OPTIMAL CONCENTRATION FOR $SU(1,1)$ COHERENT STATE TRANSFORMS AND AN ANALOGUE OF THE LIEB-WEHRL CONJECTURE FOR $SU(1,1)$

A Thesis
Presented to
The Academic Faculty

by

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the
School of Physics

Georgia Institute of Technology
August 2008

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To

Bisakha Bandyopadhyay and Raghab Bandyopadhyay,

my parents

ACKNOWLEDGEMENTS

I am greatly indebted to my advisor Prof. Eric A. Carlen for his guidance and encouragement. I would also like to thank the members of my thesis advisory committee, Prof. Predrag Cvitanovic, Prof. Michael Loss and Prof. Jean Bellissard, for many useful suggestions. Thanks also go to my fellow students in Physics department, for some wonderful moments we have spent together. I am very grateful to Prabir Bhattacharyya, Biswanath Bhattacharyya and Tirthankar Dasgupta, for their help during my stay in Atlanta. Last, but not least, I would like to thank my parents and my husband Abhyuday Mandal, for being exceptionally understanding and supportive.

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SUMMARY

We derive a lower bound for the Wehrl entropy in the setting of $SU(1,1)$. For asymptotically high values of the quantum number k, this bound coincides with the analogue of the Lieb-Wehrl conjecture for $SU(1,1)$ coherent states. The bound on the entropy is proved via a sharp norm bound. The norm bound is deduced by using an interesting identity for Fisher information of $SU(1,1)$ coherent state transforms on the hyperbolic plane $\mathbb{H}^2$ and a new family of sharp Sobolev inequalities on $\mathbb{H}^2$. To prove the sharpness of our Sobolev inequality, we need to first prove a uniqueness theorem for solutions of a semi-linear Poisson equation (which is actually the Euler-Lagrange equation for the variational problem associated with our sharp Sobolev inequality) on $\mathbb{H}^2$. Uniqueness theorems proved for similar semi-linear equations in the past do not apply here and the new features of our proof are of independent interest, as are some of the consequences we derive from the new family of Sobolev inequalities. We also prove Fisher information identities for the groups $SU(n,1)$ and $SU(n,n)$.

CHAPTER I

INTRODUCTION

Let $\mathcal{M}$ be a Riemannian manifold with volume element d$\mathcal{M}$. For a probability density ρ on $\mathcal{M}$, that is, a non-negative measurable function on $\mathcal{M}$ with $\int_{\mathcal{M}} \rho d\mathcal{M} = 1$, its entropy, if it exists, is defined as:

$$S(\rho) = -\int_{\mathcal{M}} \rho \ln \rho \, \mathrm{d}\mathcal{M}. \tag{1.1}$$

Thus defined, the entropy of a density ρ can be thought of as a measure of its "concentration". If some part of the mass of ρ is very nearly concentrated in a multiple of a Dirac mass, then $S(\rho)$ may be very negative. We shall be mainly interested in the case in which $\mathcal{M}$ is the phase space of some classical system, so that, in particular, $\mathcal{M}$ is a symplectic manifold. In that case, we shall refer to ρ as a *classical density*, and $S(\rho)$ as its *classical entropy*.

The uncertainty principle limits the extent of possible concentration in phase space: for instance, it prevents both the momentum variables p and the configuration variables q in a canonical phase space, from taking on well-defined values at the same time. A quantum mechanical density ρ^Q is a non-negative operator on the Hilbert space $\mathcal{H}$, which is the state space of the quantum system, having unit trace. Then the quantum entropy (or von Neumann entropy) of ρ^Q is defined by

$$S^Q(\rho^Q) = -\mathrm{Tr}\ \rho^Q \ln \rho^Q\ . \tag{1.2}$$

Since all of the eigenvalues of ρ^Q lie in the interval $[0, 1]$, it is clear that

$$S^Q(\rho^Q) \geq 0\ . \tag{1.3}$$

As Wehrl emphasized [Weh], when one considers a quantum system and its corresponding classical analogue, not all of the classical probability densities on the phase space $\mathcal{M}$ can correspond to physical densities for the quantum system, and one might expect a lower bound on the classical entropy of those probability densities that do correspond to actual quantum states.

There is a natural way to make the correspondence between quantum states and classical probability densities on phase space, which goes back to Schrödinger. It is based on the *coherent state transform*, which is an isometry $\mathcal{L}$ from the quantum state space $\mathcal{H}$ *into* $L^2(\mathcal{M})$, the Hilbert space of square integrable functions on the classical phase space $\mathcal{M}$. Since it is an isometry, if ψ is any unit vector in $\mathcal{H}$, $\rho_\psi = |\mathcal{L}\psi|^2$ is a probability density on $\mathcal{M}$. Wehrl [Weh] proposed defining the classical entropy of a quantum state ψ in this way (note the the corresponding density matrix has rank one, and hence the von Neumann entropy would be zero, for a "pure state"). The Wehrl entropy is defined in terms of the coherent states for the quantum system and is bounded below by the quantum entropy. It has several physically desirable features such as monotonicity, strong subadditivity, and of course, positivity (see [Weh], [Lie]).

Wehrl identified the class of probability densities arising through the coherent state transform as the class of *quantum mechanically significant* probability densities on $\mathcal{M}$, and conjectured that corresponding to (1.3), there should be a lower bound on $S(|\mathcal{L}\psi|^2)$ as ψ ranges over the unit sphere in $\mathcal{H}$.

Specifically, if $\mathcal{H}$ is $L^2(\mathbb{R}, \mathrm{d}x)$, so that the classical phase space is $\mathbb{R}^2$ with its usual symplectic and Riemannian structure, Wehrl conjectured that the lower bound on $S(|\mathcal{L}\psi|^2)$ is attained when ψ is a minimal uncertainty state $\psi_{\min}$, also known as a *Glauber coherent state*. That is:

$$\inf_{\|\psi\|_{\mathcal{H}}=1} S(|\mathcal{L}\psi|^2) = S(|\mathcal{L}\psi_{min}|^2) \ . \tag{1.4}$$

This was proved by Lieb [Lie] . There is a natural analogue of the Wehrl conjecture for other state spaces and other coherent state transforms. Lieb generalized the Wehrl conjecture to the $SU(2)$ coherent states, for which the corresponding classical phase space is $\mathbb{S}^2$, the two-dimensional sphere, with its usual Riemannian and symplectic structure. The analogues of the Glauber coherent states in this case are the *Bloch coherent states* generated by least weight vectors in the various unitary representations of $SU(2)$, indexed by the half integer *quantum number* j and Lieb conjectured the analogue of (1.4) for the $SU(2)$ coherent state transform.

Although Lieb's conjecture for $SU(2)$ is still open, it has attracted the attention of a number of researchers, and much progress has been made. The various unitary representations of $SU(2)$ are indexed by a half integer j, which is the *quantum number* in this context; for each such j there is a coherent state transform, and hence a conjectured lower bound of the Wehrl entropy. The bound is trivial for $j = 1/2$, in which case every state is a Bloch coherent state, but is already non trivial for $j = 1$. Schupp [Sch] proved the conjecture for $j = 1$ and $j = 3/2$. Later Bodmann [Bod] proved a result which may be seen as complementary to Schupp's result; he deduced a lower bound for the Wehrl entropy of $SU(2)$ coherent states, for which the high spin asymptotics coincided with Lieb's conjecture up to, but not including, terms of first and higher orders in the inverse of spin quantum number j.

Bodmann did this by proving a sharp L^p bound on the range of the coherent state transform. This led to a proof of an analogue of Lieb's conjecture for certain *Renyi entropies* (cf. [Gnu]) : for any $p > 1$ and any classical density ρ, define

$$S_p(\rho) = \frac{1}{p-1} \ln \left(\|\rho\|_p \right) \ , \tag{1.5}$$

where $\|\rho\|_p$ is the L^p norm of ρ. Then it is easy to see that, if it exists,

$$\lim_{p \to 1} S_p(\rho) = S(\rho) \ .$$

Bodmann derived his bound on Renyi entropies from a Sobolev type inequality and a Fisher information identity, which is another type of concentration bound on the range of the coherent state transform. The Fisher information $I(\rho)$ of a probabilty density ρ on M is defined by

$$I(\rho) = \int_M |\nabla \ln \rho|^2 \rho \ \mathrm{d}\mathcal{M} = 4 \int_M |\nabla \sqrt{\rho}|^2 \ \mathrm{d}\mathcal{M} \ .$$

For the Glauber coherent state transform, Carlen [Car] had proved that all classical densities on $\mathbb{R}^2$ arising through the coherent state transform had the *same* finite value of the Fisher information. He then used that together with the logarithmic Sobolev inequality (cf. [Gro1]) to give a new proof of Wehrl's conjecture, and to show that the lower bound in (1.4) is attained only for Glauber coherent states. Bodmann proved an analogue of Carlen's result for Fisher information, and used this, together with a sharp Sobolev inequality, instead of the sharp logarithmic Sobolev inequality, to obtain his Renyi information bounds.

In this thesis, we investigate the analogue of the Lieb-Wehrl conjecture for $SU(1,1)$. The representations of $SU(1,1)$ belonging to a discrete series, are labeled by a half-integer k, the relevant *quantum number* in this context. While the classical phase space for $SU(2)$ is the sphere $\mathbb{S}^2$, for $SU(1,1)$ the classical phase space is $\mathbb{H}^2$, the hyperbolic plane (cf. [Per]). It is natural to conjecture that, here too, the coherent states generated by the least-weight vector of the representation provide a lower bound on the entropy, as in Lieb's conjecture for $SU(2)$. We prove that this is indeed asymptotically true, in the semi-classical limit. To obtain these results, we prove a number of theorems concerning analysis in $\mathbb{H}^2$ that are of independent interest. Specifically, we prove a new sharp Sobolev inequality, and a sharpened energy–entropy inequality in $\mathbb{H}^2$. The Sobolev inequality is

$$\|f\|_q^q + \frac{4}{kq(kq-2)} \int |\nabla |f|^{q/2}|^2 d\nu \geq \left(\frac{2k-1}{kq-1}\right) \left(\frac{kp-1}{2k-1}\right)^{q/p} \left(\frac{kq-1}{kq-2}\right) \|f\|_p^q,$$

where $p = q + 1/k$, $q \geq 2$, $kq > 2$ and the measure $d\nu$ is a constant times the standard measure on $\mathbb{H}^2$, obtained from the Poincaré metric; we determine all of the cases of equality.

To prove the sharpness of our Sobolev inequality we need to prove and use a uniqueness result for radial solutions of a semi-linear Poisson equation on the hyperbolic plane. The nature of this equation on $\mathbb{H}^2$ is substantially different from that of similar equations which have been investigated in the past. The methods developed here may well be useful for other uniqueness problems.

We then prove the following Fisher information identity:

$$\int |\nabla|\mathcal{L}\psi|^{q/2}|^2 d\nu = \frac{1}{4}kq \int |\mathcal{L}\psi|^q d\nu,$$

where q is a positive number such that $kq > 2$.

The sharp Sobolev inequality and the Fisher information identity allow us to prove an L^p norm estimate *a la* Bodmann. This norm estimate is used to deduce a lower bound for the Wehrl entropy of coherent state transforms via a convexity argument, and the result is:

$$S(|\mathcal{L}\psi(\zeta)|^2) \geq 2k \ln\left(1 + \frac{1}{2k-1}\right).$$

It is seen that for high values (this gives us the semi-classical limit) of the quantum number k, this lower bound coincides with the analogue of the Lieb-Wehrl conjecture, up to but not including terms of first and higher order in k^{-1}.

The methods used to bound the entropy also serve to produce a new, sharpened energy–entropy inequality for functions on $\mathbb{H}^2$. An energy–entropy inequality is an inequality of the form

$$-S(\rho) \leq \Phi_M(I(\rho)), \tag{1.6}$$

for some function Φ. Since the Fisher information can be expressed in terms of an energy integral as shown above, the entropy-energy terminology is natural. For a given Riemannian manifold $\mathcal{M}$, the entropy–energy problem is to determine the least function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ for which (1.6) is true.

For example, in the case $\mathcal{M} = \mathbb{R}^2$, the optimal Φ is known:

$$-S(\rho) \leq \ln\left(\frac{4}{\pi e} I(\rho)\right).$$

Equality is achieved when ρ is an isotropic Gaussian function. For an appropriate choice of the variance of the Gaussian, $I(\rho)$ can take on any value, and this inequality is sharp for all values of $I(\rho)$. That is,

$$\Phi_{\mathbb{R}^2}(t) = \ln\left(\frac{4}{\pi e}t\right) .$$

There has been much investigation of entropy-energy inequalities for various Riemannian manifolds (see [Bec1], [Bec2], [Heb], [Rot] for example). Though there has been significant progress, many questions are still open.

In the case of $\mathbb{H}^2$, Beckner proved [Bec2] that the entropy–energy inequality for H^2 holds with the same Φ as in $\mathbb{R}^2$. That is,

$$\Phi_{\mathbb{H}^2}(t) \leq \Phi_{\mathbb{R}^2}(t),$$

for all t. This result is asymptotically sharp in the sense that

$$\lim_{t\to 0} \frac{\Phi_{\mathbb{H}^2}(t)}{\Phi_{\mathbb{R}^2}(t)} = 1 ,$$

however, $\Phi_{\mathbb{H}^2}(t) < \Phi_{\mathbb{R}^2}(t)$. We shall give a sharpened estimate on $\Phi_{\mathbb{H}^2}(t)$.

It is interesting to observe how sharp bounds on the Fisher information of coherent state transforms can lead to sharp Sobolev type inequalities in a larger function space, which can then be used to derive entropy–energy inequalities on various symplectic Riemannian manifolds that are classical phase spaces, e.g., the sphere and the hyperbolic plane. These manifolds are determined by the groups for which we construct the coherent states. It seems natural to ask: for which other groups having unitary irreducible representations in spaces of holomorphic functions, can one obtain bounds on the Fisher information of the coherent state transforms and formulate analogues of the Lieb-Wehrl conjecture? Although this is a problem we are working on right now, we can prove a Fisher information identity for the groups $SU(n,1)$ and $SU(n,n)$.

In the next chapter we give a short description of a discrete representation of $SU(1,1)$ and define the associated coherent states and coherent state transform.

Given any quantum state ψ, we denote its coherent state transform by $\mathcal{L}\psi(\zeta)$, where the complex number ζ is used to label the coherent states. We show that these coherent state transforms are actually probability amplitudes on the hyperbolic plane. We also state the analogue of the Lieb-Wehrl conjecture in this setting.

Chapter 3 contains the proof of the lower bound for the Wehrl entropy for $SU(1,1)$, and the results leading up to it. Here we prove Fisher information identity for the coherent state transforms, and the sharp Sobolev inequality. Chapter 4 contains the sharpened entropy–energy inequality for $\mathbb{H}^2$, and chapter 5 contains our uniqueness proof. Finally, in chapter 6 we construct coherent state transforms for the group $SU(p,q)$ and prove Fisher information identities for the groups $SU(n,1)$ and $SU(n,n)$.

CHAPTER II

REPRESENTATION OF THE GROUP SU(1,1) AND THE CONSTRUCTION OF COHERENT STATES

The group $SU(1,1)$ consists of unimodular 2×2 complex matrices which leave the Hermitian form $|z_1|^2 - |z_2|^2$ invariant. These matrices can be parametrized by a pair of complex numbers, α, β as follows:

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|^2 - |\beta|^2 = 1.$$

One can define a new variable $z = \dfrac{z_2}{z_1}$ and describe the action of the element $g \in SU(1,1)$ on $\mathbb{C}^1$ as:

$$z \longrightarrow z_g = \frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}.$$

However, the group action on $\mathbb{C}^1$ is not transitive; in fact the complex plane is foliated into three orbits, namely, i) the interior of the unit disk, ii) the boundary of the unit disk, and, iii) the complement of the closed unit disk in the complex plane.

We shall work with one of the two discrete series of representations of $SU(1,1)$, in the space of functions that are defined and analytical in the unit disc. The Lie algebra for $SU(1,1)$ has three generators as its basis elements, which we call K_0, K_1 and K_2 following Perelomov. The commutation relations satisfied are:

$$[K_1, K_2] = -iK_0, \qquad [K_2, K_0] = iK_1, \qquad [K_0, K_1] = iK_2.$$

Thus, in two dimensions $K_0 = \sigma_3/2, \quad K_{1,2} = \pm i\sigma_{2,1}/2$ would be the generators. There is one Casimir operator given by: $\hat{C} = -K_0^2 + K_1^2 + K_2^2$. So for any irreducible representation the operator is a multiple of the identity and we write:

$$\hat{C} = k(1-k)\hat{I}.$$

Thus, a particular irreducible representation of $SU(1,1)$ is labeled by a single number k [Per]. For the discrete series this number takes on discrete half-integral values, $k = 1/2, 1, 3/2, ...$ (cf. [Bar], [Per]). Let us call a particular representation $T^k(g)$. simultaneous eigenvectors of the Casimir operator $\hat{C}$ and K_0 to be the basis vectors. We use Dirac's bra-ket notation and denote these vectors by $|k,\mu\rangle$ where μ denotes the eigenvalue corresponding to K_0 and

$$K_0|k,\mu\rangle = \mu|k,\mu\rangle.$$

Here $\mu = k + m$, where, for the positive discrete series of representations, m is either zero or any positive integer [Bar] (the representations are infinite-dimensional). We consider a realization of $T^k(g)$ in the space $\mathcal{G}_k$ of functions $f(z)$ which are analytic inside the unit circle and have finite L^2-norm with respect to the invariant density $d\varpi_k(z) = \frac{2k-1}{\pi}(1-|z|^2)^{2k-2}d^2z$ [Bar], i.e.,:

$$\frac{2k-1}{\pi}\int_D |f(z)|^2(1-|z|^2)^{2k-2}d^2z < \infty, \qquad D = \{z : |z| < 1\}.$$

The pre-factor $\dfrac{2k-1}{\pi}$ is chosen so that we have $(f,g)_k = \displaystyle\int_D \overline{f(z)}g(z)d\varpi_k(z) \equiv 1$ when $f \equiv 1$ and $g \equiv 1$, where $(f,g)_k$ denotes the inner product of f and g in this representation. The group action on $\mathcal{G}_k$ in the multiplier representation $T^k(g)$ is given by [Bar]:

$$T^k(g)f(z) = (\beta z + \bar{\alpha})^{-2k}f(z_g), \qquad z_g = \frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}.$$

Now let us observe that:

$$1 - |z_g|^2 = |\beta z + \alpha|^{-2}(1-|z|^2) \qquad \text{and,} \quad J_g(z) = |\beta z + \alpha|^{-2},$$

where $J_g(z)$ denotes the Jacobian determinant of the transformation g. Taking inner products with respect to the invariant density $d\varpi_k(z)$, we see that, for two functions

$f_1(z),\ f_2(z)$, we have:

$$\begin{aligned}
\left(T^k(g)f_1, T^k(g)f_2\right)_k &= \int_D \overline{T^k(g)f_1(z)}T^k(g)f_2(z)d\varpi_k(z) \\
&= \frac{2k-1}{\pi}\int_D (\bar{\beta}\bar{z}+\alpha)^{-2k}\overline{f_1(z_g)}(\beta z+\bar{\alpha})^{-2k}f_2(z_g)(1-|z|^2)^{2k-2}d^2z \\
&= \frac{2k-1}{\pi}\int_D |\beta z+\alpha|^{-4k}\overline{f_1(z_g)}f_2(z_g)|\beta z+\alpha|^{4k}(1-|z_g|)^{2k-2}d^2z_g \\
&= (f_1, f_2)_k
\end{aligned}$$

Thus, the operators $T^k(g)$ with the group action defined as above furnish a unitary and irreducible representation of $SU(1,1)$.

To construct the coherent states we choose, as the stability group, subgroup H of diagonal matrices of the form $h = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$. Then any matrix in a coset gH can be represented in the following form:

$$g_{\mathbf{n}} = \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2}e^{-i\phi} \\ \sinh\frac{\tau}{2}e^{i\phi} & \cosh\frac{\tau}{2} \end{pmatrix}.$$

The factor space G/H is realized as the unit disk $\{\zeta : |\zeta| < 1\}$, or equivalently, as the hyperbolic plane $\mathbb{H}^2 = \{\mathbf{n} : |n|^2 = n_0^2 - n_1^2 - n_2^2 = 1, n_0 > 0\}$ [Per] via the following correspondence:

$$n_0 = \cosh\frac{\tau}{2}, \quad n_1 = \sinh\frac{\tau}{2}\cos\phi, \quad n_2 = \sinh\frac{\tau}{2}\sin\phi \qquad \text{and} \qquad \zeta = \tanh\frac{\tau}{2}e^{i\phi}.$$

An element of G/H determines a hyperbolic rotation and in two dimensions, we have the following decomposition:

$$\begin{aligned}
g_{\mathbf{n}} &= \begin{pmatrix} \cosh\frac{\tau}{2} & \sinh\frac{\tau}{2}e^{-i\phi} \\ \sinh\frac{\tau}{2}e^{i\phi} & \cosh\frac{\tau}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & \tanh\frac{\tau}{2}e^{-i\phi} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\cosh\frac{\tau}{2}} & 0 \\ 0 & \cosh\frac{\tau}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tanh\frac{\tau}{2}e^{i\phi} & 1 \end{pmatrix} \\
&= \exp\left(\frac{1}{2}\zeta\sigma_+\right)\exp\left(\frac{1}{2}\eta\sigma_3\right)\exp\left(\frac{1}{2}\bar{\zeta}\sigma_-\right),
\end{aligned}$$

where $\zeta = \tanh\frac{\tau}{2}e^{-i\phi}$ and $\eta = 2\ln\cosh\frac{\tau}{2}$. As mentioned before, $K_0 = \sigma_3/2$, $K_{1,2} = \pm i\sigma_{2,1}/2$ in two dimensions. Hence for a hyperbolic rotation parametrized by τ and ϕ, in the representation T^k, we have:

$$T^k(g_{\mathbf{n}}) = \exp\left(\zeta K_+\right)\exp\left(\eta K_0\right)\exp\left(-\bar{\zeta}K_-\right),$$

where $\zeta = \tanh\frac{\tau}{2}e^{-i\phi}$ and $\eta = -2\ln\cosh\frac{\tau}{2}$, as before, and $K_\pm = (K_1 \pm iK_2)$. We will use this expression for the operator $T^k(g_{\mathbf{n}})$, to construct the coherent states.

Now, in $\mathcal{G}_k$ the generators act as first order differential operators. An element of $SU(1,1)$ generated by K_1 is given by:

$$\begin{aligned} u = \exp\left(-\sigma_1\tau/2\right) &= \cosh\frac{\tau}{2}\mathbf{1} - \sinh\frac{\tau}{2}\sigma_1 \\ &= \begin{pmatrix} \cosh\frac{\tau}{2} & -\sinh\frac{\tau}{2} \\ -\sinh\frac{\tau}{2} & \cosh\frac{\tau}{2} \end{pmatrix}, \end{aligned}$$

where $\mathbf{1}$ is the identity operator. Thus, in the representation T^k, the action of the generator K_1 on a function f would be given by:

$$\begin{aligned} K_1 f(z) &= -i.\frac{d}{d\tau}T^k(u)f(z)|_{\tau=0} \\ &= -i\,\frac{d}{d\tau}\left[(-\sinh\frac{\tau}{2}z + \cosh\frac{\tau}{2})^{-2k} f\left(\frac{-\sinh\frac{\tau}{2} + \cosh\frac{\tau}{2}z}{-\sinh\frac{\tau}{2}z + \cosh\frac{\tau}{2}}\right)\right]\Bigg|_{\tau=0} \\ &= -i\left(kzf(z) - \frac{1}{2}(1-z^2)\frac{df(z)}{dz}\right) \end{aligned}$$

In a similar manner it can be seen that:

$$K_2 f(z) = -\frac{1}{2}(1+z^2)\frac{df(z)}{dz} - kzf(z) \qquad \text{and } K_0 f(z) = z\frac{df(z)}{dz} + kf(z).$$

From the form of K_0 it is clear that its eigenfunctions, are monomials in z. We denote these basis vectors spanning the representation space by $|k, k+m\rangle$, where m is a nonnegative integer (cf. [Per]). Normalized with respect to the measure $d\varpi_k(z)$ these eigenvectors are written:

$$|k, k+m\rangle = \left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right)^{\frac{1}{2}} z^m. \tag{2.1}$$

To construct the coherent states let us choose the least-weight vector $|k,k\rangle$ in $\mathcal{F}_k$. The reason behind this particular choice is that the dispersion of the Casimir operator is minimal with respect to $|k,k\rangle$. The stationary subgroup for this state is the subgroup H of diagonal matrices, as mentioned before.

We let these operators act on the chosen least-weight vector $|k,k\rangle$ to obtain an expression for the coherent states in the orthonormal basis (cf. [Per]):

$$\begin{aligned} T^k(g_{\mathbf{n}})|k,k\rangle &= \exp\left(\zeta K_+\right)\exp\left(\ln(1-|\zeta|^2)K_0\right)\exp\left(-\bar{\zeta}K_-\right)|k,k\rangle \\ &= \exp\left(\zeta K_+\right)(1-|\zeta|^2)^k|k,k\rangle \\ &= (1-|\zeta|^2)^k\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right)^{\frac{1}{2}}\zeta^m|k,k+m\rangle, \end{aligned}$$

where we have expanded the exponential in the second line and used the fact that:

$$(K_+)^m|k,k\rangle = \left(\frac{m!\Gamma(2k+m)}{\Gamma(2k)}\right)^{1/2}|k,k+m\rangle.$$

Represented as above, the coherent states are parametrized by a complex number ζ on the unit disk or equivalently, by two real parameters τ and ϕ on the hyperbolic $\mathbb{H}^2$. In what follows, we shall denote the coherent state corresponding to a particular ζ by $|\zeta\rangle$. If we now choose any arbitrary normalized vector $|\psi\rangle = \sum_{m=0}^{\infty} a_m|k,m\rangle$, then we can define its coherent state transform $\mathcal{L}\psi(\zeta)$ via the following inner product:

$$\mathcal{L}\psi(\zeta) = \langle\psi|\zeta\rangle = (1-|\zeta|^2)^k\sum_{m=0}^{\infty}\left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)}\right)^{\frac{1}{2}}\bar{a}_m\zeta^m. \tag{2.2}$$

Evidently $\mathcal{L}\psi(\zeta)$ is a function on the unit disk and so the coherent state transform maps unit vectors in our representation space $\mathcal{G}_k$ into functions on the unit disk, which vanish at the boundary of the disk. This mapping becomes an isometry if we equip the unit disk with the L^2-metric corresponding to the measure: $d\nu(\zeta) = \left(\frac{2k-1}{\pi}\frac{1}{(1-|\zeta|^2)^2}\right)d^2\zeta$. Note that $d\nu(\zeta)$ is just $\left(\frac{2k-1}{4\pi}\right)$ times the standard measure on the unit disk, that is, the measure $d\mu(\zeta) = \left(\frac{4}{(1-|\zeta|^2)^2}\right)d^2\zeta$, obtained from the Poincaré metric on the disk. With inner product defined in the usual way with respect to the measure $d\nu(\zeta)$, the space of the coherent state transforms described above is a Hilbert space [Bar]. We call this space $\mathfrak{F}_k$. The transform $\mathcal{L}$ is thus an analogue of the Bargmann-Segal transform for the Glauber coherent states based on the Heisenberg group. Since $|\psi\rangle$ is a unit vector in our representation space $\mathcal{G}_k$, its coherent state transform $\mathcal{L}\psi(\zeta)$ is a probability amplitude on the unit disk. Thus, $\mathfrak{F}_k$ is a space of probability amplitudes on the unit disk. We can now think about the Wehrl entropy $S(|\mathcal{L}\psi(\zeta)|^2)$ associated with the coherent state transform $\mathcal{L}\psi(\zeta)$. If the unit vector $|\psi\rangle$ happens to be the ground state (which, we note, is also a coherent state), we can easily compute that: $S(|\mathcal{L}\psi(\zeta)_{coherent}|^2) = \frac{2k}{2k-1}$. Due to invariance under $SU(1,1)$, this means that the entropy would be the same for any other coherent state. The analogue of the Lieb-Wehrl conjecture for $SU(1,1)$ coherent states would then be:

Conjecture 2.0.1. *For all $\mathcal{L}\psi(\zeta) \in \mathfrak{F}_k$, the Wehrl entropy is bounded below by:*

$$S(|\mathcal{L}\psi(\zeta)|^2) \geq \frac{2k}{2k-1}. \tag{2.3}$$

CHAPTER III

THE ENTROPY BOUND AND RELATED RESULTS

In this section we first present a useful Fisher information identity for functions in $\mathfrak{F}_k$, that relates the q-norm (for all positive q such that $kq > 2$) of a function to the L^2-norm of the associated gradient. We then prove a sharp Sobolev inequality for functions in a larger function space $\mathfrak{H}$, defined to be the space of bounded non-constant functions $f \in W^{1,2}(D)$ on the unit disk which vanish at the boundary; the norms here are computed with respect to the measure $d\nu(\zeta)$. Next, we prove a sharp norm estimate for functions in $\mathfrak{F}_k$ (note that $\mathfrak{F}_k$ is a subspace of $\mathfrak{H}$) by converting the gradient norm of $|f|^{q/2}$ that appears in our sharp Sobolev inequality, into the L^q-norm of the function f, via the Fisher information identity. This sharp norm estimate is then used to derive a lower bound on the entropy of functions in $\mathfrak{F}_k$.

The variational problem associated with our sharp Sobolev inequality in the function space $\mathfrak{H}$, naturally leads us to an Euler-Lagrange equation which is actually a semi-linear Poisson equation on the unit disk. We reduce the Euler-Lagrange equation to an ordinary differential equation by using radially symmetric decreasing rearrangements of functions. To prove the sharpness of the Sobolev inequality we need to prove that the ground state solution, that is to say, the solution that decays to zero at the boundary of the disk, is unique. Since the proof is somewhat involved, we present a detailed analysis of the Euler-Lagrange equation and relevant results in section 5.

3.1 A Fisher Information Identity

The Fisher information of a probability density function is a measure of its concentration. In this subsection we prove a Fisher information identity for functions in $\mathfrak{F}_k$.

Theorem 3.1.1. *For $\mathcal{L}\psi(\zeta)$ in $\mathfrak{F}_k$ the following identity holds:*

$$\int |\nabla|\mathcal{L}\psi(\zeta)|^{q/2}|^2 d\nu(\zeta) = \frac{1}{4}kq \int |\mathcal{L}\psi(\zeta)|^q d\nu(\zeta),$$

where q is a positive number such that $kq > 2$.

Proof. Using the expression (2.2) for the coherent state transforms in $\mathfrak{F}_k$, we can write:

$$|\mathcal{L}\psi(\zeta)|^{q/2} = (1-|\zeta|^2)^{kq/2} \left| \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+2k)}{m!\Gamma(2k)} \right)^{\frac{1}{2}} \bar{a}_m \zeta^m \right|^{q/2} = (1-|\zeta|^2)^{kq/2} |\Phi(\zeta)|^{q/2},$$

where $\Phi(\zeta)$ is holomorphic in ζ. Thus $\Phi(\zeta)$ satisfies the Cauchy-Riemann equations on the unit disk/hyperbolic plane. Let us do our computations in terms of the radial variable τ and the angular variable ϕ on the two-dimensional hyperbolic plane. The gradient is then given by: $\nabla = \left(\frac{\partial}{\partial \tau}, \frac{1}{\sinh\tau} \frac{\partial}{\partial \phi} \right)$.

A brief computation yields the following Cauchy-Riemann equations for an analytic function $\Phi = u + iv$ on the hyperbolic plane:

$$\frac{\partial u}{\partial \tau} = \frac{1}{\sinh\tau} \frac{\partial v}{\partial \phi} \quad , \quad \frac{\partial u}{\partial \phi} = -\sinh\tau \frac{\partial v}{\partial \tau}.$$

Using these two equations we obtain the following:

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial \tau}\frac{\partial v}{\partial \tau} + \frac{1}{\sinh^2\tau}\frac{\partial u}{\partial \phi}\frac{\partial v}{\partial \phi} = 0$$

$$\begin{aligned}
|\nabla u|^2 &= |\frac{\partial u}{\partial \tau}|^2 + \frac{1}{\sinh^2\tau}|\frac{\partial u}{\partial \phi}|^2 \\
&= |\frac{\partial v}{\partial \tau}|^2 + \frac{1}{\sinh^2\tau}|\frac{\partial v}{\partial \phi}|^2 \\
&= |\nabla v|^2.
\end{aligned}$$

We now compute some results for the non-holomorphic pre-factor $(1-|\zeta|^2)^{kq/2}$ in the expression for the coherent state transforms:

$$\nabla(1-|\zeta|^2)^{kq/2} = \left(\frac{\partial}{\partial \tau}, \frac{1}{\sinh\tau}\frac{\partial}{\partial \phi} \right) (1-\tanh^2\frac{\tau}{2})^{kq/2} = \left(-\frac{kq}{2}\tanh\frac{\tau}{2}\,\mathrm{sech}^{\,kq}\frac{\tau}{2}, 0 \right),$$

and,

$$\begin{aligned}\triangle(1-|\zeta|^2)^{kq/2} &= \left(\frac{\partial^2}{\partial\tau^2}+\coth\tau\frac{\partial}{\partial\tau}\right)\left(1-\tanh^2\frac{\tau}{2}\right)^{kq/2}\\ &= \left(\frac{kq}{2}\right)^2\tanh^2\frac{\tau}{2}\ \mathrm{sech}^{\ kq}\frac{\tau}{2}-\frac{kq}{2}\ \mathrm{sech}^{\ kq}\frac{\tau}{2}.\end{aligned}$$

As for $|\Phi|^{q/2}$, the Cauchy-Riemann equations for Φ guarantee that:

$$\begin{aligned}&\triangle|\Phi|^q\\ =\ &\triangle(u^2+v^2)^{q/2}\\ =\ &\nabla\cdot\left(\frac{q}{2}(u^2+v^2)^{\frac{q}{2}-1}(2u\nabla u+2v\nabla v)\right)\\ =\ &q\Big[(q-2)(u^2+v^2)^{\frac{q}{2}-2}(u\nabla u+v\nabla v)\cdot(u\nabla u+v\nabla v)\\ &\qquad\qquad +(u^2+v^2)^{\frac{q}{2}-1}\nabla\cdot(u\nabla u+v\nabla v)\Big]\\ =\ &q\Big[(q-2)(u^2+v^2)^{\frac{q}{2}-2}\left(u^2|\nabla u|^2+v^2|\nabla v|^2\right)\\ &\qquad\qquad +(u^2+v^2)^{\frac{q}{2}-1}\left(|\nabla u|^2+|\nabla v|^2+u\triangle u+v\triangle v\right)\Big]\\ =\ &q^2(u^2+v^2)^{\frac{q}{2}-1}|\nabla u|^2\\ =\ &4|\nabla|\Phi|^{q/2}|^2.\end{aligned}$$

Thus:

$$\begin{aligned}&|\nabla|\mathcal{L}\psi(\zeta)|^{q/2}|^2\\ =\ &(1-|\zeta|^2)^{kq}|\nabla|\Phi|^{q/2}|^2\\ &\qquad +|\nabla(1-|\zeta|^2)^{kq/2}|^2|\Phi|^q+2(1-|\zeta|^2)^{kq/2}\nabla(1-|\zeta|^2)^{kq/2}\cdot|\Phi|^{q/2}\nabla|\Phi|^{q/2}\\ =\ &(1-|\zeta|^2)^{kq}|\nabla|\Phi|^{q/2}|^2+\frac{1}{4}|\Phi|^q(1-|\zeta|^2)^{-kq}|\nabla(1-|\zeta|^2)^{kq}|^2\\ &\qquad +\frac{1}{2}\nabla(1-|\zeta|^2)^{kq}\cdot\nabla|\Phi|^q\\ =\ &(1-|\zeta|^2)^{kq}|\nabla|\Phi|^{q/2}|^2+\frac{1}{4}|\Phi|^q\left(\triangle(1-|\zeta|^2)^{kq}+kq(1-|\zeta|^2)^{kq}\right)\\ &+\frac{1}{2}\left(\nabla\cdot((1-|\zeta|^2)^{kq}\nabla|\Phi|^q)-(1-|\zeta|^2)^{kq}\triangle|\Phi|^q\right).\end{aligned}$$

We notice that the divergence term, when integrated with respect to the invariant measure $d\nu(\zeta)=\left(\frac{2k-1}{\pi}\frac{1}{(1-|\zeta|^2)^2}\right)$ yields a vanishing surface integral for $kq>2$.

Also,

$$\frac{1}{4}|\Phi|^2\triangle(1-|\zeta|^2)^{kq} = \frac{1}{4}\left(\nabla\cdot(|\Phi|^2\nabla(1-|\zeta|^2)^{kq}) - \nabla|\Phi|^2\cdot\nabla(1-|\zeta|^2)^{kq}\right).$$

We can ignore the divergence terms coming from the expression above again by the same logic as before and write

$$\frac{1}{4}\int|\Phi|^q\triangle(1-|\zeta|^2)^{kq}d\nu(\zeta) = \frac{1}{4}\int(1-|\zeta|^2)^{kq}\triangle|\Phi|^q d\nu(\zeta).$$

Putting these all together we finally arrive at:

$$\begin{aligned}\int|\nabla|\mathcal{L}\psi(\zeta)|^{q/2}|^2 d\nu(\zeta) &= \int(1-|\zeta|^2)^{kq}\left(|\nabla|\Phi|^{q/2}|^2 - \frac{1}{4}\triangle|\Phi|^q\right)d\nu(\zeta) \\ &+ \frac{1}{4}kq\int|\Phi|^q(1-|\zeta|^2)^{kq}d\nu(\zeta).\end{aligned}$$

The first term on the right hand side in the equation above, vanishes due to analyticity of Φ as we have already shown, yielding the following identity:

$$\int|\nabla|\mathcal{L}\psi(\zeta)|^{q/2}|^2 d\nu(\zeta) = \frac{1}{4}kq\int|\mathcal{L}\psi(\zeta)|^q d\nu(\zeta).$$

□

3.2 A Sharp Sobolev inequality and a Norm Estimate

We now prove a sharp Sobolev inequality for functions in $\mathfrak{H}$.

Theorem 3.2.1. *For all functions in $\mathfrak{H}$ the following inequality holds:*

$$\|f\|_q^q + \frac{4}{kq(kq-2)}\int|\nabla|f|^{q/2}|^2 d\nu(\zeta) \geq \left(\frac{2k-1}{kq-1}\right)\left(\frac{kp-1}{2k-1}\right)^{q/p}\left(\frac{kq-1}{kq-2}\right)\|f\|_p^q \quad (3.1)$$

where $p = q + 1/k$, $q \geq 2$, $kq > 2$ and the norms are computed with respect to the measure $d\nu(\zeta)$; equality is obtained only when the function f comes from a coherent state.

Proof. Proving Theorem 3.2 is equivalent to showing that the infimum of the functional

$$I[f] = \frac{\|f\|_q^q + \frac{4}{kq(kq-2)}\int|\nabla|f|^{q/2}|^2 d\nu(\zeta)}{\left(\frac{kq-1}{kq-2}\right)\|f\|_p^q},$$

is $\left(\frac{2k-1}{kq-1}\right)\left(\frac{kp-1}{2k-1}\right)^{q/p}$. Since we are in the function space $\mathfrak{H}$, the existence of the infimum is obvious. Let us take a minimizing sequence $\{f_n\}$. We can now perform a radially symmetric decreasing rearrangement (cf.[Bae1],[Bae2]), since the gradient norm can only decrease under such a rearrangement while the other norms in the functional stay constant. So each function in the minimizing sequence is replaced by its decreasing rearrangement. Functions in the new sequence $\{f_n^*\}$ thus obtained also have bounded norms and gradient norms. The sequence being monotone and bounded we can use Helly's principle to obtain a convergent subsequence. Since the functions are in $W^{1,2}$, the convergence is in the s-norm, for all finite s, by Rellich-Kondrashov theorem. We thus need to show that in a class of radially symmetric solutions the minimizer is unique. The minimizer satisfies the following Euler-Lagrange equation for our optimization problem:

$$\triangle u + kq(kq-2)(\gamma u^{1+\frac{2}{kq}} - u) = 0\ , \tag{3.2}$$

where $u = |f|^{q/2}$, $\triangle$ is the Laplacian on the hyperbolic plane (or, equivalently, the unit disk), $\gamma > 0$ is fixed by choosing the p-norm of the function f. It is readily seen that this Euler-Lagrange equation is solved by the coherent state: $f = A(1-|\zeta|^2)^k$ where A is a constant determined by fixing the p-norm. Since we are dealing with radial functions only, (3.2) is equivalent to an ordinary differential equation. We now refer to section 5, where we prove in detail that there is only one solution of this ODE, in the space of radially symmetric functions on the unit disk, which decays to zero at the boundary of the disk (or, equivalently, decays to zero as the radial coordinate on the hyperbolic plane tends to infinity). On the basis of this uniqueness result we can conclude that the coherent state $f = A(1-|\zeta|^2)^k$ is indeed the unique solution and hence furnishes the infimum. □

This sharp Sobolev inequality, coupled with our Fisher information identity, yields the following corollary:

Corollary 3.2.2. *For all functions in $\mathfrak{F}_k$ the following inequality holds:*

$$||f||_q^q \geq \left(\frac{2k-1}{kq-1}\right)\left(\frac{kp-1}{2k-1}\right)^{q/p} ||f||_p^q \quad , \tag{3.3}$$

where $q \geq 2$; equality is obtained if and only if the function f is a coherent state.

Proof. The Fisher information identity for functions in $\mathfrak{F}_k$ tells us:

$$\int |\nabla |f|^{q/2}|^2 d\nu(\zeta) = \frac{1}{4} kq \int |f|^q d\nu(\zeta).$$

We can thus re-write the left hand side of (3.1) as:

$$\|f\|_q^q + \frac{4}{kq(kq-2)} \int \|\nabla |f|^{q/2}|^2 d\nu(\zeta) = \left(\frac{kq-1}{kq-2}\right) \|f\|_q^q.$$

So now our sharp Sobolev inequality yields the following norm estimate for functions in $\mathfrak{F}_k$:

$$\|f\|_q^q \geq \left(\frac{2k-1}{kq-1}\right)\left(\frac{kp-1}{2k-1}\right)^{q/p} \|f\|_p^q.$$

□

3.3 A Lower Bound for the Wehrl Entropy of functions in $\mathfrak{F}_k$

We now derive a lower bound for the entropy of functions in $\mathfrak{F}_k$.

Theorem 3.3.1. *The Wehrl entropy associated with $\mathcal{L}\psi(\zeta) \in \mathfrak{F}_k$ has a lower bound given by:*

$$S(|\mathcal{L}\psi(\zeta)|^2) \geq 2k \ln\left(1 + \frac{1}{2k-1}\right). \tag{3.4}$$

Proof. Let us define, for any function f, $\varphi(p) = \ln ||f||_p^p = \ln \int |f|^p$. Then, we have:

$$S(|f|^2) = -2\int |f|^2 \ln |f| = -2\varphi'(2),$$

if $||f||_2 = 1$. By logarithmic convexity of the p-norm:

$$-2\varphi'(2) \geq -2k\varphi\left(2 + \frac{1}{k}\right).$$

If we now set $q = 2, p = 2 + \dfrac{1}{k}$ in Corollary 3.3, we have:

$$\|\mathcal{L}\psi(\zeta)\|_{2+\frac{1}{k}}^{2+\frac{1}{k}} \leq \left(\frac{2k-1}{2k}\right),$$

since $\|\mathcal{L}\psi(\zeta)\|_2^2 = 1$, by definition. This implies, in $\mathfrak{F}_k$:

$$\varphi\left(2 + \frac{1}{k}\right) \leq \ln\left(\frac{2k-1}{2k}\right).$$

Thus:

$$\begin{aligned} -2\varphi'(2) &\geq -2k\varphi\left(2 + \frac{1}{k}\right) \\ \text{or,} \quad S(|\mathcal{L}\psi(\zeta)|^2) &\geq 2k\ln\left(1 + \frac{1}{2k-1}\right). \end{aligned}$$

$\square$

A comparison between (2.3) and (3.4) shows that the estimate obtained above has the conjectured high-spin asymptotics up to, but not including, first and higher order terms in (k^{-1}) because $\ln\left(1 + \dfrac{1}{2k-1}\right) = 2k\left(\dfrac{1}{2k-1} - \dfrac{1}{2}\dfrac{1}{(2k-1)^2} + ...\right)$. In fact this is completely analogous to the lower bound Bodmann [Bod] obtained for coherent state transforms on the sphere $\mathbb{S}^2$.

CHAPTER IV

ENTROPY-ENERGY INEQUALITIES ON THE HYPERBOLIC PLANE H^2

We say a Riemannian manifold $\mathcal{M}$ with measure $d\mathcal{M}$ admits a logarithmic Sobolev inequality with constant C if:

$$\int_{\mathcal{M}} |f|^2 \ln |f|^2 d\mathcal{M} \leq C \int_{\mathcal{M}} |\nabla f|^2 d\mathcal{M} \qquad \text{for all } f \text{ such that } \int_{\mathcal{M}} |f|^2 d\mathcal{M} = 1. \quad (4.1)$$

Since the Fisher information associated with a function is often regarded as an "energy", one can say that logarithmic Sobolev inequalities give a bound on the entropy of a function f in terms of its energy $E(f) = \displaystyle\int_{\mathcal{M}} |\nabla |f||^2 d\mathcal{M}$.

Even if C is the best possible constant in (4.1), this is only one of a whole family of sharp inequalities, and in many applications, use of the whole family leads to more incisive results.

To obtain this family of inequalities, one must determine, for each $A > 0$, the least value of B for which

$$\int_{\mathcal{M}} |f|^2 \ln |f|^2 d\mathcal{M} \leq A \int_{\mathcal{M}} |\nabla f|^2 d\mathcal{M} + B \quad \text{for all } f \text{ such that } \int_{\mathcal{M}} |f|^2 d\mathcal{M} = 1 \; (4.2)$$

is true. Call this optimal choice $B(A)$. If one then defines an increasing concave function Φ through

$$\Phi(t) = \inf_{A>0} \{\ At + B(A)\}\ ,$$

one has

$$\int_{\mathcal{M}} |f|^2 \ln |f|^2 d\mathcal{M} \leq \Phi(E(f))$$

for all f with $\displaystyle\int_{\mathcal{M}} |f|^2 d\mathcal{M} = 1$.

Conversely, given the optimal function $\Phi(t)$, $B(A)$ can be recovered: It is just the y–intercept of the tangent line to $y = \Phi(t)$ at the value of t for which $\Phi'(t) = A$.

Thus, determining an optimal entropy energy inequality is essentially equivalent to solving an "AB" type problem in the sense of Hebey [Heb]: obviously, if (4.2) holds for some A (that is, if, given some A, one can find a constant B such that (4.2) is valid), then it holds for all $A' \geq A$. Similarly, if (4.2) is valid for some B, it remains valid for all $B' \geq B$. Thus, it is natural to ask: what is the smallest constant A (or B) for which one can find a constant B (respectively, A) such that inequality (4.2) holds? In fact, these questions arise naturally whenever one has a Sobolev-type inequality on a Riemannian manifold [Heb]. The smallest A for which (4.2) holds is called the first best constant while the smallest such B is called the second best constant with respect to the inequality (4.2). Given any Sobolev-type inequality on some Riemannian manifold, Hebey associated two parallel research programs with the notion of best constants. The A-part of the program gives priority to the first best constant while the B-part is concerned with the second best constant.

On $\mathbb{R}^2$, the optimal entropy–energy function $\Phi_{\mathbb{R}^2}(t)$ is given by

$$\Phi_{\mathbb{R}^2}(t) = \ln\left(\frac{1}{\pi e} t\right) .$$

Thus:

$$\int_{\mathbb{R}^2} |f|^2 \ln |f|^2 \leq \ln\left(\frac{1}{\pi e} E(f)\right) .$$

Equality is achieved when f is an isotropic Gaussian function. For an appropriate choice of the variance of the Gaussian, the energy $E(f)$ can take any value, so this inequality is sharp for all values of $E(f)$.

In the case of $\mathbb{H}^2$, Beckner proved [Bec2] that the entropy has the same bound as in $\mathbb{R}^2$, i.e.,

$$\int_{\mathbb{H}^2} |f|^2 \ln |f|^2 \leq \ln\left(\frac{1}{\pi e} E(f)\right) .$$

In other words,

$$\Phi_{\mathbb{H}^2} \leq \Phi_{\mathbb{R}^2} \ .$$

This result is asymptotically sharp for small t as explained in the introduction. However, the inequality is actually strict, and significantly so, for large t. Here we prove an improved bound:

For $t > 0$, define $\Phi^\star(t)$ by

$$\Phi^\star(t) = \inf_{k\in\mathbb{N}} \left\{ \frac{1}{2} \ln \left[\left(\frac{2k-2}{2k-1} \right)^{2k+1} \left(\frac{2k-1}{2k} \right)^{2k} \left(\frac{2k-1}{4\pi} \right) \left(1 + \frac{1}{k(k-1)} t \right)^{2k+1} \right] \right\} \ .$$

Notice that this is an infimum over a a family of increasing, concave functions. As such, it is increasing and concave.

While we cannot explicitly evaluate the infimum that defines $\Phi^\star(t)$, we have the following result:

Theorem 4.0.2. *For all $t > 0$,*

$$\Phi_{\mathbb{H}^2} \leq \Phi^\star(t) < \Phi_{\mathbb{R}^2} \ .$$

Proof. We start from the sharp Sobolev inequality proved in Theorem 3.2, re-written in terms of the standard measure derived from the Poincaré metric. Recall that the measures $d\mu$ and $d\nu$ are related via: $d\nu = \dfrac{2k-1}{4\pi} d\mu$.

If we rescale f in inequality (3.1) so as to make it L^2-normalized in the measure $d\mu$ and rewrite the inequality with respect to $d\mu$, we get:

$$\int f^p d\mu \leq \left(\frac{kq-1}{2k-1} \right)^{p/q} \left(\frac{2k-1}{kp-1} \right) \left(\frac{kq-2}{kq-1} \right)^{p/q} \left(\frac{2k-1}{4\pi} \right)^{p-q/q} \left[\int f^q d\mu \right.$$
$$\left. + \frac{4}{kq(kq-2)} \int |\nabla f^{q/2}|^2 d\mu \right]^{p/q} .$$

Putting $q = 2, \quad p = 2 + 1/k$ and using the logarithmic convexity of the p-norm as in the proof of theorem 3.4, we obtain the following estimate:

$$\int f^2 \ln f d\mu \leq \frac{1}{2} \ln \left[\left(\frac{2k-2}{2k-1}\right)^{2k+1} \left(\frac{2k-1}{2k}\right)^{2k} \right. \tag{4.3}$$
$$\left. \times \left(\frac{2k-1}{4\pi}\right) \left(1 + \frac{1}{k(k-1)} \int |\nabla f|^2 d\mu \right)^{2k+1} \right].$$

Since this holds for every k, we get an entropy–energy inequality by taking the infimum over k, and this amounts to the inequality $\Phi_{\mathbb{H}^2} \leq \Phi^\star(t)$.

It remains to show that $\Phi^\star(t) < \Phi_{\mathbb{R}^2}$. We shall do this using the equivalent A–B form of the inequality. To make the tangent line computation and subsequent comparison with $\Phi_{\mathbb{R}^2}$, and hence Beckner's estimate, we note that, (4.3) implies:

$$\int f^2 \ln f^2 d\mu \quad \leq \quad 2k \ln \left(\frac{k-1}{k}\right) + \ln \left(\frac{k-1}{2\pi}\right) + \frac{2k+1}{k(k-1)} \int |\nabla f|^2 d\mu. \tag{4.4}$$

Now Beckner's inequality [Bec2] on the upper half plane is:

$$\int |f|^2 \ln |f| d\mu \leq \frac{1}{2} \ln \left[\frac{1}{\pi e} \int |\nabla |f||^2 d\mu \right]. \tag{4.5}$$

Since the logarithm is a concave function of its argument, $\dfrac{\ln x - \ln x_0}{x - x_0} < \dfrac{1}{x_0}$, where $x > x_0$. If we put $x = \int |\nabla f|^2 d\mu$ in (4.5), we obtain the following inequality:

$$\int f^2 \ln f^2 d\mu \leq \frac{1}{x_0} \int |\nabla f|^2 d\mu + \ln x_0 - \ln \pi - 2. \tag{4.6}$$

Inequalities (4.4) and (4.6) have the form $\int f^2 \ln f^2 d\mu \leq C_\epsilon + \epsilon \int |\nabla f|^2 d\mu$. We would like to see how the values for the intercept C_ϵ compare for a given value of the slope ϵ. Let C_{x_0} and C_k denote the intercepts for the inequalities parametrized by x_0 and k respectively. Now, to make the comparison let us put $\dfrac{1}{x_0} = \dfrac{2k+1}{k(k-1)}$. Then, for this value of x_0 we have:

$$\begin{aligned} C_{x_0} = \ln x_0 - \ln \pi - 2 &= \ln k + \ln(k-1) - \ln(2k+1) - \ln \pi - 2 \\ &= -\ln \left(1 + \frac{1}{2k}\right) - \ln 2\pi - 2 + \ln(k-1) \\ &= -\left[\frac{1}{2k} + \frac{1}{2}\left(\frac{1}{2k}\right)^2 + \ldots\right] + \ln(k-1) - \ln 2\pi - 2. \end{aligned}$$

On the other hand:

$$
\begin{aligned}
C_k &= 2k \ln\left(\frac{k-1}{k}\right) + \ln\left(\frac{k-1}{2\pi}\right) \\
&= \ln(k-1) - \ln 2\pi - 2 - \frac{1}{k} - \frac{2}{3}\left(\frac{1}{k}\right)^2 - \frac{1}{2}\left(\frac{1}{k}\right)^3 - \ldots \quad .
\end{aligned}
$$

Thus, for $x_0 = \dfrac{k(k-1)}{2k+1}$, we have: $C_{x_0} - C_k = \dfrac{1}{2k} + \dfrac{13}{24}\dfrac{1}{k^2} + \ldots$. This means that the logarithmic Sobolev inequality (4.4) actually gives an improvement on Beckner's inequality (4.6) as regards the second best constant and $\Phi^\star(t) < \Phi_{\mathbb{R}^2}$. $\square$

Another way to see the extent to which $\Phi^\star$ is a better estimate of $\Phi_{\mathbb{H}^2}$ than is $\Phi_{\mathbb{R}^2}$ is to use them both to estimate the entropy of our coherent state transforms, since for these, $E(f) = \dfrac{k}{2} > \dfrac{k(k-1)}{2k+1}$.

Inserting the value $E(f) = \dfrac{k}{2}$ into $\Phi_{\mathbb{R}^2}$ we obtain, using Beckner's estimate with respect to the measure $d\nu(\zeta)$:

$$
-\int |f|^2 \ln |f|^2 d\nu \geq 1 - \ln\left(\frac{2k}{2k-1}\right),
$$

while inserting this value into $\Phi^\star$ (with respect to measure $d\nu(\zeta)$) yields the marginally better bound (3.4).

CHAPTER V

THE UNIQUENESS THEOREM

As shown in section 3.2 of chapter 3, the Fisher information identity leads to a sharp Sobolev inequality. In order to prove the sharpness of this Sobolev inequality, we must show that the Euler-Lagrange equation (that is, equation (3.2)) corresponding to this variational problem, has a unique ground state solution. In this chapter we study equation (3.2) written in terms of the radial hyperbolic coordinate. Similar equations in $\mathbb{R}^n$ have been investigated in the past (cf. [Pel1], [Pel2], [McLe] and [Kwo]). Our case is significantly different and here we adapt the methods described in [Kwo] to the hyperbolic setting.

We investigate the question of uniqueness of ground state solution of the equation

$$u'' + \coth\tau u' + h(u) = 0. \tag{5.1}$$

where $\tau \in (0, \infty)$ on the two-dimensional hyperbolic plane. The function $h(u)$ is given by: $h(u) = \tilde{a}u^{1+\frac{2}{kq}} - \tilde{b}u$, where $\tilde{b} = kq(kq-2)$ and $\tilde{a} = \gamma kq(kq-2)$. The boundary conditions on the solutions of interest are: $\lim_{\tau\longrightarrow\infty} u(\tau) = 0$ and $u'(0) = 0$. There exist three points ξ_0, ξ_1 and ξ_2 in $(0, \infty)$ such that:

$$\int_{u=0}^{\xi_0} h(u)du = 0; \qquad \int_{u=0}^{v} h(u)du < 0 \text{ for } v < \xi_0 \quad \text{and}$$
$$\int_{u=0}^{v} h(u)du > 0 \text{ for } v > \xi_0.$$

$$h(\xi_1) = 0; \qquad h(u) < 0 \quad \text{if} \quad u < \xi_1 \quad \text{and} \quad h(u) > 0 \text{ if } u > \xi_1.$$
$$h'(\xi_2) = 0; \qquad h'(u) < 0 \quad \text{if} \quad u < \xi_2 \quad \text{and} \quad h'(u) > 0 \text{ if } u > \xi_2.$$

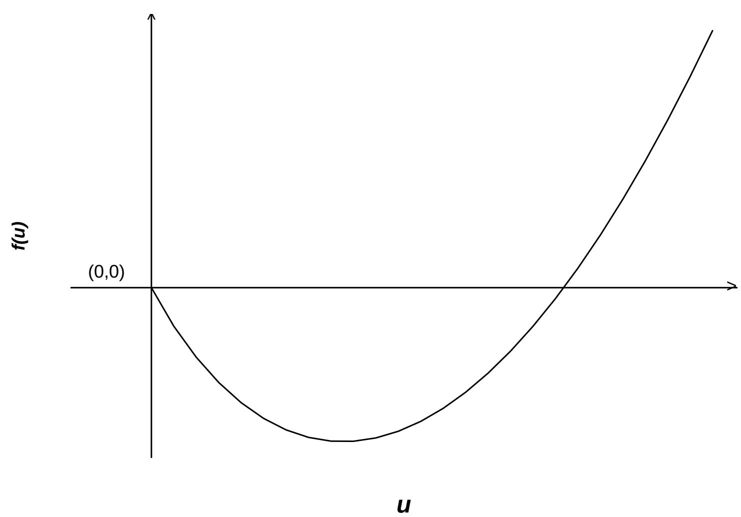

Figure 1: The function $h(u)$

Following [McLe] and [Kwo], let us consider u as a function of the initial value α and τ, and study, in stead of the boundary value problem mentioned above, the following initial value problem:

$$u'' + \coth \tau u' + h(u) = 0, \tag{5.2}$$
$$u(0) = \alpha > 0, \qquad u'(0) = 0.$$

We first divide the set of solutions into three mutually disjoint subsets, namely:

1. Solutions that have a zero at some finite τ. We call the corresponding set of initial values N. We denote the finite zero as $b(\alpha)$.

2. Positive solutions that satisfy $\lim_{\tau\to\infty} u(\tau) = 0$. We call the set of initial values G in this case.

3. Solutions that remain positive and do not belong to case 2. We let P denote the set of initial values for such solutions.

For a particular solution $u \in G \cup N$, we let τ_1 denote the zero of $h(u)$, that is to say, $u(\tau_1) = \xi_1$ (it is possible to define this point uniquely because, as we shall show momentarily, solutions $u \in G \cup N$ are monotone). Our subsequent results rely heavily on Sturm's comparison theorem (as mentioned in [lemma 1, [Kwo]] and also in chapter X, page 229 of [Inc]) and a few important corollaries that we state below.

Consider two second order differential equations:

$$U''(x) + f(x)U'(x) + g(x)U(x) = 0, \qquad x \in (a, b) \tag{5.3}$$

$$V''(x) + f(x)V'(x) + G(x)V(x) = 0, \qquad x \in (a, b) \tag{5.4}$$

Suppose that (5.3) has solutions that do not vanish in a neighborhood of point b. Then the largest neighborhood of b, (c, b), on which there exists a solution of (5.3) without any zero, is called the disconjugacy interval of (5.3). Sturm's theorem implies that no non-trivial solution can have more than one zero in (c, b). A corollary (lemma 6, [Kwo]) of Sturm's theorem is: if (c, ∞) is the discongugacy interval of (5.3), as defined above, then every solution of (5.3) with a zero in (c, ∞) is unboounded. We also have another very useful corollary (lemma 3, [Kwo]) of Sturm's theorem: if the equations (5.3) and (5.4) satisfy the comparison condition $G(x) \geq g(x)$, U is not identically equal to V in any neighborhood of b and there exists a solution V of (5.4) with a largest zero at $\rho \in (a, b)$, then the disconjugacy interval of (5.3) is a strict superset of (ρ, b).

We are now ready to state and prove our results. But first let us briefly outline our strategy in a few steps, since the proof of uniqueness is rather involved:

1. The first two lemmas state well-known facts about the structure of the sets N, P and G. As we increase α from 0 we first have solutions in P. Since the arguments are exactly similar to those used for the Euclidean case in [Kwo], we refer to the relevant lemmas in [Kwo], in stead of reiterating the proofs.

2. Next we study the variation w of a solution $u \in G \cup N$ with respect to its initial value. Lemma 5.4 states that w has to change sign before the point $u = \xi_1$. The proof of uniqueness depends crucially on the properties of w. Lemma 5.5 shows that if, for $\alpha \in G$ $\lim_{\tau \longrightarrow \infty} w(\alpha, \tau) = -\infty$, then a right neighborhood of α belongs to N. Also, if $\alpha \in N$ and $w(\alpha, b(\alpha)) < 0$, then a neighborhood of α belongs to N as well. Suppose these hypotheses are indeed true. As we continuously increase α, we shall first have solutions in P. The right boundary point will belong to G. If the corresponding α satisfies the hypothesis of lemma 5.5, then a right neighborhood of the corresponding α will be in N. Then, if for all $\alpha \in N$, $w(\alpha, b(\alpha)) < 0$, we would continue to remain in N as we increase α further. Thus the proof of uniqueness of the ground state will be complete. Hence we just need to prove that for $\alpha \in G$, $\lim_{\tau \longrightarrow \infty} w(\alpha, \tau) = -\infty$, while for $\alpha \in N$, $w(\alpha, b(\alpha)) < 0$. In fact, if we can prove that w has only one zero for initial values in $G \cup N$ and w is unbounded for initial values in G, uniqueness will be guaranteed. Initial values satisfying these two conditions are called strict admissible.

3. To prove that w can have no more than one zero and that it is unbounded, we construct a comparison function v for w.The zero of w is then shown to belong to the disconjugacy interval of the differential equation satisfied by w, which in turn implies unboundedness of w. The idea of constructing a comparison

function like this was used in [Kwo] to prove uniqueness of positive solutions of a semi-linear Poisson equation in a bounded or unbounded annular region in $\mathbb{R}^n$, for $n > 1$. It is in this crucial step, right after lemma 5.5 in this paper, that our proof of uniqueness differs from that of [Kwo]. This happens because we are dealing with a semi-linear Poisson equation on the hyperbolic plane $\mathbb{H}^2$. The difference in geometry manifests itself in the form of the comparison function and, more importantly, in the subsequent analysis. Proofs of lemma 5.6 through lemma 5.8 are thus specific to the hyperbolic case. As we go along we point out these differences in detail.

The main result of this section is:

Theorem 5.0.3. *The initial value $\alpha \in G \cup N$ is strictly admissible.*

Let us construct an "energy" function corresponding to (5.2):

$$E(\tau) = \frac{u'^2(\tau)}{2} + \frac{\tilde{a}u^{2+\frac{2}{kq}}}{2+\frac{2}{kq}} - \frac{\tilde{b}u^2}{2}.$$

It is readily seen that $E'(\tau) = -\coth\tau u'^2(\tau) \leq 0$. Thus E is a non-increasing function of τ.

Lemma 5.0.4. *The set $(0, \xi_0]$ of initial values belongs to the set P. [lemma 8, [Kwo]]*

For solutions in N, the function E decreases to a positive constant while for solutions in G, $E(\infty) = 0$. This fact leads us to the following lemma:

Lemma 5.0.5. *If $u \in G \cup N$, then $u'(\tau) < 0$ in $(0, b(\alpha))$ (if $u \in b(\alpha)$) or $(0, \infty)$ (if $u \in G$). [lemma 11, [Kwo]]*

The fact that the sets N and P are open subsets of $(0, \infty)$ [lemma 13, [Kwo]; lemma 1.1, [Ber]] is crucial but easy to observe.

We concern ourselves only with solutions that are either in G or in N . Let us define: $w = w(\tau, \alpha) = \left.\dfrac{\partial u}{\partial \alpha}\right|_{\tau,\alpha}$. We study the function w for such solutions. First of all let us note that $w = 0$ means two nearby solutions (i.e. solutions having nearby initial values) can intersect.

Evidently w satisfies the following equation (the derivatives are taken with respect to τ):

$$w'' + \coth \tau w' + h'(u)w = 0 \tag{5.5}$$

$$w(0) = 1, \qquad w'(0) = 0.$$

We first prove that w has its first zero before ξ_1.

Lemma 5.0.6. *For $u \in G \cup N$, w has to change sign before ξ_1. [lemma 17, [Kwo]]*

Following Kwong, we call the initial value $\alpha \in G$ strictly admissible if the corresponding $w(\alpha, \tau)$ has only one zero in $(0, \infty)$ and $\lim_{\tau \longrightarrow \infty} w(\alpha, \tau) = -\infty$. We call the initial value $\alpha \in N$ strictly admissible if the corresponding $w(\alpha, \tau)$ has only one zero in $(0, \infty)$ and $w(\alpha, b(\alpha)) < 0$.

It is easy to see that if for a particular $\alpha \in N$, $w(b(\alpha)) = \dfrac{\partial u}{\partial \alpha}(b(\alpha), \alpha) < 0$, then in a right neighborhood of α, $b(\alpha)$ is a strictly decreasing function of α and thus that neighborhood belongs to N.

Lemma 5.0.7. *If for $\alpha \in G$, $\lim_{\tau \longrightarrow \infty} w(\alpha, \tau) = -\infty$, in particular if $w(\alpha, \tau)$ is strictly admissible, then there exists a right neighborhood of α that belongs to N. [lemma 19, [Kwo]]*

We now need to prove that every initial value $\alpha \in G \cup N$ is strictly admissible. The strategy is to construct a comparison function $v(\tau)$ (to be compared with w), which has the following properties:

1. $v(\tau)$ has only one zero in $(0, \infty)$.

   and

2. $v(\tau)$ is a strict Sturm majorant of $w(\alpha, \tau)$ in both $(0, \rho)$ and (ρ, ∞), where ρ is the first zero of $w(\alpha, \tau)$.

If we are able to construct such a function, then by property (2) the zero of v occurs before that of w and by property (1) w cannot have another zero in $(0, b(\alpha))$. Here $b(\alpha)$ is the zero of the solution $u \in G \cup N$. If $u \in G$ then $b(\alpha)$ is to be interpreted as the point $\tau = \infty$. If $b(\alpha)$ is finite then of course the corresponding u is in N and $w(\alpha, b(\alpha)) < 0$, i.e., α is strictly admissible. On the other hand if $b(\alpha) = \infty$, w has a zero in the disconjugacy interval of v, and hence in the disconjugacy interval of the differential equation satisfied by w itself. This happens because w being a strict Sturm minorant of v in $(0, \infty)$, the disconjugacy interval of (5.3) is bigger than that of the differential equation satisfied by v. This means w is unbounded. Hence the corresponding α is strictly admissible.

It is helpful to first construct an auxiliary function $\theta(\tau)$ and then use it to deduce that v has the necessary properties described above. In the Euclidean case [Kwo], the auxiliary function $\theta(r)$ is given by: $\theta(r) = -\dfrac{ru'(r)}{u(r)}$. For the hyperbolic case we define the auxiliary function for all solutions $u \in G \cup N$ as:

$$\theta(\tau) = \frac{-\sinh \tau u'(\tau)}{u(\tau)}. \tag{5.6}$$

The auxiliary functions and the comparison functions in the Euclidean and hyperbolic cases have different forms but similar properties. Thus lemmas that follow are basically hyperbolic analogues of lemmas proved by Kwong in the Euclidean case.

The function $\theta(\tau)$ is obviously continuous in $(0, \infty)$ for $u \in G$; for $u \in N$ $\theta(\tau)$ is continuous in $(0, b(\alpha))$ where $b(\alpha)$ is the zero of $u(\alpha)$.

Lemma 5.0.8. *For solutions $u \in G \cup N$, $\theta(0) = 0$ and $\lim_{\tau \longrightarrow b(\alpha)} \theta(\tau) = \infty$. If $u \in N$ $b(\alpha)$ is interpreted to be the zero of u and if $u \in G$, $b(\alpha) = \infty$.*

Proof. The first claim is easy to verify since for all $u \in G \cup N$, $u'(0) = 0$; since $u'(\tau) < 0$, $\theta(\tau) > 0$ in $(0, \infty)$.

For $u \in N$, $u'(b(\alpha)) \neq 0$ and the second assertion of the lemma automatically follows.

Let us consider the case: $u \in G$.

Let $R = -\dfrac{u'}{u}$.

Then $R \geq 0$ and $R' = -\dfrac{u''}{u} + \dfrac{u'^2}{u^2} = R^2 - R\coth\tau + \dfrac{h(u)}{u}$.

Now we know that $\lim\limits_{\tau \longrightarrow \infty} \dfrac{h(u)}{u} = -\tilde{b}$. We assert that for large values of τ we would always have: $R(\tau) > \sqrt{\dfrac{\tilde{b}}{2}}$. If not, then $R(\tau) \leq \sqrt{\dfrac{\tilde{b}}{2}}$ for some τ. Then:

$R'(\tau) = R^2 - \coth\tau R + \dfrac{h(u)}{u} < R^2 + \dfrac{h(u)}{u} \leq -\dfrac{\tilde{b}}{2}$.

Thus R' will remain strictly and hugely negative eventually causing R to change sign.

Thus $-\dfrac{u'(\tau)}{u(\tau)} > \sqrt{\dfrac{\tilde{b}}{2}}$ for large values of τ. This in turn means $\lim_{\tau \longrightarrow \infty} \theta(\tau) = \infty$. $\square$

We next define the comparison function $v_\beta(\tau) = \sinh\tau u' + \beta u$ (in the Euclidean case it is defined as $v_\beta(r) = ru'(r) + \beta u(r)$) . It is readily seen that $v_\beta(\tau) = (<,>)0$ if and only if θ intersects (is above, is below) the straight line $y(\tau) = \beta$. Also, $v_\beta(\tau)$ is tangent to the τ-axis at some point $\hat{\tau}$ if and only if $\theta(\tau)$ is tangent to the straight line $y(\tau) = \beta$ at $\hat{\tau}$.

The function $v_\beta(\tau)$ satisfies the following differential equation:

$$v'' + \coth\tau v' + h'(u)v = \Phi(\tau) = \beta(uh'(u) - h(u)) - 2\cosh\tau h(u) \qquad (5.7)$$

$$v(0) > 0, \quad v'(0) = 0$$

Now,

$$\begin{aligned}
\Phi &= \beta\left(uh'(u) - h(u)\right) - 2\cosh\tau h(u) \\
&= \beta\left[\tilde{a}\left(1+\frac{2}{kq}\right)u^{1+\frac{2}{kq}} - \tilde{b}u - \tilde{a}u^{1+\frac{2}{kq}} + \tilde{b}u\right] - 2\cosh\tau h(u) \\
&= \frac{2}{kq}\beta\tilde{a}u^{1+\frac{2}{kq}} - 2\cosh\tau h(u).
\end{aligned}$$

It is not really obvious that one can choose a β such that Φ has only one zero and the position of that zero has a continuous dependence on β. However our next lemma proves that this can indeed be achieved.

Lemma 5.0.9. *There exists some $\bar{\beta}$ such that for $0 < \beta < \bar{\beta}$ the function $\Phi(u, \tau)$ has only one zero, say at $\tau = \sigma$ in $(0, \infty)$ such that:*

$$\begin{aligned}
\Phi(u,\tau) &< 0 \qquad \textit{for } \tau < \sigma \\
\Phi(u,\tau) &> 0 \qquad \textit{for } \tau > \sigma.
\end{aligned}$$

The point σ is a continuous monotone function of β.

Proof. First, we note that $\Phi(\tau) > 0$ in $[\tau_1, \infty)$ by definition; so its zeros must be concentrated in $(0, \tau_1)$. At a zero of the function Φ we have:

$$\frac{2}{kq}\beta\tilde{a}u^{1+\frac{2}{kq}} \qquad = 2\cosh\tau h(u).$$

Thus at $\Phi = 0$ we have:

$$\begin{aligned}
\Phi' &= \beta\frac{2}{kq}\tilde{a}\left(1+\frac{2}{kq}\right)u^{\frac{2}{kq}}u' - 2\sinh\tau h(u) - 2\cosh\tau h'(u)u' \\
&= \left(1+\frac{2}{kq}\right)u'\frac{2}{kq}\beta\tilde{a}u^{\frac{2}{kq}} - 2\sinh\tau h(u) - 2\cosh\tau h'(u)u' \\
&= \left(1+\frac{2}{kq}\right)\frac{2\cosh\tau h(u)}{u}u' - 2\cosh\tau h'(u)u' - 2\sinh\tau h(u) \\
&= \frac{2u'\cosh\tau}{u}\left[\left(1+\frac{2}{kq}\right)h(u) - uh'(u)\right] - 2\sinh\tau h(u).
\end{aligned}$$

So, if at $\Phi = 0$, $\Phi' > 0$, then:

$$\frac{2u' \cosh \tau}{u} \left[\left(1 + \frac{2}{kq} \right) h(u) - uh'(u) \right] > 2 \sinh \tau h(u)$$
$$\text{or} \ , \quad -\frac{2\tilde{b}}{kq} u' > \tanh \tau h(u),$$

which in turn implies

$$\frac{2\tilde{b}}{kq}(-\sinh \tau u') > \sinh \tau \tanh \tau h(u) \tag{5.8}$$

Similarly if $\Phi' < 0$ at $\Phi = 0$, then:

$$\frac{2\tilde{b}}{kq}(-\sinh \tau u') < \sinh \tau \tanh \tau h(u). \tag{5.9}$$

Now the differential equation (5.2) satisfied by u can be rewritten as:

$$(-\sinh \tau u')' = \sinh \tau h(u).$$

If at the first zero of the function $\Phi(\tau)$, $\Phi'(\tau) > 0$ then inequality (5.8) holds at that point and we also know that the left hand side of the inequality is positive and increasing at the rate $\left(\frac{2\tilde{b}}{kq}(-\sinh \tau u') \right)' = \frac{2\tilde{b}}{kq} \sinh \tau h(u)$. As for the right hand side, we have, in the interval $(0, \tau_1)$:

$$(\sinh \tau \tanh \tau h(u))' = \sinh \tau h(u) + \sinh \tau \text{sech}^2 \tau h(u) + \sinh \tau \tanh \tau h'(u) u' < 2 \sinh \tau h(u).$$

The inequality above holds because $h'(u) > 0$ in $(0, \tau_1)$ and $u' < 0$. Since in our case $\frac{2\tilde{b}}{kq} = 2(kq - 2)$ and k is chosen so that $kq > 1$, it turns out that $\frac{2\tilde{b}}{kq} \sinh \tau h(u) > 2 \sinh \tau h(u)$. This in turn implies that the left hand side of (5.8) increases more rapidly than the right hand side. So if inequality (5.8) holds at some point in $(0, \tau_1)$ then it prevails at all subsequent points in this interval. We can thus conclude that if $\Phi(0) < 0$, then $\Phi(\tau)$ can have only one zero in $(0, \tau_1)$.

Now for a particular solution having initial value α ,$\Phi(\tau = 0) =$

$\beta(\alpha f'(\alpha) - f(\alpha)) - 2f(\alpha)$. Putting in the specific form of $h(u)$ we obtain the condition that $\Phi(\tau)$ has a negative initial value:

$$\begin{aligned} \frac{\beta}{kq}\tilde{a}\alpha^{1+2/kq} &< \tilde{a}\alpha^{1+2/kq} - \tilde{b}\alpha \\ \text{or,} \quad \beta < kq &\left[1 - \frac{\tilde{b}}{\tilde{a}\alpha^{2/kq}}\right]. \end{aligned}$$

We let $\bar{\beta}$ denote the upper limit set on β by the condition above. Then for $\beta \in (0, \bar{\beta})$, the function $\Phi(\tau)$ has a negative initial value and consequently only one zero in $(0, \tau_1)$. We denote that zero by σ.

Let us now find out how σ depends on β. We have:

$$\frac{\tilde{a}\beta}{kq} = \cosh\sigma \frac{f(u(\sigma))}{u(\sigma)^{1+2/kq}}.$$

Evidently then β depends continuously on σ. Also:

$$\beta'(\sigma) = \frac{kq}{\tilde{a}} u^{-1-2/kq} \left[\frac{2\tilde{b}}{kq} u'(\sigma)\cosh\sigma + h(u)\sinh\sigma\right].$$

Now for $\beta \in (0, \bar{\beta})$, (5.8) holds at σ, as proved before. Thus $\left[\frac{2\tilde{b}}{kq} u'(\sigma)\cosh\sigma + h(u)\sinh\sigma\right] < 0$, and hence $\beta'(\sigma) < 0$ for all β in this range. This means there exists a continuous inverse function in a neighborhood of $\beta(\sigma)$. Thus σ depends continuously on β. In fact σ is a decreasing function of β. When $\beta = 0$ the only zero of $\Phi(\tau)$ is at τ_1. As we increase β the zero shifts continuously to the left. $\square$

Let ρ_β be the first zero of $v_\beta(\tau)$ (we do not yet know how many zeros v can have). Then for $\beta = 0$, $\rho = 0$. As we increase β, ρ_β moves to the right. In order to prove that we can control β such that ρ_β and σ_β can be made to coincide, we need to show that ρ_β continuously depends on β. We first show that actually, given any β, $v_\beta(\tau)$ can have only one zero and then prove the continuous dependence of that zero on the parameter β.

Lemma 5.0.10. *The function $v_\beta(\tau)$ has only one zero in $(0, \infty)$.*

Proof. In the interval $[0, \tau_1]$,

$$(-\sinh \tau u'(\tau))' = h(u) \sinh \tau \geq 0.$$

Thus $(-\sinh \tau u'(\tau))$ is non-decreasing in $[0, \tau_1]$. Since $u(\tau)$ is decreasing, $\theta(\tau) = \dfrac{-\sinh \tau u'(\tau)}{u(\tau)}$ is non-decreasing in $[0, \tau_1]$. Thus for any β it can intersect the straight line $y(\tau) = \beta$ no more than once in this interval and the corresponding $v_\beta(\tau)$ can have at most one zero.

Since $\lim_{\tau \longrightarrow \infty} \theta(\tau) = \infty$, if $\theta(\tau)$ is not non-decreasing in the entire interval (τ_1, ∞), then it has to have local minima. Suppose the lowest of all such minima occurs at ω and has height β_0. Then in (ω, ∞), $v_{\beta_0}(\tau)$ is negative and has a double zero at ω. Also $v_{\beta_0}(\tau)$ satisfies the following differential inequality in (ω, ∞):

$$v'' + \coth \tau v' + h'(u) v \geq 0.$$

But this is impossible (since, if v satisfies the second-order differential equation above, then it cannot have a double zero; cf. lemma 5, [Kwo]).

Thus we conclude that $\theta(\tau)$ is non-decreasing in $(0, \infty)$, which in turn implies that for any value of β, $v_\beta(\tau)$ can have only one zero in $(0, \infty)$. $\square$

To prove that one can choose β such that $\rho_\beta = \sigma_\beta$ it is sufficient to show that ρ_β as a function of β does not have any discontinuity in $(0, \tau_1)$. Since v_β has a zero at ρ_β if and only if θ intersects the straight line $y(\tau) = \beta$ at $\tau = \rho_\beta$, we just need to show $\theta'(\rho_\beta) \neq 0$. As shown in the preceding lemma, $\theta'(\tau) > 0$ in $(0, \tau_1)$. As we increase β, the height of the horizontal straight line $y(\tau) = \beta$ increases. This results in a continuous shift of the point of intersection ρ_β to the right. Thus we can conclude that in $(0, \tau_1)$ ρ_β is a continuous increasing function of β. For $\beta = 0$, $\rho = 0$ and $\sigma = \tau_1$. When we increase β, ρ_β moves continuously to the right even as σ_β shifts continuously to the left until it is at the origin $\tau = 0$ for $\beta = \bar{\beta}$, as shown before. It follows that there exists a $\beta_0 \in (0, \bar{\beta})$ for which we would have: $\rho_{\beta_0} = \sigma_{\beta_0}$. Let us then fix the parameter β by choosing that value β_0.

We are now in a position to prove Theorem 5.1.

Proof. Let us use $v_{\beta_0}(\tau)$ as a comparison function for $w(\tau)$. The differential equations to be compared are:

$$w'' + \coth\tau w' + h'(u)w = 0$$
$$w(0) = 1, \qquad w'(0) = 0,$$

and

$$v'' + \coth\tau v' + \left[h'(u) - \frac{\Phi(\tau)}{v}\right] v = 0$$
$$v(0) > 0, \qquad v'(0) = 0.$$

Since in $(0, \rho)$, $\Phi < 0$ and $v > 0$ the coefficient of v is larger than that of w. Thus v is a strict Sturm majorant of w and its zero ρ occurs before the first zero of w, say c. But at c, $\Phi > 0$ and $v < 0$, thus the coefficient of v is still larger than that of w. Moreover, since $w(c) = 0$, $\dfrac{w'(c)}{w(c)} = +\infty$ and $\dfrac{w'(c)}{w(c)} > \dfrac{v'(c)}{v(c)}$. Thus v again is a strict Sturm majorant of w. But v does not have a zero in $[c, \infty)$. Then w cannot have a zero in this interval either. So if $u \in N$ then $w(b(\alpha)) < 0$ and α is strictly admissible. Let us consider the case $u \in G$ now. Evidently, c belongs to the disconjugacy interval of (5.7). Since v is a strict Sturm majorant of w in $(0, \infty)$, the disconjugacy interval of (5.5) is a superset of the disconjugacy interval of (5.7). Thus w has a zero in the disconjugacy interval of the differential equation it satisfies. Hence it must be unbounded.

Thus for $u \in G \cup N$ the corresponding initial value is strictly admissible (and this ensures uniqueness of the corresponding solution, as shown before). □

CHAPTER VI

COHERENT STATE TRANSFORMS FOR THE GROUP $SU(P,Q)$ AND FISHER INFORMATION IDENTITIES FOR $SU(N,1)$ AND $SU(N,N)$

In this chapter, we prove analogs of the Fisher information identity proved in the case of $SU(1,1)$, for the groups $SU(n,1)$ and $SU(n,n)$. We mainly follow the construction of coherent states given in [Mon]. We first obtain an expression for coherent state transforms for the group $SU(p,q)$ for generic p and q. The representations for $SU(p,q)$ are constructed on "generalized unit disks" . To obtain the formulae for the metric tensors and differential operators on these domains, we follow [Sto], [Hua] and [Mit]. We then prove two new, interesting Fisher information identities for the groups $SU(n,1)$ and $SU(n,n)$.

Let us consider the realization of the group $SU(p,q)$ as the group of automorphisms of domain D, defined to be the space of $p \times q$ complex matrices Z satisfying:

$$I_q - Z^\dagger Z > 0,$$

where I_q is the $q \times q$ identity matrix. We write an element of $SU(p,q)$ as:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are $p \times p$, $p \times q$, $q \times p$ and $q \times q$ matrices respectively, such that:

$$AA^\dagger - BB^\dagger = I_p, \qquad AC^\dagger = BD^\dagger, \qquad DD^\dagger - CC^\dagger = I_q.$$

Throughout this note we will use the symbols A', $\bar{A}$ and $A^\dagger$ to denote the transpose, complex conjugate and conjugate transpose, respectively, of the matrix A. Let us choose any point Z of D. Consider the following action of a group element g of $SU(p,q)$:

$$\begin{pmatrix} Z' & I_q \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} Z'A + C & Z'B + D \end{pmatrix}.$$

We now define:

$$\begin{pmatrix} Z_g \\ I_q \end{pmatrix} = \begin{pmatrix} Z'A + C & Z'B + D \end{pmatrix}' (B'Z + D')^{-1}.$$

Thus, we say that the group $SU(p,q)$ acts on D according to the formula:

$$Z \longrightarrow Z.g = (A'Z + C')(B'Z + D')^{-1}.$$

With its action on D defined thus, $SU(p,q)$ indeed is the automorphism group of D. To verify this, we note that:

$$\begin{aligned}
I_q - Z_g^\dagger Z_g &= \begin{pmatrix} Z_g \\ I_q \end{pmatrix}^\dagger \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} Z_g \\ I_q \end{pmatrix} \\
&= ((B'Z + D')^\dagger)^{-1} \begin{pmatrix} A'Z + C' \\ B'Z + D' \end{pmatrix}^\dagger \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A'Z + C' \\ B'Z + D' \end{pmatrix} (B'Z + D')^{-1} \\
&= ((B'Z + D')^\dagger)^{-1} \begin{pmatrix} Z \\ I_q \end{pmatrix}^\dagger \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}^\dagger \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix} \begin{pmatrix} Z \\ I_q \end{pmatrix} \\
&\qquad \times (B'Z + D')^{-1} \\
&= ((B'Z + D')^\dagger)^{-1} (I_q - Z^\dagger Z)(B'Z + D')^{-1}.
\end{aligned}$$

The quantity in the last line is positive definite because $(I_q - Z^\dagger Z)$ is positive definite.

We now need to determine the invariant measure on D. We know the following invariant measure exists:

$$d\mu(z) = \rho(z, \bar{z}) \prod_{j=1,k=1}^{j=p,q=k} dx_{jk} dy_{jk}, \qquad z = (z_{11}, z_{12}, ..., z_{pq}) \in D,$$

where $z_{jk} = x_{jk} + iy_{jk}$, for $j = 1, 2, ..., p$, $k = 1, 2, ..., q$. For our case, z would be a point in the pq-dimensional domain D (so the point z is the matrix Z).

Let $J_g(z)$ denote the Determinant of the Jacobian matrix for the transformation $z \longrightarrow zg$. The invariance of the measure $d\mu(z)$ implies:

$$\rho(zg, \overline{zg}) = |J_g(z)|^{-2} \rho(z, \bar{z}).$$

We use the normalization condition $\rho(0,0) = 1$. Let $J_{g_1,z}(0)$ be the Jacobian determinant for the transformation that translates the origin to the point z. Then we have:

$$\rho(z, \bar{z}) = |J_{g_1,z}(0)|^{-2} \rho(0,0).$$

Then:

$$\begin{aligned} \rho(zg, \overline{zg}) &= |J_{g,zg}(z)|^{-2} \rho(z, \bar{z}) \\ &= |J_{g,zg}(z)|^{-2} |J_{g_1,z}(0)|^{-2} \rho(0,0) \end{aligned}$$

Again:

$$\rho(zg, \overline{zg}) = |J_{g_2,zg}(0)|^{-2} \rho(0,0).$$

Thus,

$$|J_{g,zg}(z)| = |J_{g_1,z}(0)|^{-1} |J_{g_2,zg}(0)|. \tag{6.1}$$

Equation 6.1 implies that in order to obtain the expressions for $J_{g,zg}(z)$ and $\rho(z, \bar{z})$ it suffices to compute only $J_g(0)$.

Let us calculate the Jacobian determinants for the transformation $z \longrightarrow zg$. Recall that:

$$Z_1 = Z.g = (A'Z + C')(B'Z + D')^{-1}, \quad i.e., \quad Z_1(B'Z + D') = (A'Z + C').$$

Differentiating the last equality at the origin (i.e. the point $Z = 0$), we get:

$$dZ_1 D' = (A' - C'(D')^{-1} B') dZ.$$

To determine the Jacobian determinant, let us think of Z as a pq-dimensional vector $(z_{11}, z_{12}, ..., z_{pq})$. Of course, to every matrix $Z \in D$, we can associate such a point uniquely. The transformation matrices would then be $pq \times pq$. For example the $p \times p$ matrix $(A' - C'(D')^{-1}B')$ would be represented as a $pq \times pq$ block-diagonal matrix where each block is the $p \times p$ matrix $(A' - C'(D')^{-1}B')$ and there are q such blocks. Similarly, the matrix D' is represented by a $pq \times pq$ block-diagonal matrix, where each of the p blocks is the $q \times q$ matrix D'. Then, we have:

$$J_{g,zg}(0) = [\det\ (A - BD^{-1}C)]^q [\det\ (D)]^{-p}.$$

But,

$$[\det\ (A - BD^{-1}C)] = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} (\det\ D)^{-1} = (\det\ D)^{-1}.$$

The last step follows because

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1.$$

Thus:

$$J_{g,zg}(0) = [\det\ D]^{-(p+q)}.$$

It is easily seen that the general transformation that translates $Z \longrightarrow 0$ has the form:

$$g = \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} I_p & -\bar{Z} \\ -Z' & I_q \end{pmatrix},$$

where

$$A_1^\dagger A_1 = (I_p - \bar{Z}Z')^{-1}, \quad D_1^\dagger D_1 = (I_q - Z'\bar{Z})^{-1}.$$

This in turn means that:

$$\rho(z, \bar{z}) = [\det\ D_1]^{2(p+q)} = [\det\ (I_q - Z'\bar{Z})^{-1}]^{p+q} = [\det\ [(I_q - Z^\dagger Z)]]^{-(p+q)}.$$

Similarly,

$$J_{g,zg}(z) = |J_{g_1,z}(0)|^{-1} J_{g_2,zg}(0) = [\det\ D_1]^{p+q}[\det\ D_2]^{-(p+q)},$$

where $J_{g_2,zg}(0) = [\det\ D_2]^{-(p+q)}$. Now, since

$$I_q - Z_g^\dagger Z_g = ((B'Z + D')^\dagger)^{-1}(I_q - Z^\dagger Z)(B'Z + D')^{-1},$$

we can take the determinant of both sides, and obtain:

$$[\det\ (I_q - Z_g^\dagger Z_g)][\det\ (I_q - Z^\dagger Z)]^{-1} = [|\det\ (B'Z + D')|]^{-2}.$$

Thus:

$$\begin{aligned} J_{g,zg}(z) &= [\det\ D_1]^{p+q}[\det\ D_2]^{-(p+q)} \\ &= [\det\ (I_q - Z_g^\dagger Z_g)^{-1}]^{-(p+q)/2}[\det\ (I_q - Z^\dagger Z)^{-1}]^{(p+q)/2} \\ &= [|\det\ (B'Z + D')|]^{-(p+q)}. \end{aligned}$$

Now let us consider an irreducible representation (cf. [Mon] and [Per]) of $SU(p,q)$ in the space of holomorphic functions on the domain D, where the group action is defined as:

$$T_g\psi(z) = [J_{g,zg}(z)]^k \psi(zg),$$

where k is an integral positive number. Let us introduce the norm of a function according to the formula

$$||\psi||_k^2 = \int |\psi(z)|^2 d\mu_k(z).$$

Unitarity of the representation then fixes the measure $d\mu_k(z)$ as:

$$d\mu_k(z) = [\rho(z,\bar{z})]^{-k} d\mu(z).$$

So, we consider the representation of $SU(p,q)$ in the space $\mathcal{F}_k$ of holomorphic functions on the domain D, that have finite norm with respect to the measure $d\mu_k$. Let us choose the function $\psi_0 \equiv 1$ as the 'least-weight vector' which would generate the coherent states. Clearly, the stability group for this function is $S(U(p) \times U(q))$, an element of which has the form

$$\begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix},$$

where $U_1 \in U(p)$, $U_2 \in U(q)$ and $(\det U_1)(\det U_2) = 1$. Let us choose a transformation g on the state $\psi_0(z)$, which takes a point $\zeta \in D$ to the origin and has the form

$$g = U\tilde{\zeta} = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} I_p & -\bar{\zeta} \\ -\zeta' & I_q \end{pmatrix}.$$

Thus the matrix $\tilde{\zeta}$ belongs to a left coset of $SU(p,q)$ by $S(U(p) \times U(q))$, and all such cosets are labeled by points in D. If we take a member of one such coset labeled by ζ and let it act on our 'least-weight' vector $\psi_0(z)$, we get:

$$\begin{aligned} \psi_\zeta(z) = T^k_\zeta \psi_0(z) &= [J_\zeta(z)]^k \\ &= [\det(I_q - \zeta^\dagger Z)]^{-(p+q)k} N(\zeta), \end{aligned}$$

where $N(\zeta)$ is the normalization factor which is easily seen to be $N(\zeta) = [\det\,(I_q - \zeta^\dagger\zeta)]^{(p+q)k/2}$. Thus the coherent states are given by:

$$\psi_\zeta = \frac{[\det\,(I_q - \zeta^\dagger\zeta)]^{(p+q)k/2}}{[\det(I_q - \zeta^\dagger Z)]^{(p+q)k}}.$$

Now, in the holomorphic, irreducible, unitary representations of $SU(p,q)$ the orthonormal basis functions are properly normalized homogeneous polynomials in the entries z_{ij} of the matrix Z, where $1 \leq i \leq p, \quad 1 \leq j \leq q$. Thus, given a unit vector $\phi(z) = \sum_{[m]} a_{[m]} z^{[m]}$, where $[m]$ denotes a multi-index, such that $z^{[m]} = z_{11}^{m_1} z_{12}^{m_2}...$ Evidently, the inner product of $\phi(z)$ and $\psi_\zeta(z)$, with respect to the measure $d\mu_k(z)$, is a function of the following form:

$$< \psi_\zeta(z)|\phi(z) >= \mathcal{L}\phi(\zeta) = [\det\,(I_q - \zeta^\dagger\zeta)]^{(p+q)k/2}\Phi(\bar{\zeta}),$$

where Φ is holomorphic in $\bar{\zeta}$. These coherent state transforms are normalized with respect to the norm

$$d\nu(\zeta) = N_k[\det(I_q - \zeta^\dagger\zeta)]^{-(p+q)} \prod_{j,k} d\zeta_{jk} d\bar{\zeta}_{jk},$$

where N_k is the appropriate normalization factor.

6.1 A Fisher Information Identity for $\mathbf{SU(n,1)}$ coherent states:

Let us now consider the special case of $SU(n,1)$. The complex bounded domain in this case is the unit ball in $\mathbb{C}^n$. The expressions obtained above, for the Bergman kernel $\rho(z,\bar{z})$, the invariant measure and the form of coherent states become easier to explicitly compute. Bergman's kernel in this case, is given by:

$$\rho(z,\bar{z}) = [\det(I_q - Z^\dagger Z)]^{-(n+1)} = \left(1 - \sum_k |z_k|^2\right)^{-(n+1)}.$$

The metric tensor is then obtained as:

$$\begin{aligned} h_{lj} = \frac{\partial^2}{\partial z_l \partial \bar{z}_j} \ln \rho(z,\bar{z}) &= -(n+1)\frac{\partial}{\partial z_l}\left(\frac{-z_j}{1 - \sum_k |z_k|^2}\right) \\ &= \frac{(n+1)}{(1 - \sum_k |z_k|^2)^2}\left[\left(1 - \sum_k |z_k|^2\right)\delta_{ij} + \bar{z}_l z_j\right]. \end{aligned}$$

It is easy to see that the inverse of the metric tensor is given by:

$$h^{lj} = \frac{1}{(n+1)}\left(1 - \sum_k |z_k|^2\right)[\delta_{lj} - \bar{z}_l z_j].$$

We are now in a position to compute the Laplacian Δ, given by:

$$\Delta = \frac{2}{h}\sum_{l,j}\left[\frac{\partial}{\partial \bar{z}_l}\left(h h^{ij}\frac{\partial}{\partial z_j}\right) + \frac{\partial}{\partial z_l}\left(h \bar{h}^{lj}\frac{\partial}{\partial \bar{z}_j}\right)\right],$$

where $h = \det\,(h_{lj})$. It is easy to compute that:

$$\begin{aligned}
& \sum_i \frac{\partial}{\partial \bar{z}_l}(hh^{lj}) \\
= & \sum_i \frac{\partial}{\partial \bar{z}_i}\left[(n+1)^{n-1}(\delta_{lj} - \bar{z}_l z_j)(1 - \sum_k |z_k|^2)^{-n}\right] \\
= & \sum_i (n+1)^{n-1}\left[-z_j(1 - \sum_k |z_k|^2)^{-n} + n(\delta_{lj} - \bar{z}_l z_j)(1 - \sum_k |z_k|^2)^{-n-1} z_i\right] \\
= & (n+1)^{n-1}(1 - \sum_k |z_k|^2)^{-n-1} \\
& \quad \left[n z_j(1 - |z_j|^2) - n z_j(1 - |z_j|^2) - n \sum_{l, l \neq j} |z_l|^2 z_j + n z_j \sum_{k, k \neq j} |z_k|^2\right] \\
= & 0.
\end{aligned}$$

A similar computation shows that $\sum_i \frac{\partial}{\partial z_l}(h\bar{h}^{lj}) = 0$. Thus, the Laplacian contains no linear term and is given by:

$$\begin{aligned}
\Delta &= \frac{2}{h}\sum_{l,j}\left[\frac{\partial}{\partial \bar{z}_l}\left(hh^{lj}\frac{\partial}{\partial z_j}\right) + \frac{\partial}{\partial z_l}\left(h\bar{h}^{lj}\frac{\partial}{\partial \bar{z}_j}\right)\right] \\
&= \frac{2}{h}\sum_{l,j}\left(hh^{lj}\frac{\partial^2}{\partial \bar{z}_l \partial z_j} + h\bar{h}^{lj}\frac{\partial^2}{\partial z_l \partial \bar{z}_j}\right).
\end{aligned}$$

Since the metric tensor h is hermitian, we have, for any function $U(z, \bar{z})$, which is twice continuously differentiable in its arguments:

$$\Delta U = 4\sum_{l,j} h^{lj}\frac{\partial^2}{\partial \bar{z}_l \partial z_j} U.$$

We define the gradient and inner product as follows, for real-valued functions f and g:

$$\nabla f \cdot \nabla g = 2\sum_{l,j} h^{lj}\left[\frac{\partial f}{\partial \bar{z}_l}\frac{\partial g}{\partial z_j} + \frac{\partial g}{\partial \bar{z}_l}\frac{\partial f}{\partial z_j}\right].$$

We can now prove a Fisher Information Identity for coherent state transforms of the group $SU(n,1)$. In what follows, we use ζ to denote a point in the open unit ball in $\mathbb{C}^n$.

Theorem 6.1.1. *For $SU(n,1)$ coherent transforms $\mathcal{L}\psi(\zeta)$ the following identity holds:*

$$\int |\nabla|\mathcal{L}\psi(\zeta)|^{q/2}|^2 d\nu(\zeta) = \frac{1}{4}kq \int |\mathcal{L}\psi(\zeta)|^q d\nu(\zeta),$$

where q is a positive number such that $kq > 2$.

Proof. We first notice that:

$$\begin{aligned}
\frac{\partial}{\partial \bar{\zeta}_l} e^{-\frac{kq}{4}\ln\rho} &= -\frac{kq}{4} e^{-\frac{kq}{4}\ln\rho}\left(\frac{\partial \ln\rho}{\partial \bar{\zeta}_l}\right). \\
\text{Thus, } \frac{\partial^2}{\partial \zeta_j \partial \bar{\zeta}_l} e^{-\frac{kq}{4}\ln\rho} &= \left(\frac{kq}{4}\right)^2 e^{-\frac{kq}{4}\ln\rho}\left(\frac{\partial}{\partial \zeta_j}\ln\rho\right)\left(\frac{\partial}{\partial \bar{\zeta}_l}\ln\rho\right) - \frac{kq}{4} e^{-\frac{kq}{4}\ln\rho}\frac{\partial^2}{\partial \zeta_j \partial \bar{\zeta}_l}\ln\rho \\
&= 4\left(\frac{\partial}{\partial \zeta_j}\rho^{\frac{-kq}{8}}\right)\left(\frac{\partial}{\partial \bar{\zeta}_l}\rho^{\frac{-kq}{8}}\right) - \frac{kq}{4}\rho^{-\frac{kq}{4}} h_{jl}.
\end{aligned}$$

Multiplying both sides by $4h^{lj}$ and summing over l, j we get:

$$\begin{aligned}
4\sum_{l,j} h^{lj}\frac{\partial^2}{\partial \bar{\zeta}_l \partial \zeta_j}\rho^{-\frac{kq}{4}} &= \Delta\rho^{-\frac{kq}{4}} \\
&= \sum_{l,j}\left[-kq\rho^{-\frac{kq}{4}} h^{lj} h_{jl} + 16 h_{lj}\left(\frac{\partial}{\partial \zeta_j}\rho^{\frac{-kq}{8}}\right)\left(\frac{\partial}{\partial \bar{\zeta}_i}\rho^{\frac{-kq}{8}}\right)\right] \\
&= -kq\rho^{-\frac{kq}{4}} + 4|\nabla\rho^{-\frac{kq}{8}}|^2.
\end{aligned}$$

We know that a coherent state transform has the form:

$$\mathcal{L}\phi(\zeta) = \rho^{-\frac{k}{2}}\Phi(\zeta),$$

where Φ is holomorphic in each of the variables ζ_l. This means, Φ can be represented as: $\Phi = u + iv$, where u and v are real functions which obey the Cauchy-Riemann equations such that, for any particular duplet $\zeta_l = (x_l, y_l), \zeta_j = (x_j, y_j)$ we have:

$$\begin{aligned}
\frac{\partial u}{\partial x_l} &= \frac{\partial v}{\partial y_l}, & \frac{\partial u}{\partial x_j} &= \frac{\partial v}{\partial y_j}, \\
\frac{\partial u}{\partial y_l} &= -\frac{\partial v}{\partial x_l}, & \frac{\partial u}{\partial y_j} &= -\frac{\partial v}{\partial x_j}.
\end{aligned}$$

Thus:

$$\frac{\partial^2}{\partial \bar{z}_l \partial z_j} u = \frac{\partial^2}{\partial \bar{z}_l \partial z_j} v = 0.$$

It is easy to see that the Cauchy-Riemann equations imply:

$$\Delta u = \Delta v = 0 \quad , \quad |\nabla u|^2 = |\nabla v|^2 \quad \text{and } \nabla u \cdot \nabla v = 0.$$

Thus:

$$\begin{aligned} |\nabla |\Phi|^q|^2 &= |\nabla (u^2 + v^2)^{q/2}|^2 \\ &= q^2 (u^2 + v^2)^{q-2} \left[u^2 |\nabla u|^2 + v^2 |\nabla v|^2 + 2uv \nabla u \cdot \nabla v \right] \\ &= q^2 (u^2 + v^2)^{q-1} |\nabla u|^2. \end{aligned}$$

A similar computation shows,

$$\Delta |\Phi|^q = q^2 |\Phi|^{q-2} |\nabla u|^2 = 4 |\nabla |\Phi|^{q/2}|^2.$$

Thus:

$$\begin{aligned} \int |\nabla |\mathcal{L}\phi(\zeta)|^{q/2}|^2 N_k d\nu(\zeta) &= N_k \int |\nabla \rho^{-kq/4} |\Phi|^{q/2}|^2 d\nu(\zeta) \\ &= \frac{1}{4} \int \left[\Delta \rho^{-kq/2} + 2kq \rho^{-kq/2} \right] |\Phi|^q N_k d\nu(\zeta) \\ &+ \frac{1}{4} \int \rho^{-kq/2} \Delta |\Phi|^q N_k d\nu(\zeta) \\ &+ \frac{1}{2} \int \left(\nabla \cdot \left(\rho^{-kq/2} \nabla |\Phi|^q \right) - \rho^{-kq/2} \Delta |\Phi|^q \right) N_k d\nu(\zeta) \\ &= \frac{1}{2} kq \int \rho^{-kq/2} |\Phi|^q N_k d\nu(\zeta). \end{aligned}$$

Here we have made use of the fact that the divergence term in the third line vanishes, when integrated with respect to the measure $d\nu(z)$, for $kq > 2$.

Thus, we obtain:

$$\int |\nabla |\mathcal{L}\phi(\zeta)|^{q/2}|^2 N_k d\nu(\zeta) = \frac{1}{4} kq \int |\mathcal{L}\phi(\zeta)|^q d\nu(\zeta).$$

□

6.2 A Fisher Information Identity for $\mathbf{SU(n,n)}$ coherent states:

In case of $SU(n,n)$, the complex bounded domain is the generalized unit disk in $\mathbb{C}^{n^2}$. The metric is given by:

$$ds^2 = \sum_{i,j,k,l} h_{ijkl} dZ_{ij} d\bar{Z}_{kl},$$

where

$$h_{ijkl} = -2n \frac{\partial^2}{\partial Z_{ij} \partial \overline{Z}_{kl}} \ln \det(I - Z^\dagger Z),$$

where $Z^\dagger$ and Z are $n \times n$ nonsingular matrices such that $I - Z^\dagger Z > 0$. Thus:

$$\begin{aligned}
\sum_{i,j,k,l} h_{ijkl} dZ_{ij} d\overline{Z}_{kl} &= -2n \sum_{i,j,k,l} \frac{\partial^2}{\partial Z_{ij} \partial \overline{Z}_{kl}} \ln \det(I - Z^\dagger Z) dZ_{ij} d\overline{Z}_{kl} \\
&= -2n \sum_{i,j} \frac{\partial}{\partial Z_{ij}} \left[\bar{\partial} (\ln \det(I - Z^\dagger Z)) \right] dZ_{ij} \\
&= -2n \sum_{i,j} \frac{\partial}{\partial Z_{ij}} \left[\frac{1}{\det(I - Z^\dagger Z)} \bar{\partial} \det(I - Z^\dagger Z) \right] dZ_{ij}.
\end{aligned}$$

Now, for any nonsingular matrix A:

$$\bar{\partial}(\det A) = \mathrm{Tr}((\mathrm{adjoint} A)\bar{\partial} A) = \det A \mathrm{Tr}(A^{-1} \bar{\partial} A),$$

where Tr denotes the trace. Thus we can write:

$$\begin{aligned}
& \sum_{i,j} \frac{\partial}{\partial Z_{ij}} \left[\frac{1}{\det(I - Z^\dagger Z)} \bar{\partial} \det(I - Z^\dagger Z) \right] dZ_{ij} \\
= \; & \partial \left[-\mathrm{Tr} \left((I - Z^\dagger Z)^{-1} \bar{\partial} (Z^\dagger Z) \right) \right] \\
= \; & -\partial \left[\mathrm{Tr} \left(dZ^\dagger Z (I - Z^\dagger Z)^{-1} \right) \right] \\
= \; & -\mathrm{Tr} \left[\partial \left(dZ^\dagger Z (I - Z^\dagger Z)^{-1} \right) \right] \\
= \; & -\mathrm{Tr} \left[dZ^\dagger dZ (I - Z^\dagger Z)^{-1} + dZ^\dagger Z (I - Z^\dagger Z)^{-1} Z^\dagger dZ (I - Z^\dagger Z)^{-1} \right] \\
= \; & -\mathrm{Tr} \left[dZ^\dagger \left(I + Z (I - Z^\dagger Z)^{-1} Z^\dagger \right) dZ (I - Z^\dagger Z)^{-1} \right].
\end{aligned}$$

We now notice that:

$$\left(I + Z(I - Z^\dagger Z)^{-1} Z^\dagger\right) = (I - ZZ^\dagger)^{-1}.$$

Thus, we have:

$$h_{ijkl} dZ_{ij} d\overline{Z}_{kl} = (2n)\mathrm{Tr}\left[dZ^\dagger (I - ZZ^\dagger)^{-1} dZ (I - Z^\dagger Z)^{-1}\right],$$

and

$$ds^2 = 2n\ dZ^\dagger_{ij}((I - ZZ^\dagger)^{-1})_{jk} dZ_{kl}((I - Z^\dagger Z)^{-1})_{li},$$

where the subscripts denote matrix elements and repeated indices are summed over. Thus:

$$h_{ijkl} = 2n((I - ZZ^\dagger)^{-1})_{ki}((I - Z^\dagger Z)^{-1})_{jl}.$$

It is easy to see that, since $h_{ijkl} h^{ijpq} = \delta^p_k \delta^q_l$, by definition, where repeated indices are summed over, we have the following expression for h^{ijkl}:

$$h^{ijkl} = \frac{1}{2n}(I - Z^\dagger Z)_{lj}(I - ZZ^\dagger)_{ik}.$$

Then the Laplacian for any real-valued function V is given by:

$$\begin{aligned}
\Delta V &= \frac{2}{h}\sum_{ijkl}\left[\frac{\partial}{\partial \overline{Z}_{kl}}\left(h h^{ijkl}\frac{\partial V}{\partial Z_{ij}}\right) + \frac{\partial}{\partial Z_{kl}}\left(h \bar{h}^{ijkl}\frac{\partial V}{\partial \overline{Z}_{ij}}\right)\right] \\
&= \frac{4}{h}\mathrm{Re}\left[\frac{\partial}{\partial \overline{Z}_{kl}}\left(h h^{ijkl}\frac{\partial V}{\partial Z_{ij}}\right)\right],
\end{aligned}$$

where $h = \det\, h_{ijkl}$. We note that, since the metric is given as the direct product of two matrices,

$$h = \left(\det\ ((I - ZZ^\dagger)\right)^{-n}\left(\det\ ((I - Z^\dagger Z)\right)^{-n} = \left(\det\ ((I - Z^\dagger Z)\right)^{-2n}.$$

Let us now evaluate the part $L(\Delta V)$ of the Laplacian of V, that contains only linear terms. We have:

$$L(\Delta V) = 4\left(\det\ ((I - ZZ^\dagger)\right)^{2n} Re\left[\left(\frac{\partial}{\partial \overline{Z}_{ij}} h h^{ijkl}\right)\frac{\partial V}{\partial Z_{kl}}\right].$$

Now:

$$\begin{aligned} & 2n\left[\left(\frac{\partial}{\partial \overline{Z}_{kl}} hh^{ijkl}\right)\frac{\partial V}{\partial Z_{ij}}\right] \qquad (6.2)\\ = & \left[\frac{\partial}{\partial \overline{Z}_{kl}}\left(\det(I - Z^\dagger Z)\right)^{-2n}\right](I - Z^\dagger Z)_{lj}(I - ZZ^\dagger)_{ik}\frac{\partial V}{\partial Z_{ij}} \\ + & \left(\det(I - Z^\dagger Z)\right)^{-2n}\left[\frac{\partial}{\partial \overline{Z}_{kl}}(I - Z^\dagger Z)_{lj}(I - ZZ^\dagger)_{ik}\right]\frac{\partial V}{\partial Z_{ij}}. \end{aligned}$$

In what follows we will not use Einstein's convention of summation over repeated indices, rather we will show the summations explicitly, for clarity. We have:

$$\bar{\partial}\ \det(I - Z^\dagger Z) \;=\; -\det(I - Z^\dagger Z)\mathrm{Tr}\left((I - Z^\dagger Z)^{-1}\bar{\partial}(Z^\dagger Z)\right),$$

which implies,

$$\begin{aligned} \frac{\partial}{\partial \overline{Z}_{kl}}\det(I - Z^\dagger Z) &= -\det(I - Z^\dagger Z)\sum_{p,q,r}\left[((I - Z^\dagger Z)^{-1})_{pq}\frac{\partial Z^\dagger_{qr}}{\partial \overline{Z}_{kl}}Z_{rp}\right] \\ &= -\det(I - Z^\dagger Z)\sum_{p,q,r}\left[((I - Z^\dagger Z)^{-1})_{pq}\delta_{kr}\delta_{lq}Z_{rp}\right] \\ &= -\det(I - Z^\dagger Z)\sum_{p}\left[((I - Z^\dagger Z)^{-1})_{pl}Z_{kp}\right]. \end{aligned}$$

Thus:

$$\begin{aligned} & \sum_{ijkl}\left(\frac{\partial}{\partial \overline{Z}_{kl}}\left(\det(I - Z^\dagger Z)\right)^{-2n}\right)(I - Z^\dagger Z)_{lj}(I - ZZ^\dagger)_{ik}\frac{\partial V}{\partial Z_{ij}} \\ = & (2n)\det(I - Z^\dagger Z)^{-2n}\sum_{i,j,p,k,l} Z_{kp}((I - Z^\dagger Z)^{-1})_{pl}(I - Z^\dagger Z)_{lj}(I - ZZ^\dagger)_{ik}\frac{\partial V}{\partial Z_{ij}} \\ = & (2n)\det(I - Z^\dagger Z)^{-2n}\sum_{i,j,p,k} Z_{kp}\delta_{pj}(I - ZZ^\dagger)_{ik}\frac{\partial V}{\partial Z_{ij}} \\ = & (2n)\det(I - Z^\dagger Z)^{-2n}\sum_{i,j,k}(I - ZZ^\dagger)_{ik}Z_{kj}\frac{\partial V}{\partial Z_{ij}} \\ = & (2n)\det(I - Z^\dagger Z)^{-2n}\mathrm{Tr}\left[(I - ZZ^\dagger)Z\left(\frac{\partial V}{\partial Z_{ij}}\right)'\right], \end{aligned}$$

where $\left(\frac{\partial V}{\partial Z_{ij}}\right)'$ denotes the transpose of the matrix $\left(\frac{\partial V}{\partial Z_{ij}}\right)$. Now:

$$\begin{aligned}
&\left(\det(I - Z^\dagger Z)\right)^{-2n} \sum_{ijkl} \left[\frac{\partial}{\partial \overline{Z}_{kl}}(I - Z^\dagger Z)_{lj}(I - ZZ^\dagger)_{ik}\right] \frac{\partial V}{\partial Z_{ij}} \\
= \;& -\left(\det(I - Z^\dagger Z)\right)^{-2n} \sum_{ijkl} \left[Z_{kj}(I - ZZ^\dagger)_{ik}\frac{\partial V}{\partial Z_{ij}} + (I - Z^\dagger Z)_{lj} Z_{il}\frac{\partial V}{\partial Z_{ij}}\right] \\
= \;& -n\left(\det(I - Z^\dagger Z)\right)^{-2n} \sum_{ij} \left[\left((I - ZZ^\dagger)Z\right)_{ij}\frac{\partial V}{\partial Z_{ij}} + \left(Z(I - Z^\dagger Z)\right)_{ij}\frac{\partial V}{\partial Z_{ij}}\right] \\
= \;& -2n\left(\det(I - Z^\dagger Z)\right)^{-2n} \operatorname{Tr}\left[Z(I - Z^\dagger Z)\left(\frac{\partial V}{\partial Z_{ij}}\right)'\right].
\end{aligned}$$

Hence, referring back to equation 6.2 we can see that:

$$\left[\left(\frac{\partial}{\partial \overline{Z}_{kl}} h h^{ijkl}\right)\frac{\partial V}{\partial Z_{ij}}\right] = 0,$$

with summation over repeated indices and this, in turn implies

$$L(\Delta V) = 0.$$

Thus the Laplacian of a real-valued function is given by:

$$\Delta V = h^{ijkl}\frac{\partial^2}{\partial \overline{Z}_{kl}\partial Z_{ij}} V.$$

We define the inner product of the gradients of two real-valued functions u and v as:

$$\nabla u \cdot \nabla v = 2h^{ijkl}\left[\frac{\partial u}{\partial \overline{Z}_{kl}}\frac{\partial v}{\partial Z_{ij}} + \frac{\partial v}{\partial \overline{Z}_{kl}}\frac{\partial u}{\partial Z_{ij}}\right].$$

Now we prove a Fisher information identity for $SU(n,n)$ coherent states. As in the case of $SU(n,1)$, in what follows, we denote a point inside the generalized unit disk, by ζ.

Theorem 6.2.1. *For $SU(n,n)$ coherent transforms $\mathcal{L}\psi(\zeta)$, the following identity holds:*

$$\int |\nabla|\mathcal{L}\psi(\zeta)|^{q/2}|^2 d\nu(\zeta) = \frac{1}{4}kq\int |\mathcal{L}\psi(\zeta)|^q d\nu(\zeta),$$

where q is a positive number such that $kq > 2$.

Proof. We first notice that:

$$\frac{\partial e^{-\frac{kq}{4}\ln\rho}}{\partial\overline{\zeta}_{lm}} = -\frac{kq}{4}e^{-\frac{kq}{4}\ln\rho}\left(\frac{\partial\ln\rho}{\partial\overline{\zeta}_{lm}}\right).$$

Thus:

$$\begin{aligned}
&\frac{\partial^2}{\partial\zeta_{jk}\partial\overline{\zeta}_{lm}}e^{-\frac{kq}{4}\ln\rho} \\
&= -\frac{kq}{4}\frac{\partial}{\partial\zeta_{jk}}\left[e^{-\frac{kq}{4}\ln\rho}\left(\frac{\partial\ln\rho}{\partial\overline{\zeta}_{lm}}\right)\right] \\
&= \left(\frac{kq}{4}\right)^2 e^{-\frac{kq}{4}\ln\rho}\left(\frac{\partial\ln\rho}{\partial\zeta_{jk}}\right)\left(\frac{\partial\ln\rho}{\partial\overline{\zeta}_{lm}}\right) - \frac{kq}{4}e^{-\frac{kq}{4}\ln\rho}\left(\frac{\partial^2\ln\rho}{\partial\zeta_{jk}\partial\overline{\zeta}_{lm}}\right) \\
&= \left(\frac{kq}{4}\right)^2 e^{-\frac{kq}{4}\ln\rho}\left(\frac{\partial\ln\rho}{\partial\zeta_{jk}}\right)\left(\frac{\partial\ln\rho}{\partial\overline{\zeta}_{lm}}\right) - \frac{kq}{4}e^{-\frac{kq}{4}\ln\rho}h_{jklm}.
\end{aligned}$$

Multiplying both sides by $4h^{jklm}$, we obtain:

$$\begin{aligned}
\triangle\rho^{\frac{-kq}{4}} &= 4h^{jklm}\frac{\partial^2}{\partial\zeta_{jk}\partial\overline{\zeta}_{lm}}e^{-\frac{kq}{4}\ln\rho} \\
&= 16h^{jklm}\left(\frac{\partial\rho^{-\frac{kq}{8}}}{\partial\zeta_{jk}}\right)\left(\frac{\partial\rho^{-\frac{kq}{8}}}{\partial\overline{\zeta}_{lm}}\right) - kq\rho^{-\frac{kq}{4}}.
\end{aligned}$$

Thus:

$$\triangle\rho^{\frac{-kq}{4}} = 4|\nabla\rho^{-\frac{kq}{8}}|^2 - kq\rho^{-\frac{kq}{4}}. \tag{6.3}$$

We know that a coherent state transform has the form:

$$\mathcal{L}\phi(\zeta) = \rho^{-\frac{k}{2}}\Phi(\zeta),$$

where Φ is holomorphic in each of the variables ζ_{lm}. This means, Φ can be represented as: $\Phi = u + iv$, where u and v are real functions which obey the Cauchy-Riemann equations such that, for any particular duplet $\zeta_{lm} = (x_{lm}, y_{lm}), \zeta_{jk} = (x_{jk}, y_{jk})$ we have:

$$\begin{aligned}
\frac{\partial u}{\partial x_{lm}} &= \frac{\partial v}{\partial y_{lm}}, & \frac{\partial u}{\partial x_{jk}} &= \frac{\partial v}{\partial y_{jk}}, \\
\frac{\partial u}{\partial y_{lm}} &= -\frac{\partial v}{\partial x_{lm}}, & \frac{\partial u}{\partial y_{jk}} &= -\frac{\partial v}{\partial x_{jk}}.
\end{aligned}$$

As in the previous case (for $SU(n,1)$), the Cauchy-Riemann equations guarantee that:

$$|\nabla u|^2 = |\nabla v|^2 \quad , \quad \nabla u \cdot \nabla v = 0 \quad \text{and,} \quad \triangle u = \triangle v = 0.$$

Hence: Thus:

$$\begin{aligned} |\nabla |\Phi|^q|^2 &= |\nabla (u^2+v^2)^{q/2}|^2 \\ &= q^2(u^2+v^2)^{q-2}\left[u^2|\nabla u|^2 + v^2|\nabla v|^2 + 2uv\nabla u \cdot \nabla v\right] \\ &= q^2(u^2+v^2)^{q-1}|\nabla u|^2. \end{aligned}$$

Also,

$$\Delta|\Phi|^q = q^2|\Phi|^{q-2}|\nabla u|^2 = 4|\nabla|\Phi|^{q/2}|^2. \tag{6.4}$$

Thus:

$$\begin{aligned} & \int |\nabla|\mathcal{L}\phi(\zeta)|^{q/2}|^2 N_k d\nu(\zeta) \\ = & \int ||\Phi|^{q/2}\nabla\rho^{-\frac{kq}{4}} + \rho^{-\frac{kq}{4}}\nabla|\Phi|^{q/2}|^2 d\nu(\zeta) \\ = & \int |\Phi|^q|\nabla\rho^{-\frac{kq}{4}}|^2 d\nu(\zeta) + \int \rho^{-\frac{kq}{2}}|\nabla|\Phi|^q|^2 d\nu(\zeta) \\ & +\frac{1}{2}\int \nabla\cdot\left(\rho^{-\frac{kq}{2}}\nabla|\Phi|^q\right) d\nu(\zeta) - \frac{1}{2}\int \rho^{-\frac{kq}{2}}\triangle|\Phi|^q d\nu(\zeta) \\ = & \frac{1}{4}\int \rho^{\frac{-kq}{2}}|\Phi|^q d\nu(\zeta), \end{aligned}$$

Here we have made use of the fact that the divergence term in the fourth line vanishes, when integrated with respect to the measure $d\nu(z)$, for $kq > 2$. This implies:

$$\int |\nabla|\mathcal{L}\psi(\zeta)|^{q/2}|^2 d\nu(\zeta) = \frac{1}{4}kq \int |\mathcal{L}\psi(\zeta)|^q d\nu(\zeta).$$

□

A Remark about Future Research: Our results regarding the coherent state transforms for the groups $SU(n,1)$ and $SU(n,n)$ suggest the possibility of deriving sharp Sobolev inequalities on the unit ball in $\mathbb{C}^n$ and the generalized unit disk in $\mathbb{C}^{n^2}$. Exploring this is a significant part of our plan for future research.

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