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Atomistic and orthoatomistic effect algebras

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We characterize atomistic effect algebras, prove that a weakly orthocomplete Archimedean atomic effect algebra is orthoatomistic and present an example of an orthoatomistic orthomodular poset that is not weakly orthocomplete.

1. Introduction

One of the basic concepts in the foundation of quantum physics is the quantum effect that plays an important role in the theory of the so-called unsharp measurements [1, 2]. Quantum effects are studied within a general algebraic framework called the effect algebra [2, 3, 5].

An important role in quantum structures play atoms (minimal nonzero elements) especially if every element of the structure can be built up from atoms, i.e., if the structure is atomistic or orthoatomistic—hence these properties are of particular interest [3, 6, 7, 8, 9].

In this paper we generalize some results concerning atomistic and orthoatomistic quantum structures and present a few illustrating examples.

2. Basic notions and properties

Definition 2.1: An *effect algebra* is an algebraic structure $(E, \oplus, \mathbf{0}, \mathbf{1})$ such that E is a set, $\mathbf{0}$ and $\mathbf{1}$ are different elements of E, and $\oplus$ is a partial binary operation on E such that for every $a, b, c \in E$ the following conditions hold:

(1) $a \oplus b = b \oplus a$ if $a \oplus b$ exists,
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if $(a \oplus b) \oplus c$ exists,
(3) there is a unique $a' \in E$ such that $a \oplus a' = \mathbf{1}$ (*orthosupplement*),
(4) $a = \mathbf{0}$ whenever $a \oplus \mathbf{1}$ is defined.

For simplicity, we use the notation E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \le b$ if there is a $c \in E$ such that $b = a \oplus c$. Such an element c is unique (if it exists) and is denoted by $b \ominus a$. $\mathbf{0}$ ($\mathbf{1}$, respectively) is the least (the greatest, respectively) element of E with respect to this partial ordering. For every $a, b \in E$, $a'' = a$ and $b' \le a'$ whenever $a \le b$. It can be shown that $a \oplus \mathbf{0} = a$ for every $a \in E$ and that a *cancellation law* is valid: for every $a, b, c \in E$ with $a \oplus b \le a \oplus c$ we have $b \le c$. An *orthogonality* relation on E is defined by $a \perp b$ if $a \oplus b$ exists (if $a \le b'$). See, e.g., [2, 3].

Obviously, if $a \perp b$ and $a \vee b$ exist in an effect algebra, then $a \vee b \le a \oplus b$. The reverse inequality need not be true (it holds in orthomodular posets).

Definition 2.2: Let E be an effect algebra. An element $a \in E$ is *principal* if $b \oplus c \le a$ for every $b, c \in E$ such that $b, c \le a$ and $b \perp c$.

Definition 2.3: An *orthoalgebra* is an effect algebra E in which, for every $a \in E$, $a = \mathbf{0}$ whenever $a \oplus a$ is defined.

An *orthomodular poset* is an effect algebra in which every element is principal.

An *orthomodular lattice* is an orthomodular poset that is a lattice.

Every orthomodular poset is an orthoalgebra. Indeed, if $a \oplus a$ is defined then $a \oplus a \leq a = a \oplus \mathbf{0}$ and, according to the cancellation law, $a \leq \mathbf{0}$ and therefore $a = \mathbf{0}$.

Orthomodular posets are characterized as effect algebras such that $a \oplus b = a \vee b$ for every orthogonal pair a, b (see [3, 4]). Let us remark that an orthomodular poset is usually defined as a bounded partially ordered set with an orthocomplementation in which the orthomodular law is valid.

Definition 2.4: Let E be an effect algebra. The *isotropic index* of an element $a \in E$ is $\sup\{n \in \mathbb{N} \colon na \text{ is defined}\}$, where $na = \bigoplus_{i=1}^{n} a$ is the sum of n copies of a.

An effect algebra is *Archimedean* if every its nonzero element has a finite isotropic index.

The isotropic index of $\mathbf{0}$ is ∞. In an orthoalgebra, we have that $a \oplus a$ is defined only for $a = \mathbf{0}$, hence the isotropic index of every nonzero element is 1. Therefore we obtain:

Proposition 2.5: Every orthoalgebra is Archimedean.

Definition 2.6: Let E be an effect algebra. A system $(a_i)_{i \in I}$ of (not necessarily distinct) elements of E is called *orthogonal*, if $\bigoplus_{i \in F} a_i$ is defined for every finite set $F \subset I$. We define $\bigoplus_{i \in I} a_i = \bigvee\{\bigoplus_{i \in F} a_i \colon F \subset I \text{ is finite}\}$ if the supremum exists.

An effect algebra E is *orthocomplete* if $\bigoplus_{i \in I} a_i$ is defined for every orthogonal system $(a_i)_{i \in I}$ of elements of E.

An effect algebra E is *weakly orthocomplete* if for every orthogonal system $(a_i)_{i \in I}$ of elements of E either $\bigoplus_{i \in I} a_i$ exists or there is no minimal upper bound of the set $\{\bigoplus_{i \in F} a_i \colon F \subset I \text{ is finite}\}$ in E.

Every pair of elements of an orthogonal system is orthogonal. On the other hand, there are mutually orthogonal elements that do not form an orthogonal system if the effect algebra is not an orthomodular poset. Since only the zero element is orthogonal to itself in an orthoalgebra, we may consider sets instead of systems in orthoalgebras.

Proposition 2.7: Every orthocomplete effect algebra is Archimedean.

Proof: Let E be an orthocomplete effect algebra and let $a \in E$ has an infinite isotropic index. There is an element $b \in E$ such that $b = \bigoplus_{n \in \mathbb{N}} a = \bigvee_{n \in \mathbb{N}} na$. Since $a \leq b$, there is an element $c \in E$ such that $b = a \oplus c$. For every $n \in \mathbb{N}$ we have $a \oplus c = b \geq (n+1)a = a \oplus na$ and therefore, according to the cancellation law, $c \geq na$. Hence, $c \oplus \mathbf{0} = c \geq \bigvee_{n \in \mathbb{N}} na = b = c \oplus a$ and, according to the cancellation law, $\mathbf{0} \geq a$ and therefore $a = \mathbf{0}$.

Definition 2.8: An *atom* of an effect algebra E is a minimal element of $E \setminus \{\mathbf{0}\}$.

An effect algebra is *atomic* if every nonzero element dominates an atom (i.e., there is an atom less than or equal to it).

An effect algebra is *atomistic* if every nonzero element is a supremum of a set of atoms (i.e., of the set of all atoms it dominates).

An effect algebra is *orthoatomistic* if every nonzero element is a sum of a set of atoms.

It is easy to see that every atomistic and every orthoatomistic effect algebra is atomic and that every orthoatomistic orthomodular poset is atomistic. There are atomic orthomodular posets that are not atomistic [6], atomistic orthomodular posets that are not orthoatomistic [7] and orthoatomistic orthoalgebras that are not atomistic—e.g., the so-called Wright triangle [4, Example 2.13].

3. Results

First, let us present a characterization of atomistic effect algebras that generalizes the result of [7] stated for orthomodular posets.

Definition 3.1: An effect algebra E is *disjunctive* if for every $a, b \in E$ with $a \not\leq b$ there is a nonzero element $c \in E$ such that $c \leq a$ and $c \wedge b = \mathbf{0}$.

Theorem 3.2: *An effect algebra is atomistic if and only if it is atomic and disjunctive.*

Proof: Let E be an effect algebra and let us for every $x \in E$ denote by A_x the set of atoms dominated by x.

$\Rightarrow$: Obviously, every atomistic effect algebra is atomic. Let $a, b \in E$ such that $a \not\leq b$. Then there is an atom $c \in A_a \setminus A_b$, hence $c \leq a$ and $c \wedge b = \mathbf{0}$.

$\Leftarrow$: Let us prove that $a \leq b$ for every nonzero $a \in E$ and for every upper bound $b \in E$ of A_a (hence, $a = \bigvee A_a$). Let us suppose that $a \not\leq b$ and seek a contradiction. Since E is disjunctive, there is a nonzero element $c \in E$ such that $c \leq a$ and $c \wedge b = \mathbf{0}$. Since E is atomic, there is an atom $d \in E$ such that $d \leq c$. Hence, $d \leq a$ and $d \wedge b = \mathbf{0}$. Since d is an atom, $d \not\leq b$ and therefore $d \in A_a \setminus A_b$—a contradiction.

Before stating the second main result of this paper, let us discuss relations of some properties.

Proposition 3.3: Let E be an effect algebra fulfilling at least one of the following conditions:

(OC) *E is orthocomplete.*
(L) *E is a lattice.*

Then E is weakly orthocomplete.

Proof: (OC): Obvious.

(L): Let $(a_i)_{i \in I}$ be an orthogonal system of elements of E. Let us show that if a minimal upper bound a of the set $A = \{\bigoplus_{i \in F} a_i \colon F \subset I \text{ is finite}\}$ exists then $a = \bigvee A$. Let b be an upper bound of A. Then $b \wedge a \leq a$ is an upper bound of A and, since a is minimal, $b \wedge a = a$. Hence, $a \leq b$.

Let us present examples showing that the scheme of implications in the previous proposition cannot be improved.

Example 3.4: Let X be a countable infinite set. Let E be a family of finite and cofinite subsets of X with the $\oplus$ operation defined as the union of disjoint sets. Then $(E, \oplus, \emptyset, X)$ is an orthomodular lattice (it forms a Boolean algebra) that is not orthocomplete.

Example 3.5: Let X be a 6-element set. Let E be the family of even-element subsets of X with the $\oplus$ operation defined as the union of disjoint sets from E. Then $(E, \oplus, \emptyset, X)$ is a finite (hence orthocomplete) orthomodular poset that is not a lattice.

Example 3.6: Let X_1, X_2, X_3, X_4 be mutually disjoint infinite sets, $X = \bigcup_{i=1}^{4} X_i$,

$$E_0 = \{\emptyset, X_1 \cup X_2, X_2 \cup X_3, X_3 \cup X_4, X_4 \cup X_1, X\},$$
$$E = \{(A \setminus F) \cup (F \setminus A) \colon F \subset X \text{ is finite}, \ A \in E_0\},$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X)$ is a weakly orthocomplete orthomodular poset that is neither orthocomplete (e.g., $\bigvee\big\{\{x\} \colon x \in X_1\big\}$ does not exist) nor a lattice (e.g., $(X_1 \cup X_2) \wedge (X_2 \cup X_3)$ does not exist).

Theorem 3.7: *Every weakly orthocomplete Archimedean atomic effect algebra is orthoatomistic.*

Proof: Let E be a weakly orthocomplete Archimedean atomic effect algebra and let $a \in E \setminus \{\mathbf{0}\}$. Let us consider a set $\mathcal{M}$ of orthogonal systems of atoms such that their finite sums are dominated by a. Since E is atomic, $\mathcal{M} \neq \emptyset$. Since E is Archimedean, the number of occurences of every element of E at orthogonal systems is bounded by its finite isotropic index. Let us define an equivalence relation on $\mathcal{M}$ by $M_1 \sim M_2$ if every element of E occurs in M_2 with the same multiplicity as in M_1, and a partial ordering $\preceq$ on $\mathcal{M}|_\sim$ by $M_1 \preceq M_2$ if every element of E occurs in M_2 with at least the same multiplicity as in M_1. Every chain in $\mathcal{M}|_\sim$ has an upper bound in $\mathcal{M}_\sim$ (we can take every element of E with the maximal multiplicity that appears in the elements of the chain). According to Zorn's lemma, there is a maximal element of $\mathcal{M}_\sim$ and therefore an $M \in \mathcal{M}$ such that there is no atom $c \in E$ with $c \oplus \bigoplus F$ defined for every finite subsystem F of M. Let us show that a is a minimal upper bound of the set $A = \{\bigoplus F :$ F is a finite subsystem of $M\}$. Indeed, if there is an upper bound $b \in E$ of A such that $b < a$ then $a \ominus b \neq \mathbf{0}$, there is an atom $c \in E$ such that $c \leq a \ominus b$ and therefore $c \oplus \bigoplus F \leq a$ for every finite subsystem F of M—this contradicts to the property of M. Since E is weakly orthocomplete, $a = \bigvee A = \bigoplus M$.

The previous theorem generalizes the result of [7] stated for weakly orthocomplete atomic orthomodular posets, the result of [3, Proposition 4.11] stated for chain finite effect algebras and the result of [8, Theorem 3.1] stated for lattice Archimedean atomic effect algebras.

None of the assumptions in Theorem 3.7 can be omitted. Indeed, there are atomistic orthomodular posets that are not orthoatomistic [7], Boolean algebras that are not atomic (e.g., $\exp \mathbb{N}|_{F(\mathbb{N})}$ where $F(\mathbb{N})$ denotes the family of finite subsets of the set $\mathbb{N}$ of natural numbers), and, as the following example shows, weakly orthocomplete atomic effect algebras that are not orthoatomistic.

Example 3.8: Let $E = \{0, 1, 2, \ldots, n, \ldots, n', \ldots, 2', 1', 0'\}$ with the $\oplus$ operation defined by $m \oplus n = m + n$ for every $m, n \in \mathbb{N}$ and $m \oplus n' = (n - m)'$ for every $m, n \in \mathbb{N}$ with $m \leq n$. Then $(E, \oplus, 0, 0')$ is an atomic effect algebra (it forms a chain) that is weakly orthocomplete. Indeed, if an orthogonal system M of nonzero elements of E is finite then $\bigoplus M$ is defined; if M is infinite then the set of finite sums of elements of M forms an unbounded set of natural numbers and, therefore, does not have a minimal upper bound. The effect algebra is not orthoatomistic because no element n', $n \in \mathbb{N}$, is a sum of atoms.

Let us present an example that an orthoatomistic orthomodular poset need not be weakly orthocomplete.

Example 3.9: Let X, Y be disjoint infinite countable sets,

$$E_0 = \{A \subset (X \cup Y) \colon \ \mathrm{card}(A \cap X) = \mathrm{card}(A \cap Y) \text{ is finite}\},$$
$$E = E_0 \cup \{(X \cup Y) \setminus A \colon \ A \in E_0\},$$

$A \oplus B = A \cup B$ for disjoint $A, B \in E$. Then $(E, \oplus, \emptyset, X \cup Y)$ is an orthomodular poset. It is orthoatomistic because for every nonempty $A \in E$ we have $\mathrm{card}(A \cap X) = \mathrm{card}(A \cap Y)$, there is a bijection $f \colon (A \cap X) \to (A \cap Y)$ and $A = \bigoplus\big\{\{x, f(x)\} \colon \ x \in (A \cap X)\big\}$. The orthomodular poset is not weakly orthocomplete because for $x_0 \in X$, $y_0 \in Y$ there is a bijection $f \colon X \to (Y \setminus \{y_0\})$ and the orthogonal set $\big\{\{x, f(x)\} \colon x \in X \setminus \{x_0\}\big\}$ has different minimal upper bounds $(X \cup Y) \setminus \{x_0, f(x_0)\}$ and $(X \cup Y) \setminus \{x_0, y_0\}$.

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