),(2,2,1)\}, \\ a_{22} & \text{if } x \in \{(1,2,2),(2,2,2),(3,1,1),(3,1,2),(3,2,1)\}, \end{cases}
$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$
\begin{aligned}
D \;=\; &\{\omega((0,1,2),(1,2)),\omega((0,2,1),(2,1)),\omega((0,2,2),(1,2)), \\
&\omega((1,1,1),(1,2)),\omega((1,1,2),(1,2)),\omega((1,2,1),(2,1)),\omega((1,2,2),(2,2)), \\
&\omega((2,1,1),(1,2)),\omega((2,1,2),(1,2)),\omega((2,2,1),(2,1)),\omega((2,2,2),(2,2)), \\
&\omega((3,1,1),(2,2)),\omega((3,1,2),(2,2)),\omega((3,2,1),(2,2))\}.
\end{aligned}
$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2),(0,2,1)$, and $(0,2,2)$, we have :

$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{11})} - c_{w((0,2,1),a_{11})}$$
$$= c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})}$$
$$\frac{-2\gamma_1\gamma_2^2 - 3\gamma_1^2\gamma_2 - \gamma_2^3 + 6c\gamma_1^3 + 14c\gamma_1^2\gamma_2 + 8c\gamma_1\gamma_2^2 + 4c\gamma_2^3}{3\gamma_1^3 + 6\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 2\gamma_2^3},$$
$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = 2c\gamma_1,$$
$$c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} = 0.$$

These quantities are nonnegative because $c > \frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2}$. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})}$$
$$= \frac{\gamma_1\gamma_2(3\gamma_1^2 + 4\gamma_1\gamma_2 + 2\gamma_2^2 - 4c\gamma_1^2 - 4c\gamma_1\gamma_2)}{3\gamma_1^3 + 6\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 2\gamma_2^3},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = \frac{2\gamma_1\gamma_2(-2\gamma_1\gamma_2 - \gamma_2^2 + 2c\gamma_1^2 + 4c\gamma_1\gamma_2)}{3\gamma_1^3 + 6\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 2\gamma_2^3},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = 0,$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})}$$
$$= \frac{2\gamma_2(2\gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 + 4c\gamma_1^3 + 8c\gamma_1^2\gamma_2 + 8c\gamma_1\gamma_2^2 + 4c\gamma_2^2)}{3\gamma_1^3 + 6\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 2\gamma_2^3},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\gamma_1\gamma_2(3\gamma_1^2 + 4\gamma_1\gamma_2 + 2\gamma_2^2 - 4c\gamma_1^2 - 4c\gamma_1\gamma_2)}{3\gamma_1^3 + 6\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 2\gamma_2^3},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\gamma_1^2\gamma_2(3\gamma_1 + 4c\gamma_2)}{3\gamma_1^3 + 6\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 2\gamma_2^3},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})}$$
$$= \frac{2\gamma_1(-2\gamma_1\gamma_2^2 - \gamma_2^3 + 6c\gamma_1^3 + 14c\gamma_1^2\gamma_2 + 12c\gamma_1\gamma_2^2 + 4c\gamma_2^2)}{3\gamma_1^3 + 6\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 2\gamma_2^3},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = 2c(\gamma_1 + \gamma_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})}$$
$$= \frac{2\gamma_2(2\gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 + 4c\gamma_1^3 + 8c\gamma_1^2\gamma_2 + 8c\gamma_1\gamma_2^2 + 4c\gamma_2^2)}{3\gamma_1^3 + 6\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 2\gamma_2^3},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\gamma_1^2\gamma_2(3\gamma_1+4c\gamma_2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\gamma_1\gamma_2(3\gamma_1^2+4\gamma_1\gamma_2+2\gamma_2^2-4c\gamma_1^2-4c\gamma_1\gamma_2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})}$$
$$= \frac{2\gamma_1(-2\gamma_1\gamma_2^2-\gamma_2^3+6c\gamma_1^3+14c\gamma_1^2\gamma_2+12c\gamma_1\gamma_2^2+4c\gamma_2^2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = 2c(\gamma_1+\gamma_2),$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})}$$
$$= \frac{2\gamma_2(2\gamma_1^2\gamma_2+\gamma_1\gamma_2^2+4c\gamma_1^3+8c\gamma_1^2\gamma_2+8c\gamma_1\gamma_2^2+4c\gamma_2^2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})},$$
$$= \frac{\gamma_1^2(\gamma_1\gamma_2-\gamma_2^2+2c\gamma_1^2+4c\gamma_1\gamma_2+4c\gamma_2^2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})}$$
$$= \frac{\gamma_1(-2\gamma_1^2\gamma_2-3\gamma_1\gamma_2^2-\gamma_2^3+8c\gamma_1^3+18c\gamma_1^2\gamma_2+12c\gamma_1\gamma_2^2+4c\gamma_2^2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})}$$
$$= \frac{\gamma_1(\gamma_1+\gamma_2)(-2\gamma_1\gamma_2-\gamma_2^2+2c\gamma_1^2+4c\gamma_1\gamma_2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

For state $(2,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})}$$
$$= \frac{\gamma_1\gamma_2(3\gamma_1^2+\gamma_1\gamma_2+2\gamma_2^2-4c\gamma_1^2-4c\gamma_1\gamma_2-4c\gamma_2^2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{2\gamma_1\gamma_2(\gamma_1\gamma_2-\gamma_2^2+2c\gamma_1^2+4c\gamma_1\gamma_2+4c\gamma_2^2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = 0,$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})}$$
$$= \frac{2\gamma_2(\gamma_1^2\gamma_2+2\gamma_1\gamma_2^2+2c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

for state $(2,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\gamma_1\gamma_2(3\gamma_1^2+\gamma_1\gamma_2+2\gamma_2^2-4c\gamma_1^2-4c\gamma_1\gamma_2-4c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\gamma_1\gamma_2(\gamma_1+\gamma_2)(3\gamma_1^2+4c\gamma_2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})}$$
$$= \frac{2\gamma_1(\gamma_1\gamma_2^2-\gamma_2^3+6c\gamma_1^3+14c\gamma_1^2\gamma_2+12c\gamma_1\gamma_2^2+8c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = 2c(\gamma_1+\gamma_2),$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})}$$
$$= \frac{2\gamma_2(\gamma_1^2\gamma_2+2\gamma_1\gamma_2^2+2c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\gamma_1\gamma_2(\gamma_1+\gamma_2)(3\gamma_1^2+4c\gamma_2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{\gamma_1\gamma_2(3\gamma_1^2-\gamma_1\gamma_2-2\gamma_2^2+4c\gamma_1^2+4c\gamma_1\gamma_2+4c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} =$$
$$= \frac{2\gamma_1(\gamma_1\gamma_2^2-\gamma_2^3+6c\gamma_1^3+14c\gamma_1^2\gamma_2+12c\gamma_1\gamma_2^2+8c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = 2c(\gamma_1+\gamma_2),$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})}$$
$$= \frac{2\gamma_2(\gamma_1^2\gamma_2+2\gamma_1\gamma_2^2+2c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

and for state $(2,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})},$$
$$= \frac{\gamma_2(2\gamma_1^2\gamma_2+\gamma_1\gamma_2^2+4c\gamma_1^3+8c\gamma_1^2\gamma_2+8c\gamma_1\gamma_2^2+4c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})}$$
$$= \frac{2\gamma_1(-\gamma_1^2\gamma_2-\gamma_1\gamma_2^2-\gamma_2^3+10c\gamma_1^3+22c\gamma_1^2\gamma_2+16c\gamma_1\gamma_2^2+8c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})}$$
$$= \frac{\gamma_1(\gamma_1+\gamma_2)(-\gamma_1\gamma_2-2\gamma_2^2+4c\gamma_1^2+8c\gamma_1\gamma_2+4\gamma_2^2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},$$

These quantities are nonnegative because $\frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2} < c \leq \frac{3\gamma_1^2+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$. Finally, for states $(3,1,1),(3,1,2)$ and $(3,2,1)$ we have

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{12})} - c_{w((3,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{12})} - c_{w((3,1,2),a_{12})} \\
&= c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})} \\
&\quad = \frac{\gamma_2(3\gamma_1^3+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2+4c\gamma_1^3+4c\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4c\gamma_2^3)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3}, \\
&c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} \\
&\quad = \frac{\gamma_1\gamma_2(3\gamma_1^2+\gamma_1\gamma_2-\gamma_2^2+2c\gamma_1^2+8c\gamma_1\gamma_2+4c\gamma_2^2)}{3\gamma_1^3+6\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+2\gamma_2^3},
\end{aligned}
$$

These quantities are nonnegative because $\gamma_1^2 > \gamma_1\gamma_2+\gamma_2^2$ and $c > \frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(0,2,2)$, $(1,1,2)$, $(1,2,2)$, $(2,1,2)$, $(2,2,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\frac{2\gamma_1\gamma_2+\gamma_2^2}{2\gamma_1^2+4\gamma_1\gamma_2} < c \leq \frac{3\gamma_1^2+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^2+4\gamma_1^2\gamma_2+4\gamma_1^2\gamma_2+4\gamma_2^2}$.

Finally, let $\gamma_1^2 > \gamma_1\gamma_2+\gamma_2^2$ and $c > \frac{3\gamma_1^2+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$
d(x) = \begin{cases}
a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2),(2,1,1),(2,1,2), \\
& \qquad\qquad (3,1,1),(3,1,2)\}, \\
a_{21} & \text{if } x \in \{(0,2,1),(1,2,1),(2,2,1)\}, \\
a_{22} & \text{if } x \in \{(1,2,2),(2,2,2),(3,2,1)\},
\end{cases}
$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$
\begin{aligned}
D \ = \ & \{\omega((0,1,2),(1,2)), \omega((0,2,1),(2,1)), \omega((0,2,2),(1,2)), \\
& \omega((1,1,1),(1,2)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(2,2)), \\
& \omega((2,1,1),(1,2)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\
& \omega((3,1,1),(1,2)), \omega((3,1,2),(1,2)), \omega((3,2,1),(2,2))\}.
\end{aligned}
$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2),(0,2,1)$, and $(0,2,2)$, we have :

$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})}$$
$$= \frac{\gamma_1(-\gamma_1^2\gamma_2 - \gamma_1\gamma_2^2 - \gamma_2^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})}$$
$$= \frac{\gamma_1(-3\gamma_1^3 - \gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 + 8c\gamma_1^3 + 8c\gamma_1^2\gamma_2 + 8c\gamma_1\gamma_2^2 + 8c\gamma_2^3)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{11})} - c_{w((0,2,1),a_{11})} = \frac{\gamma_1(\gamma_1-\gamma_2)(3\gamma_1^2+3\gamma_1\gamma_2+\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{\gamma_1^2(3\gamma_1^2+\gamma_1\gamma_2-\gamma_2^2)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})}$$
$$= \frac{\gamma_1(-3\gamma_1^3 - \gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

These quantities are nonnegative because $c > \frac{3\gamma_1^2+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = \frac{\gamma_2^2(\gamma_1^2+3\gamma_1\gamma_2+\gamma_2^2)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)}$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})}$$
$$= \frac{\gamma_2(4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3 - 2\gamma_1^3 - \gamma_1^2\gamma_2 + 3\gamma_1\gamma_2^2 + \gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = \frac{\gamma_1\gamma_2(\gamma_1^2-\gamma_1\gamma_2-\gamma_1^2)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})}$$
$$= \frac{-3\gamma_1^3 - \gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3}{2(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})}$$
$$= \frac{\gamma_2(-\gamma_1^3 + \gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\gamma_1\gamma_2^2}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\gamma_1^3\gamma_2}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})}$$
$$= \frac{\gamma_1(\gamma_1^2\gamma_2 - \gamma_1\gamma_2^2 - \gamma_2^2 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})}$$
$$= \frac{-3\gamma_1^3 - \gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 + 8c\gamma_1^3 + 8c\gamma_1^2\gamma_2 + 8c\gamma_1\gamma_2^2 + 8c\gamma_2^3}{2(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})}$$
$$= \frac{\gamma_1(-\gamma_1^3 + \gamma_1^2\gamma_2 + \gamma_1\gamma_2^2 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\gamma_1^3\gamma_2}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\gamma_1\gamma_2^2}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{\gamma_1(3\gamma_1^2 - \gamma_1\gamma_2 - \gamma_2^2)}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = \frac{\gamma_1(3\gamma_1^2 - \gamma_1\gamma_2 - \gamma_2^2)}{2(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{2\gamma_1^2\gamma_2}{\gamma_1^2+\gamma_2^2},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = \frac{\gamma_1(2c\gamma_1^3 + 2c\gamma_1^2\gamma_2 + 2c\gamma_1\gamma_2^2 + 2c\gamma_2^3 - \gamma_1^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} = \frac{\gamma_1^3\gamma_2}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})}$$
$$= \frac{\gamma_1(\gamma_1^2 - \gamma_1\gamma_2 - \gamma_2^2 + 4c\gamma_1^2 + 4c\gamma_1\gamma_2 + 4c\gamma_2^2)}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = \frac{(\gamma_1+\gamma_2)(2c\gamma_1^2 + 2c\gamma_2^2 - \gamma_1^2)}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})}$$
$$= \frac{\gamma_2(\gamma_1\gamma_2 - \gamma_1^2\gamma_2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)}.$$

For state $(2,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = \frac{\gamma_1\gamma_2^3}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})}$$
$$= \frac{\gamma_1(2c\gamma_1^3+2c\gamma_1^2\gamma_2+2c\gamma_1\gamma_2^2+2c\gamma_2^3+\gamma_1\gamma_2^2-\gamma_1^2\gamma_2-\gamma_1^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{\gamma_1\gamma_2(\gamma_1^2-\gamma_1\gamma_2+\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})}$$
$$= \frac{-3\gamma_1^3-\gamma_1^2\gamma_2^2+\gamma_1\gamma_2^2+4c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3}{2(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = \frac{2\gamma_2(-\gamma_1^2+\gamma_1\gamma_2+2c\gamma_1^2+2c\gamma_2^2)}{\gamma_1^2+\gamma_2^2},$$

for state $(2,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\gamma_1\gamma_2^3}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\gamma_1^2\gamma_2}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})}$$
$$= \frac{\gamma_1(\gamma_1^2\gamma_2+\gamma_1\gamma_2^2-\gamma_2^3+4c\gamma_1^3+4c\gamma_1^2\gamma_2+4c\gamma_1\gamma_2^2+4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})}$$
$$= \frac{-3\gamma_1^3-\gamma_1^2\gamma_2^2+\gamma_1\gamma_2^2+8c\gamma_1^3+8c\gamma_1^2\gamma_2+8c\gamma_1\gamma_2^2+8c\gamma_2^3}{2(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})} = \frac{2\gamma_2(-\gamma_1^2+\gamma_1\gamma_2+2c\gamma_1^2+2c\gamma_2^2)}{\gamma_1^2+\gamma_2^2},$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\gamma_1^2\gamma_2}{\gamma_1^2+\gamma_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{\gamma_1\gamma_2^3}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} = \frac{\gamma_1(3\gamma_1^3+2\gamma_1^2\gamma_2-\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{\gamma_1(3\gamma_1^2+\gamma_1\gamma_2-\gamma_2^2)}{2(\gamma_1^2+\gamma_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} = \frac{\gamma_1\gamma_2(\gamma_1^2+\gamma_1\gamma_2+\gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$

and for state $(2,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})}$$
$$= \frac{\gamma_1(\gamma_1\gamma_2^2 + \gamma_1^2\gamma_2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})} = \frac{\gamma_1^2(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})}$$
$$= \frac{\gamma_1(2\gamma_1^3 + \gamma_1^2\gamma_2^2 - \gamma_1\gamma_2^2 - \gamma_2^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{2(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = \frac{\gamma_1(\gamma_1-\gamma_2)(\gamma_1+\gamma_1)}{\gamma_1^2+\gamma_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})}$$
$$= \frac{-\gamma_1\gamma_2^2 - \gamma_1^2\gamma_2 - \gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3}{2(\gamma_1^2+\gamma_2^2)},$$

These quantities are nonnegative because $\gamma_1^2 > \gamma_1\gamma_2 + \gamma_2^2$ and $c > \frac{3\gamma_1^2+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$. Finally, for states $(3,1,1),(3,1,2)$ and $(3,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})}$$
$$= \frac{\gamma_2(\gamma_1\gamma_2^2 - \gamma_1^2\gamma_2 - 3\gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{22})} - c_{w((3,1,1),a_{22})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{22})} - c_{w((3,1,2),a_{22})}$$
$$= \frac{\gamma_2(\gamma_1\gamma_2^2 - \gamma_1^2\gamma_2 - 3\gamma_1^3 + 4c\gamma_1^3 + 4c\gamma_1^2\gamma_2 + 4c\gamma_1\gamma_2^2 + 4c\gamma_2^3)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})}$$
$$= \frac{\gamma_2(\gamma_1\gamma_2^2 - \gamma_1^2\gamma_2 - 3\gamma_1^3 + 8c\gamma_1^3 + 8c\gamma_1^2\gamma_2 + 8c\gamma_1\gamma_2^2 + 8c\gamma_2^3)}{2(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} = \frac{\gamma_1\gamma_2(3\gamma_1^2 + \gamma_1\gamma_2 - \gamma_2^2)}{(\gamma_1+\gamma_2)(\gamma_1^2+\gamma_2^2)}$$
$$c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})} = 0.$$

These quantities are nonnegative because $c > \frac{3\gamma_1^2+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, $(2,1,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\gamma_1^2 > \gamma_1\gamma_2 + \gamma_2^2$ and $c > \frac{3\gamma_1^2+\gamma_1^2\gamma_2-\gamma_1\gamma_2^2}{4\gamma_1^3+4\gamma_1^2\gamma_2+4\gamma_1\gamma_2^2+4\gamma_2^3}$. Hence the proof is complete. $\square$

Proof of Theorem 5.4.3: We will use the LP approach for communicating Markov decision processes using the notation in the proof of Theorem 5.4.1.

Recall that we assumed that $\mu_1 \geq \mu_2$ since the servers can be relabeled otherwise. We first prove the optimality of the policy for $0 \leq c \leq \frac{\mu_2}{4\mu_1+2\mu_2}$. Consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,1,2),(0,2,1),(0,2,2)\}, \\ a_{12} & \text{if } x \in \{(1,1,1),(1,1,2),(1,2,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1)\}, \\ a_{22} & \text{if } x \in \{(2,1,1),(2,1,2),(2,2,1)\}, \end{cases}$$

More specifically, consider the basic solution ω corresponding to the policy $\pi = (d)^\infty$ with the basis

$$\begin{aligned} D \ = \ & \{\omega((0,1,2),(1,1)),\omega((0,2,1),(1,1)),\omega((0,2,2),(1,1)), \\ & \omega((1,1,1),(1,2)),\omega((1,1,2),(1,2)),\omega((1,2,1),(2,1)),\omega((1,2,2),(1,2)), \\ & \omega((2,1,1),(2,2)),\omega((2,1,2),(2,2)),\omega((2,2,1),(2,2))\}. \end{aligned}$$

For states $(0,1,2),(0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{12})} - c_{w((0,1,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{12})} - c_{w((0,2,2),a_{12})} \\ & \quad = \frac{1}{2}\mu_2 - 2c\mu_1 - c\mu_2, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = \frac{1}{2}\mu_1(1-2c), \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{1}{2}\mu_2(1-2c), \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} \\ & \quad = \frac{1}{2}\mu_1 - c\mu_1 - 2c\mu_2. \end{aligned}$$

It is clear that these quantities are nonnegative when $0 \leq c \leq \frac{\mu_2}{4\mu_1+2\mu_2}$. For state

$(1,1,1)$ we have :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = \frac{1}{2}\mu_1(1-2c),\\
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} = \frac{1}{2}\mu_2(1-2c),\\
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = 0,\\
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = 2c(\mu_1+\mu_2),
\end{aligned}
$$

for state $(1,1,2)$ we obtain :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{1}{2}\mu_1(1-2c),\\
&c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{1}{2}\mu_2(1-2c),\\
&c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})}\\
&= c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = 2c(\mu_1+\mu_2),
\end{aligned}
$$

for state $(1,2,1)$ we obtain :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{1}{2}\mu_1(1-2c),\\
&c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{1}{2}\mu_2(1-2c),\\
&c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})}\\
&= c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,12),a_{22})} = 2c(\mu_1+\mu_2),
\end{aligned}
$$

and for state $(1,2,2)$ we have :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = \frac{1}{2}\mu_1(1-2c),\\
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} = \frac{1}{2}\mu_1(1-2c),\\
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = 2c(\mu_1+\mu_2),\\
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{22})} - c_{w((1,2,2),a_{22})} = 0.
\end{aligned}
$$

Finally, for states $(2,1,1),(2,1,2)$ and $(2,2,1)$ we have

$$
c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{12})} - c_{w((2,1,2),a_{12})}
$$

$$= \frac{1}{2}\mu_1 - c\mu_1 - 2c\mu_2,$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{21})} - c_{w((2,2,1),a_{21})}$$

$$= \frac{1}{2}\mu_2 - 2c\mu_1 - c\mu_2,$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{1}{2}\mu_2(1-2c),$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{1}{2}\mu_1(1-2c).$$

These quantities are also nonnegative when $0 \le c \le \frac{\mu_2}{4\mu_1+2\mu_2}$. Hence we have shown that the inequality (55) is satisfied for all nonbasic variables. We can conclude that D is an optimal basis for LP (54), and consequently $\pi = (d)^\infty$ is an optimal policy when $0 \le c \le \frac{\mu_2}{4\mu_1+2\mu_2}$. We see that the recurrent states are $(0,1,2)$, $(1,1,1)$, $(1,2,2)$, and $(2,1,2)$ under this policy. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $0 \le c \le \frac{\mu_2}{4\mu_1+2\mu_2}$.

Next, let $\frac{\mu_2}{4\mu_1+2\mu_2} < c \le \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,1,1)\}, \\ a_{22} & \text{if } x \in \{(1,2,2),(2,1,2),(2,2,1)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D \;=\; & \{\omega((0,1,2),(1,2)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,2)), \\ & \omega((1,1,1),(1,2)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(2,2)), \\ & \omega((2,1,1),(2,1)), \omega((2,1,2),(2,2)), \omega((2,2,1),(2,2))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2),(0,2,1)$, and $(0,2,2)$, we have :

$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})}$$
$$= \frac{(\mu_1+\mu_2)(2\mu_1+\mu_2)(4c\mu_1+2c\mu_2-\mu_2)}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})}$$
$$= \frac{2\mu_1^3+\mu_1^2\mu_2-3\mu_1\mu_2^2-\mu_2^3-2c\mu_1^3+12c\mu_1^2\mu_2+10c\mu_1\mu_2^2+2c\mu_2^3}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{2\mu_1+\mu_2-2c\mu_1}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} = \frac{2\mu_1^2-\mu_1^2\mu_2-2c\mu_1^3-2c\mu_1\mu_2-2c\mu_2^2}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})}$$
$$= \frac{2\mu_1^3+\mu_1^2\mu_2-3\mu_1\mu_2^2-\mu_2^3-2c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2}.$$

These quantities are nonnegative because $\frac{\mu_2}{4\mu_1+2\mu_2} < c \leq \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = \frac{2\mu_1^2+\mu_1^2\mu_2+\mu_2^2+2c\mu_1^2-4c\mu_1\mu_2-2c\mu_2^2}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} = \frac{\mu_1\mu_2+4\mu_1^2+4\mu_1\mu_2+2\mu_2^2}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = \frac{\mu_1(\mu_1+\mu_2)(4c\mu_1+2c\mu_2-\mu_2)}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = 0,$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{\mu_1\mu_2+4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{2\mu_1^2+\mu_1^2\mu_2+\mu_2^2+c\mu_1^2-4c\mu_1\mu_2-2c\mu_2^2}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\mu_1(\mu_1+\mu_2-3c\mu_1-c\mu_2)}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = 2c(\mu_1+\mu_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = \frac{12c\mu_1^2+8c\mu_1\mu_2+2c\mu_2^2-\mu_1\mu_2}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{\mu_1\mu_2+4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2}{4\mu_1^2+3\mu_1\mu_2+\mu_2^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{2\mu_1^2 + \mu_1^2\mu_2 + \mu_2^2 - 2c\mu_1^3 - 4c\mu_1\mu_2 - 2c\mu_2^2}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{2\mu_1\mu_2(\mu_1 + c\mu_1 + c\mu_2)}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})}$$

$$= \frac{\mu_1\mu_2 + 4c\mu_1^2 + 4c\mu_1\mu_2 + 2c\mu_2^2}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{2\mu_2(\mu_1 + \mu_2)(\mu_1 + c\mu_1 + c\mu_2)}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2}$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = \frac{\mu_1\mu_2 + 2\mu_2^2 + 4c\mu_1^2 - 4c\mu_1\mu_2 - 2c\mu_2^2}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} == \frac{2\mu_1^2 - \mu_1^2\mu_2 - 2c\mu_1^3 - 2c\mu_1\mu_2 - 2c\mu_2^2}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{(\mu_1 + \mu_2)(12c\mu_1^2 + 8c\mu_1\mu_2 + 2c\mu_2^2 - \mu_1\mu_2)}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})}$$

$$= \frac{2\mu_1(\mu_1 + \mu_2)(4c\mu_1 + 2c\mu_2 - \mu_2)}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2}.$$

These quantities are nonnegative because $\frac{\mu_2}{4\mu_1+2\mu_2} < c \leq \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$. Finally, for states $(2,1,1), (2,1,2)$ and $(2,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})} = \frac{2\mu_1^3 + \mu_1^2\mu_2 - \mu_1\mu_2^2 - \mu_2^3 - 2c\mu_1^3 - 4c\mu_1^2\mu_2}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})}$$

$$= \frac{\mu_2(\mu_1 + \mu_2)(4c\mu_1 + 2c\mu_2 - \mu_2)}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{12})} - c_{w((2,1,2),a_{12})} = \frac{2\mu_1^2 - \mu_1\mu_2 - 2c\mu_1^2 - 2c\mu_1\mu_2 - 2c\mu_2^2}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{\mu_2^2 + 8c\mu_1^2 + 2c\mu_1\mu_2}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})}$$

$$= \frac{2\mu_1^3 + \mu_1^2\mu_2 - \mu_1\mu_2^2 - \mu_2^3 - 2c\mu_1^3 + 4c\mu_1^2\mu_2 + 6c\mu_1\mu_2^2 + 2c\mu_2^3}{4\mu_1^2 + 3\mu_1\mu_2 + \mu_2^2}.$$

These quantities are nonnegative because $\frac{\mu_2}{4\mu_1+2\mu_2} < c \leq \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are

$(0,1,2)$, $(0,2,2)$, $(1,1,2)$, $(1,2,2)$, and $(2,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\frac{\mu_2}{4\mu_1+2\mu_2} < c \leq \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$.

Next, let $\frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2} < c \leq \frac{2\mu_1^2-\mu_2^2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,1,1)\}, \\ a_{22} & \text{if } x \in \{(1,2,2),(2,2,1)\}, \end{cases}$$

The basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D \;=\; & \{\omega((0,1,2),(1,2)),\omega((0,2,1),(1,1)),\omega((0,2,2),(1,2)), \\ & \omega((1,1,1),(1,2)),\omega((1,1,2),(1,2)),\omega((1,2,1),(2,1)),\omega((1,2,2),(2,2)), \\ & \omega((2,1,1),(2,1)),\omega((2,1,2),(1,2)),\omega((2,2,1),(2,2))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2),(0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\ & \quad = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1\mu_2+2c\mu_2^2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = \frac{\mu_2(4c\mu_1^2+4c\mu_1^2\mu_2+4c\mu_2^2-\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} = 0, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} = \frac{\mu_2(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}. \end{aligned}$$

These quantities are nonnegative because $c > \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} = \frac{\mu_1(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-2\mu_1^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_1^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_1^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\mu_1(4c\mu_1^2+4c\mu_1^2\mu_2+4c\mu_2^2-4\mu_1^2+\mu_1\mu_2+2\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1^2\mu_2+4c\mu_2^2-2\mu_1^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1^2\mu_2+4c\mu_2^2-\mu_1^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_1^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\mu_1^2\mu_2}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_1^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = \frac{\mu_1^2(\mu_1+\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{\mu_1\mu_2(\mu_1+\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = 2c\mu_2,$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} = \frac{\mu_1(4c\mu_1^2 + 4c\mu_1^2\mu_2 + 4c\mu_2^2 - 3\mu_1^2 + 2\mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{(\mu_1 + \mu_2)(4c\mu_1^2 + 4c\mu_1^2\mu_2 + 4c\mu_2^2 - \mu_1^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = \frac{\mu_1(\mu_1 + \mu_2)(\mu_1 - \mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = \frac{(\mu_1 + \mu_2)(2c\mu_1^2 + 2c\mu_1^2\mu_2 + 2c\mu_2^2 - \mu_1\mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}.$$

These quantities are nonnegative because $c > \frac{2\mu_1^2 - \mu_1\mu_2}{2\mu_1^2 + 2\mu_1\mu_2 + 2\mu_2^2}$. Finally, for states $(2,1,1), (2,1,2)$ and $(2,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})} = \frac{\mu_2(2\mu_1^2 - \mu_2^2 - 2c\mu_1^2 - 2c\mu_1\mu_2 - 2c\mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})}$$
$$= \frac{\mu_2(\mu_1 + \mu_2)(\mu_1 - \mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{\mu_1(4c\mu_1^2 + 4c\mu_1^2\mu_2 + 4c\mu_2^2 - 2\mu_1^2 + \mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})}$$
$$= \frac{(\mu_1 + \mu_2)(2c\mu_1^2 + 2c\mu_1^2\mu_2 + 2c\mu_2^2 - 2\mu_1^2 + \mu_1\mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{\mu_2(2\mu_1^2 - \mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}.$$

These quantities are nonnegative because $\frac{2\mu_1^2 - \mu_1\mu_2}{2\mu_1^2 + 2\mu_1\mu_2 + 2\mu_2^2} < c \leq \frac{2\mu_1^2 - \mu_2^2}{2\mu_1^2 + 2\mu_1\mu_2 + 2\mu_2^2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, and $(2,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\frac{2\mu_1^2 - \mu_1\mu_2}{2\mu_1^2 + 2\mu_1\mu_2 + 2\mu_2^2} < c \leq \frac{2\mu_1^2 - \mu_2^2}{2\mu_1^2 + 2\mu_1\mu_2 + 2\mu_2^2}$.

Finally, let $c > \frac{2\mu_1^2 - \mu_2^2}{2\mu_1^2 + 2\mu_1\mu_2 + 2\mu_2^2}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2),(2,1,1),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1)\}, \\ a_{22} & \text{if } x \in \{(1,2,2),(2,2,1)\}, \end{cases}$$

The basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D \;=\; & \{\omega((0,1,2),(1,2)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,2)), \\ & \omega((1,1,1),(1,2)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(2,2)), \\ & \omega((2,1,1),(1,2)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,2))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2)$, $(0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\ & \quad = \frac{(\mu_1+\mu_2)(2c\mu_1^2 + 2c\mu_1^2\mu_2 + 2c\mu_2^2 - \mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = \frac{\mu_2(4c\mu_1^2 + 4c\mu_1^2\mu_2 + 4c\mu_2^2 - \mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{\mu_1\mu_2^2}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} = 0, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} = \frac{\mu_2(2c\mu_1^2 + 2c\mu_1^2\mu_2 + 2c\mu_2^2 - \mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}. \end{aligned}$$

These quantities are nonnegative because $c > \frac{2\mu_1^2 - \mu_1\mu_2}{2\mu_1^2 + 2\mu_1\mu_2 + 2\mu_2^2}$. For state $(1,1,1)$ we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = \frac{\mu_1\mu_2^2}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} = \frac{\mu_1(2c\mu_1^2 + 2c\mu_1^2\mu_2 + 2c\mu_2^2 - \mu_1^2 + \mu_1\mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}, \\ & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = \frac{\mu_2(\mu_1+\mu_2)(\mu_1-\mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}, \end{aligned}$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_1^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_1^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

for state $(1, 1, 2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\mu_1(4c\mu_1^2+4c\mu_1^2\mu_2+4c\mu_2^2-4\mu_1^2+\mu_1\mu_2+2\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1^2\mu_2+4c\mu_2^2+\mu_1\mu_2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1^2\mu_2+4c\mu_2^2-\mu_1^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_1^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1, 2, 1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\mu_1^2\mu_2}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{(\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = \frac{\mu_1^2(\mu_1+\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{\mu_1\mu_2(\mu_1+\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

and for state $(1, 2, 2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = 2c\mu_2,$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} = \frac{\mu_1(4c\mu_1^2+4c\mu_1^2\mu_2+4c\mu_2^2-3\mu_1^2+2\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2+\mu_1^2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = \frac{\mu_1(\mu_1+\mu_2)(\mu_1-\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1^2\mu_2+2c\mu_2^2-\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

These quantities are nonnegative because $c > \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$. Finally, for states $(2,1,1), (2,1,2)$ and $(2,2,1)$ we have

$$\begin{aligned}
c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} &= \frac{\mu_2(2c\mu_1^2 + 2c\mu_1^2\mu_2 + 2c\mu_2^2 - 2\mu_1^2 + \mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},\\
c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} &= c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})}\\
= \frac{(\mu_1+\mu_2)(2c\mu_1^2 + 2c\mu_1^2\mu_2 + 2c\mu_2^2 - 2\mu_1^2 + \mu_1\mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}&,\\
c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} &= \frac{\mu_1(4c\mu_1^2 + 4c\mu_1^2\mu_2 + 4c\mu_2^2 - 2\mu_1^2 + \mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},\\
c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} &= \frac{\mu_2(2\mu_1^2 - \mu_2^2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2},\\
c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} &= \frac{\mu_2(\mu_1+\mu_2)(\mu_1-\mu_2)}{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}.
\end{aligned}$$

These quantities are nonnegative because $c > \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, and $(2,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $c > \frac{2\mu_1^2-\mu_1\mu_2}{2\mu_1^2+2\mu_1\mu_2+2\mu_2^2}$. Hence the proof is complete. □

Proof of Theorem 5.4.4: We will use the LP approach for communicating Markov decision processes using the notation in the proof of Theorem 5.4.1.

Here, we prove the result when $\mu_1 \geq \mu_2$ since the servers can be relabeled otherwise. We first prove the optimality of the policy for $0 \leq c \leq \frac{4\mu_1^2\mu_2+5\mu_1\mu_2^2+3\mu_2^3}{12\mu_1^3+20\mu_1^2\mu_2+12\mu_1\mu_2^2+4\mu_2^3}$. Consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,1,2),(0,2,1),(0,2,2),(1,1,1)\},\\ a_{12} & \text{if } x \in \{(1,1,2),(1,2,2),(2,1,2)\},\\ a_{21} & \text{if } x \in \{(1,2,1),(2,1,1),(2,2,1)\},\\ a_{22} & \text{if } x \in \{(2,2,2),(3,1,1),(3,1,2),(3,2,1)\},\end{cases}$$

More specifically, consider the basic solution ω corresponding to the policy $\pi = (d)^\infty$

with the basis

$$
\begin{aligned}
D = \{&\omega((0,1,2),(1,1)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,1)),\\
&\omega((1,1,1),(1,1)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(1,2)),\\
&\omega((2,1,1),(2,1)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)),\\
&\omega((3,1,1),(2,2)), \omega((3,1,2),(2,2)), \omega((3,2,1),(2,2))\}.
\end{aligned}
$$

For states $(0,1,2),(0,2,1)$, and $(0,2,2)$, we have :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{12})} - c_{w((0,1,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{12})} - c_{w((0,2,2),a_{12})}\\
&\quad = \frac{4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 3\mu_2^3 - 12c\mu_1^3 - 20c\mu_1^2\mu_2 - 12c\mu_1\mu_2^2 - 4c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = \frac{\mu_1(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2 - 4c\mu_1^2 + 4c\mu_1\mu_2)}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{4\mu_1^3 + 4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 4c\mu_1^3 - 4c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})}\\
&\quad = \frac{4\mu_1^3 + 5\mu_1^2\mu_2 + 3\mu_1\mu_2^2 - 4c\mu_1^3 - 12c\mu_1^2\mu_2 - 20c\mu_1\mu_2^2 - 12c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)}.
\end{aligned}
$$

It is clear that these quantities are nonnegative when $0 \le c \le \frac{4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 3\mu_2^3}{12\mu_1^3 + 20\mu_1^2\mu_2 + 12\mu_1\mu_2^2 + 4\mu_2^3}$.

For state $(1,1,1)$ we have :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})}\\
&\quad = \frac{4\mu_1^3 + 5\mu_1^2\mu_2 + 3\mu_1\mu_2^2 - 4c\mu_1^3 + 4c\mu_1^2\mu_2 + 8c\mu_1\mu_2^2 + 4c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} = \frac{4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 3\mu_2^3 + 8c\mu_1^3 + 8c\mu_1^2\mu_2 + 4c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{12})} - c_{w((1,1,1),a_{12})} = \frac{2c(\mu_1 + \mu_2)(\mu_1^2 + \mu_1\mu_2 + \mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},\\
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = \frac{2c\mu_1(\mu_1 + \mu_2)(2\mu_1 + \mu_2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},\\
&c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{4c(\mu_1 + \mu_2)(3\mu_1^2 + 4\mu_1\mu_2 + 2\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},
\end{aligned}
$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2-4c\mu_1^2+4c\mu_2^2}{2(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2-8c\mu_1^2-12c\mu_1\mu_2-4c\mu_2^2}{2(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = \frac{2c(\mu_1+\mu_2)(3\mu_1^2+4\mu_1\mu_2+2\mu_2^2)}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = \frac{2c(\mu_1+\mu_2)(5\mu_1^2+6\mu_1\mu_2+2\mu_2^2)}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{2c(\mu_1+\mu_2)(5\mu_1^2+7\mu_1\mu_2+3\mu_2^2)}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\mu_1(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2-4c\mu_1^2+4c\mu_2^2)}{2(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})}$$
$$= \frac{\mu_2(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2-4c\mu_1^2-12c\mu_1\mu_2-8c\mu_2^3)}{2(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{2c(\mu_1+\mu_2)(2\mu_1^2+4\mu_1\mu_2+3\mu_2^2)}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = \frac{2c(\mu_1+\mu_2)(3\mu_1^2+5\mu_1\mu_2+4\mu_2^2)}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{2c(\mu_1+\mu_2)(4\mu_1^2+7\mu_1\mu_2+4\mu_2^2)}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})}$$
$$= \frac{4\mu_1^3+5\mu_1^2\mu_2+3\mu_1\mu_2^2-4c\mu_1^3-8c\mu_1^2\mu_2-8c\mu_1\mu_2^2-4c\mu_2^3}{2(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})}$$
$$= \frac{\mu_2(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2-4c\mu_1^2-12c\mu_1\mu_2-8c\mu_2^3)}{2(4\mu_1^2+5\mu_1\mu_2+3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{2c(\mu_1+\mu_2)(3\mu_1^2+4\mu_1\mu_2+2\mu_2^2)}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = \frac{2c(\mu_1-\mu_2)(\mu_1+\mu_2)^2}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{22})} - c_{w((1,2,2),a_{22})} = \frac{2c\mu_1(\mu_1+\mu_2)(\mu_1+2\mu_2)}{4\mu_1^2+5\mu_1\mu_2+3\mu_2^2},$$

For state $(2,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = \frac{4\mu_1^3 + 5\mu_1^2\mu_2 + 3\mu_1\mu_2^2 - 4c\mu_1^3 - 8c\mu_1^2\mu_2 - 8c\mu_1\mu_2^2 - 4c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})} = \frac{\mu_2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2 - 4c\mu_1^2 - 12c\mu_1\mu_2 - 8c\mu_2^3)}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{2c\mu_1(\mu_1 + \mu_2)(\mu_1 + 2\mu_2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})} = \frac{2c(\mu_1 - \mu_2)(\mu_1 + \mu_2)^2}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = \frac{2c(\mu_1 + \mu_2)(3\mu_1^2 + 4\mu_1\mu_2 + 2\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

for state $(2,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\mu_2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2 + 4c\mu_1^2 - 4c\mu_2^3)}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\mu_2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2 - 4c\mu_1^2 - 12c\mu_1\mu_2 - 8c\mu_2^3)}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})} = \frac{2c(\mu_1 + \mu_2)(4\mu_1^2 + 7\mu_1\mu_2 + 4\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{2c(\mu_1 + \mu_2)(3\mu_1^2 + 5\mu_1\mu_2 + 4\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})} = \frac{2c(\mu_1 + \mu_2)(3\mu_1^2 + 4\mu_1\mu_2 + 2\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\mu_2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2 - 4c\mu_1^2 + 4c\mu_2^3)}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{\mu_2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2 - 8c\mu_1^2 - 12c\mu_1\mu_2 - 4c\mu_2^3)}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} = \frac{2c(\mu_1 + \mu_2)(5\mu_1^2 + 7\mu_1\mu_2 + 3\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{2c(\mu_1 + \mu_2)(5\mu_1^2 + 5\mu_1\mu_2 + 2\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} = \frac{2c(\mu_1 + \mu_2)(3\mu_1^2 + 4\mu_1\mu_2 + 2\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$

and for state $(2,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})} = \frac{4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 3\mu_2^3 + 8c\mu_1^3 - 8c\mu_1^2\mu_2 - 4c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})}$$
$$= \frac{4\mu_1^3 + 5\mu_1^2\mu_2 + 3\mu_1\mu_2^2 - 4c\mu_1^3 + 4c\mu_1^2\mu_2 + 8c\mu_1\mu_2^2 + 4c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})} = \frac{2c(\mu_1+\mu_2)(3\mu_1^2 + 4\mu_1\mu_2 + 2\mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = \frac{2c\mu_1(\mu_1+\mu_2)(2\mu_1+\mu_2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((2,2,2),a_{21})} = \frac{2c(\mu_1+\mu_2)(\mu_1^2 + \mu_1\mu_2 + \mu_2^2)}{4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2}.$$

Finally, for states $(3,1,1), (3,1,2)$ and $(3,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{12})} - c_{w((3,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{12})} - c_{w((3,1,2),a_{12})}$$
$$= \frac{4\mu_1^3 + 5\mu_1^2\mu_2 + 3\mu_1\mu_2^2 - 4c\mu_1^3 - 12c\mu_1^2\mu_2 - 20c\mu_1\mu_2^2 - 12c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})}$$
$$= \frac{4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 3\mu_2^3 - 12c\mu_1^3 - 20c\mu_1^2\mu_2 - 12c\mu_1\mu_2^2 - 4c\mu_2^3}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})} = \frac{4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 3\mu_2^2 + 4c\mu_1^2 - 4c\mu_2^2}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} = \frac{\mu_1(4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 3\mu_2^2 - 4c\mu_1^2 + 4c\mu_1\mu_2)}{2(4\mu_1^2 + 5\mu_1\mu_2 + 3\mu_2^2)}.$$

These quantities are also nonnegative when $0 \leq c \leq \frac{4\mu_1^2\mu_2+5\mu_1\mu_2^2+3\mu_2^3}{12\mu_1^3+20\mu_1^2\mu_2+12\mu_1\mu_2^2+4\mu_2^3}$. Hence we have shown that the inequality (55) is satisfied for all nonbasic variables. We can conclude that D is an optimal basis for LP (54), and consequently $\pi = (d)^\infty$ is an optimal policy when $0 \leq c \leq \frac{4\mu_1^2\mu_2+5\mu_1\mu_2^2+3\mu_2^3}{12\mu_1^3+20\mu_1^2\mu_2+12\mu_1\mu_2^2+4\mu_2^3}$. We see that the recurrent states are $(0,1,2)$, $(0,2,1)$, $(1,1,1)$, $(1,1,2)$, $(1,2,1)$, $(1,2,2)$, $(2,1,1)$, $(2,1,2)$, $(2,2,1)$, $(2,2,2)$, $(3,1,2)$ and $(3,2,1)$ under this policy. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $0 \leq c \leq \frac{4\mu_1^2\mu_2+5\mu_1\mu_2^2+3\mu_2^3}{12\mu_1^3+20\mu_1^2\mu_2+12\mu_1\mu_2^2+4\mu_2^3}$.

Next, let $\frac{4\mu_1^2\mu_2+5\mu_1\mu_2^2+3\mu_2^3}{12\mu_1^3+20\mu_1^2\mu_2+12\mu_1\mu_2^2+4\mu_2^3} \leq c < \frac{\mu_2}{2\mu_1+2\mu_2}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1),(1,1,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,2),(1,2,2),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,1,1),(2,2,1)\}, \\ a_{22} & \text{if } x \in \{(2,2,2),(3,1,1),(3,1,2),(3,2,1)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D \;=\; & \{\omega((0,1,2),(1,2)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,2)), \\ & \omega((1,1,1),(1,1)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(1,2)), \\ & \omega((2,1,1),(2,1)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\ & \omega((3,1,1),(2,2)), \omega((3,1,2),(2,2)), \omega((3,2,1),(2,2))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2), (0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\ & \quad = \frac{-4\mu_1^2\mu_2 - 5\mu_1\mu_2^2 - 3\mu_2^3 + 12c\mu_1^3 + 20c\mu_1^2\mu_2 + 12c\mu_1\mu_2^2 + 4c\mu_2^3}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} \\ & \quad = \frac{2\mu_1^4 + 2\mu_1^3\mu_2 + \mu_1^2\mu_2^2 - \mu_1\mu_2^3 - 2c\mu_1^4 - 4c\mu_1^3\mu_2 - 4c\mu_1^2\mu_2^2 - 4c\mu_1\mu_2^3 - 2c\mu_2^4}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} \\ & \quad = \frac{(\mu_1+\mu_2)(-4\mu_1^2\mu_2 - 5\mu_1\mu_2^2 - 3\mu_2^3 + 12c\mu_1^3 + 20c\mu_1^2\mu_2 + 12c\mu_1\mu_2^2 + 4c\mu_2^3)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} = \\ & \quad = \frac{(\mu_1+\mu_2)(2\mu_1^3 - 5\mu_1\mu_2^2 - 3\mu_2^3 - 2c\mu_1^3 + 10c\mu_1^2\mu_2 + 6c\mu_1\mu_2^2 + 2c\mu_2^3)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)}. \end{aligned}$$

. For state $(1,1,1)$ we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} \\ & \quad = \frac{\mu_2(2\mu_1^3 + 4\mu_1^2\mu_2 + 4\mu_1\mu_2^2 + \mu_2^3 - 4c\mu_1^3 - 6c\mu_1^2\mu_2 - 4c\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \end{aligned}$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{12})} - c_{w((1,1,1),a_{12})}$$
$$= \frac{(\mu_1+\mu_2)(2\mu_1^2\mu_2+3\mu_1\mu_2^2+2\mu_2^3-4c\mu_1^3-10c\mu_1^2\mu_2-6c\mu_1\mu_2^2-2c\mu_2^3)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = \frac{\mu_1(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2-\mu_2^2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{2\mu_2(\mu_1+\mu_2)(2\mu_1^2+2\mu_1\mu_2+\mu_2^2-2c\mu_1^2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\mu_1\mu_2(\mu_1+\mu_2-2c\mu_1)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\mu_1^2\mu_2(\mu_1^2+2c\mu_2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})}$$
$$= \frac{(\mu_1+\mu_2)(-2\mu_1^2\mu_2-3\mu_1\mu_2^2-2\mu_2^3+12c\mu_1^3+18c\mu_1^2\mu_2+12c\mu_1\mu_2^2+4c\mu_2^3)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})} = 2c(\mu_1+\mu_2)$$
$$= \frac{2(\mu_1+\mu_2)(-\mu_1^2\mu_2-2\mu_1\mu_2^2-\mu_2^3+8c\mu_1^3+11c\mu_1^2\mu_2+7c\mu_1\mu_2^2+2c\mu_2^3)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1\mu_2+2c\mu_2^2+\mu_1\mu_2)}{2\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\mu_1^2\mu_2(\mu_1^2+2c\mu_2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\mu_1\mu_2(\mu_1+\mu_2-2c\mu_1)}{2\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})}$$
$$= \frac{2(\mu_1+\mu_2)(\mu_1\mu_2^2+4c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+4c\mu_2^3)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = \frac{2(\mu_1+\mu_2)(-3c\mu_1^2-c\mu_1\mu_2+\mu_2^2+2\mu_1\mu_2+\mu_1^2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})}$$
$$= \frac{(\mu_1+\mu_2)(4\mu_1^2\mu_2+5\mu_1\mu_2^2+2\mu_2^3-4c\mu_1^3-8c\mu_1^2\mu_2-2c\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

and for state $(1,2,2)$ we have :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})},\\
&\quad = \frac{\mu_1^2\mu_2(\mu_1+2c\mu_2)}{2\mu_1^2+\mu_1\mu_2+\mu_2^2},\\
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})}\\
&\quad = \frac{(\mu_1+\mu_2)(-2\mu_1^2\mu_2-3\mu_1\mu_2^2-2\mu_2^3+12c\mu_1^3+18c\mu_1^2\mu_2+12c\mu_1\mu_2^2+4c\mu_2^3)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})}\\
&\quad = \frac{2(\mu_1+\mu_2)^2(-\mu_1\mu_2-\mu_2^2+4c\mu_1^2+3c\mu_1\mu_2+c\mu_2^2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{22})} - c_{w((1,2,2),a_{22})} = \frac{\mu_1(\mu_1+\mu_2)(\mu_2-2c\mu_1)}{2\mu_1^2+\mu_1\mu_2+\mu_2^2}.
\end{aligned}
$$

For state $(2,1,1)$ we have :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})}\\
&\quad = \frac{\mu_1\mu_2(1-2c)}{2\mu_1^2+\mu_1\mu_2+\mu_2^2},\\
&c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{\mu_1(\mu_1+\mu_2)(\mu_2-2c\mu_1)}{2\mu_1^2+\mu_1\mu_2+\mu_2^2},\\
&c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})} = \frac{(\mu_1+\mu_2)(\mu_2-2c\mu_1-2c\mu_2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = \frac{\mu_2(\mu_1+\mu_2)(2\mu_1^2+2\mu_1\mu_2+\mu_2^2-2c\mu_1^2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},
\end{aligned}
$$

for state $(2,1,2)$ we obtain :

$$
\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\mu_1\mu_2(1-2c)}{2\mu_1+\mu_2},\\
&c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\mu_1\mu_2(\mu_1^2+2c\mu_2)}{2\mu_1^2+\mu_1\mu_2+\mu_2^2},\\
&c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})}\\
&\quad = \frac{(\mu_1+\mu_2)(8c\mu_1^3+12c\mu_1^2\mu_2+10c\mu_1\mu_2^2+4c\mu_2^3-\mu_2^3)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{(\mu_1+\mu_2)(-\mu_2+6c\mu_1+4c\mu_2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})}\\
&\quad = \frac{(\mu_1+\mu_2)(4c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3+\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},
\end{aligned}
$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\mu_1\mu_2(\mu_2 + 2c\mu_1 + 4c\mu_2)}{2\mu_1 + \mu_2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{\mu_1\mu_2(\mu_1^2 + 2c\mu_2)}{2\mu_1^2 + \mu_1\mu_2 + \mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} = \frac{(\mu_1 + \mu_2)(2c\mu_1^2 + 2c\mu_1\mu_2 + 2c\mu_2^2 + \mu_1\mu_2)}{(2\mu_1 + \mu_2)(2\mu_1^2 + \mu_1\mu_2 + \mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{(\mu_1 + \mu_2)(2c\mu_1 + \mu_2)}{2\mu_1 + \mu_2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} = \frac{\mu_2(\mu_1 + \mu_2)(2\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2 - 2c\mu_1\mu_2)}{(2\mu_1 + \mu_2)(2\mu_1^2 + \mu_1\mu_2 + \mu_2^2)},$$

and for state $(2,2,2)$ we have :

$$\begin{aligned}
c_B\mathbf{B}^{-1}&v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})},\\
&= \frac{\mu_1(2\mu_1^2\mu_2 + \mu_1\mu_2^2 + 2c\mu_1^3 + 2c\mu_1^2\mu_2 + 8c\mu_1\mu_2^2 + 4c\mu_2^3)}{(2\mu_1 + \mu_2)(2\mu_1^2 + \mu_1\mu_2 + \mu_2^2)},\\
c_B\mathbf{B}^{-1}&v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})}\\
&= \frac{(\mu_1 + \mu_2)(-\mu_1\mu_2^2 - \mu_2^3 + 12c\mu_1^3 + 16c\mu_1^2\mu_2 + 12c\mu_1\mu_2^2 + 4c\mu_2^3)}{(2\mu_1 + \mu_2)(2\mu_1^2 + \mu_1\mu_2 + \mu_2^2)},\\
c_B\mathbf{B}^{-1}&v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = \frac{(\mu_1 + \mu_2)(4c\mu_1^2 + 4c\mu_1\mu_2 + 2c\mu_2^2 - \mu_2)^2}{(2\mu_1 + \mu_2)(2\mu_1^2 + \mu_1\mu_2 + \mu_2^2)},\\
c_B\mathbf{B}^{-1}&v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})}\\
&= \frac{(\mu_1 + \mu_2)(-2\mu_1^2 - 2\mu_1\mu_2^2 - \mu_2^3 + 8c\mu_1^3 + 10c\mu_1^2\mu_2 + 6c\mu_1\mu_2^2 + 2c\mu_2^3)}{(2\mu_1 + \mu_2)(2\mu_1^2 + \mu_1\mu_2 + \mu_2^2)},
\end{aligned}$$

Finally, for states $(3,1,1), (3,1,2)$ and $(3,2,1)$ we have

$$\begin{aligned}
c_B\mathbf{B}^{-1}&v_{w((3,1,1),a_{12})} - c_{w((3,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{12})} - c_{w((3,1,2),a_{12})}\\
&= \frac{2\mu_1^4 + 2\mu_1^3\mu_2 + \mu_1^2\mu_2^2 - \mu_1\mu_2^3 - 2c\mu_1^4 - 4c\mu_1^3\mu_2 - 4c\mu_1^2\mu_2^2 - 4c\mu_1\mu_2^3 - 2c\mu_2^4}{(2\mu_1 + \mu_2)(2\mu_1^2 + \mu_1\mu_2 + \mu_2^2)},\\
c_B\mathbf{B}^{-1}&v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})} = 0,\\
c_B\mathbf{B}^{-1}&v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})} = 2c\mu_1,\\
c_B\mathbf{B}^{-1}&v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})}\\
&= \frac{2\mu_1^3 + 2\mu_1^2\mu_2 + \mu_1\mu_2^2 - \mu_2^3 - 2c\mu_1^3 + 4c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 2c\mu_2^3}{(2\mu_1 + \mu_2)(2\mu_1^2 + \mu_1\mu_2 + \mu_2^2)},
\end{aligned}$$

These quantities are nonnegative because $\frac{4\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 3\mu_2^3}{12\mu_1^3 + 20\mu_1^2\mu_2 + 12\mu_1\mu_2^2 + 4\mu_2^3} \leq c < \frac{\mu_2}{2\mu_1 + 2\mu_2}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, $(1,2,2)$, $(2,1,2)$, $(2,2,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\frac{4\mu_1^2\mu_2+5\mu_1\mu_2^2+3\mu_2^3}{12\mu_1^3+20\mu_1^2\mu_2+12\mu_1\mu_2^2+4\mu_2^3} \leq c < \frac{\mu_2}{2\mu_1+2\mu_2}$.

Next, let $\frac{\mu_2}{2\mu_1+2\mu_2} < c \leq \min\{\frac{\mu_2}{2\mu_1}, \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}\}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1),(1,1,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,2),(1,2,2),(2,1,1),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,2,1),(3,1,1),(3,2,1)\}, \\ a_{22} & \text{if } x \in \{(2,2,2),(3,1,2)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D \;=\; & \{\omega((0,1,2),(1,2)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,2)), \\ & \omega((1,1,1),(1,1)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(1,2)), \\ & \omega((2,1,1),(1,2)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\ & \omega((3,1,1),(2,1)), \omega((3,1,2),(2,2)), \omega((3,2,1),(2,1))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2), (0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\ & \quad = \frac{2(\mu_1+\mu_2)^2(4c\mu_1^2+3c\mu_1^2\mu_2+\mu_2^2-\mu_1\mu_2-\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} \\ & \quad = \frac{2\mu_1^4+2\mu_1^3\mu_2-\mu_1^2\mu_2^2-5\mu_1\mu_2^3-2\mu_1^4+12c\mu_1^3\mu_2+18c\mu_1^2\mu_2^2+10c\mu_1\mu_2^3+2c\mu_2^4}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} = \frac{\mu_1\mu_2(\mu_1^2+2\mu_1\mu_2+\mu_2^2-3c\mu_1^2-c\mu_1\mu_2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} = \\ & \quad = \frac{(\mu_1+\mu_2)(2\mu_1^3+2\mu_1^2\mu_2-5\mu_1\mu_2^2-2c\mu_1^3+10c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)}. \end{aligned}$$

. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})}$$
$$= \frac{\mu_2(\mu_1+\mu_2)(2c\mu_1+\mu_2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{12})} - c_{w((1,1,1),a_{12})} = \frac{(\mu_1+\mu_2)(\mu_2-2c\mu_1)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} = \frac{\mu_1(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_2^2+\mu_1\mu_2+\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\mu_1(\mu_1^2-\mu_1\mu_2+2c\mu_2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\mu_1^2\mu_2(\mu_1^2+2c\mu_2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})}$$
$$= \frac{(\mu_1+\mu_2)(3\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2-\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_2^2+\mu_1\mu_2+\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\mu_1\mu_2(\mu_1+\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})}$$
$$= \frac{(\mu_1+\mu_2)(4c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3+\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})}$$
$$= \frac{(\mu_1+\mu_2)(4c\mu_1^3+2c\mu_1\mu_2^2+2c\mu_2^3+3\mu_1\mu_2^2+\mu_2^3)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})}$$
$$= \frac{\mu_2(\mu_1+\mu_2)(-2c\mu_1^3+2c\mu_1\mu_2+2c\mu_2^2+2\mu_1^2+4\mu_1\mu_2+\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

and for state $(1,2,2)$ we have :

$$
\begin{aligned}
& c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})}, \\
& \quad = \frac{\mu_1\mu_2(\mu_1+\mu_2)^2}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \\
& c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2-\mu_2^2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \\
& c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = \\
& \quad = \frac{(\mu_1+\mu_2)(4c\mu_1^3+8c\mu_1^2\mu_2+4c\mu_1\mu_2^2-3\mu_1\mu_2^2-\mu_2^3)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\
& c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})} = \frac{\mu_1(\mu_1+\mu_2)(\mu_2-2c\mu_1)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}.
\end{aligned}
$$

For state $(2,1,1)$ we have :

$$
\begin{aligned}
& c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})} \\
& \quad = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1^2\mu_2+2c\mu_1\mu_2^2+\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\
& c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{\mu_1(\mu_1+\mu_2)(\mu_2-2c\mu_1)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \\
& c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})} = 0, \\
& c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} \\
& \quad = \frac{(\mu_1+\mu_2)(4c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3+\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},
\end{aligned}
$$

for state $(2,1,2)$ we obtain :

$$
\begin{aligned}
& c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \\
& c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\mu_1\mu_2(\mu_1^2+2c\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \\
& c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})} = \frac{(\mu_1+\mu_2)(42c\mu_1^2+2c\mu_1\mu_2+2c\mu_2^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2}, \\
& c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = 2c(\mu_1+\mu_2), \\
& c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})} \\
& \quad = \frac{(\mu_1+\mu_2)(4c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3+\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},
\end{aligned}
$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\mu_1\mu_2(\mu_1^2+2c\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} = \frac{(\mu_1+\mu_2)(2c\mu_1^2+2c\mu_1\mu_2+2c\mu_2^2+\mu_1\mu_2)}{\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = 2c(\mu_1+\mu_2),$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} = \frac{(\mu_1+\mu_2)(4c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3+\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

and for state $(2,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})}, = \frac{\mu_1(\mu_1\mu_2^2-\mu_2^3+c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})} = \frac{(\mu_1+\mu_2)(4c\mu_1^3+5c\mu_1^2\mu_2+4c\mu_1\mu_2^2+c\mu_2^3+\mu_1^2\mu_2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})} = \frac{\mu_1(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+4c\mu_2^2-\mu_2^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)}.$$

Finally, for states $(3,1,1),(3,1,2)$ and $(3,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{12})} - c_{w((3,1,1),a_{12})} = \frac{(\mu_1-\mu_2)(\mu_1+\mu_2)(2\mu_1^2+2\mu_1\mu_2+\mu_2^2-2\mu_1^2)}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{22})} - c_{w((3,2,1),a_{22})} = \frac{(\mu_1+\mu_2)(2c\mu_1+2c\mu_2-\mu_2)}{2\mu_1+\mu_2},$$

$$c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{12})} - c_{w((3,1,2),a_{12})} = \frac{\mu_1(2c\mu_1+\mu_2)}{2\mu_1+\mu_2},$$

$$c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})} = \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3-2c\mu_1^4-4c\mu_1^3\mu_2-4c\mu_1^2\mu_2^2-4c\mu_1\mu_2^3-2c\mu_2^4}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} = \frac{2\mu_1^4+2\mu_1^3\mu_2-\mu_1^2\mu_2^2-2\mu_1\mu_2^3+2c\mu_1^4+8c\mu_1^3\mu_2+10c\mu_1^2\mu_2^2+6\mu_1\mu_2^3+2c\mu_2^4}{(2\mu_1+\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

These quantities are nonnegative if $\frac{\mu_2}{2\mu_1+2\mu_2} < c \le \min\{\frac{\mu_2}{2\mu_1}, \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}\}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, $(1,2,2)$, $(2,1,2)$, $(2,2,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $\frac{\mu_2}{2\mu_1+2\mu_2} < c \le \min\{\frac{\mu_2}{2\mu_1}, \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}\}$.

Let $2\mu_1^5+\mu_1^4\mu_2 \ge \mu_1^3\mu_2^2+3\mu_1^2\mu_2^3+2\mu_1\mu_2^4+\mu_2^5$ and $\frac{\mu_2}{2\mu_1} < c \le \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1),(2,1,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2),(2,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,2,1),(3,1,1),(3,2,1)\}, \\ a_{22} & \text{if } x \in \{(1,2,2),(2,2,2),(3,1,2)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D \;=\; & \{\omega((0,1,2),(1,2)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,2)), \\ & \omega((1,1,1),(1,2)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(2,2)), \\ & \omega((2,1,1),(2,1)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\ & \omega((3,1,1),(1,1)), \omega((3,1,2),(2,2)), \omega((3,2,1),(2,1))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2), (0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\ & \quad = \frac{-2\mu_1^2\mu_2 - 2\mu_1\mu_2^2 - \mu_2^3 + 12c\mu_1^3 + 12c\mu_1^2\mu_2 + 6c\mu_1\mu_2^2 + 2c\mu_2^3}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} = c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} \\ & \quad = \frac{3\mu_1^4 + 2\mu_1^3\mu_2 - 3\mu_1^2\mu_2^2 - 4\mu_1\mu_2^3 - 2c\mu_1^4 + 20c\mu_1^3\mu_2 + 18c\mu_1^2\mu_2^2 + 8c\mu_1\mu_2^3 + 2c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} = \frac{\mu_1\mu_2(3\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2 - 2c\mu_1^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \end{aligned}$$

$$c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} =$$
$$= \frac{(\mu_1+\mu_2)(-3\mu_1^2\mu_2 - 2\mu_1\mu_2^2 - \mu_2^3 + 12c\mu_1^3 + 12c\mu_1^2\mu_2 + 6c\mu_1\mu_2^2 + 2c\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}.$$

. For state $(1,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})}$$
$$= \frac{\mu_2(2\mu_1^3 + 5\mu_1^3\mu_2 + +\mu_2^3 - 4c\mu_1^3 + c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = 0,$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})}$$
$$= \frac{\mu_1(\mu_1+\mu_2)(8c\mu_1^2 + 6c\mu_1\mu_2 + 2c\mu_2^2 - \mu_1\mu_2 - 2\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2 + 2c\mu_1\mu_2 + 2c\mu_2^2 + \mu_1\mu_2}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\mu_1(\mu_1^2 + \mu_1\mu_2 + 2\mu_2^2 - c\mu_1)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\mu_1^2\mu_2(\mu_1^2 + 2c\mu_2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})} = 2c(\mu_1+\mu_2),$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})}$$
$$= \frac{(\mu_1+\mu_2)(-\mu_1^2\mu_2 - 2\mu_1\mu_2^2 + 20c\mu_1^3 + 16c\mu_1^2\mu_2 + 8c\mu_1\mu_2^2 + 2c\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2 + 2c\mu_1\mu_2 + 2c\mu_2^2 + \mu_1\mu_2}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\mu_1^2\mu_2(\mu_1^2 + 2c\mu_2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\mu_1\mu_2(\mu_1+\mu_2+2c\mu_1)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})}$$
$$= \frac{2(\mu_1+\mu_2)(\mu_1^2\mu_2 + 2\mu_1\mu_2^2 + 4c\mu_1^3 + 4c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 2c\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{\mu_2(\mu_1+\mu_2)^2(3\mu_1+2c\mu_1+2c\mu_2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})},$$
$$= \frac{\mu_1^2\mu_2(\mu_1+2c\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})} = \frac{(\mu_1+\mu_2)(8c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2-\mu_1\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = \frac{\mu_1(\mu_1+\mu_2)(2c\mu_1-\mu_2)}{(2\mu_1+\mu_2)(2\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})}$$
$$= \frac{\mu_1(\mu_1+\mu_2)(12c\mu_1^2+8c\mu_1\mu_2+2c\mu_2^2-3\mu_1\mu_2-3\mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2}.$$

For state $(2,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})}$$
$$= \frac{\mu_1(\mu_1+\mu_2)(4c\mu_1^2+3c\mu_1\mu_2+c\mu_2^2+2\mu_1\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})}$$
$$= \frac{\mu_1(\mu_1+\mu_2)(2c\mu_1-\mu_2)}{2\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})}$$
$$= \frac{(\mu_1+\mu_2)(10c\mu_1^3+13c\mu_1^2\mu_2+9c\mu_1\mu_2^2+3c\mu_2^2+\mu_1^2\mu_2-\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

for state $(2,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\mu_2(\mu_1+\mu_2)(2c\mu_1+\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\mu_1\mu_2(\mu_1^2+2c\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})} = \frac{(\mu_1+\mu_2)(3c\mu_1^2 - c\mu_1\mu_2 + c\mu_2^2 + \mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{(\mu_1+\mu_2)(8c\mu_1^2 + 2c\mu_1\mu_2 + 2c\mu_2^2 - \mu_1\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})} = \frac{(\mu_1+\mu_2)(10c\mu_1^3 + 9c\mu_1^2\mu_2 + 7c\mu_1\mu_2^2 + 3c\mu_2^2 + 2\mu_1^2\mu_2 + 2\mu_1\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\mu_2(\mu_1+\mu_2)(2c\mu_1+\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{\mu_1\mu_2(\mu_1^2+2c\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} = \frac{(\mu_1+\mu_2)(c\mu_1^2 + c\mu_1\mu_2 + c\mu_2^2 + \mu_1\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2 + 4c\mu_1\mu_2 + 2c\mu_2^2 + \mu_1\mu_2 - \mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} = \frac{(\mu_1+\mu_2)(6c\mu_1^3 + 11c\mu_1^2\mu_2 + 9c\mu_1\mu_2^2 + 3c\mu_2^2 + 3\mu_1^2\mu_2 + \mu_1\mu_2^2 - \mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

and for state $(2,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})} = c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})}, = \frac{\mu_1(2\mu_1^2\mu_2 + \mu_1\mu_2^2 + c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 4c\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})} = \frac{(\mu_1+\mu_2)(6c\mu_1^3 + 11c\mu_1^2\mu_2 + 9c\mu_1\mu_2^2 + 3c\mu_2^2 + 3\mu_1^2\mu_2 + \mu_1\mu_2^2 - \mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = \frac{\mu_1(\mu_1+\mu_2)(8c\mu_1^2 + 6c\mu_1\mu_2 + 2c\mu_2^2 - \mu_1\mu_2 - 2\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})} = \frac{(\mu_1+\mu_2)(2\mu_1\mu_2^2 - \mu_2^3 + 4c\mu_1^3 + 6c\mu_1^2\mu_2 + 6c\mu_1\mu_2^2 + 2c\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

Finally, for states $(3,1,1),(3,1,2)$ and $(3,2,1)$ we have

$$\begin{aligned}
&c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{12})} - c_{w((3,1,1),a_{12})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{12})} - c_{w((3,1,2),a_{12})}\\
&\quad = \frac{-2c\mu_1^4 - 4c\mu_1^3\mu_2 - 4c\mu_1^2\mu_2^2 - 4c\mu_1\mu_2^3 - 2c\mu_2^4 + 3\mu_1^4 + 3c\mu_1^3\mu_2 - 2c\mu_1\mu_2^3}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})} = c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})}\\
&\quad = \frac{\mu_1(12c\mu_1^3 + 2c\mu_1^2\mu_2 + \mu_1\mu_2^2 + 2\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})}\\
&\quad = \frac{3\mu_1^4 + 2\mu_1^3\mu_2 - 3\mu_1\mu_2^3 - 2\mu_2^4 + 2c\mu_1^4 + 8c\mu_1^3\mu_2 + 14c\mu_1\mu_2^3 + 2c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},\\
&c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} = \frac{8c\mu_1^2 + 6c\mu_1\mu_2 + 2c\mu_2^2 - 2\mu_1\mu_2 - 2\mu_2^2}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},
\end{aligned}$$

These quantities are nonnegative because $2\mu_1^5 + \mu_1^4\mu_2 \geq \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$ and $\frac{\mu_2}{2\mu_1} < c \leq \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(0,2,2)$, $(1,1,2)$, $(1,2,2)$, $(2,1,2)$, $(2,2,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $2\mu_1^5 + \mu_1^4\mu_2 \geq \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$ and $\frac{\mu_2}{2\mu_1} < c \leq \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$.

Now, let $2\mu_1^5+\mu_1^4\mu_2 \geq \mu_1^3\mu_2^2+3\mu_1^2\mu_2^3+2\mu_1\mu_2^4+\mu_2^5$ and $c > \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1),(2,1,1)\},\\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,1),(1,1,2),(2,1,2),(3,1,1),(3,1,2)\},\\ a_{21} & \text{if } x \in \{(1,2,1),(2,2,1),(3,2,1)\},\\ a_{22} & \text{if } x \in \{(1,2,2),(2,2,2)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D &= \{\omega((0,1,2),(1,2)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,2)), \\ &\quad \omega((1,1,1),(1,2)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(2,2)), \\ &\quad \omega((2,1,1),(1,1)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\ &\quad \omega((3,1,1),(1,2)), \omega((3,1,2),(1,2)), \omega((3,2,1),(2,1))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2), (0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned}
& c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\
&\quad = \frac{12c\mu_1^3 + 12c\mu_1^2\mu_2 + 6c\mu_1\mu_2^2 + 2c\mu_2^3 - 3\mu_1^2\mu_2 - 2\mu_1\mu_2^2 - \mu_2^3}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\
& c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} \\
&\quad = \frac{3\mu_1^4 + 2\mu_1^3\mu_2 - 4\mu_1\mu_2^3 - \mu_2^4 - 2c\mu_1^4 + 20c\mu_1^3\mu_2 + 18c\mu_1^2\mu_2^2 + 8c\mu_1\mu_2^3 + 2c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\
& c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{11})} - c_{w((0,2,1),a_{11})} = \frac{\mu_1\mu_2(3\mu_1^2 + 2\mu_1\mu_2 + \mu_2^2 - 2c\mu_1^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\
& c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{\mu_1^2(3\mu_1^2 + \mu_1\mu_2 - \mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\
& c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} \\
&\quad = \frac{3\mu_1^4 + 2\mu_1^3\mu_2 - 4\mu_1\mu_2^3 - \mu_2^4 - 2c\mu_1^4 + 8c\mu_1^3\mu_2 + 8c\mu_1^2\mu_2^2 + 2c\mu_1\mu_2^3 + 2c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},
\end{aligned}$$

For state $(1,1,1)$ we have :

$$\begin{aligned}
& c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = \frac{\mu_2(\mu_1^3 + \mu_1^2\mu_2 + \mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)} \\
& c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} \\
&\quad = \frac{\mu_2(2c\mu_1^3 + 2c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 4c\mu_2^3 + 2\mu_1^3 - \mu_1^2\mu_2 + 3\mu_1\mu_2^2 + \mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\
& c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} \\
&\quad = \frac{(\mu_1-\mu_2)(\mu_1+\mu_2)(3\mu_1^3 + 5\mu_1^2\mu_2 + 5\mu_1\mu_2^2 + 2\mu_2^3 - 2c\mu_1^3 - 6c\mu_1^2\mu_2 - 2c\mu_1\mu_2^2)}{\mu_2(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},
\end{aligned} \tag{56}$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})}$$
$$= \frac{\mu_1(\mu_1+\mu_2)(-\mu_1\mu_2 - \mu_2^2 + 8c\mu_1^2 + 6c\mu_1\mu_2 + 2c\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2 + 2c\mu_1\mu_2 + 2c\mu_2^2 + \mu_1\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\mu_1\mu_2(2c\mu_1+\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\mu_1\mu_2(\mu_1^2+2c\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})}$$
$$= \frac{3\mu_1^4 + 2\mu_1^3\mu_2 - 3\mu_1\mu_2^3 - 2\mu_2^3 - 2c\mu_1^4 + 8c\mu_1^3\mu_2 + 14c\mu_1^2\mu_2^2 + 8c\mu_1\mu_2^3 + 2c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})}$$
$$= \frac{-\mu_1^2\mu_2 - 2\mu_1\mu_2^2 + 20c\mu_1^3 + 16c\mu_1^2\mu_2 + 8c\mu_1\mu_2^2 + 2c\mu_2^3}{2(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2 + 2c\mu_1\mu_2 + 2c\mu_2^2 + \mu_1\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\mu_1^3\mu_2}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\mu_1\mu_2^2}{\mu_1^2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})}$$
$$= \frac{3\mu_1^4 + 2\mu_1^3\mu_2 - \mu_1\mu_2^3 - 2\mu_2^3 - 2c\mu_1^4 + 8c\mu_1^2\mu_2^2 + 6c\mu_1\mu_2^3 + 2c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})}$$
$$= \frac{\mu_1^2\mu_2 + 2\mu_1\mu_2^2 + 4c\mu_1^3 + 4c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 2c\mu_2^3}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{\mu_2(\mu_1+\mu_2)^2(2c\mu_1+2c\mu_2+3\mu_1)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = \frac{\mu_1(c\mu_1^3 + 4c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 2c\mu_2^3 + \mu_1^2\mu_2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} = \frac{\mu_1^2\mu_2^2}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})}$$
$$= \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1^2\mu_2^2-4\mu_1\mu_2^3-2c\mu_1^4+12c\mu_1^3\mu_2+16c\mu_1^2\mu_2^2+8c\mu_1\mu_2^3+2c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{12})} - c_{w((1,2,2),a_{12})} = \frac{\mu_1(\mu_1+\mu_2)(2c\mu_1-\mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})}$$
$$= \frac{\mu_1(\mu_1+\mu_2)(-3\mu_1\mu_2-3\mu_2^2-\mu_1^3+12c\mu_1^2+8c\mu_1\mu_2+2c\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}.$$

For state $(2,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = \frac{\mu_1\mu_2^3}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})}$$
$$= \frac{\mu_1(2c\mu_1^3+2c\mu_1^2\mu_2+2c\mu_1\mu_2^2+2c\mu_2^3+\mu_1\mu_2^2-\mu_1^2\mu_2-\mu_1^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{12})} - c_{w((2,1,1),a_{12})}$$
$$= \frac{(\mu_1+\mu_2)(-3\mu_1^4-2\mu_1^3\mu_2-\mu_1\mu_2^3+\mu_2^4+2c\mu_1^4+4c\mu_1^3\mu_2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})}$$
$$= \frac{-3\mu_1^4-2\mu_1^3\mu_2-2\mu_1^2\mu_2^2+2\mu_1\mu_2^3+2\mu_2^4+2c\mu_1^4+8c\mu_1^3\mu_2-2c\mu_1^2\mu_2^2-2c\mu_1\mu_2^3}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})} = \frac{2\mu_2(-\mu_1^2+\mu_1\mu_2+2c\mu_1^2+2c\mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

for state $(2,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\mu_1\mu_2^3}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\mu_1^2\mu_2}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})} = \frac{(\mu_1+\mu_2)(3c\mu_1^2-c\mu_1\mu_2+c\mu_2^2+\mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})} = \frac{(\mu_1+\mu_2)(8c\mu_1^2+2c\mu_2^2-\mu_1\mu_2+\mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})}$$
$$= \frac{3\mu_1^4 + 2\mu_1^3\mu_2 + \mu_1^2\mu_2^2 - 2\mu_2^4 - 2c\mu_1^4 + 6c\mu_1^3\mu_2 + 13c\mu_1^2\mu_2^2 + 9c\mu_1\mu_2^3 + 3c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\mu_1^2\mu_2}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{\mu_1\mu_2^3}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} = \frac{(\mu_1+\mu_2)(c\mu_1^2+c\mu_1\mu_2+c\mu_2^2+\mu_1\mu_2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{(\mu_1+\mu_2)(4c\mu_1^2+4c\mu_1\mu_2+2c\mu_2^2+\mu_1\mu_2-\mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})}$$
$$= \frac{3\mu_1^4 + 2\mu_1^3\mu_2 + 3\mu_1^2\mu_2^2 - 3\mu_2^4 - 2c\mu_1^4 + 2c\mu_1^3\mu_2 + 15c\mu_1^2\mu_2^2 + 11c\mu_1\mu_2^3 + 3c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

and for state $(2,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})}$$
$$= \frac{\mu_1(\mu_1\mu_2^2+\mu_1^2\mu_2-\mu_1^3+2c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})} = \frac{\mu_1^2(\mu_1^2+2\mu_1\mu_2+\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})}$$
$$= \frac{\mu_1(2\mu_1^3+\mu_1^2\mu_2^2-\mu_1\mu_2^2-\mu_2^3+2c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+2c\mu_2^3)}{2(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = \frac{(\mu_1+\mu_2)(8c\mu_1^2+6c\mu_1\mu_2+2c\mu_2^2-\mu_1\mu_2-\mu_2^2)}{3\mu_1^2+\mu_1\mu_2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})}$$
$$= \frac{3\mu_1^4 + 3\mu_1^3\mu_2 + 2\mu_1\mu_2^3 - 3\mu_2^4 - 2c\mu_1^4 + 2c\mu_1^3\mu_2 + 14c\mu_1^2\mu_2^2 + 8c\mu_1\mu_2^3 + 3c\mu_2^4}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

Finally, for states $(3,1,1),(3,1,2)$ and $(3,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})}$$
$$= \frac{\mu_2(2\mu_1\mu_2^2+2\mu_1^2\mu_2-3\mu_1^3+4c\mu_1^3+4c\mu_1^2\mu_2+4c\mu_1\mu_2^2+4c\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$

$$= c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{22})} - c_{w((3,1,1),a_{22})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{22})} - c_{w((3,1,2),a_{22})}$$
$$= \frac{\mu_1(-\mu_1\mu_2^2 - \mu_2^3 + 6c\mu_1^3 + 13c\mu_1^2\mu_2 + 9c\mu_1\mu_2^2 + 3c\mu_2^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})}$$
$$= \frac{3\mu_1^4 + 2\mu_1^3\mu_2 - 2\mu_1\mu_2^3 + 2c\mu_1^4 + 4\mu_1^3\mu_2 + 4\mu_1^2\mu_2^2 + 4\mu_1\mu_2^3 + 2\mu_2^4)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} = \frac{\mu_1\mu_2(3\mu_1^2+\mu_1\mu_2-\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}$$
$$c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})}$$
$$= \frac{\mu_2(\mu_1+\mu_2)(-\mu_1\mu_2 - \mu_2^2 + 8c\mu_1^2 + 6c\mu_1\mu_2 + 2c\mu_2^2)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}.$$

These quantities are nonnegative because $2\mu_1^5 + \mu_1^4\mu_2 \geq \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$ and $c > \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, $(2,1,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $2\mu_1^5 + \mu_1^4\mu_2 \geq \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$ and $c > \frac{3\mu_1^4+2\mu_1^3\mu_2-2\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$.

Finally, let $2\mu_1^5+\mu_1^4\mu_2 < \mu_1^3\mu_2^2+3\mu_1^2\mu_2^3+2\mu_1\mu_2^4+\mu_2^5$ and $c > \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$, and consider the decision rule d, where $d(x)$ is defined as follows for all $x \in S$:

$$d(x) = \begin{cases} a_{11} & \text{if } x \in \{(0,2,1),(1,1,1)\}, \\ a_{12} & \text{if } x \in \{(0,1,2),(0,2,2),(1,1,2),(1,2,2),(2,1,1),(2,1,2), \\ & \qquad (3,1,1),(3,1,2)\}, \\ a_{21} & \text{if } x \in \{(1,2,1),(2,2,1),(3,2,1)\}, \\ a_{22} & \text{if } x \in \{(2,2,2)\}, \end{cases}$$

Then, the basic solution ω corresponding to the policy $\pi = (d)^\infty$ has the basis

$$\begin{aligned} D &= \{\omega((0,1,2),(1,2)), \omega((0,2,1),(1,1)), \omega((0,2,2),(1,2)), \\ &\quad \omega((1,1,1),(1,1)), \omega((1,1,2),(1,2)), \omega((1,2,1),(2,1)), \omega((1,2,2),(1,2)), \\ &\quad \omega((2,1,1),(1,1)), \omega((2,1,2),(1,2)), \omega((2,2,1),(2,1)), \omega((2,2,2),(2,2)), \\ &\quad \omega((3,1,1),(1,2)), \omega((3,1,2),(1,2)), \omega((3,2,1),(2,1))\}. \end{aligned}$$

Proceeding as before, we will show that inequality (55) holds for every nonbasic variable. More specifically, for states $(0,1,2), (0,2,1)$, and $(0,2,2)$, we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{11})} - c_{w((0,1,2),a_{11})} = c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{11})} - c_{w((0,2,2),a_{11})} \\ &\quad = \frac{2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 2c\mu_2^3 + \mu_1^3\mu_2 - \mu_1^2\mu_2 - 2\mu_1\mu_2^3 - 2\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,1,2),a_{21})} - c_{w((0,1,2),a_{21})} \\ &\quad = \frac{\mu_2(4c\mu_1^4 + 8c\mu_1^3\mu_2 + 8c\mu_1^2\mu_2 + 8c\mu_1\mu_2^2 + 4c\mu_2^3 + \mu_1^3\mu_2 - \mu_1^2\mu_2 - 2\mu_1\mu_2^3 - 2\mu_2^4)}{(\mu_1+\mu_2)^2(\mu_1^2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{12})} - c_{w((0,2,1),a_{12})} = \frac{\mu_1\mu_2(\mu_1^2\mu_2 + 2\mu_1\mu_2^2 + 2\mu_2^3 - \mu_1^3)}{(2\mu_1+\mu_2)(3\mu_1^2+\mu_1\mu_2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,1),a_{21})} - c_{w((0,2,1),a_{21})} = 0, \\ & c_B\mathbf{B}^{-1}v_{w((0,2,2),a_{21})} - c_{w((0,2,2),a_{21})} \\ &\quad = \frac{\mu_2(2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 2c\mu_2^3 + \mu_1^3\mu_2 - \mu_1^2\mu_2 - 2\mu_1\mu_2^3 - 2\mu_2^4)}{(\mu_1+\mu_2)^2(\mu_1^2+\mu_2^2)}, \end{aligned}$$

For state $(1,1,1)$ we have :

$$\begin{aligned} & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{02})} - c_{w((1,1,1),a_{02})} = \frac{\mu_1^3 + 2\mu_1^2\mu_2 + 2\mu_2^3}{\mu_1^2+\mu_2^2} \\ & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,1,1),a_{20})} \\ &\quad = \frac{\mu_2(2c\mu_1^3 + 2c\mu_1^2\mu_2 + 4c\mu_1\mu_2^2 + 4c\mu_2^3 + 2\mu_1^3 - \mu_1^2\mu_2 + 3\mu_1\mu_2^2 + \mu_2^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{11})} - c_{w((1,1,1),a_{11})} = \frac{\mu_2(-\mu_1^3 + 5\mu_1^2\mu_2 + \mu_2^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)}, \\ & c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{21})} - c_{w((1,1,1),a_{21})} \\ &\quad = \frac{-\mu_1^4 - \mu_1^3\mu_2 - \mu_1^2\mu_2^2 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)}, \end{aligned}$$

$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{22})} - c_{w((1,1,1),a_{22})}$$
$$= \frac{-\mu_1^3 + \mu_1\mu_2^2 + \mu_2^3 + 2c\mu_1^3 + 4c\mu_1^2\mu_2 + 2c\mu_1\mu_2^2 + 2c\mu_2^3}{\mu_1^2 + \mu_2^2},$$

for state $(1,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{02})} - c_{w((1,1,2),a_{02})} = \frac{\mu_1^3 + \mu_1\mu_2^2 - 2\mu_2^3 + 2c\mu_1^3 + 2c\mu_1\mu_2^2 + 2c\mu_2^3}{\mu_1^2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{10})} - c_{w((1,1,2),a_{10})} = \frac{\mu_1^3 + \mu_1\mu_2^2 + \mu_2^3 + c\mu_1^3 + 4c\mu_1\mu_2^2 + 3c\mu_1^2\mu_2}{\mu_1^2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{11})} - c_{w((1,1,2),a_{11})}$$
$$= \frac{\mu_1^3\mu_2 - \mu_1\mu_2^3 - \mu_2^4 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1 + \mu_2)(\mu_1^2 + \mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{21})} - c_{w((1,1,2),a_{21})}$$
$$= \frac{-\mu_1^4 - \mu_1^2\mu_2^2 - \mu_1\mu_2^3 - \mu_2^4 + 4c\mu_1^4 + 8c\mu_1^3\mu_2 + 8c\mu_1^2\mu_2^2 + 8c\mu_1\mu_2^3 + 4c\mu_2^4}{(\mu_1 + \mu_2)(\mu_1^2 + \mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,2),a_{22})} - c_{w((1,1,2),a_{22})}$$
$$= \frac{-\mu_1^4 + \mu_1^2\mu_2^2 + \mu_1\mu_2^3 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1 + \mu_2)(\mu_1^2 + \mu_2^2)},$$

for state $(1,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{01})} - c_{w((1,2,1),a_{01})} = \frac{\mu_1^2\mu_2}{\mu_1^2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,1,1),a_{20})} - c_{w((1,2,1),a_{20})} = \frac{\mu_1\mu_2^2}{\mu_1^2 + \mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{11})} - c_{w((1,2,1),a_{11})} = \frac{\mu_1^2(\mu_1^2 + \mu_1\mu_2 + \mu_2^2)}{(\mu_1 + \mu_2)(\mu_1^2 + \mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{12})} - c_{w((1,2,1),a_{12})} = \frac{\mu_1^4 + \mu_1^2\mu_2^2 + \mu_1\mu_2^3 + \mu_2^4}{(\mu_1 + \mu_2)(\mu_1^2 + \mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,1),a_{22})} - c_{w((1,2,1),a_{22})} = \frac{\mu_2^2(2\mu_1^2 + 2\mu_1^2\mu_2 + \mu_2^2)}{(\mu_1 + \mu_2)(\mu_1^2 + \mu_2^2)},$$

and for state $(1,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{01})} - c_{w((1,2,2),a_{01})} = \frac{\mu_1(2c\mu_1^3 + c\mu_1^2\mu_2 + 2c\mu_1\mu_2^2 + 2c\mu_2^3 + \mu_1^3)}{(\mu_1 + \mu_2)(\mu_1^2 + \mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{10})} - c_{w((1,2,2),a_{10})} = \frac{\mu_1^3\mu_2}{(\mu_1 + \mu_2)(\mu_1^2 + \mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{11})} - c_{w((1,2,2),a_{11})}$$
$$= \frac{\mu_1^3\mu_2 - \mu_1\mu_2^3 - \mu_2^4 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{21})} - c_{w((1,2,2),a_{21})}$$
$$= \frac{-\mu_1^4 - \mu_1^2\mu_2^2 - \mu_1\mu_2^3 - \mu_2^4 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((1,2,2),a_{22})} - c_{w((1,2,2),a_{22})} = \frac{\mu_2(-\mu_1^3 + 5\mu_1^2\mu_2 + \mu_2^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)}.$$

For state $(2,1,1)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{02})} - c_{w((2,1,1),a_{02})} = \frac{\mu_1\mu_2^3}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,1,1),a_{20})}$$
$$= \frac{\mu_1(2c\mu_1^3 + 2c\mu_1^2\mu_2 + 2c\mu_1\mu_2^2 + 2c\mu_2^3 + \mu_1\mu_2^2 - \mu_1^2\mu_2 - \mu_1^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{11})} - c_{w((2,1,1),a_{11})} = \frac{\mu_2(\mu_1^3 + \mu_1^2\mu_2 - \mu_2^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{21})} - c_{w((2,1,1),a_{21})}$$
$$= \frac{-2\mu_1^4 - 2\mu_1^3\mu_2 - \mu_1^2\mu_2^2 + \mu_1\mu_2^3 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{22})} - c_{w((2,1,1),a_{22})}$$
$$= \frac{-\mu_1^4 - \mu_1^3\mu_2 + \mu_1\mu_2^3 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$

for state $(2,1,2)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{02})} - c_{w((2,1,2),a_{02})} = \frac{\mu_1\mu_2(2\mu_1 + c\mu_2)}{\mu_1^2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{10})} - c_{w((2,1,2),a_{10})} = \frac{\mu_1(\mu_1 + 2\mu_2)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{11})} - c_{w((2,1,2),a_{11})}$$
$$= \frac{\mu_1^3\mu_2 + \mu_1\mu_2^3 - \mu_2^4 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{21})} - c_{w((2,1,2),a_{21})}$$
$$= \frac{-2\mu_1^4 - 2\mu_1^3\mu_2 - \mu_1^2\mu_2^2 + \mu_1\mu_2^3 + 4c\mu_1^4 + 8c\mu_1^3\mu_2 + 8c\mu_1^2\mu_2^2 + 8c\mu_1\mu_2^3 + 4c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((2,1,2),a_{22})} - c_{w((2,1,2),a_{22})}$$
$$= \frac{-\mu_1^4 - \mu_1^3\mu_2 + \mu_1\mu_2^3 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$

for state $(2,2,1)$ we obtain :

$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{01})} - c_{w((2,2,1),a_{01})} = \frac{\mu_1^2\mu_2}{\mu_1^2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,1,1),a_{20})} - c_{w((2,2,1),a_{20})} = \frac{\mu_1\mu_2^3}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{11})} - c_{w((2,2,1),a_{11})} = \frac{2\mu_1^4 + 3\mu_1^3\mu_2 + 2\mu_1^2\mu_2^2 - \mu_1\mu_2^3 - \mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{12})} - c_{w((2,2,1),a_{12})} = \frac{\mu_1(2\mu_1^3 + 2\mu_1^2\mu_2 + \mu_1\mu_2^2 - \mu_2^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,1),a_{22})} - c_{w((2,2,1),a_{22})} = \frac{\mu_1^2(\mu_1^2 + \mu_1\mu_2 + \mu_2^2)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$

and for state $(2,2,2)$ we have :

$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{01})} - c_{w((2,2,2),a_{01})} = \frac{\mu_1(2\mu_1\mu_2^2 + \mu_1^3 + 4c\mu_1^3 + 4c\mu_1^2\mu_2 + 2c\mu_2^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{10})} - c_{w((2,2,2),a_{10})} = \frac{\mu_1^2\mu_2^2}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{11})} - c_{w((2,2,2),a_{11})}$$
$$= \frac{2\mu_1^3 + \mu_1^2\mu_2 - \mu_2^3 + 2c\mu_1^3 + 2c\mu_1^2\mu_2 + 2c\mu_1\mu_2^2 + 2c\mu_2^3)}{\mu_1^2+\mu_2^2},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{12})} - c_{w((2,2,2),a_{12})} = \frac{\mu_1(\mu_1^3 + \mu_1^2\mu_2 - \mu_2^3)}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((2,2,2),a_{21})} - c_{w((2,2,2),a_{21})}$$
$$= \frac{-\mu_1^4 - \mu_1^3\mu_2 - \mu_1^2\mu_2^2 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$

Finally, for states $(3,1,1), (3,1,2)$ and $(3,2,1)$ we have

$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{21})} - c_{w((3,1,1),a_{21})}$$
$$= \frac{-3\mu_1^4 - 3\mu_1^3\mu_2 - \mu_1^2\mu_2^2 + 2\mu_1\mu_2^3 + 2c\mu_1^4 + 4c\mu_1^3\mu_2 + 4c\mu_1^2\mu_2^2 + 4c\mu_1\mu_2^3 + 2c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,1,1),a_{22})} - c_{w((3,1,1),a_{22})} = c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{22})} - c_{w((3,1,2),a_{22})}$$
$$= \frac{-2\mu_1^4 - 2\mu_1^3\mu_2 - \mu_1^2\mu_2^2 + \mu_1\mu_2^3 + 4c\mu_1^4 + 8c\mu_1^3\mu_2 + 8c\mu_1^2\mu_2^2 + 8c\mu_1\mu_2^3 + 4c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$

$$c_B\mathbf{B}^{-1}v_{w((3,1,2),a_{21})} - c_{w((3,1,2),a_{21})}$$
$$= \frac{-3\mu_1^4 - 3\mu_1^3\mu_2 - \mu_1^2\mu_2^2 + 2\mu_1\mu_2^3 + 4c\mu_1^4 + 8c\mu_1^3\mu_2 + 8c\mu_1^2\mu_2^2 + 8c\mu_1\mu_2^3 + 4c\mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{12})} - c_{w((3,2,1),a_{12})} = \frac{3\mu_1^4 + 3\mu_1^3\mu_2 + 2\mu_1^2\mu_2^2 - 3\mu_1\mu_2^3 - \mu_2^4}{(\mu_1+\mu_2)(\mu_1^2+\mu_2^2)},$$
$$c_B\mathbf{B}^{-1}v_{w((3,2,1),a_{21})} - c_{w((3,2,1),a_{21})} = \frac{(\mu_1-\mu_2)(\mu_1^2+\mu_1\mu_2+\mu_2^2)}{\mu_1^2+\mu_2^2}.$$

These quantities are nonnegative because $2\mu_1^5 + \mu_1^4\mu_2 < \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$ and $c > \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$. Hence, the policy $\pi = (d)^\infty$ is an optimal policy and the recurrent states under this policy are $(0,1,2)$, $(1,1,2)$, $(2,1,2)$, and $(3,1,2)$. In the transient states (i.e., states in $S \setminus S_{w^*}$) we can select an action that will take the process to one of the recurrent states and this shows that the policy π^* described in the theorem is optimal when $2\mu_1^5 + \mu_1^4\mu_2 < \mu_1^3\mu_2^2 + 3\mu_1^2\mu_2^3 + 2\mu_1\mu_2^4 + \mu_2^5$ and $c > \frac{2\mu_1^4+2\mu_1^3\mu_2+\mu_1^2\mu_2^2-\mu_1\mu_2^3}{2\mu_1^4+4\mu_1^3\mu_2+4\mu_1^2\mu_2^2+4\mu_1\mu_2^3+2\mu_2^4}$. Hence the proof is complete. □

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