PROBLEMS AND RESULTS IN PARTIALLY ORDERED SETS, GRAPHS AND GEOMETRY

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PROBLEMS AND RESULTS IN PARTIALLY ORDERED SETS, GRAPHS AND GEOMETRY

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To my family,

my wife, Melinda and my children, Esther and Flora

without their love and understanding I could not have made this achievement.

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SUMMARY

The thesis consist of three independent parts. In the first part, we investigate the height sequence of an element of a partially ordered set. Let x be an element of the partially ordered set P. Then $h_i(x)$ is the number of linear extensions of P in which x is in the ith lowest position. The sequence $\{h_i(x)\}$ is called the height sequence of x in P. Stanley proved in 1981 that the height sequence is log-concave, but no combinatorial proof has been found, and Stanley's proof does not reveal anything about the deeper structure of the height sequence. In this part of the thesis, we provide a combinatorial proof of a special case of Stanley's theorem. The proof of the inequality uses the Ahlswede–Daykin Four Functions Theorem.

In the second part, we study two classes of segment orders introduced by Shahrokhi. Both classes are natural generalizations of interval containment orders and interval orders. We prove several properties of the classes, and inspired by the observation, that the classes seem to be very similar, we attempt to find out if they actually contain the same partially ordered sets. We prove that the question is equivalent to a stretchability question involving certain sets of pseudoline arrangements. We also prove several facts about continuous universal functions that would transfer segment orders of the first kind into segments orders of the second kind.

In the third part, we consider the lattice whose elements are the subsets of $\{1, 2, \ldots, n\}$. Trotter and Felsner asked whether this subset lattice always contains a monotone Hamiltonian path. We make progress toward answering this question by constructing a path for all n that satisfies the monotone properties and covers every set of size at most 3. This portion of thesis represents joint work with David M. Howard.

CHAPTER I

INTRODUCTION

In this chapter, we introduce the basic concepts and background material for this thesis. In the first section we give a very brief overview of the key definitions in graph theory. In the second section, we give a slightly more detailed introduction to the theory of partially ordered sets. We will mention the basic results and examples to which we refer later in this work.

1.1 Basics of graphs and partially ordered sets

1.1.1 Graphs

A *graph* is a ordered triple of sets (V, E, I), where $I \subseteq V \times E$ is an incidence relation between elements of V (which are called *vertices* or less frequently *points*) and the elements of E (which are called *edges* or less frequently *arcs*) with the restriction that every edge is in relation with one or two vertices. An edge that is incident with only one vertex is called a *loop*. Two edges that are incident with the same set of vertices are called *parallel edges*. If a graph has no loops and parallel edges, we call it a *simple* graph. In this case we often refer to G as the ordered pair (V, E), where $E \subseteq V \times V$. Here the two components of an $e \in E$ are the two vertices that the edge is incident with. From now on we will only deal with simple graphs with finitely many vertices, and we will make use of the second notation for graphs. We will also use $V(G)$ to denote the set of vertices of G and $E(G)$ to denote the set of edges of G.

We say two vertices v_1 and v_2 are adjacent (in notation $v_1 \sim v_2$), if $(v_1, v_2) \in E$. Let G and H be two graphs, and let $f : V(G) \to V(H)$ be a function that preserves adjacency: for $u, v \in V(G)$, $u \sim v$ if and only if $f(u) \sim f(v)$. We say that two graphs are *isomorphic.* We do not differentiate isomorphic graphs in the theory. For example, if a graph has n vertices, and every vertex is adjacent to every other vertex, we call that graph K_n, *the* complete graph on n vertices, when in fact we talk about a isomorphy class of graphs.

A sequence of vertices $v_1, \dots, v_k$ is called a *walk*, if $v_1 \sim v_2 \sim \cdots \sim v_k$. A walk with distinct vertices is called a *path*. A sequence of vertices $v_1, \dots, v_k$ is called a cycle, if $v_1 = v_k$ and there are no other repeated vertices, i.e. the walk $v_1, \dots, v_{k-1}$ is a path. A cycle is *Hamiltonian*, if every vertex of the graph appears among the vertices of the cycle. Similarly, a path is Hamiltonian, if it covers every vertex of the graph.

If a graph has a Hamiltonian cycle it obviously has a Hamiltonian path, but not vice versa. If a graph has a Hamiltonian path, we call it a *traceable* graph, if it also has a cycle, we say it is *Hamiltonian* graph.

The questions: "Is G traceable?" and "Is G Hamiltonian?" are computationally difficult: they are NP-complete. These question are not only important for this reason: finding Hamiltonian or "almost" Hamiltonian paths in graphs is one of the flourishing areas of combinatorial optimization. Nevertheless there are some simple examples. Let V be the subsets of the set $\{1, \dots, n\}$. Let two vertices u and v be adjacent, if $u \subseteq v$ and $|u|+1 = |v|$ or if $v \subseteq u$ and $|v| + 1 = |u|$. The resulting graph is frequently called the n-dimensional hypercube. It is easy to show that the n-dimensional hypercube is Hamiltonian for all $n \geq 2$.

A function $c : V(G) \to \mathbb{N}$ is called a *coloring* if $c(u) = c(v)$ implies $u \not\sim v$, in other words, adjacent vertices never get the same color. The least number of colors necessary to color a graph G is called the *chromatic number* of G, denoted by $\chi(G)$. For a fixed $k \geq 3$, to determine if $\chi(G) \leq k$ is an NP-complete problem. On the other hand, for $k = 2$, the question is polynomial: $\chi(G) \leq 2$ if and only if it contains no odd cycles.

Graphs with large chromatic numbers obviously exist: $\chi(K_n) = n$. However, a graph does not have to contain a large K_n to have large chromatic numbers. There are K_3-free graphs with arbitrarily high chromatic number.

1.1.2 Partially ordered sets

Let A be an arbitrary set. A *relation* R on is a subset of $A \times A$. Some basic characteristics of relations are the following.

- If $(a, a) \in R$ for all $a \in A$, then R is *reflexive*.

- If $(a,a) \notin R$ for any $a \in A$, then R is *irreflexive*.
- If $(a,b) \in R$ implies $(b,a) \in R$, then R is *symmetric*.
- If $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$, then R is *antisymmetric*.
- If $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$, then R is *transitive*.

A relation that is reflexive and transitive is called a *quasiorder*. If a quasiorder is also symmetric, we say it is *equivalence*. If a quasiorder is also antisymmetric, we says it is a *partial order*. In a partial order R, if $(a,b) \in R$, we say a is *less than or equal to* b. We frequently write aRb, $a \leq_R b$ or simply $a \leq b$ is it does not cause confusion. If $(a,b) \in R$ but $a \neq b$, then we say a is *less than* b, and write $a <_R b$ or simply $a < b$. If either $(a,b) \in R$ or $(b,a) \in R$, we say a and b are *comparable*, other we call them *incomparable*. To denote the latter, we can write $a\|_R b$ or $a\|b$.

Irreflexivity and transitivity together imply antisymmetry. This has an interesting consequence that makes it possible to define partial orders in a slightly different way. Suppose that S is an irreflexive and transitive relation and let $R = S \cup \{(a,a) : a \in A\}$. Then R is a partial order. Some authors call an irreflexive and transitive relation a *strict partial order*. In this way, a partial order is strict partial order with all the diagonal pairs added.

A *partially ordered set* or *poset*[1] P is a pair (A,R), where A is a set (called the *ground set*) and R is a partial order on the set. Sometimes, to make notations simpler, we will write $x \in P$, when in fact we mean $x \in A$. Thereby we will not need to name the ground set and the relation.

We say x is *covered* by y or y *covers* x in P (notation: $x <: y$), if $x < y$ and there is no $z \in P$ such that $x < z < y$.

Let $P = (A,R)$ be a poset and let $P' = (A', R')$ such that $A' \subseteq A$ and for all $a,b \in A'$ $a <'_R b$ if and only if $a <_R b$. We say P' is a *subposet* of P.

An element a of a poset P is called *maximal* (*minimal*), if for all $x \in P$, $x \not> a$ ($x \not< a$). An element a is called *maximum* (*minimum*), if for all $x \in P$, $x \leq a$ ($x \geq a$). A poset

[1] The name "order" is also commonly used. We discourage the usage of the ambiguous name. However, this explains the names of some subclasses of posets, like "interval orders".

can only have at most one maximum and at most one minimum element, and they are also maximal and minimal respectively. However, there may be several maximal or minimal elements, that are not maximum or minimum.

A subposet X of P is called a *chain*, if any two elements of X are comparable. A subposet X is called an *antichain*, if no two elements of X are comparable. A chain C is *maximal*, if no chain of P contains X. A chain C is *maximum*, if it is of maximum cardinality. Minimal and minimum antichains can be defined in a similar way. The cardinality of a maximum chain is called the *height* of the poset and the cardinality of a maximum antichain is called the *width* of the poset. Posets that are chains themselves are called *total orders* or *linear orders*.

One of the most classical theorem of posets is by Dilworth [8].

Theorem 1.1.1. *Every poset of width w can be partitioned into w chains. Dually, every poset of height h can be partitioned into h antichains.*

These are in fact two quite different statements, and they are not equivalent. The dual version is very easy, and the first statement is what is usually referred as Dilworth's Theorem. It belongs to an elegant family of combinatorial theorems, all with a common linear programming flavor; these include Hall's Theorem, the Kőnig–Egerváry Theorem, Menger's Theorem and Tutte's Theorem.

Let P and Q be two posets on the same ground set. Suppose that $a <_P b$ implies $a <_Q b$. We say Q is an *extension* of P. If Q is a linear order, we say Q is a *linear extension* of P. The linear extensions are frequently denoted by enumerating the elements as an ordered sequence from the least to the greatest. The number of linear extensions of a poset P will be denoted by $e(P)$.

It is not completely obvious that every poset has a linear extension, and in fact this statement is equivalent to the axiom of choice (see [24]). For finite posets, the axiom of choice is not necessary. The proof is based on the elementary fact that if $x \| y$, then there is a linear extension in which $x < y$. Then we can find a sequence of extension, each of them contains one fewer incomparable pair than the previous one, and since there are only finitely many pairs, we must end at a linear extension.

It is frequently useful to visualize posets. Let $P = (A, R)$. In an affine plane fix a straight directed line and call its direction "up". (This is equivalent of a Cartesian plane with the x-axis removed.) To each point of A, assign a point of the plane in such manner, that for $a, b \in A$, $a < b$, the point assigned to b is higher than the point assigned to a. It is clear that this condition is satisfied if the order of the points upwards form a linear extension of P, therefore, it is always possible to make such an assignment. Now connect the points a and b with a straight line segment, if and only if $a <: b$. The set of points and segments together is called the *Hasse diagram* of P. It is also not unusual to draw the Hasse diagram of a poset with curves between the points instead of straight line segments.

The Hasse diagram also defines a graph in a natural way. The vertex set is the set of points, and two vertices a and b are adjacent, if $a <: b$.

This form of visualization explains the name of the following sets. For an element $a \in A$, the set $U(x) = \{x \in A : a < x\}$ is called the *upset* of a, and the set $D(x) = \{x \in A : x < a\}$ is the *downset* of a. The set $U[x] = \{x \in A : a \leq x\}$ is referred as the *closed upset* and the set $D[x] = \{x \in A : x \leq a\}$ is the *closed downset* of a.

The set of elements that are neither in the upset nor in the downset of x is called the *incomparable set* of x, in notation: $I(x) = \{x \in A : x \| a\}$.

1.1.3 Dimension of partially ordered sets

From now on, we will always make the assumptions that our partially ordered sets are finite. The following definition occupies central place in the theory of posets.

Definition 1.1.2. *Let P be a poset. Let $R = \{R_1, R_2, \ldots, R_t\}$ is a set of linear extensions. If $R_1 \cap \cdots \cap R_t = P$, we say R is a* realizer *of P. The minimum cardinality of a realizer is called the* dimension *of P, denoted by* $\dim(P)$.

Every poset has a realizer, because the set of all linear extensions is always a realizer. But in general it is difficult to find an optimal realizer with minimum cardinality.

Chains are of dimension 1 and they are the only 1-dimensional posets. Antichains are of dimension 2. Dilworth [8] proved that the dimension can not exceed the width, but

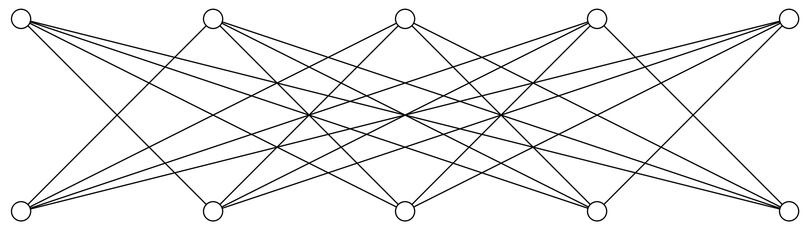


Figure 1: S_5, the 5-dimensional standard example

antichains show that the dimension can be much less than the width. Hiraguchi [16] proved that the dimension can not exceed half of the number of points.

There are posets of arbitrarily large dimension. Since the following examples are commonly used examples for large dimensional posets, they were named "standard examples".

Definition 1.1.3. *Let $U = \{u_1, \ldots, u_n\}$ and $D = \{d_1, \ldots, d_n\}$. Let $A = U \cup D$ the ground set. Let $d_i < u_j$ if and only if $i \neq j$. The poset defined this way is called the* n-dimensional standard example*, denoted by S_n.*

Proposition 1.1.4. *For $n \geq 2$, $\dim(S_n) = n$.*

Proof. Let $L_i = (d_1, d_2, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n, u_i, d_i, u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$. Then $\{L_1, \ldots, L_n\}$ is a realizer so $\dim(P) \leq n$. Now consider the pair $\{d_i, u_i\}$. Since they are incomparable, there must be a linear extension in every realizer where they are ordered $u_i < d_i$. Now suppose there is linear extension where for $i \neq j$ $u_i < d_i$ and $u_j < d_j$. Then $d_j < u_i < d_i < u_j$, which contradicts $u_j < d_j$. Therefore one linear extension can only reverse one $\{d_i, u_i\}$ pair, so there must be at least n linear extension in the realizer, i.e. $\dim(P) \geq n$. □

Dimension of partially ordered sets is sometimes considered the analogue of chromatic number for graphs. In this sense, S_n is analogous to K_n. And just as there are graphs of high chromatic number with no K_n subgraph, there are posets of high dimension with no S_n subposet (see interval orders later).

There are several ways of constructing new posets from existing one. We will mention two of these. The first one is the following.

Definition 1.1.5. *Let $P = (A, \leq_P)$ and $Q = (B, \leq_Q)$ be two posets. The* Cartesian product *of P and Q is*

$$P \times Q = (A \times B, \leq_{A\times B}),$$

where $(a_1, b_1) \leq_{A\times B} (a_2, b_2)$ if and only if $a_1 \leq_A a_2$ and $b_1 \leq_B b_2$.

The second way is the following.

Definition 1.1.6. *Let (A, P) a poset and (B_i, Q_i) posets for $i \in A$. The lexicographic sum of the posets (B_i, Q_i) over P is the poset (C, R) where $C = \cup_{i\in A} B_i$ and for $a \in B_i$ and $b \in B_j$ we have $a <_R b$ if either $i <_P j$ or $i = j$ and $a <_{Q_i} b$. In notation:*

$$(C, R) = \sum_{i\in(A,P)} (B_i, Q_i)$$

1.1.4 Special poset classes

Let $P = (A, R)$ be a poset and let $S \subseteq A$. A point $x \in A$ is an *upper bound* of S, if $s \leq x$ for all $s \in S$. One can define *lower bounds* similarly. An upper bound x is a *least upper bound*, if every other upper bound is greater than x. Similarly we can define *greatest lower bound*.

Definition 1.1.7. *A poset is a* lattice*, if every par of elements $\{x, y\}$ has a unique least upper bound (also called* join*, denoted by $x \vee y$) and a unique greatest lower bound (also called* meet*, denoted by $x \wedge y$).*

A finite lattice always contains a least and a greatest element, they are sometimes called "zero" and "one".

The lattice can be viewed as an algebraic structure with two binary operations. These operations are commutative, associative, idempotent, and they satisfy the so called absorption laws ($a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ for all a, b). These properties also characterize lattices (in fact idempotence is implied by the others).

A lattice $P = (A, \vee, \wedge)$ is a *sublattice* of a lattice $P' = (A', \vee', \wedge')$, if $A \subseteq A'$ and A is closed under the lattice operations. Observe, that a subposet of P' that is a lattice itself for the partial ordering of P' is not necessarily a sublattice. In this sense, "sublattice" is an algebraic notion, not combinatorial.

Observe, that the two operations are not necessarily distributive.

Definition 1.1.8. *A lattice is a* distributive lattice *if the two operations are distributive with each other, in other words*

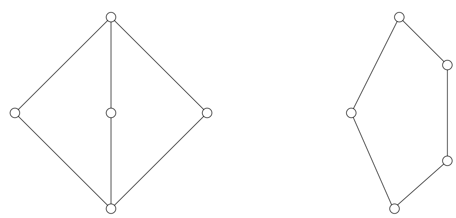
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for all x, y, z.[2]

There are nice characterizations of distributive lattices, the most famous being the following.

Theorem 1.1.9 (Birkhoff). *A lattice is distributive if and only if it has no sublattice isomorphic to one of the following lattices.*



It is important to remark that a sublattice is not simply a subposet that happens to be a lattice, but rather a subset that is closed under the lattice operations.

A *Boolean lattice* is a distributive lattice in which every element has a unique complement. From the point of view of algebra, a Boolean lattice is an algebraic structure $(A, \wedge, \vee, \neg)$, with $(A, \wedge, \vee)$ being a distributive lattice, and the complement operation $\neg$ satisfies the following properties:

$$a \vee \neg a = 1, \qquad a \wedge \neg a = 0 \qquad \forall a \in A.$$

The *Boolean lattice of subsets*, or simply the *subset lattice* is the lattice whose ground set consists of all the subsets of the set $\{1, \ldots, n\}$, with $\wedge$ is the $\cap$ operation $\vee$ is the $\cup$

[2] *The two distributivity conditions are in fact equivalent.*

operation and $\neg$ is complement. The number n is the *dimension* of the subset lattice. This does not cause confusion, because it is also the poset-dimension of the lattice. Every finite Boolean lattice is isomorphic to the subset lattice of dimension n for some n. In the subset lattice, $X \leq Y$ if and only if $X \subseteq Y$.

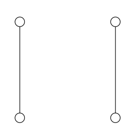
Now we will talk about another important subclass of posets. Let $P = (A, R)$ be a poset. Suppose there is a bijection $f : A \to I$, where is a set of closed intervals on the real line. Suppose furthermore, that $a <_R b$ if and only if the right endpoint of $f(a)$ is less than the left endpoint of $f(b)$. We call I an *interval representation* of P.

Definition 1.1.10. *We say P is an* interval order *if P has an interval representation.*

The interval representation of an interval order is of course not unique.

Interval orders have a nice characterization in terms of forbidden subposets. The following theorem was first given explicitly by Fishburn in [11].

Theorem 1.1.11. *A poset is an interval order if and only if it does not contain the subposet below.*



This immediately implies the following.

Corollary 1.1.12. *If $n \geq 2$, then S_n is not an interval order.*

The theory of interval orders is extremely rich. Here we will only discuss two basic facts that are important for us later in this work.

Let I_n be the *canonical interval order*, whose representation consist of every interval with integer endpoints from $\{1, \ldots, n\}$.

Theorem 1.1.13 ([14]). $\dim(I_n) = \lg\lg n + \left(\frac{1}{2} + o(1)\right) \lg\lg\lg n$

The previously mentioned theorem of Hiraguchi inspired a conjecture that became one of the longest standing open problems in the dimension theory of posets. The following statement would imply Hiraguchi's Theorem directly.

Conjecture 1.1.14 (Removable Pair Conjecture). *For every finite partially ordered set on at least three points, there is a pair whose removal decreases the dimension of the poset by at most one.*

It has been proven for several special cases of posets. The following theorem has been proven in [4].

Theorem 1.1.15. *The Removable Pair Conjecture holds for posets of dimension at most* 3*, as well as interval orders.*

1.2 Previous and related work

1.2.1 Correlation

Let P be finite partially ordered set, and let Ω be a sample space that consist of the all the linear extensions of P. Define a probability measure on Ω such that every $L \in \Omega$ has equal probability, namely $\mathbf{Pr}(L) = \frac{1}{|\Omega|}$.

For $x, y \in P$, the probability of the event

$$A_{x<y} = \{L \in \Omega : x <_L y\}$$

is denoted by $\mathbf{Pr}[x < y]$. In more simple words, it is the number of linear extensions of P in which $x < y$, divided by the total number of linear extensions of P.

One of the most famous conjectures of the theory of partially ordered sets is the so-called 1/3–2/3 conjecture.

Conjecture 1.2.1. *Every finite partially ordered set that is not chain contains a pair x, y, such that*

$$\frac{1}{3} \leq \mathbf{Pr}[x < y] \leq \frac{2}{3}.$$

Kahn and Saks [18] proved the existence of a pair x, y, such that $3/11 \leq \mathbf{Pr}[x < y] \leq 8/11$. Brightwell, Felsner and Trotter [6] improved the result by showing that there exists a pair for which

$$\frac{5-\sqrt{5}}{10} \leq \mathbf{Pr}[x < y] \leq \frac{5+\sqrt{5}}{10}.$$

The result is best possible for infinite posets.[3] The authors pose a very interesting conjecture, whose special case, which they prove, implies their result. We describe the conjecture in the following after a few necessary definitions.

Let P be a finite poset. Let $h_L(x)$ denote the position of the element x in L counted from below, more precisely

$$h_L(x) = |\{y \in P : y \leq_L x\}|.$$

Now fix x, y, z elements of P. Let $L(i, j)$ be the number of linear extensions L in which $h_L(y) - h_L(x) = i$ and $h_L(z) - h_L(y) = j$.

Conjecture 1.2.2 (Cross Product Conjecture). *For any* $i, j \geq 1$

$$L(i, j)L(i + 1, j + 1) \leq L(i, j + 1)L(i + 1, j)$$

So far we said nothing about correlation. The first major conditional type theorem was by Shepp [22]. He proved the following.

Theorem 1.2.3 (XYZ theorem).

$$\mathbf{Pr}[x < y] \leq \mathbf{Pr}[x < y | x < z]$$

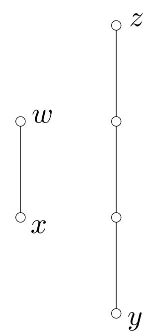
Intuitively, this shows that if we know that $x < z$, or in other words, there is an element z that is greater than x, then it becomes more likely that x is less than a fixed element y.

The theorem might seem to be trivial, but the following example shows we have to be careful with these kind of statements. Consider the following inequality.

$$\mathbf{Pr}[x < y < w] \leq \mathbf{Pr}[x < y < w | x < z < w] \tag{1}$$

One might argue, that if we know that there is an element that is greater than x and less than w, it would make more likely x to be "small" and w to be "large", so it would make it more likely that y is between them. However, C. L. Mallows showed that (1) is *false* by providing the following simple counterexample, in which $\mathbf{Pr}[x < y < w] = 4/15$ and $\mathbf{Pr}[x < y < w | x < z < w] = 1/4$.

[3] Of course, for this remark to make sense, we have to be able to define the probability of the event $A_{x<y}$. That is possible for certain countable posets, but the discussion of this is beyond the scope of this work.



Fishburn [12] proved that the inequality of the XYZ theorem is strict if and only if x, y and z form a 3-element antichain.

Most of these results has some non-combinatorial part in the argument, for example a tool that is used frequently in these results is Aleksandrov–Fenchel inequalities for mixed volumes. On the other hand, most of these questions can be translated as an inequality between cardinalities of two finite sets of linear extensions. This latter approach suggests that it may be possible to find a injection from the smaller set to the larger set directly. The main benefit of this approach would be that it would be much simpler to handle the questions when these inequalities are strict, and it would be possible to say something about the error if they are strict.

The main advance to this direction is a paper by Brightwell and Trotter [7] in which proved the strong form of XYZ inequality (and another correlation result) using purely combinatorial argument.

1.2.2 Geometric containment orders

One of the most basic examples of partially ordered sets is the following. Let A be a set whose elements are also sets.[4] Then $P = (A, \subseteq)$ forms a partially ordered set.

This example is especially nice, because it is *universal*, that is, every partially ordered set can be represented as a set inclusion order.

The question becomes more interesting, if we restrict what kind of sets can be the elements of the ground set. If the sets are geometric objects, we arrive to geometric containment orders.

[4] Of course this property is true for *every* set under ZFC, but here we would like to emphasize this property.

One basic example when the elements of the ground set are intervals of the real line. These posets are called *interval containment orders*. It is not difficult to show that they are precisely the two dimensional posets.

Now consider the following objects. Fix n positive integer and consider the n-tuple $(x_1, x_2, \ldots, x_n)$. Let $X \subseteq \mathbb{R}^n$ such that $(y_1, \ldots, y_n) \in X$ if and only if $0 \leq y_1 \leq x_1, \ldots, 0 \leq y_n \leq x_n$. This way every n-tuple defines an n-dimensional rectangle. If we restrict the ground set to contain a set of these n-dimensional rectangles and the ordering is set containment, then we get the class of all n-dimensional posets. This fact is not completely trivial, but it is an easy exercise to see. This explains why the name of "dimension".

Fishburn and Trotter [13] posed the definition of *angle orders*. The ground set consists of angular regions. An angle is a point of the plane with and ordered pair of half-lines starting from the point. The ordering is important to distinguish inner and outer angles. then the angular region is the region bounded by the two half lines swept from the first one to the second one.

Fishburn and Trotter [13] showed that every 4-dimensional order, just like every interval order is an angle order. They constructed a 7-dimensional order that is not an angle order. It can be shown using degrees of freedom (see [2]) that there exits a 5-dimensional order that is not an angle order. These techniques are very similar to the ones we will use to study segment orders in Chapter 3 of this thesis.

Let A consist of circular disk in the Euclidean plane. The poset $P = (A, \subseteq)$ is called a circle order. A degrees of freedom argument shows that there exist a 4-dimensional poset that is not a circle order. It is a simple exercise to show that every 2-dimensional order is a circle order.

Fix a positive integer n. If A consist of regular polygons with n-sides, then $P = (A, \subseteq)$ is called an *n-gon order*.

Proposition 1.2.4. *Every finite 3-dimensional poset is an n-gon order for every $n \geq 3$.*

Since intuitively, if n is very large, an n-gon much like a circle, this result suggested that every 3-dimensional poset is a circle order. Until the following result surfaced.

Theorem 1.2.5 (Scheinerman and Weirman [20]). *The infinite poset $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ is not a circle order.*

Nevertheless, Felsner, Trotter and Fishburn [9] constructed a finite 3-dimensional poset that is not a circle order. The proof is very sophisticated and uses deep techniques.

1.2.3 α-sequences

The Hasse diagram of a poset can be viewed as a graph, with two vertices adjacent if and only one covers the other. It is interesting to study the chromatic number of these graphs and their relations to the poset structure.

The dual of Dilworth's Theorem implies that the chromatic number can not be greater than the height of the poset. Nešetřil and Rödl [19] proved that for every $h \geq 1$ there exists a poset of height h such that the chromatic number of the diagram is h.

Definition 1.2.6. *Let t be a positive integer. $C(t)$ is the largest integer h such that whenever P is an interval order of height h, the chromatic number of the diagram is at most t.*

It might seem strange that the first couple of paragraph of a section titled "α-sequences" only talked about chromatic numbers of various poset diagrams. As strange as it seems to be, this notion has a strong connection to the sequences mentioned in the title.

Definition 1.2.7. *A sequence of sets $(S_0,\ldots,S_h)$ of sets is an α-sequence, if $S_0 \not\subseteq S_1$, and $S_j \not\subseteq S_i \cup S_{i+1}$ when $j > i+1$.*

Let $F(t)$ be the largest integer h for which there is an α-sequence $(S_0,\ldots,S_h)$ with $S_i \subseteq \{1,\ldots t\}$ for all i.

Proposition 1.2.8. *The following example is the longest α-sequence with the subsets of $\{1,2,3,4\}$.*

$$(\emptyset,\{1\},\{2\},\{3\},\{4\},\{2,4\},\{1,4\},\{1,3\},\{1,2,3\},\{2,3,4\})$$

Corollary 1.2.9. $F(4) = 9$

Theorem 1.2.10 (Felsner and Trotter [10]). *For every $t \geq 1$*

$$C(t) = F(t).$$

Consider the Boolean lattice of subsets of $\{1, \dots, n\}$. The Hasse diagram of this lattice is isomorphic to the n-dimensional hypercube. We mentioned previously that this graph is Hamiltonian. Let $\{S_1, \dots, S_{2^n}\}$ be a Hamiltonian path. We say it is *monotone*, if for every i, either (a) every subset of S_i appears among the sets $S_1, \dots, S_{i-1}$, or (b) only one (say S) does not, furthermore $S_{i+1} = S$.

It is not known if the diagram of the subset lattice of $\{1, \dots, n\}$ is Hamiltonian for all n.

Felsner and Trotter also showed the following theorem.

Theorem 1.2.11.

$$F(t) \leq 2^{t-1} + \left\lfloor \frac{t-1}{2} \right\rfloor$$

with equality holding if and only if there is a monotone Hamiltonian path in the subset lattice of $\{1, \dots, t\}$.

CHAPTER II

HEIGHT-SEQUENCE OF PARTIALLY ORDERED SETS

2.1 Introduction

In the following, we will study the *height-sequence* of a partially ordered set. We will use the following notation.

Definition 2.1.1. *Let x be an element of the partially ordered set P. Then $h_i(x)$ is the number of linear extensions of P in which x is in the ith lowest position, in other words, those linear extensions L, for which*

$$|\{y \in P : y \leq_L x\}| = i.$$

By this definition, $h_i = 0$ if $i \leq 0$ or $i > |P|$. If it is clear from the context, we will frequently omit x, simply writing h_i.

Stanley proved in [23] that for any partially ordered set, the height-sequence is log concave. More precisely

Theorem 2.1.2. *Let P be a partially ordered set and $x \in P$ an element. Then*

$$h_i h_{i+2} \leq h_{i+1}^2.$$

for all $i \in \mathbb{Z}$.

In his argument, he used the Aleksandrov–Fenchel inequalities from the theory of mixed volumes. The proof is short and elegant, however it does not provide any information about the difference of the two sides of the inequality. In particular, it is not known when equality holds.

In the following, we will provide a fully combinatorial argument to prove a special case of the inequality. We hope, that the techniques used here are deep enough that they can provide a proof the theorem in whole generality.

Theorem 2.1.3. *Let P be a partially ordered set such that the height sequence has at most 3 nonzero elements. Then*

$$h_i(x)h_{i+2}(x) \leq h_{i+1}^2(x)$$

for all $i \in \mathbb{Z}$.

Before we provide a proof, we explain an another motivation why it would be important to know more about the error term in the inequality.

Definition 2.1.4. *The* average *height of $x \in P$ is*

$$h(x) = \frac{\sum_{i=1}^{n} ih_i(x)}{\sum_{i=1}^{n} h_i(x)}$$

Kahn conjectured the following.

Conjecture 2.1.5. *Let $x, y \in P$. Let $n = |P|$ and $m = |D[x] \cup D[y]|$. Then*

$$\max\{h(x), h(y)\} \geq m - 1$$

Kahn observed that log-concavity of the height sequence implies the conjecture when $n = m$, i.e. when x and y are the only two maximal elements. Applying the same technique for the general conjecture, one can only derive a weaker result.

Proposition 2.1.6. *Let $x, y \in$. Let $n = |P|$ and $m = |D[x] \cup D[y]|$. Then*

$$\max\{h(x), h(y)\} \geq m \ln 2 \approx 0.7m$$

If we could understand the behavior of the difference of the two sides in Staley's inequality we may have a chance to prove the general conjecture, or at least something stronger than the proposition above.

2.2 Proof of Theorem 2.1.3

First we discuss the case when y_1 and y_2 are comparable. Without loss of generality we may assume $y_1 < y_2$.

We will use the following theorem by Ahlswede and Daykin (see [1]).

Theorem 2.2.1. *Let L be a distributive lattice. For sets $X, Y \subseteq L$ and a function $f : L \to \mathbb{R}$, let*

$$f(X) = \sum_{x \in X} f(x)$$

$$X \wedge Y = \{x \wedge y : x \in X, y \in Y\}$$

$$X \vee Y = \{x \vee y : x \in X, y \in Y\}$$

Let α, β, γ and δ four functions mapping L to the non-negative reals. If

$$\alpha(x)\beta(y) \leq \gamma(x \vee y)\delta(x \wedge y)$$

for all $x, y \in L$, then

$$\alpha(X)\beta(Y) \leq \gamma(X \vee Y)\delta(X \wedge Y)$$

for all $X, Y \subseteq L$.

In this proof, some of the ideas are similar to that are used in [21] and [22]. We will define a distributive lattice whose elements are 4-tuples of order-preserving functions from P to a long chain. When the chain is very long, these functions will almost look like linear extensions of P.

As promised, let us define a distributive lattice L. The elements of the ground set are 4-tuples of functions (f_1, f_2, f_3, f_4) such that

$$f_i : P \cup \{u\} \to \{0, \ldots, k-1\}$$

are order preserving functions, where k is some fixed positive integer (parameter of L) and u is an artificial element that is not in the ground set of P. Also, when we require order preserving functions, we consider u to be incomparable with all other elements of P, so in effect, $f_i(u)$ can be assigned freely.

To complete the definition of the lattice, we have to define the lattice operations.

$$(f_1, f_2, f_3, f_4) \wedge (g_1, g_2, g_3, g_4) = (f_1 \wedge g_1, f_2 \wedge g_2, f_3 \wedge g_3, f_4 \wedge g_4)$$

$$(f_1, f_2, f_3, f_4) \vee (g_1, g_2, g_3, g_4) = (f_1 \vee g_1, f_2 \vee g_2, f_3 \vee g_3, f_4 \vee g_4)$$

where

$$(f \wedge g)(u) = \max\{f(u), g(u)\}$$
$$(f \wedge g)(x) = \max\{f(u), g(u)\}+ \qquad (2)$$
$$+ \min\{f(x) - f(u), g(x) - g(u)\} \qquad \text{for all } x \in P,$$

and similarly,

$$(f \vee g)(u) = \min\{f(u), g(u)\}$$
$$(f \vee g)(x) = \min\{f(u), g(u)\}+ \qquad (3)$$
$$+ \max\{f(x) - f(u), g(x) - g(u)\} \qquad \text{for all } x \in P.$$

Intuitively, the meet operation pushes u up, and everything else is down, relative to u. The join operation pushes u down, and everything else is up, relative to u.

Lemma 2.2.2. *L is a distributive lattice.*

Proof. Observe that $L = L' \times L' \times L' \times L'$, with L' having order preserving functions $P \cup \{u\} \to \{0, \dots, k-1\}$ as elements, and operations are as described in (2) and (3). It is sufficient to prove that L' is a distributive lattice, because products of distributive lattices are distributive lattices.

First we show L' is a lattice. For this, note that L' is closed under the lattice operations. Then note that the operations are trivially commutative. Associativity and absorption follow from simple calculations.

Now we show that the operations are distributive. For this, note that for any three real numbers a, b and c

$$\max\{a, \min\{b, c\}\} = \min\{\max\{a, b\}, \max\{a, c\}\} \qquad (4)$$

and dually

$$\min\{a, \max\{b, c\}\} = \max\{\min\{a, b\}, \min\{a, c\}\}. \qquad (5)$$

Then

$$
\begin{aligned}
f \wedge (g \vee h)(x) = \max\{f(u), \min\{g(u), h(u)\} + \min\{f(x) - f(u), \\
\min\{g(u), h(u)\} + \max\{g(x) - g(u), h(x) - h(u)\} - \min\{g(u), h(u)\}\} = \\
= \max\{f(u), \min\{g(u), h(u)\} + \min\{f(x) - f(u), \\
\max\{g(x) - g(u), h(x) - h(u)\}\}
\end{aligned}
$$

$$
\begin{aligned}
(f \wedge g) \vee (f \wedge h)(x) = \min\{\max\{f(u), g(u)\}, \max\{f(u), h(u)\}\}+ \\
+ \max\{\max\{f(u), g(u)\} + \min\{f(x) - f(u), g(x) - g(u)\} - \max\{f(u), g(u)\}, \\
\max\{f(u), h(u)\} + \min\{f(x) - f(u), h(x) - h(u)\} - \max\{f(u), h(u)\}\} = \\
\min\{\max\{f(u), g(u)\}, \max\{f(u), h(u)\}\}+ \\
+ \max\{\min\{f(x) - f(u), g(x) - g(u)\}, \\
\min\{f(x) - f(u), h(x) - h(u)\}\}
\end{aligned}
$$

Applying (4) and (5) we get that the two expressions above are equal. $\square$

We define $X, Y \subseteq L$ formally. The definitions are illustrated on Figure 2. In the following definitions, A denotes the the upset of x and B denotes the downset of x in P.

$$
\begin{aligned}
X = \{(f_1, f_2, f_3, f_4) : \\
& f_1(u) = f_1(y_1), \\
& f_1(B) = f_2(B) = f_2(y_1) = f_2(y_2) = 0, \\
& f_2(u) < f_2(x) \qquad \text{for all } x \in A, \\
& f_3(A) = f_4(A) = f_3(y_1) = f_3(y_2) = k - 1, \\
& f_3(u) > f_3(x) \qquad \text{for all } x \in B, \\
& f_4(u) = f_4(y_1) = f_4(y_2), \\
& f_i(x) \neq f_i(y) \quad \forall x, y \in P, i = 1, 2, 3, 4, \\
& \qquad \text{other than when required by the above}\}
\end{aligned}
$$

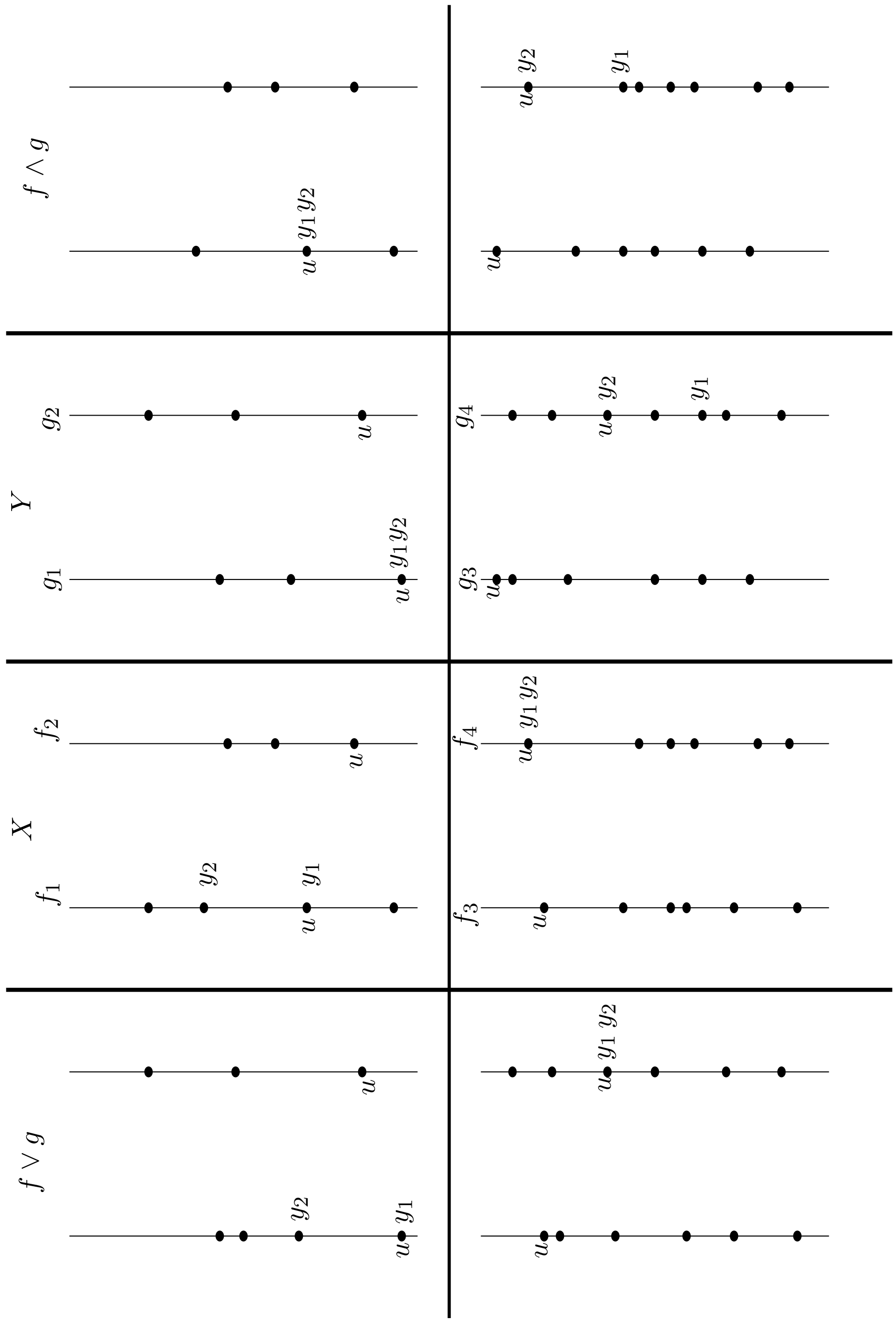


Figure 2: Subsets X, Y, $X \vee Y$ and $X \wedge Y$

$$
\begin{aligned}
Y = \{(f_1, f_2, f_3, f_4) :& \\
& f_4(u) = f_4(y_2), \\
& f_4(A) = f_3(A) = f_3(y_1) = f_3(y_2) = k-1, \\
& f_3(u) > f_3(x) \qquad \text{for all } x \in B, \\
& f_1(B) = f_2(B) = f_2(y_1) = f_2(y_2) = 0, \\
& f_2(u) < f_2(x) \qquad \text{for all } x \in A, \\
& f_1(u) = f_1(y_1) = f_1(y_2), \\
& f_i(x) \neq f_i(y) \quad \forall x, y \in P, i = 1,2,3,4, \\
& \qquad \text{other than when required by the above}\}
\end{aligned}
$$

To apply Theorem 2.2.1, we need to determine the cardinality of $|X|$, $|Y|$, $|X \vee Y|$ and $|X \wedge Y|$. This is relatively easy. In the following, $a = |A|$ and $b = |B|$. Also e will denote the number of linear extensions of the poset $A \cup B$, in which the relation is inherited from P.

$$
|X| = h_i e \binom{k}{a+2}\binom{k}{b+1}\binom{k}{a+1}\binom{k}{b+1}
$$
$$
|Y| = h_{i+2} e \binom{k}{a+1}\binom{k}{a+1}\binom{k}{b+1}\binom{k}{b+2}
$$

We can not exactly count $|X \vee Y|$ and $|X \wedge Y|$ with our method. However, the following is clear:

$$
|X \vee Y| = h^*_{i+1} e \binom{k}{a+2}\binom{k}{a+1}\binom{k}{b+1}\binom{k}{b+1} + R_\vee,
$$

where h^*_{i+1} is the number of linear extensions of P such that x is in the ith lowest position *and* y_2 is in a position where y_1 could also go. Clearly $h^*_{i+1} \leq h_{i+1}$.

$R_\vee$ is the number of elements of $X \vee Y$ in which "collision" happens. This is when $f_i(x) \neq f_i(y)$ or $g_i(x) \neq g_i(y)$, but $(f_i \vee g_i)(x) = (f_i \vee g_i)(y)$ for some i. Clearly, these case are not included in the first term, that is why we need the adjustment.

Using a similar counting method and notation

$$
|X \wedge Y| = h^{**}_{i+1} e \binom{k}{a+1}\binom{k}{a+1}\binom{k}{b+1}\binom{k}{b+2} + R_\wedge.
$$

with $h^{**}_{i+1} \leq h_{i+1}$.

Now apply Theorem 2.2.1 with all four functions being constant 1. Then simplify by e and the binomial coefficients. We have

$$h_i h_{i+2} \leq \left(h^*_{i+1} + r_\vee\right)\left(h^{**}_{i+1} + r_\wedge\right)$$

where $r_\vee = \frac{R_\vee}{e\binom{k}{a+2}\binom{k}{a+1}\binom{k}{b+1}\binom{k}{b+1}}$ and $r_\wedge = \frac{R_\wedge}{e\binom{k}{a+1}\binom{k}{a+1}\binom{k}{b+1}\binom{k}{b+2}}$.

Now let $k \to \infty$. The probability of a collision tends to zero, more precisely $r_\vee \to 0$ and $r_\wedge \to 0$. Then applying $h_{i+1} \leq h^*_{i+1}$ and $h_{i+1} \leq h^{**}_{i+1}$ implies the inequality.

It remains to discuss the case when $y_1 \| y_2$.

Let the number of linear extensions in which x is in the ith position and $y_1 < y_2$ be denoted by $h_i(y_1 < y_2)$, and the number of linear extensions in which $y_2 < y_1$ be denoted by $h_i(y_2 < y_1)$.The inequality we need to show is

$$[h_i(y_1 < y_2) + h_i(y_2 < y_1)][h_{i+2}(y_1 < y_2) + h_{i+2}(y_2 < y_1)] \leq$$
$$[h_{i+1}(y_1 < y_2) + h_{i+1}(y_2 < y_1)]^2.$$

Since $h_i(y_1 < y_2)h_{i+2}(y_1 < y_2) \leq h_{i+1}(y_1 < y_2)^2$ and $h_i(y_2 < y_1)h_{i+2}(y_2 < y_1) \leq h_{i+1}(y_2 < y_1)^2$ by the first case, we only need to show $h_i(y_2 < y_1)h_{i+2}(y_1 < y_2) + h_i(y_1 < y_2)h_{i+2}(y_2 < y_1) \leq 2h_{i+1}(y_1 < y_2)h_{i+1}(y_2 < y_1)$. This can be done by showing $h_i(y_2 < y_1)h_{i+2}(y_1 < y_2) \leq h_{i+1}(y_2 < y_1)h_{i+1}(y_1 < y_2)$ and $h_i(y_1 < y_2)h_{i+2}(y_2 < y_1) \leq h_{i+1}(y_1 < y_2)h_{i+1}(y_2 < y_1)$ similarly to the first case.

CHAPTER III

SEGMENT ORDERS

3.1 Introduction

The containment order of closed intervals on the real line is a very natural partially ordered set that one wants to investigate. Interval I is less than interval J, if and only if $I \subset J$. It is easy to show the following.

Proposition 3.1.1. *A partially ordered set is a containment order of real intervals if and only its dimension is at most 2.*

Farhad Shahrokhi proposed a definition of two families of partially ordered sets. These came up for him with relation of a graph theory problem. He originally asked if the families contain an arbitrary large dimensional partially ordered sets. It is not difficult to show that the answer is “yes”. However, the classes turned out to be interesting on their own right. In particular, Shahrokhi, Trotter and his students realized that it is not clear if the two families are actually the same or not. This inspired this line of research, in which we will define Shahrokhi's two families and we propose the definition of two other families. They are all very similar, and actually they all might be the same (though it seems unlikely).

First we will define two kinds of partial orders between certain closed line segments of a two-dimensional Cartesian coordinate system. The segments subject to this order have an endpoint $(x_1, 0)$ on the x-axis, and the other endpoint (x_2, x_3) satisfies $x_1 < x_2$ and $x_3 > 0$.

First we define the partial order of the first kind. We say, the line segment x (whose endpoints are $(x_1, 0)$ and (x_2, x_3)) is greater than the line segment y (whose endpoints are $(y_1, 0)$ and (y_2, y_3)), if $x_1 < y_1$, $x_2 > y_2$ and every vertical line that intersects both x and y intersects x strictly above y. In particular, if $x \cap y \neq \emptyset$ then they are incomparable.

For the partial order of the second kind, we say, the line segment x is greater than the line segment y, if $x_1 < y_1$, $x_2 < y_2$ and every vertical line that intersects both x and y intersects x strictly above y. In other words, we reverse the ordering rule on the right end of the line segment.

Let P be a poset. If there is a function l_1 such that for $x, y \in P$, $x < y$ if and only is $l_1(x)$ is less than $l_2(x)$ in the first kind of line segment ordering, then we say P is a *segment order of the first kind*. We denote the class of segment orders of the fist kind by $\mathfrak{P}_1$. We can also say P is a $\mathfrak{P}_1$-order, and the set of corresponding line segment is a $\mathfrak{P}_1$-representation of P. We use similar definitions to define the class $\mathfrak{P}_2$, the class of *segment orders of the second kind*.

Shahrokhi's two families were not these two. He added another restriction to both classes producing subfamilies of these families.

Let $P \in \mathfrak{P}_1$ a poset. If P has a $\mathfrak{P}_1$-representation L, such that every line segment in L intersects the y-axis, we say P is a *central segment order of the first kind*. We use $\mathfrak{p}_1$ to denote this class. Similarly, we can define $\mathfrak{p}_2$, the class of *central segment orders of the second kind*.

It is clear that $\mathfrak{p}_1 \subseteq \mathfrak{P}_1$ and $\mathfrak{p}_2 \subseteq \mathfrak{P}_2$, but it is not at all clear if these are proper containments or not.

Every interval containment order is in $\mathfrak{P}_1$. This is trivial, because one can easily turn an interval containment representation into a $\mathfrak{P}_1$-representation by adding 1 to the y-coordinate to the right endpoint of each interval. In this sense, $\mathfrak{P}_1$ is a natural generalization of interval containment orders.

Using a similar analogy, $\mathfrak{P}_2$ is a natural generalization of interval orders. Again, just lifting the right endpoints of the intervals by 1 will produce a $\mathfrak{P}_2$-representation from any interval representation of an interval order. Interval orders and interval containment orders are two very different classes, nevertheless, it is not clear how different $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are.

Theorem 3.1.2. *Every interval containment order is both in $\mathfrak{p}_1$ and in $\mathfrak{p}_2$.*

Proof. To see that every interval containment order is in $\mathfrak{p}_1$, consider an interval representation $L = \{[a_1, b_1], \ldots, [a_n, b_n]\}$ of the order P. There exists an $m \in \mathbb{R}$ such that the set of intervals $L' = \{[a_i - m, b_i + m] : [a_i, b_i] \in L\}$ is pairwise intersecting. Observe, that L' is still a representation of P. Now let $L'' = \{\text{line segments from } (a_i, 0) \text{ to } (b_i, 1) : [a_i, b_i] \in L'\}$.

Then L'' is a $\mathfrak{p}_1$-representation of P. The second part of the proof is quite similar to the first, so we omit it. $\square$

We will prove a more general statement later, namely, that every 3-dimensional poset, as well as every interval order is both in $\mathfrak{p}_1$ and $\mathfrak{p}_2$. On the other hand, the fraction of 4 and higher dimensional posets that are in the classes is very small (we will make this statement rigorous later). This makes these classes even more interesting: as we know that the Removable Pair Conjecture holds for 3-dimensional posets and interval orders, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ seem to be the next ideal choices to test the conjecture.

We will show many other small results about the classes $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{P}_1$, $\mathfrak{P}_2$. For a substantial part of the chapter, statements are equally true for both centered and regular classes of segment orders, and depending on the statement, we will claim and prove the form that is stronger. The main question of this discussion (which will appear even more natural after understanding these classes a little better) is if $\mathfrak{p}_1 = \mathfrak{p}_2$. We conjecture that the answer is negative and we will be able to prove a statement that says that there is no continuous function of line segments to line segments that turns a $\mathfrak{P}_1$ order into a $\mathfrak{P}_2$ order. This statement has a corollary on the central classes. Though these results might seem to be weak, they are surprisingly difficult to prove. The discretization of these proofs are also beyond our understanding at this point.

Since our ultimate goal is to prove $\mathfrak{p}_1 \neq \mathfrak{p}_2$, the reader might think that eventually somebody will draw a $\mathfrak{p}_1$ poset with less than a hundred lines and will show it is not in $\mathfrak{p}_2$. While we can't rule out the possibility completely, the following discussion will show that it is indeed very improbable. It seems very likely the deep techniques are required to construct such a poset, and it will contain at least several billion points.

3.1.1 A note about tied endpoints

Suppose we want to redefine the definition of $\mathfrak{P}_1$ or $\mathfrak{P}_2$ by allowing equality in the x-coordinates of the endpoints. Or similarly, if x is the segment from $(0,0)$ to $(1,1)$ and y is the segment from $(-1,0)$ to $(3,1)$, why don't we define $x < y$ in $\mathfrak{P}_1$?

Observe, that if we make any or all of these changes, we actually don't change the class.

For example, if a *finite* number of line segments in a $\mathfrak{P}_1$-representation have a the same x-coordinate for their right endpoints, the segments are pairwise incomparable. One can make the ith segment (ordering them increasingly according their y-coordinates) ε/i longer to the right, using an ε so small that this addition will not affect the relationship of the tied segments with the others. If we modify the definition making the above mentioned line segments pairwise comparable, add ε/i to their lengths in reverse order.

Similarly, one can break all the ties that arise in any of the four classes. Therefore no matter how we define the classes in terms of ties, we get equivalent definitions. We will extensively use this fact by changing our definition of the classes at will, depending of what definition is the easiest to use.

3.2 Basic properties of regular and central segment orders

The following proposition answers Shahrohki's question.

Proposition 3.2.1. *For every positive integer n, the "standard examples" S_n are in both $\mathfrak{p}_1$ and $\mathfrak{p}_2$.*

Proof. Let C be the quarter circle whose equation is $(x-1)^2+y^2=4$ with $x \leq 1$ and $y \geq 0$. Let $p_1, p_2, \ldots, p_n$ be the points on C, such that the x-coordinate of p_i is $i/(n+1)$. For each $i=1,\ldots,n$, let a_i be a line segment whose left endpoint is of the form $(x,0)$ with $-1<x<0$ and right endpoint is p_i. Let b_i be a segment of the tangent of C at point p_i, such that its left endpoint lies on the x-axis and its right endpoint has x-coordinate 2. Then $\{a_1,\ldots,a_n,b_1,\ldots,b_n\}$ is a $\mathfrak{p}_1$-representation of S_n, hence $S_n \in \mathfrak{p}_1$. (See Figure 3.)

To show that $S_n \in \mathfrak{p}_2$ is very similar, but the choice of C is slightly more complicated. Let C be the arc $y=e^x$ with $x \in [0,1]$. Let $p_1, p_2, \ldots, p_n$ be the points on C, such that the x-coordinate of p_i is $i/(n+1)$. For each $i=1,\ldots,n$, let b_i be a line segment whose left endpoint is of the form $(x,0)$ with $x<-1$ and right endpoint is p_i. Let a_i be a segment of the tangent of C at point p_i, such that its left endpoint lies on the x-axis and its right endpoint has x-coordinate 2. Then $\{a_1,\ldots,a_n,b_1,\ldots,b_n\}$ is a $\mathfrak{p}_2$-representation of S_n, hence $S_n \in \mathfrak{p}_2$. (See Figure 4.) $\square$

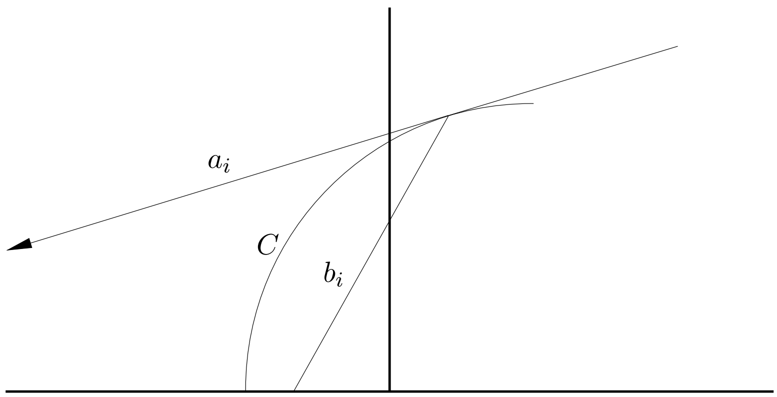


Figure 3: $\mathfrak{p}_1$-representation of S_n

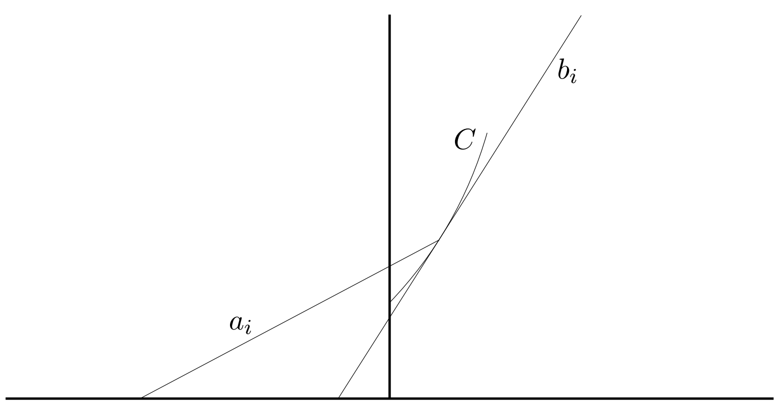


Figure 4: $\mathfrak{p}_2$-representation of S_n

Theorem 3.2.2. *Every poset of dimension at most* 3 *is in both* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$*.*

Proof. Let P be a three-dimensional poset. Let L_1, L_2 and L_3 be linear extensions such that $L_1 \cap L_2 \cap L_3 = P$. For each $x \in P$, let x_i be the position of x in L_i, counting from low to high. For each $x \in P$, draw a line segment from $(-x_1, 0)$ to $(0, x_2)$. There is an $\varepsilon > 0$ such that if we change the x-coordinate of the right endpoint of every line segment to any number between 0 and ε (from the original 0), then the intersection properties will not change, in other words, two line segments will intersect, if and only if they intersected before.

Then change the right endpoint of line segment x to $\frac{\varepsilon x_3}{|P|}$. Clearly line segment x is greater than line segment y in $\mathfrak{p}_2$ if and only if $x < y$ in L_1, L_2, L_3. So $P \in \mathfrak{p}_2$.

To prove that $P \in \mathfrak{p}_1$, we can start with the same line segments and ε as above. Then change the right endpoint of line segment x to $\varepsilon - \frac{\varepsilon(x_3-1)}{|P|}$. Now line segment x is greater than line segment y in $\mathfrak{p}_1$ if and only if $x < y$ in L_1, L_2, L_3. Therefore $P \in \mathfrak{p}_1$.

□

Before we start to think that maybe $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are universal, we discover the following theorem:

Theorem 3.2.3. *For every* $d > 3$*, there is a* d*-dimensional poset that is neither in* $\mathfrak{P}_1$ *nor in* $\mathfrak{P}_2$*. Furthermore, for every* $0 < p < 1$ *there is a positive integer* n*, such that out of all* d*-dimensional posets on* n *elements, at most their* p*-fraction is in* $\mathfrak{P}_1$ *or* $\mathfrak{P}_2$*.*

Before we discuss the proof, we need a few definitions and a theorem.

Definition 3.2.4. *Let* $\mathcal{F}$ *be a family of sets. Let* P *be a poset. We say* P *is an* $\mathcal{F}$*-order, if there is a function* $f : P \to \mathcal{F}$ *such that* $f(x) \subseteq f(y)$ *if and only if* $x \leq y$ *in* P*.*

Definition 3.2.5. *A family of sets* $\mathcal{F}$ *has* k *degrees of freedom, if there is an injective function* $f : \mathcal{F} \to \mathbb{R}^k$*, such that there exist polynomials* $p_1, \ldots, p_t$ *of* $2k$ *variables such that for two sets* $S, T \in \mathcal{F}$*, the question if* $S \subseteq T$ *can be decided by the signs of* $p_i(f_1(S), \ldots, f_k(S), f_1(T), \ldots, f_k(T))$ *for* $i = 1, \ldots, t$*.*

Theorem 3.2.6 (Alon and Scheinerman [2]). *Let $\mathcal{F}$ be any family of sets. If $\mathcal{F}$ has at most k degrees of freedom, there is some $k+1$-dimensional poset which is not an $\mathcal{F}$-order.*

Proof. We will define $\mathcal{F}_1$ (or $\mathcal{F}_2$) to be an infinite family of sets that represents the $\mathfrak{P}_1$ (or $\mathfrak{P}_2$) ordering. For $\mathfrak{P}_1$ this can be done as the following: to a line segment $(a, 0)$ to (b, c), assign the closed triangle with vertices $(a, 0)$, $(b, 0)$, (b, c). Containment of these triangles will result in an ordering that is not quite the same as in the original definition of $\mathfrak{P}_1$, but it defines an equivalent class (see Section 3.1.1). For $\mathfrak{P}_2$ the assignment is the following: to a segment $(a, 0)$, (b, c), assign the union of the closed triangle with vertices $(a, 0)$, $(b, 0)$, (b, c), and the set $\{(x, y) : x \geq b, y \geq 0\}$. Similar remark applies as in the previous case.

So every poset of $\mathfrak{P}_1$ ($\mathfrak{P}_2$) is an $\mathcal{F}_1$-order ($\mathcal{F}_2$-order). Then we show that $\mathcal{F}_1$ ($\mathcal{F}_2$) has at most 3 degrees of freedom by providing finitely many (that will be 3) polynomials, whose sign pattern will determine the relationship between two sets of $\mathcal{F}_1$ ($\mathcal{F}_2$). We will provide the polynomials for $\mathcal{F}_1$ here as an example and leave the polynomials for $\mathcal{F}_2$ as an exercise.

Let a line segment have the endpoints $(x, 0)$, (y, z). Then we will assign the triple (x, y, z) to the set of $\mathcal{F}_1$ that was assigned to the line segment. Let two line segments have the endpoints $(x_1, 0)$, (y_1, z_1) and $(x_2, 0)$, (y_2, z_2).

$$p_1(x_1, y_1, z_1, x_2, y_2, z_2) = x_2 - x_1$$

$$p_2(x_1, y_1, z_1, x_2, y_2, z_2) = y_1 - y_2$$

$$p_3(x_1, y_1, z_1, x_2, y_2, z_2) = -z_1x_2 + z_1y_2 - z_2y_1 + x_2z_2$$

The first line segments is less than the second if and only if all three polynomials are negative. Observe, that p_1 and p_2 determines the ordering of the projections of the endpoints to the x-axis, and p_3 determines if the endpoint of the second line segment stays under line determined by the first line segment.

This shows that $\mathcal{F}_1$ has at most 3 degrees of freedom. Therefore, by Theorem 3.2.6, there is a 4-dimensional poset that is not an $\mathcal{F}_1$-order, hence it not in $\mathfrak{P}_1$.

The statement of the theorem is stronger than this, but this is not a problem. The proof of Theorem 3.2.6 in [2] implies a stronger statement than the theorem, and that is exactly what we need to deduce the stronger statement of our theorem. $\square$

Corollary 3.2.7. *For every $d > 3$, there is a d-dimensional poset that is neither in $\mathfrak{p}_1$ nor in $\mathfrak{p}_2$. Furthermore, for every $0 < p < 1$ there is a positive integer n, such that out of all d-dimensional posets on n elements, at most their p-fraction is in $\mathfrak{p}_1$ or $\mathfrak{p}_2$.*

Theorems 3.2.2 and 3.2.7 suggest that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ strongly overlap with the low dimensional posets. Perhaps with the exception of a few obscure examples, these classes contain exactly the low-dimensional posets. In that case, the classes are not very interesting. Fortunately it turns out that it is very far from being true.

As Proposition 3.2.1 shows, there is no bound for the dimension of posets in either classes. However, one may wonder, if the standard examples are the only high dimensional examples in $\mathfrak{p}_1$ or $\mathfrak{p}_2$. The following theorem shows that this is not the case.

Theorem 3.2.8. *Every interval order is in both $\mathfrak{p}_1$ and $\mathfrak{p}_2$.*

Proof. It is clearly sufficient to prove that $I_n = \{[a,b] : a = 1, \ldots, n, \quad b = 1, \ldots, n\}$ is both in $\mathfrak{p}_1$ and $\mathfrak{p}_2$. We will first show that $I_n \in \mathfrak{p}_1$.

Draw a quarter of a circle with radius 1 and center $(1,0)$. For every interval of I_n, place a point to the circle, in the way it is shown on the left of Figure 5. For every point $[i,j]$, if $i \geq 3$, draw a half-line from the point representing $[i,j]$ so that it intersects the circle between the point representing $[i-2, i-1]$ and the next point up on the circle. Drop the portion of this half-line that is under the x-axis to create a line segment. This line segment will be assigned to the interval $[i,j]$.

If $i \leq 2$ draw a line segment from $(0,\varepsilon)$ to the point representing $[i,j]$, where ε is a small positive real, such that $(0,\varepsilon)$ will be the point closest to the origin on the x-axis of all the endpoints of the segments.

It is clear that the $\mathfrak{p}_1$ ordering among these segments is the same as the interval ordering among the intervals.

If we change the quarter circle as it is shown on the right of Figure 5, then the same argument gives a proof that every interval order is a $\mathfrak{p}_2$ order. $\square$

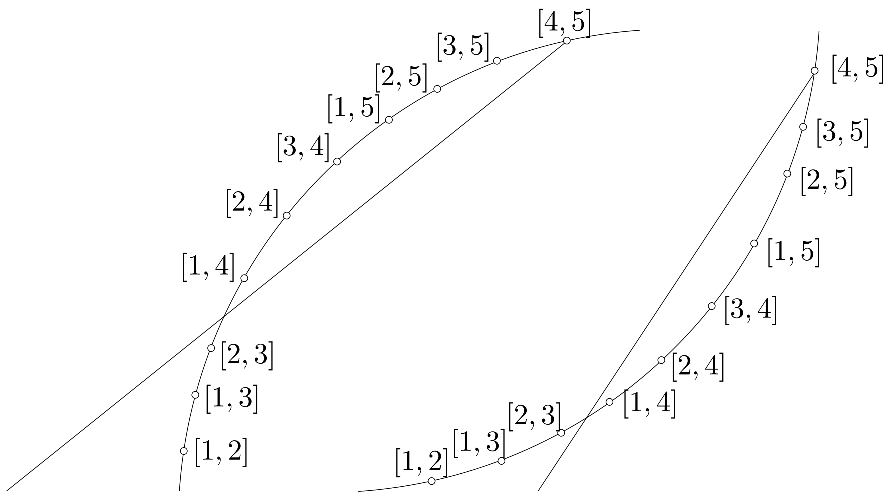


Figure 5: Right endpoints of segments and the segment assigned to $[4, 5]$

3.3 Goals and motivations

We were not able to find a $\mathfrak{p}_1$ poset that is not $\mathfrak{p}_2$ or vice versa. The previous discussion shows that if there is one, it is of dimension at least 4, not an interval order and not a standard example. If the two classes are the same, one would expect that there is short proof, which would be something like a nice, bijective function from the line segments of $\mathbb{R}^2$ to itself that maps the line segments of a $\mathfrak{p}_1$ poset to another set of line segments, so that it will be a $\mathfrak{p}_2$-representation of the same poset. Or maybe its is even true for $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Momentarily, we will show, that at least the latter statement is false.

From now on we will focus on the question whether $\mathfrak{p}_1 = \mathfrak{p}_2$.

Definition 3.3.1. *A* universal function *is a function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that for every L $\mathfrak{p}_1$-representation of a poset P, the set of line segments $\{f(l) : l \in L\}$ forms a $\mathfrak{p}_2$-representation of P.*

Remark: $f(l)$ denotes the line segment, whose left endpoint is defined by the first coordinate of $f(l)$ and right endpoint defined by the third and forth coordinates of $f(l)$; $f(l)$ is computed by transforming the line segment l to a triple using the x coordinate of the left endpoint of l and the x and y coordinates of the right endpoint of l, in order.

Definition 3.3.2. *A* shift insensitive universal function *is a function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that for every L $\mathfrak{p}_1$-representation of a poset P and for every $s \in \mathbb{R}$, there exists an s', such* *that the set of line segments $\{f(l+s) : l \in L\} + s'$ forms a $\mathfrak{p}_2$-representation of P.*

Remark: for a line segment l from $(x,0)$ to (y,z), $l+s$ denotes the line segment from $(x+s,0)$ to $(y+s,z)$. For a set of line segments L' and $s' \in \mathbb{R}$, $L'+s' = \{l'+s' : l' \in L'\}$.

Clearly, not every universal function is shift insensitive universal, but also, not every shift insensitive universal function is universal.

The partial result that we achieved is the following.

Theorem 3.3.3. *There is no continuous shift insensitive universal function.*

This is one reason why we make the following conjecture.

Conjecture 3.3.4. $\mathfrak{p}_1 \neq \mathfrak{p}_2$. *Furthermore,* $\mathfrak{p}_1 \not\subset \mathfrak{p}_2$ *and* $\mathfrak{p}_2 \not\subset \mathfrak{p}_1$.

In this whole section we will assume that the intersection of two line segments never occurs in the endpoint of any of them. We can do this because of the remarks we made in Section 3.1.1.

Definition 3.3.5. *Let P be a poset in $\mathfrak{p}_1$ with a representation P_1. We say, L is the left linear extension of P (defined by P_1), if L is defined by the regular ordering of the absolute values of the x-coordinates of the left endpoints of the line segments of P_1. Similarly, we say, R is the right linear extension of P (defined by P_1), if R is defined by the regular ordering of the absolute values of the x-coordinates of the right endpoints of the line segments of P_1.*

The following lemma will allow us to make strong assumptions about how a poset is represented.

Lemma 3.3.6. *Let P_1 be a $\mathfrak{p}_1$-representation of a poset P. Then there exists a $Q_1 \supseteq P_1$ $\mathfrak{p}_1$-representation of a poset Q, such that for any Q_2 $\mathfrak{p}_2$-representation of Q, there exists $P_2 \subseteq Q_2$ such that P_2 is a $\mathfrak{p}_2$-representation of P[1], and additionally the following is true:*

L_1 is the left linear extension of P (defined by P_1), R_1 is the right linear extension of P (defined by P_1). Similarly, L_2 is the left linear extension of P (defined by P_2), R_2 is the right linear extension of P (defined by P_2).

[1] *This far, the statement is totally trivial, as P_2 can be chosen to be set of line segments corresponding to the line segments in P_1. The question is, if we can force this to behave as it is described in the second part of the statement. In fact, probably we can not, and in the proof, P_2 will be slightly different than just the image of P_1.*

Then

i) $L_1 = R_2^d$

ii) $L_2 = R_1$.

Let P_1 be a $\mathfrak{p}_1$-representation of a poset and l and m be two line segments. Let l_1 be the absolute value of the x-coordinate of the left endpoint of l and define l_2 similarly for the right endpoint. Similarly define m_1 and m_2. Assume further that $l_1 > m_1$ and $l_2 > m_2$.

Now try to represent the poset in $\mathfrak{p}_2$. Do we know anything about the order of the projections of the endpoints of the line segments corresponding to l and m? If in P_1, l and m are comparable (then $l > m$), then yes, these orders are fixed. However, l and m may very well be incomparable in P_1 in which case there is no restriction how the endpoints are ordered in the $\mathfrak{p}_2$-representation. The power of the lemma, is that we can actually force some order on the endpoints, by building a set of "helper" lines above P_1 first to form Q_1, then find the the representation of Q_1 in $\mathfrak{p}_2$, and find the correct subset. This will essentially tie our hands how to create $\mathfrak{p}_2$-representations.

3.4 Proof of Lemma 3.3.6

The idea of the proof is the following. First we will provide a construction of Q_1 that only proves i). Then we start over and provide another construction of Q_1 that proves ii), but then we assume that i) already holds. So in reality, if one wants to force the desired behavior, he has build the helper segments that force ii), then above that he builds the helpers that force i).

We will dedicate a section to each part of the proof.

3.4.1 First part

In this section we will construct Q_1 that forces i).

Let the elements of $P = \{p_1, p_2, \ldots, p_{|P|}\}$. Let $n = 2^{|P|+1}$. We will add two sets of line segments to P_1:

$$A = \{a_1, \ldots, a_n\}, \qquad B = \{b_1, \ldots, b_n\}.$$

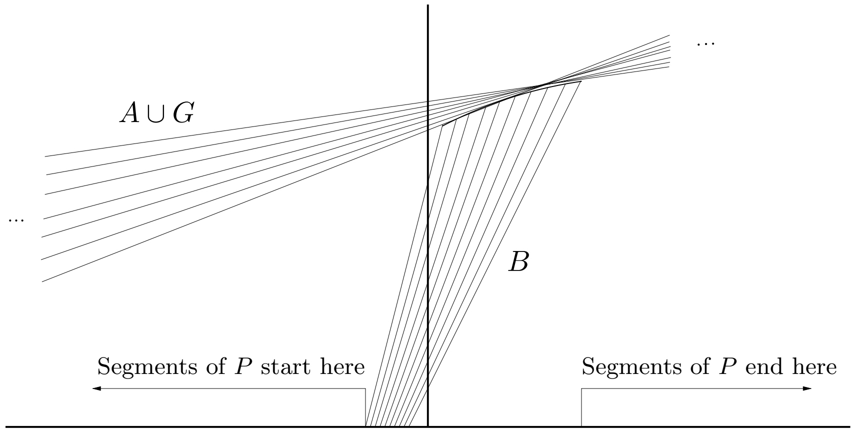


Figure 6: Arrangement of Q_1

They form a standard example, like in Figure 3. Specially, the elements of B are confined to a narrow space: each of them starts after the last element of P (i.e. the left endpoints of the elements of P are left from the left endpoints of the elements of B), and each of the ends before the first ending element of P. Also, $p_i \cap a_j = \emptyset$ for all i, j.

Additionally, we construct a set of line segments G.

$$G = \{g_S : S \text{ is a set of consecutive integers not greater than } n\}.$$

We define G so that $G \| A$, and for each S for which $g_S \in G$, it holds that $g_S \| \{b \in B : b \in S\}$ but $g_S > \{b \in B : b \notin S\}$. Note, that this does not define how to draw the line segments of G, but clearly, it is easy to do it ("intermix" the segments with the elements of A). See Figure 6.

Suppose that the x-coordinates of the right endpoints of the elements of $p_1, \ldots, p_{|P|}$ are $p_1^{\mathrm{R}}, \ldots, p_{|P|}^{\mathrm{R}}$. Without loss of generality, $p_1^{\mathrm{R}} < \cdots < p_{|P|}^{\mathrm{R}}$. Note, this is equivalent to the statement $R_1 = (p_1, \ldots, p_n)$. Then the x-coordinates of the right endpoints of the elements of A, call them $a_1^{\mathrm{R}}, \ldots, a_n^{\mathrm{R}}$ will follow the following rule:

$$a_i^{\mathrm{R}} = \frac{p_j^{\mathrm{R}} + p_{j+1}^{\mathrm{R}}}{2} \text{ if } i = \left(k - \frac{1}{2}\right) 2^j \text{ for some } k. \tag{6}$$

Some a_i^{R} are not defined by this rule (namely, iff $i | 2^{|P|}$); they will be such that $a_i^{\mathrm{R}} > p_j^{\mathrm{R}}$ for all j, but otherwise arbitrary.

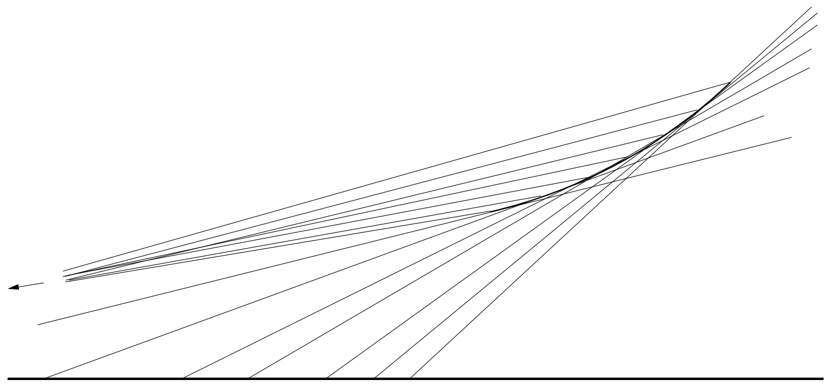


Figure 7: $\mathfrak{p}_2$-representation of S_n

Now we will show that Q_2 has to attain a very specific form, and so does P_2. From now on, we change the notation, and p_i, a_i, b_i denote line segments of the $\mathfrak{p}_2$-representation, and we use again a_i^{L}, a_i^{R} for the absolute values of the x-coordinates of the left and right endpoints of the segments. However, we maintain the indices, so in this way a_i is the image of the original a_i[2].

In the following, we will study the structure of $A \cup B$ in the $\mathfrak{p}_2$-representation. Since this will be useful later, too, we state it as a separate lemma.

Lemma 3.4.1. *Every $\mathfrak{p}_2$-representation of S_n (standard example) has the arrangement as in Figure 7, with the exception of at most two pairs of line segments.*

Proof. The sets A and B are the maximum antichains in S_n. $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$ and $a_i > b_j$ if and only if $i \neq j$.

Hence in the representation,

$$a_i^{\mathrm{L}} > b_j^{\mathrm{L}}$$

for all i, j except maybe for one pair. Let us ignore that pair now and concentrate on the remaining. Now let a_k be such that a_k^{R} is maximal. Then each $b_i < a_k$ with $i \neq k$, so each $b_i^{\mathrm{R}} > a_k^{\mathrm{R}}$ with $i \neq k$. If we ignore a_k and b_k now, too, then the remaining

$$\{a_i^{\mathrm{R}} < b_j^{\mathrm{R}}\} \text{ for all } i, j, \text{ except for the ignored.}$$

[2]We believe that the confusion this might cause is less than the confusion that would be caused by the hordes of indices.

Still, for every i, $a_i \| b_i$ somehow: the only way this can be if, $a_i \cap b_i \neq \emptyset$. So the $\mathfrak{p}_2$-representation must look like Figure 7. $\square$

We can specify an ordering on B with the aid of the left linear extension (defined by Q_2), call it $L_2(Q_2)|_B$. Also, denote the linear order specified by the left linear extension (defined by Q_1) by $L_1(Q_1)|_B$. We claim that if we keep ignoring the pairs in the previous paragraph, then either

$$L_1(Q_1)|_B = L_2(Q_2)|_B \quad \text{or} \quad L_1(Q_1)|_B = (L_2(Q_2)|_B)^d. \tag{7}$$

To see this, pick b_x and b_y consecutive elements in $L_1(Q_1)|_B$. We know that $y = x + 1$ unless some elements were ignored between them, in which case every $i : x < i < y$ is an index of an ignored element. Let $S = \{x, \ldots, y\}$ a set of (at least two) integers, and consider the image of g_S, call it g^*. Now $g^* > b_i$ for all i except for the images of b_x and b_y (and the ignored elements of course). This is only possible if the images of b_x and b_y are consecutive line segments on Figure 7, or more rigorously, the images of b_x and b_y are consecutive in $L_2(Q_2)|_B$. So we deduced that the consecutivity property is preserved between $L_1(Q_1)|_B$ and $L_2(Q_2)|_B$, which implies (7)[3].

Note that the proposition above has no consequence to the right linear extension of B. However, it immediately implies a similar statement on the right linear extension of A. Let $L_2(Q_2)|_A$ be the right linear extension (defined by Q_2). Then

$$L_2(Q_2)|_A = (a_1, \ldots, a_n) \quad \text{or} \quad (a_n, \ldots, a_1). \tag{8}$$

(Again, recall that we may have missing elements from these sets, but that will not affect our argument.)

Recall that $R_1 = (p_1, \ldots, p_{|P|})$, so we need to show that $L_2 = (p_1, \ldots, p_{|P|})$.

Now recall the definition of Q_1, specifically that all a_i had its left endpoint left from all p_i, and the ordering of their right endpoints by (6). In particular it implies that $p_1 < a_i$ for all i. Therefore it must be that $p_1 <_{L_2} a_i$ for all i. Also from (6), $p_2 < a_i$ if and only if i is even, and $p_2 \| a_i$ if and only if i is odd. Therefore using (8), the only possible arrangements

[3] In fact, this is all we need the set G for.

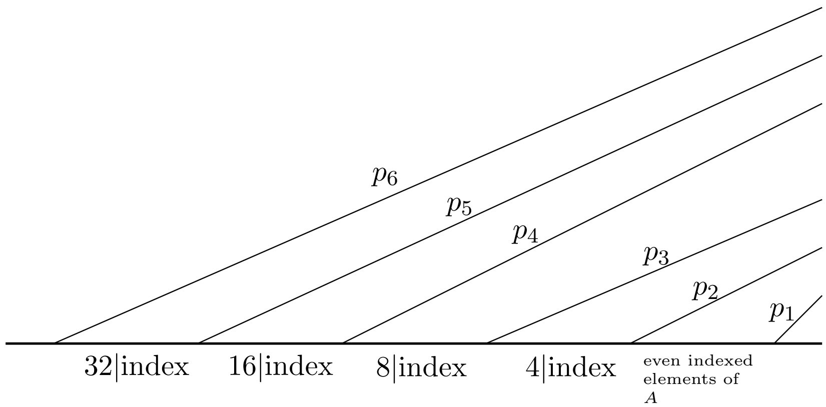


Figure 8: Left endpoints of A and P in the $\mathfrak{p}_2$-representation

for A in Q_2 is that the even indexed left endpoints are grouped together, and also the odd indexed left endpoints are grouped together, and p_2's left endpoints is between the two groups. In particular, $p_1 <_{L_2} p_1$. See Figure 8.

We can continue this argument by considering p_3, p_4, and so on, in every step concluding, that the given $p_j < a_i$ if and only if $i|2^{j-1}$, otherwise $p_j \| a_i$. So the left endpoints of A are grouped in a manner shown in Figure 8, and the left endpoints of P must be between the groups. That forces an order on the left endpoints of P, namely

$$p_1 <_{L_2} p_2 <_{L_2} \cdots <_{L_2} p_{|P|}.$$

Q.E.D.

As a last remark, let us add that we needed all $2^{|P|+1}$ elements of A and B. If we didn't have to "ignore" elements early in the argument, then $2^{|P|-1}$ would have been enough, but in the current situation, we had to defend us against the situation when crucial elements, that would force ordering on P, are ignored. Using $2^{|P|+1}$ intervals ensures that there are at least 4 elements of A in every important group on Figure 8 (3 would have been enough, but it is just simpler to increase the exponent of two by 1), so even if we ignore 2 of those, at least 1 is left do its job.

3.4.2 Second part

Surprisingly, repeating the techniques of the first part of the proof does not seem to work here. Naturally, we tried to do that first, but we could not work it out. So we needed something different. The ideas are somewhat simpler here, but the description of proof is more technical, and we heavily use the first part of the lemma.

As we mentioned earlier, in this part of the proof, we will assume that Q_1 and Q_2 already obeys (i) of the lemma.

For every pair of incomparable line segments in P_1, we will add some extra line segments to form a portion of Q_1. The configuration of the new line segments depends on the configuration of the two original line segments. According to this, we will separate several cases.

In the following cases, the two line incomparable segments will always be denoted by r and b, and always so that the left endpoint of r is left from the left endpoint of b in P_1. We will also use the notation a^{L}, a^{R} for a line segment a as in the previous section, and we will add the notation a^{U} for the y-coordinate of the right endpoint of a. Also, we keep the convention that we will use the same letter for the line segment in Q_2 (or specially P_2) as for the line segment in Q_1 (or P_1) and we will make it clear with the context which line segment is the argument about.

Using these notation and assumptions, in every case we will prove that

$$r^{\mathrm{L}} > b^{\mathrm{L}} \text{ in } P_1 \Longrightarrow r_0^{\mathrm{R}} < b_0^{\mathrm{R}} \text{ in } P_2,$$

where most of the time $r_0 = r$ and $b_0 = b$, and other times $r_0 \neq r$, but r_0 has the same relation to any other element of P_1 as r, and similar statement is true for b.

Since we eventually build helpers above every pair of incomparable line segments, this statement is going to be proven for every pair of incomparable line segments, and therefore the second part of the lemma follows.

3.4.2.1 Case 1

$$b^{\mathrm{R}} < r^{\mathrm{R}} \text{ and } b^{\mathrm{U}} < r^{\mathrm{U}} \text{ in } Q_1$$

Relabel r to r_1 and b to b_1, and add extra line segments b_2, b_3, r_2, r_3 so that $b_1^{\mathrm{R}} < b_2^{\mathrm{R}} < b_3^{\mathrm{R}} < r_1^{\mathrm{R}} < r_2^{\mathrm{R}} < r_3^{\mathrm{R}}$ and $b_1^{\mathrm{U}} < b_2^{\mathrm{U}} < b_3^{\mathrm{U}} < r_1^{\mathrm{U}} < r_2^{\mathrm{U}} < r_3^{\mathrm{U}}$ and $b_1^{\mathrm{L}} < b_2^{\mathrm{L}} < b_3^{\mathrm{L}} < r_1^{\mathrm{L}} < r_2^{\mathrm{L}} < r_3^{\mathrm{L}}$, and make sure that b_2 and b_3 has the same relation to every element of the poset as b_1, similarly for r_2, r_3 and r_1.

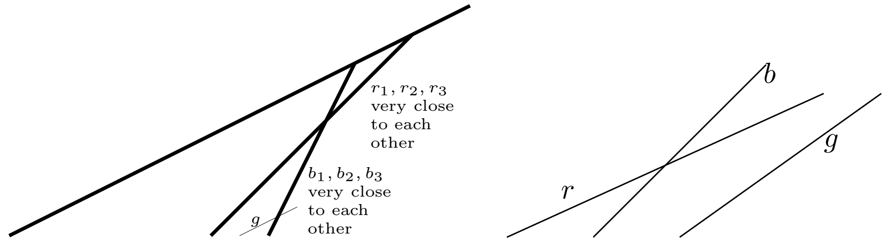


Figure 9: Case 1

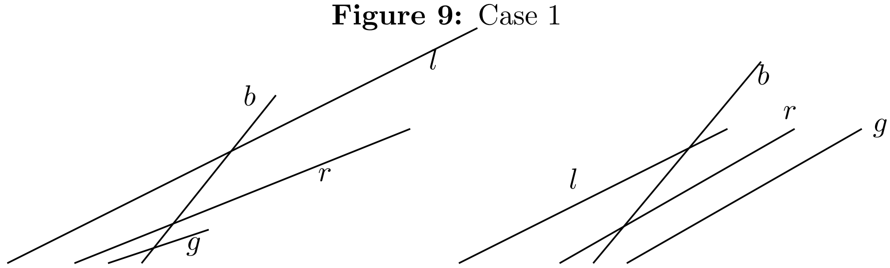


Figure 10: Case 2

Add 6 additional segments so that they now form S_6. Also add the line segment g so that $b_3^{\mathrm{L}} < g^{\mathrm{L}} < r_1^{\mathrm{L}}$ and $g^{\mathrm{R}} < b_1^{\mathrm{R}}$. Also $g <_Q r_i$ for $i = 1, 2, 3$ and g intersects b_i for $i = 1, 2, 3$ in Q_1.

Apply Lemma 3.4.1 to the poset induced by r_1, r_2, r_3 and b_1, b_2, b_3. In Q_2, there must be a pair, call them r_0 and b_0 that intersect. Where is g in Q_2? Due to the first part of the lemma, $g^{\mathrm{L}} < b_0^{\mathrm{L}} < r_0^{\mathrm{L}}$. Suppose that $b_0^{\mathrm{R}} < r_0^{\mathrm{R}}$; then by $g <_Q r_0$, g can not possibly intersect b_0 in Q_2, so $g <_Q b_0$, a contradiction. We showed that $r_0^{\mathrm{R}} < b_0^{\mathrm{R}}$. (This is the only part of the proof where r may not be identical to r_0 and b may not be identical to b_0.)

3.4.2.2 Case 2

$$b^{\mathrm{R}} < r^{\mathrm{R}} \text{ and } r^{\mathrm{U}} < b^{\mathrm{U}} \text{ in } Q_1$$

We need to add two extra line segments, g and l, so that $b^{\mathrm{L}} < g^{\mathrm{L}} < r^{\mathrm{L}} < l^{\mathrm{L}}$, $g^{\mathrm{R}} < b^{\mathrm{R}} < r^{\mathrm{R}} < l^{\mathrm{R}}$, $g^{\mathrm{U}} < r^{\mathrm{U}} < b^{\mathrm{U}} < l^{\mathrm{U}}$ in Q_1. By the first part, $g^{\mathrm{L}} < b^{\mathrm{L}} < r^{\mathrm{L}} < l^{\mathrm{L}}$ in Q_2. Also $g <_Q r <_Q l$. By Case 1, we may assume that b intersects l. If $b^{\mathrm{R}} < r^{\mathrm{R}}$, then $b > g$, a contradiction. So $r^{\mathrm{R}} < b^{\mathrm{R}}$.

3.4.2.3 Case 3

$$r^{\mathrm{R}} < b^{\mathrm{R}} \text{ and } r \cap b \neq \emptyset \text{ in } Q_1$$

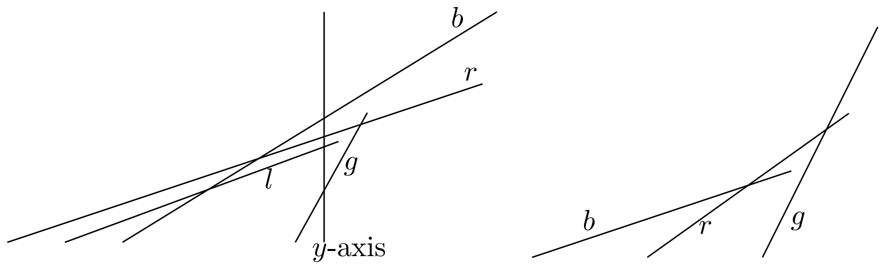


Figure 11: Case 3

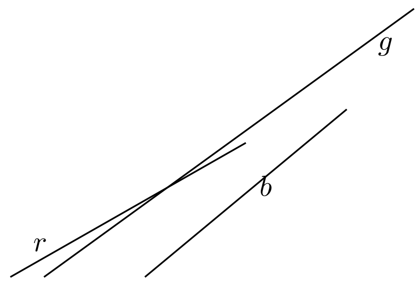


Figure 12: Case 4

Add two line segments, g and l, so that in Q_1

- $g <_Q b$
- $g \|_Q r$, and g and r are in the same relation as b and r in Case 1
- $g^{\mathrm{R}} < r^{\mathrm{R}}$ and $g^{\mathrm{R}} < b^{\mathrm{R}}$
- $l <_Q r$
- $l \|_Q b$
- l^{R} is less than all of g^{R}, r^{R}, b^{R}

In Q_2, $b^{\mathrm{L}} > r^{\mathrm{L}}$ by the first part of the lemma. Suppose that $b^{\mathrm{R}} < r^{\mathrm{R}}$. Then b and r must intersect in Q_2 in order to be incomparable. $g^{\mathrm{L}} < r^{\mathrm{L}}$, $g <_Q b$ and $g \|_Q r$. Also, $g \cap r \neq \emptyset$ and $g^{\mathrm{R}} > r^{\mathrm{R}}$ by the Case 1 construction. Then it should happen that $l <_Q r$, $l \|_Q b$ and $l^{\mathrm{L}} < g^{\mathrm{L}}$. This is impossible, so $r^{\mathrm{R}} < b^{\mathrm{R}}$.

3.4.2.4 Case 4

$$r^{\mathrm{R}} < b^{\mathrm{R}} \text{ and } r \cap b = \emptyset \text{ in } Q_1$$

Add a line segment g such that $g >_Q b$ and g and r are in the same relation as b and r in Case 3. In Q_2, $g^{\mathrm{R}} < b^{\mathrm{R}}$, because $g >_Q b$. If $b^{\mathrm{R}} < r^{\mathrm{R}}$, then $g^{\mathrm{R}} < r^{\mathrm{R}}$, contradicting Case 3.

This concludes the proof of the main lemma.

Although we worked out the proof for $\mathfrak{p}_1$ and $\mathfrak{p}_2$, observe that position of the y-axis did not really play any role in the proof. In the points when it came up in the discussion it only made the proofs harder, so if we drop the assumption that every line segments intersects the y-axis, the proof still stands. Therefore we can also conclude the following:

Lemma 3.4.2. *The statement of Lemma 3.3.6 holds for $\mathfrak{P}_1$ and $\mathfrak{P}_2$, too.*

3.5 Connections with pseudoline arrangements

In this section, unless otherwise noted, we will work on on the real projective plane $\mathbb{P}^2$.

Definition 3.5.1. *A* pseudoline *is a simple closed curve whose removal does not disconnect the plane.*

Consider a set of pseudolines on the plane. Maybe we are not interested the exact paths the pseudolines describe, or they analytical properties. If we are only interested how they connect certain points of the plane, then we talk about graph theory. In this case, we define abstract object—graphs—that contain only these properties of the lines. If we are also interested the intersection properties of these lines, then we will define a different kind of abstract object: a pseudoline arrangement.

Definition 3.5.2. *An* arrangement of pseudolines *is a set of pseudolines such that any two intersects at exactly one point, and not all of them intersect in the same point.*

Definition 3.5.3. *Consider the sphere model of $\mathbb{P}^2$, in which the points are pairs of antipodal points, and the lines are the great circles. Let a and b be two points in $\mathbb{P}^2$, and (a_1, a_2), (b_1, b_2) be the corresponding pairs of antipodal points. The* distance *of a and b is* $\min\{d(a_1, b_1), d(a_1, b_2), d(a_2, b_1), d(a_2, b_2)\}$, *where $d(a_i, b_j)$ is the length of the shortest arc*

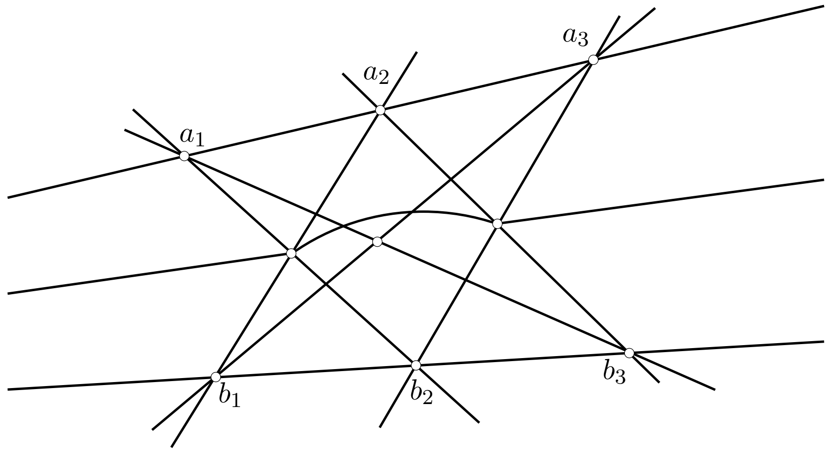


Figure 13: Non-stretchable arrangement with 9 pseudolines

between a_i and b_j on the surface of the sphere. A function $\mathbb{P}^2 \to \mathbb{P}^2$ is continuous*, if it is continuous with respect to the metric defined by the distance above. A function $\mathbb{P}^2 \to \mathbb{P}^2$ is a* homeomorphism*, if it is bijective, continuous and its inverse is also continuous.*

Definition 3.5.4. *Two pseudoline arrangements are* isomorphic*, if there is a homeomorphism that maps one to the other.*

Definition 3.5.5. *A pseudoline arrangement is* stretchable *if it is isomorphic to a pseudoline arrangement, in which every pseudoline is a straight line.*

Not every pseudoline arrangement is stretchable. To show a counterexample, recall the classical geometrical theorem by Pappus.

Theorem 3.5.6 (Pappus). *Let a_1, a_2, a_3 be collinear points, and b_1, b_2, b_3 be another set of collinear points. Let (a_i, b_j) denote the straight line that passes through the points a_i and b_j. Then the points $(a_1, b_2) \cap (a_2, b_1)$, $(a_1, b_3) \cap (a_3, b_1)$, $(a_2, b_3) \cap (a_3, b_2)$ are collinear.*

Corollary 3.5.7. *The pseudoline arrangement on Figure 13 is not stretchable.*

Desargues' Theorem can be used to produce a non-stretchable configuration on 10 lines. There are several other famous examples. Bokowski and Sturmfels provided a "minor-minimal" infinite family of non-stretchable arrangements. This already suggests that that it is difficult to determine if a pseudoline arrangement is stretchable. Indeed, Mnëv proved that the problem determining if a pseudoline arrangement is stretchable is NP-complete.

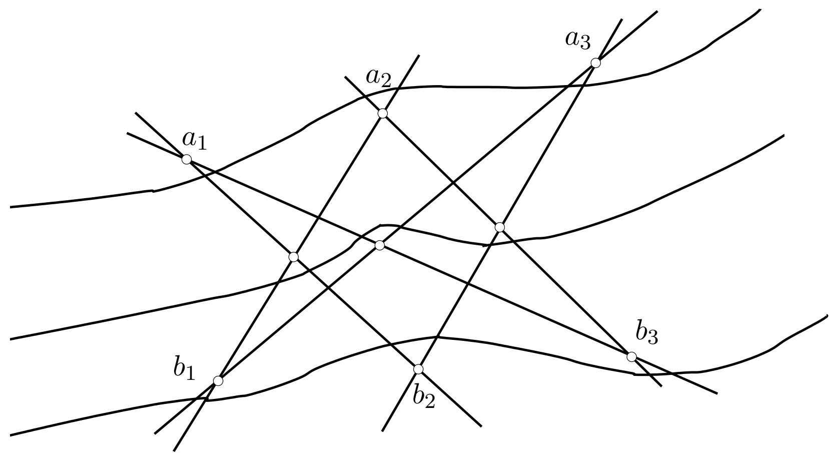


Figure 14: Non-stretchable simple arrangement with 9 pseudolines

Definition 3.5.8. *A pseudoline-arrangement is* simple*, if no three pseudolines cross in the same point.*

Simple pseudoline arrangements will be central points of interest to us, because the method that we will develop will produce arrangements that are simple, or close to simple. It is easy to transform the non-Pappus arrangement into a simple pseudoline arrangement, as shown on Figure 14. We encourage the reader to reconstruct the proof that the arrangement is non-stretchable using Pappus's Theorem.

For more information on pseudoline arrangements, see [15], or for more detailed exposition from the point of view of oriented matroids, see [3].

In the following, we will see how Conjecture 3.3.4 is related to stretchability of pseudoline arrangements. We will define a sequence of posets $U_n \in \mathfrak{p}_1$ for every positive integer n. We will assign a family of pseudoline arrangements to each U_n, and we will ask if it is true that there is a stretchable arrangement in each family.

Let $\hat{U}_n = \{1, \ldots, n\}^3$, the set of ordered triples of positive integers not greater than n. To every element (l, r, u) of $\hat{U}_n$, assign the line segment from $(-l, 0)$ to (r, u). Consider these line segments as a $\mathfrak{p}_1$-representation of a poset. Call this poset U_n.

If there is an n such that $U_n \notin \mathfrak{p}_2$, then $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$, specifically, Conjecture 3.3.4 is true. On the other hand, if $U_n \in \mathfrak{p}_2$ for every n positive integer, then $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. Indeed, every poset in $\mathfrak{p}_1$ can be represented as a subset of $\hat{U}_n$ for some large n.

The subsets of $\hat{U}_n$ that is determined by the triples

$$P(r,u) = \{(i,r,u) : i \in \{1,\ldots,n\}\}$$

are called *pencils*.

We say a pencil $P(r_1,u_1)$ *dominates* a different pencil $P(r_2,u_2)$ at level i, if

- $r_1 \le r_2$ and $u_1 \ge u_2$, or
- $r_1 > r_2$ and $u_1 > u_2$ and the line segment $(-i,0)$ to (r_1,u_1) passes above the point (r_2,u_2), or
- $r_1 < r_2$ and $u_1 < u_2$ and the line segment $(-i,0)$ to (r_2,u_2) passes below the point (r_1,u_1).

The following proposition is a straightforward consequence of the definition.

Proposition 3.5.9. *The domination relation of pencils at a fixed level i is a total order on the pencils.*

In the following, we describe a pseudoline arrangement, that is associated with the poset U_n. For simplicity, we we will use a Cartesian coordinate system, and we add the ideal line. We will only specify ordered sequences of points in the coordinate system that the pseudoline passes through, in the prescribed order. Doing so, we actually do not fully specify the arrangement, but we do specify many (intersection) properties of it. Then we consider all pseudoline arrangements that satisfy these properties to get our family of arrangement.

Start with straight lines $v_1,\ldots,v_n$, such that v_i is the straight line passing through $(i,0)$ and $(i,1)$. Add a new pseudoline $p_{r,u}$ for each pencil $P(r,u)$. Let $l_i(r,u)$ be the position of the pencil $P(r,u)$ at level i in the total order defined by domination, so that $l_i(r,u)$ is 1 for the smallest pencil and n^2 for the largest. Let the pseudoline $p_{r,u}$ pass through $(-r,0)$, then for $i = 1,\ldots,n$, $(i, l_i(r,u))$. See Figure 15.

Observe, that we did not specify how the pseudolines intersect left from v_1, but it is very strictly defined how they intersect between any v_i and v_{i+1}. This way, taking every

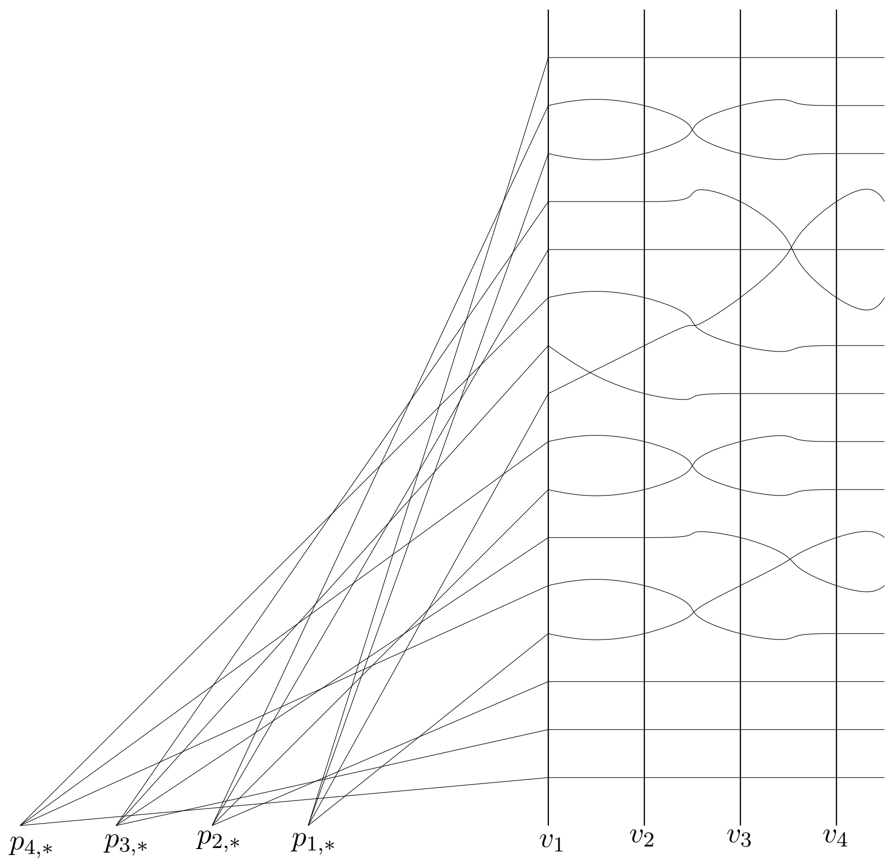


Figure 15: Pseudoline representation of U_4

possible arrangement (up to isomorphism), we have defined a family of arrangements. Call it $\mathcal{F}_n$.

Proposition 3.5.10. *If there exists an $n \in \mathbb{N}$ such that no pseudoline arrangement in $\mathcal{F}_n$ is stretchable, then $\mathfrak{p}_1 \not\subseteq \mathfrak{p}_2$.*

Proof. Consider $\hat{U}_n$, a $\mathfrak{p}_1$-representation of U_n. It is clear that we can break the ties with the endpoint coordinates in $\hat{U}_n$ without changing the underlying poset. Also, we can move the line segments so that no line segment endpoint lies on another line segment, still without changing the underlying poset.

For each set of line segment that used to be pencils, create two other identical "pencils" (these are not really pencils any more after the tiebreaking), overlapping the original. Now move one replica's right endpoints to left, and move the other's right endpoints to the right. Do this so that the replicas have the same relationship with every other line segments that are not in any copy on this pencil. Also, do this for every pencil, and call this new $\mathfrak{p}_1$-representation $\tilde{U}_n$.

We will apply Lemma 3.3.6 on $\tilde{U}_n$; recall that this involves building a huge helper poset above $\tilde{U}_n$.

In $\hat{U}_n$, there were only n values attained by x-coordinates of left endpoints of line segments (namely $-1, \ldots, -n$). In $\tilde{U}_n$, the x-coordinates of the left endpoints of the corresponding segments are grouped to n disjoint intervals. By Lemma 3.3.6, this property is preserved in the $\mathfrak{p}_2$-representation of the poset. A similar property holds for the x-coordinates of the right endpoints.

Also by the lemma, the image of a pencil in the $\mathfrak{p}_2$-representation resembles a ray (a half-line). It is so that

- the left endpoints are confined to an interval in which no other line segments start, except the ones with identical x-coordinates of right endpoints,
- the right endpoints hit the groups mentioned in the previous paragraph: each segment hits exactly one group, and the order is also preserved,

- the “shorter” line segments do not intersect any line segments that the “longer” ones do not.[4]

For each pencil, concentrate only the “longest” line segments. We know where these start on the x-axis (recall how they are confined into disjoint intervals).

We will add vertical lines to this $\mathfrak{p}_2$-representation. First consider the replicas of pencils $P(i,u)$ for a fixed i that were moved to the left. These are confined to an interval I_i^{L} in the $\mathfrak{p}_2$-representation. The replicas that were moved to the right are confined to the interval I_i^{R}. Similar holds to the original copies of pencils, call their interval I_i.

I_i^{L}, I_i and I_i^{R} are pairwise disjoint, and I_i is the one in the middle. So there is a real number r_i strictly between I_i^{L} and I_i^{R}.

In the stripe $I_i^{\mathrm{L}} \times \mathbb{R}$ there is a pseudoline, and we know the order it intersects the “longest” line segments defined above. This order is l_i, the total order defined by pencil domination. There is a similar pseudoline in the stripe $I_i^{\mathrm{R}} \times \mathbb{R}$, on which the intersection order is the same. So the order must still be same on the vertical line with $x = r_i$. Keep only these vertical lines along with the “longest” line segments, and turn the “longest” line segments into lines by continuing them to infinity at both ends.

What we got is a straight line arrangement that adhere to all the defining requirements of the members of $\mathcal{F}_n$, contradicting the assumption that no pseudoline arrangement in $\mathcal{F}_n$ is stretchable. $\square$

3.6 Properties of continuous universal functions

In this section, we will again heavily use Lemma 3.3.6. We will frequently make statements that will imply that certain arrangements of line segments must hold in a $\mathfrak{p}_2$-representation of a certain poset. What we mean in these cases, is that *we may assume* that the segments are arranged in the specified way, because by Lemma 3.3.6 there exists a superposet of the current poset that forces the current posets arrangement in the specified way. In other

[4] The “shorter” segments here are those, whose right endpoint has smaller x-coordinate, although their “arc length” can technically be longer.

words, we assume the statement of Lemma 3.3.6 in these cases. It would be really cumbersome to make the precise statement every time in each argument, so we chose to use this "shortcut".

Lemma 3.6.1. *Let f be a continuous universal function function and r, b two line segments with $r^{\mathrm{L}} > b^{\mathrm{L}}$.*

i) If the right endpoint of r lies on the line defined by b, then the right endpoint of $f(b)$ lies on the line defined by $f(r)$.

ii) If the right endpoint of b lies on the line defined by r, then the right endpoint of $f(r)$ lies on the line defined by $f(r)$ (specifically in the interior of $f(r)$).

The proof of this lemma relies on a sequence of basic statements that do not require continuity and are somewhat similar to Lemma 3.3.6. The proof of these are very technical, but not hard. The lemma will be a relatively straightforward consequence of these. The statements are claimed together in the following lemma.

Lemma 3.6.2. *Let r_1 and b_1 two line segments of a representation of a $\mathfrak{p}_1$ poset P such that $r_1^{\mathrm{L}} > b_1^{\mathrm{L}}$. Let b and r be the corresponding elements of P. Find a $\mathfrak{p}_2$-representation of P. In this, the line segment corresponding to r_1 and b_1 will be called r_2 and b_2 respectively. Let the lines defined by r_i be called R_i and the lines defined by l_i be called L_i respectively.*

Then

1. $(b_1^{\mathrm{R}} < r_1^{\mathrm{R}}) \wedge (r_1 \cap b_1 \neq \emptyset) \Longrightarrow (r_2 \cap b_2 \neq \emptyset)$.

2. $(b_1^{\mathrm{R}} < r_1^{\mathrm{R}}) \wedge (r_1 \cap b_1 = \emptyset) \wedge (r_1 \cap B_1 \neq \emptyset) \Longrightarrow (r_2 \cap b_2 = \emptyset) \wedge (R_2 \cap b_2 \neq \emptyset)$.

3. $(b_1^{\mathrm{R}} < r_1^{\mathrm{R}}) \wedge (r_1 \cap b_1 = \emptyset) \wedge (r_1 \cap B_1 = \emptyset) \Longrightarrow (r_2 \cap b_2 = \emptyset) \wedge (R_2 \cap b_2 = \emptyset)$.

4. $(r_1^{\mathrm{R}} < b_1^{\mathrm{R}}) \wedge (r_1 \cap b_1 \neq \emptyset) \Longrightarrow (r_2 \cap b_2 = \emptyset) \wedge (R_2 \cap b_2 = \emptyset)$.

5. $(r_1^{\mathrm{R}} < b_1^{\mathrm{R}}) \wedge (r_1 \cap b_1 = \emptyset) \wedge (R_1 \cap b_1 \neq \emptyset) \Longrightarrow (r_2 \cap b_2 = \emptyset) \wedge (R_2 \cap b_2 \neq \emptyset)$.

6. $(r_1^{\mathrm{R}} < b_1^{\mathrm{R}}) \wedge (r_1 \cap b_1 = \emptyset) \wedge (R_1 \cap b_1 = \emptyset) \Longrightarrow (r_2 \cap b_2 \neq \emptyset)$.

Proof. 1) is trivial, because if $r_2 \cap b_2 = \emptyset$, then $r > b$ would follow, while $r \| b$.

In 2), the part that $r_2 \cap b_2 = \emptyset$ is trivial for similar reasons: we must have $r > b$. To prove the other part, add a new line segment g_1 such that $b_1^{\mathrm{L}} < g_1^{\mathrm{L}} < r_1^{\mathrm{L}}$, $b_1^{\mathrm{R}} < g_1^{\mathrm{R}} < r_1^{\mathrm{R}}$ and $g_1 \cap b_1 \neq \emptyset$ and $g_1 \cap r_1 \neq \emptyset$. Lemma 3.3.6 implies $b_2^{\mathrm{L}} < g_2^{\mathrm{L}} < r_2^{\mathrm{L}}$ and $r_2^{\mathrm{R}} < g_2^{\mathrm{R}} < b_2^{\mathrm{R}}$. Then $g_2 \cap r_2 \neq \emptyset$, because we need $g \| r$, and $g_2 \cap b_2 \neq \emptyset$, because we need $g \| b$. This implies $R_2 \cap b_2 \neq \emptyset$.

3) will be proven at the end of the proof.

To prove 4), add a new line segment g_1 such that $g_1^{\mathrm{L}} < b_1^{\mathrm{L}} < r_1^{\mathrm{R}}$, $g_1^{\mathrm{R}} < r_1^{\mathrm{R}} < b_1^{\mathrm{R}}$ and $g_1 \cap r_1 \neq \emptyset$. Lemma 3.3.6 implies $g_2^{\mathrm{L}} < r_2^{\mathrm{L}} < b_2^{\mathrm{L}}$ and $r_2^{\mathrm{R}} < b_2^{\mathrm{R}} < g_2^{\mathrm{R}}$. Since $g < b$, we have $g_2 \cap b_2 = \emptyset$. Since $r \| g$, we must have $r_2 \cap g_2 \neq \emptyset$. Therefore the right endpoint of r_2 in Q_2 can not stay in the triangle defined by the left endpoints of b_2 and r_2 and the right endpoint of b_2. This implies both statements.

To prove 5, add two new line segments g_1 and w_1 in the following way.

- $g_1^{\mathrm{L}} < b_1^{\mathrm{L}} < r_1^{\mathrm{L}} < w_1^{\mathrm{L}}$
- $r_1^{\mathrm{R}} < g_1^{\mathrm{R}} < w_1^{\mathrm{R}} < b_1^{\mathrm{R}}$
- $g_1 \cap b_1 \neq \emptyset$, $w_1 \cap r_1 \neq \emptyset$ and $w_1 \cap b_1 \neq \emptyset$
- $w_1 \cap g_1 = \emptyset$

By Lemma 3.3.6 we may assume that $r_2^{\mathrm{L}} < g_2^{\mathrm{L}} < w_2^{\mathrm{L}} < b_2^{\mathrm{L}}$ and $w_2^{\mathrm{R}} < r_2^{\mathrm{R}} < b_2^{\mathrm{R}} < g_2^{\mathrm{R}}$. Since $g \| b$, we have $g_2 \cap b_2 \neq \emptyset$. Similarly, $r_2 \cap w_2 \neq \emptyset$. But $w > g$, therefore $r_2 \cap g_2 \neq \emptyset$. This implies the statement of the lemma.

To show 6, add g_1 so that $b_1^{\mathrm{L}} < r_1^{\mathrm{L}} < g_1^{\mathrm{L}}$, $r_1^{\mathrm{R}} < b_1^{\mathrm{R}} < g_1^{\mathrm{R}}$ and $g_1 \cap r_1 \neq \emptyset$ and $g_1 \cap b_1 = \emptyset$. By Lemma 3.3.6 $r_2^{\mathrm{L}} < b_2^{\mathrm{L}} < g_2^{\mathrm{L}}$ and $g_2^{\mathrm{R}} < r_2^{\mathrm{R}} < b_2^{\mathrm{R}}$. Since $g \| r$ it follows that $g_2 \cap r_2 \neq \emptyset$. Since $g > b$ it follows that $g_2 \cap b_2 = \emptyset$. This implies the statement.

Now let us return to 3). We get the first part trivially form $r > g$. To see the second part, add a new line segment g_1 such that $g_1^{\mathrm{L}} < b_1^{\mathrm{L}} < r_1^{\mathrm{L}}$, $b_1^{\mathrm{R}} < r_1^{\mathrm{R}} < g_1^{\mathrm{R}}$ and $g_1 \cap b_1 \neq \emptyset$ and $g_1 \cap R_1 = \emptyset$. Lemma 3.3.6 implies $b_2^{\mathrm{L}} < r_2^{\mathrm{L}} < g_2^{\mathrm{L}}$ and $r_2^{\mathrm{R}} < b_2^{\mathrm{R}} < g_2^{\mathrm{R}}$. Because of the already proven part 4, $g_2 \cap b_2 = \emptyset$. Because of part 6, $g_2 \cap r_2 \neq \emptyset$. This implies the second part. $\square$

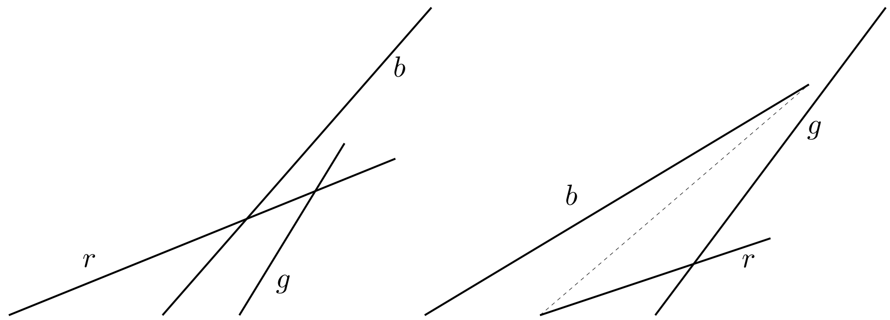


Figure 16: Proof of Lemma 3.6.2 case 4

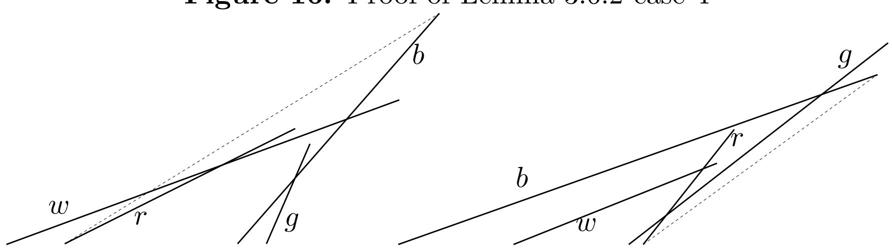


Figure 17: Proof of Lemma 3.6.2 case 5

Now we are ready to prove Lemma 3.6.1.

Proof. The lemma contains four statements altogether. Part i) has two statements enclosed: one if the right endpoint of r lies in the interior of b, and the other, if it lies outside of b. Similarly, part ii) includes two statements. All four statements have very similar proofs, so we only include the proof of the case of part i) when the right endpoint of r lies in the interior of b. The other statements are proven similarly.

Consider a sequence of segments $\{r_i\}_{i=1}^{\infty}$ such that all r_i have their left endpoints at the left endpoint of r, and their right endpoints converge to the right endpoint of r, but the relation of r_i and b is such as r and b in part 4 of Lemma 3.6.2. Also consider another sequence of segments $\{R_i\}_{i=1}^{\infty}$ such that their left endpoints are also at the left endpoint of r, their right endpoints also converge to the right endpoint of r, however, the relation of R_i and b is such as r and b in part 5 of Lemma 3.6.2.

Let T be the triangle determined by the line segment $f(b)$ as a side and the left endpoint of $f(r)$ as opposite vertex. The images $f(r_i)$ all lie outside of T (left endpoints being at a vertex), and the images $f(R_i)$ lie inside of T (left endpoints being at the same vertex).

The right endpoints of these segments converge to the right endpoint of $f(r)$. So the right endpoint of $f(r)$ must lie on the boundary of T, which implies the statement of lemma. $\square$

Nothing that has been done in this section depends on the position of the y-axis. Since Lemma 3.3.6 works also in the absence of the y-axis, and we didn't use the y-axis in Lemma 3.6.2, it follows that Lemma 3.6.1 works also for shift insensitive universal functions. We emphasize this in the following corollary.

Corollary 3.6.3. *Let f be a continuous shift insensitive universal function function and r, b two line segments with $r^{\mathrm{L}} > b^{\mathrm{L}}$.*

i) If the right endpoint of r lies on the line defined by b, then the right endpoint of $f(b)$ lies on the line defined by $f(r)$.

ii) If the right endpoint of b lies on the line defined by r, then the right endpoint of $f(r)$ lies on the line defined by $f(r)$ (specifically in the interior of $f(r)$).

3.7 Proof of Theorem 3.3.3

Lemma 3.7.1. *Let r be a line segment from $(-x, 0)$ to (y, z) and b be a segment from $(-x, 0)$ to (y, w) with $x, y, z, w > 0$ and $z > w$. Let f be a continuous universal function. Then $f(r)$ and $f(b)$ has a common left endpoint and their right endpoints have equal x-coordinate, but different y-coordinates.*

Proof. Lemma 3.3.6 with the continuity of f imply that the x-coordinates of the left endpoints of $f(r)$ and $f(b)$ are equal, therefore the left endpoints are identical. Also, for similar reasons, the x-coordinates of the right endpoints are equal. So unless $f(r) = f(b)$, the statement is true. Suppose $f(r) = f(b)$.

Let g be a line segment starting at $(-x-1, 0)$ passing through (y, z) ending at $(y+1, z(x+y+2)/(x+y+1))$. Due to Lemma 3.6.1, the right endpoint of $f(z)$ lies on $f(r) = f(b)$. Considering Lemma 3.3.6, it is clear that the right endpoint of $f(z)$ must lie in the interior of $f(b)$. This makes z and b incomparable in the underlying poset, when the construction of b and g implies $g > b$. $\square$

Corollary 3.7.2. *Let r be a line segment from $(x, 0)$ to (y, z) and b be a segment from $(x, 0)$ to (y, w) with $z > w$. Let f be a continuous shift insensitive universal function. Then $f(r)$ and $f(b)$ has a common left endpoint and their right endpoints have equal x-coordinate, but different y-coordinates.*

Proof. The argument is the same as for the previous lemma. Instead of using Lemma 3.3.6 and Lemma 3.6.1, we have to use their shift insensitive versions, and the argument can be repeated without change. □

Now we are ready to prove Theorem 3.3.3.

Consider two non-intersecting line segments in a $\mathfrak{P}_1$ or $\mathfrak{P}_2$-representation of a poset. Let $d(x_0)$ is the distance between the points that are defined by the intersection of the line segments with the horizontal line $y = x_0$. The function $d(x)$ is defined on at least on some interval $[0, \varepsilon]$ for some $\varepsilon > 0$. If $d(x)$ is monotonously increasing, then we say they *diverge*. Similarly, if $d(x)$ is decreasing, we say they *converge*. If they neither diverge nor converge, we say they are *parallel* (actually this is not our definition).

Let r be the line segment $(0, 0)$ to $(1, 2)$. Let b be the line segment $(0, 0)$ to $(1, 1)$. For $\varepsilon > 0$ define b_ε to be the segment from $(0, 0)$ to $(1, 1 - \varepsilon)$.

Because of Lemma 3.7.2, $f(r)$ and $f(b)$ diverge. Since f is continuous, there exists $\varepsilon > 0$ such that $f(r)$ and $f(b_\varepsilon)$ diverge. Let $r' = f(r)$ and $b' = f(b_\varepsilon)$.

Define a new line segment l such that it passes through the points $(1, 2)$ and $(1, 1 - \varepsilon)$ and its left endpoint is on the x-axis. This does not define l' completely, because it does not describe the right endpoint, but that is in fact not necessary: the location of the right endpoint does not matter. Let $l' = f(l)$.

Applying Lemma 3.6.1 to r and l we conclude that the right endpoint of l' lies on the line defined by r'. Similarly, looking at b_ε and l, we conclude that the right endpoint of l' lies on the line defined by b'. Hence the lines defined by r' and b' intersect at the right endpoint of l'. This contradicts the fact that they diverge. Q.E.D.

3.8 Some known configurations

Naturally we attempted to find an integer n, such that no pseudoline arrangement of $\mathcal{F}_n$ is stretchable. This actually would imply that for all $m > n$, no pseudoline arrangement of $\mathcal{F}_m$ is stretchable, because $\hat{U}_n \subseteq \hat{U}_m$. So we do not need to find the smallest n, and it is certainly more natural to try to prove the statement, that for n large enough, no pseudoline arrangement of $\mathcal{F}_n$ is stretchable.

We know that the problem of stretchability is difficult is general. However, we may be able to find a subset of $\hat{U}_n$ for large n, that is a known example of a non-stretchable arrangement. The simplest known non-stretchable arrangement is the non-Pappus arrangement. Unfortunately we can come to the following conclusion.

Theorem 3.8.1. *The non-Pappus pseudoline arrangement is* not *a subset of $\hat{U}_n$ for any n.*

This theorem is actually quite hard to prove, but since it is not closely related to the topic, we omit the proof.

Two more attempts to find the so called *bad pentagon* (see [15]) and the the *Desargues* configuration also resulted in failures.

All we can really do here is to describe certain arrangements whose image under a hypothetical continuous universal function is very restricted. They provide some hope that a contradiction will arise from them.

Fix the positive real numbers $x_6 < x_3$ and $y_6 < y_5$. We will define 8 points in $\mathbb{R}^2$:

- $P_{-1}(-1, 0)$
- $P_0(0, 0)$
- $P_6(x_6, y_6)$
- $P_5(x_6, y_5)$
- $P_4\left(x_3, \frac{x_3 y_5}{x_6}\right)$
- $P_3\left(x_3, \frac{(x_3+1)y_6}{x_6+1}\right)$

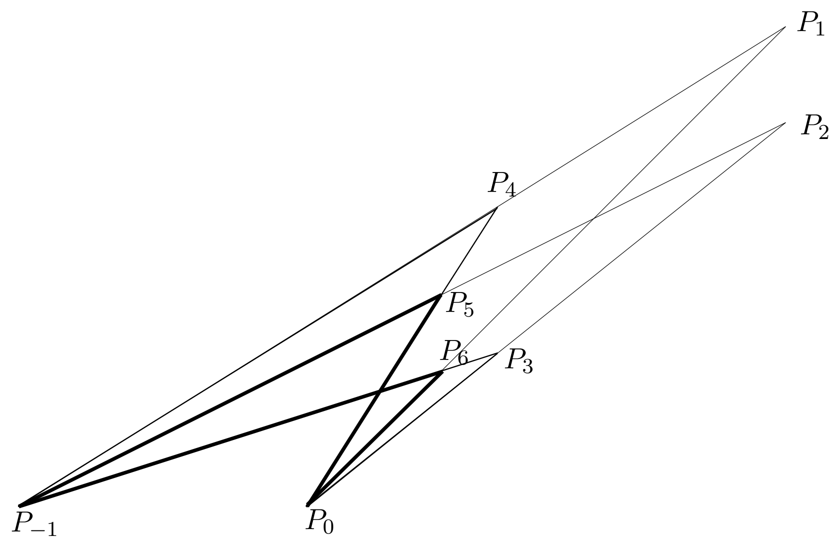


Figure 18: $\mathfrak{p}_1$ poset whose image is the Pappus configuration

- P_2 is the intersection of the line determined by the points P_{-1} and P_5 and the line determined by the points P_0 and P_3.
- P_1 is the intersection of the line determined by the points P_{-1} and P_4 and the line determined by the points P_0 and P_6.

An easy calculation gives that the x-coordinates of P_1 and P_2 are both

$$\frac{x_3 y_5}{x_3 y_6 - x_3 y_5 + y_6}.$$

Now let the line segment $l_{i,j}$ connect points P_i and P_j for $i = 1, \ldots, 6$ and $j = -1, 0$. See Figure 18. Since this representation is full of ties, we can not directly apply Lemma 3.3.6 to it. However, it is fairly intuitive, that the lemma can be applied to representations that are "close" to this one, and therefore get a strong conclusion about the image of $\hat{P}_0$. Therefore, its is easy to show the following.

Proposition 3.8.2. *Let f be a continuous universal function. Then f maps Figure 18 into a Pappus-like configuration, shown on Figure 19.*

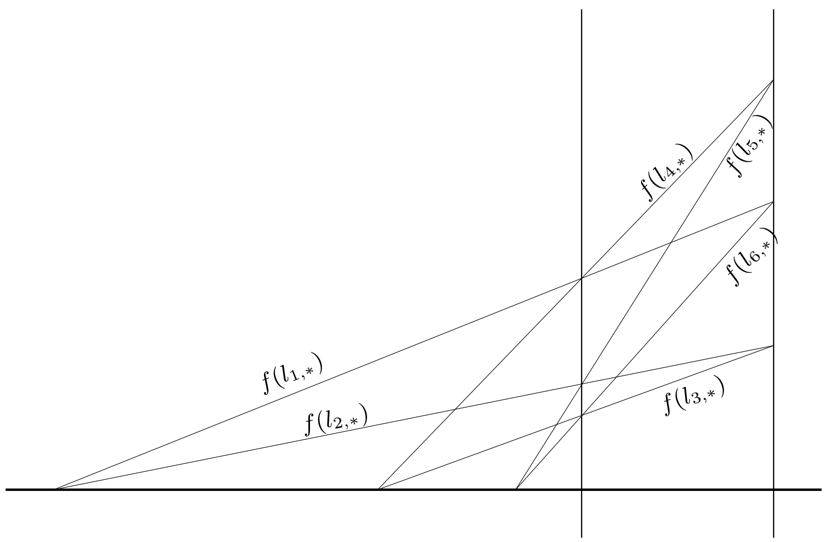


Figure 19: The Pappus $\mathfrak{p}_2$ poset

CHAPTER IV

MONOTONE HAMILTONIAN PATHS IN THE BOOLEAN LATTICE OF SUBSETS

4.1 Introduction

The n-dimensional hypercube, as a graph is one of the first standard examples of a Hamiltonian graph in college graph theory classes; a simple induction shows that a Hamiltonian cycle exists for every $n \geq 2$. However, if we follow any cycle like this, we will find, that the sizes of the sets along the cycle goes up and down and does not show any kind of monotone property. The question occurs: if we give up having a cycle, and we just want a Hamiltonian path, can we find such a path, for which every time we cover a set, we will have covered all the subsets of this set? This kind of path would start from the empty set, end in the set $\{1, \ldots, n\}$, and it would traverse the diagram of the subset lattice "from bottom to top".

It is immediately clear that this is impossible even for $n = 2$. So we relax the conditions even more. For this, we change our point of view, and from this point on, we consider the subset lattice of the set $\{1, \ldots, n\}$. The diagram of this lattice, as a graph, is isomorphic to the n-dimensional hypercube. We are searching for a Hamiltonian path $\{S_1, \ldots, S_{2^n}\}$ such that for every i, either

- every subset of S_i appears among the sets $S_1, \ldots, S_{i-1}$, or
- only one (say S) does not; in this case $S_{i+1} = S$.

If a Hamiltonian path has the above properties, we call it a *monotone* Hamiltonian path. If a path is not Hamiltonian, but it starts at the empty set and has the above properties, we call it a *monotone stub* or simply *stub* of a monotone Hamiltonian path. If a stub covers all the sets of at most size k, we call is a k-stub. Obviously, the existence of an n-stub is equivalent to the existence of a monotone Hamiltonian path. The answer for the following questions is not known:

1. Is there a monotone Hamiltonian path for every n in the n-cube?

2. If the answer is no, what is the largest k such that a k-stub of a monotone Hamiltonian path exists?

Trotter and Felsner [10] found some striking combinatorial connections related to these questions. Without going into much details, let us just mention the results. For basic definitions on interval orders and partial orders, see [25], and on graph theory, see [5].

Definition 4.1.1. *$C(t)$ is the largest integer h so that whenever P is an interval order of height h, the chromatic number of the diagram of P is at most t.*

Definition 4.1.2. *A sequence $(S_0, S_1, \ldots, S_h)$ of sets is an* α-sequence *if*

- *$S_0 \not\subseteq S_1$, and*
- *$S_j \not\subseteq S_i \cup S_{i+1}$ when $j > i + 1$*

$F(t)$ is the largest h for which there exists an α-sequence of subsets of $\{1, \ldots, t\}$.

Theorem 4.1.3. *For every $t \geq 1$*

$$C(t) = F(t) \leq 2^{t-1} + \left\lfloor \frac{t-1}{2} \right\rfloor,$$

with equality holding if and only there is monotone Hamiltonian path in the t-cube.

Another closely related problem is the famous "middle two level" conjecture posed by Ivan Havel. That problem is almost solved, see [17]. We do not know any specific connections, but both problems deal with the same kind of objects, so we can hope that solving one might help solving the other.

Trotter conjectured, that if n is sufficiently large, there is no monotone Hamiltonian path in the n-cube. He also conjectured that if n is sufficiently large, there is no 3-stub in the n-cube.

The existence of a k-stub for $k = 0, 1$ is absolutely trivial, and it is really easy to show that 2-stubs exist. The number of 2-stubs is enormous, but not all of them are prefixes of 3-stubs. In our construction, we have to be really careful how we cover the 2-sets, so that we can continue it to construct a 3-stub.

4.2 The existence of 3-stubs

Theorem 4.2.1. *For every n positive integer, a 3-stub exists in the subset lattice of $\{1, \ldots, n\}$.*

The existence of a monotone Hamiltonian path for $n \leq 10$ was checked with computers by Trotter and it has been rechecked by the authors. So in the following, we may assume that $n \geq 9$.

4.2.1 Construction of a 2-stub

Start the path with the following sequence: $\emptyset$, $\{1\}$, $\{1, 2\}$, $\{2\}$, $\{2, 3\}$, $\{3\}$, $\ldots$, $\{n\}$. So far, this is a 1-stub.

Now we will use a table to demonstrate how to cover the remaining 2-sets. In the table, we just list the 2-sets that are not covered so far without curly braces and commas to save space. When we connect two 2-sets in this table, say $\{a, b\}$ and $\{b, c\}$ (which will be denoted by `ab` and `bc`), we mean that the path goes from $\{a, b\}$ to $\{a, b, c\}$ and then to $\{b, c\}$. Observe, that this is only possible if the set $\{a, c\}$ has been covered. We will be careful to only connect subsets, if that is allowed. Here is our table:

```
13
14   24
15   25  35
...
1n   2n  3n  ...  n-2 n
```

Later we will refer to the rows of this table. When we refer to the first, second, etc. row, we will call them row 3, row 4, etc. respectively. So row k is the row that contains the 2-sets, whose largest element is k.

Observe, that if two sets are next to each other (up-down or left-right), it is always possible to connect them, because the missing 2-set that must have been covered is of the form $\{\alpha, \alpha + 1\}$, and these were covered in the 1-stub. Additional opportunities to connect sets arise as we go along the path. It is now fairly obvious, that we can construct 2-stubs

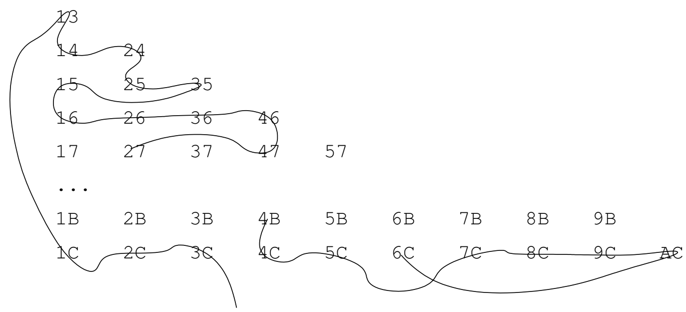


Figure 20: Construction of a 2-stub

in many ways. We will do it as demonstrated in Figure 20. In this figure we used $n = 12$ and we used the letters A, B and C in place of the numbers 10, 11 and 12.

In general, the demonstrated path is the following: after the set $\{n\}$, we cover $\{3, n\}$, $\{2, 3, n\}$, $\{2, n\}$, $\{1, 2, n\}$, $\{1, n\}$, $\{1, 3, n\}$ (it is allowed, because $\{3, n\}$ has been covered), $\{1, 3\}$, $\{1, 3, 4\}$, $\{1, 4\}$ and so on as demonstrated in Figure 20. After $\{2, 7\}$, we do the following. We cover the table row-by-row, going down to the next row only after we covered every 2-set in a given row. The order of the 2-sets within a row is arbitrary, with the following restrictions ($k \geq 7$, the $*$ sign denotes an arbitrary element):

- If a row ends with the sets $\{a, k\}$, $\{*, k\}$, $\{b, k\}$, then the next row starts with the sets $\{b, k + 1\}$, $\{a, k + 1\}$.
- The row before the last one ends with the sets $\{5, n - 1\}$, $\{*, n - 1\}$, $\{4, n - 1\}$.
- The last row ends with $\{6, n\}$.

When we cover row k, every row of index less than k has been covered, and this makes it possible that every jump within the row is allowed. So just by this fact, each row could be covered in an arbitrary order. The restrictions given above are important though in the construction of the 3-stub.

It is clear, that for large n there are still exponentially many ways to build a 2-stub that satisfies the requirements. For clarity, let us list the 2-sets covered in order for $n = 12$ in the *lexicographically first* stub. We will omit the curly braces, commas and we will not mention the implied 3-sets. We will use the letters A, B, C as before.

`3C, 2C, 1C, 13, 14, 24, 25, 35, 15, 16, 26, 36, 46, 47, 27, 17, 37, 57, 58, 18, 28, 38, 48, 68, 69, 39, 19, 29, 49, 59, 79, 7A, 4A, 1A, 2A, 3A, 6A, 5A, 8A, 8B, 6B, 1B, 2B, 3B, 6B, 5B, 7B, 4B, 4C, 5C, 7C, 8C, 9C, AC, 6C`

4.2.2 Construction of a 3-stub

The idea of this construction is the following. We will partition the non-covered 3-sets into "tables". Table k contains the set of non-covered 3-sets, whose largest element is k. So Table 3 contains only one element: $\{1, 2, 3\}$, Table 4 contains one again ($\{2, 3, 4\}$ – the sets $\{1, 2, 4\}$ and $\{1, 3, 4\}$ were covered during the beginning of the 2-stub), Table 5 consists of $\{1, 2, 5\}$, $\{1, 4, 5\}$ and $\{3, 4, 5\}$. We will traverse the tables one by one and in an increasing order.

Of course one problem is that the last 2-set is $\{6, n\}$, so we can not make a jump to $\{1, 2, 3\}$. This is how we go back: $\{1, 6, n\}$, $\{1, 2, 6, n\}$, $\{2, 6, n\}$, $\{2, 3, 6, n\}$, $\{3, 6, n\}$, $\{1, 3, 6, n\}$, $\{1, 3, 6\}$, $\{1, 2, 3, 6\}$, $\{1, 2, 3\}$. We encourage the reader to check that every step is valid. Note, that this is basically a "greedy" path; we swap out the elements of the set $\{6, n\}$ to the elements of the set $\{1, 2, 3\}$ as fast as possible.

Another important thing to observe that we only used some elements of Table n and one element of Table 6, so tables from 7 to $n - 1$ are unaffected.

Here is how we cover the 3-sets from Table 3 to Table 6. We will omit the the 4-sets between each pair of 3-sets for simplicity. The reader is encouraged to check the validity.

$\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 5\}$, $\{1, 4, 5\}$, $\{1, 2, 5\}$, $\{2, 5, 6\}$, $\{3, 5, 6\}$, $\{4, 5, 6\}$, $\{1, 4, 6\}$, $\{2, 4, 6\}$

Now we will show how to cover tables from 7 to $n - 1$. In order to do this, we will introduce a nice, compact way to write down the element of a table in an actual table. When we consider Table k, each element will contain k, so we will omit that element. In effect we will just write down 2-sets, without curly braces and commas.

We will use custom row and column indices. An element is row i and column j will be the element `ij`, denoting the set $\{i, j, k\}$.

Consider the way we covered row k in the construction of the 2-stub. Say the order of the 2-sets in that row is $\{\alpha_1, k\}, \{\alpha_2, k\}, \ldots, \{\alpha_{k-2}, k\}$. Index the columns in order with

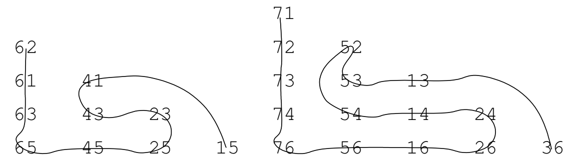


Figure 21: Table 7 on the left and Table 8 on the right

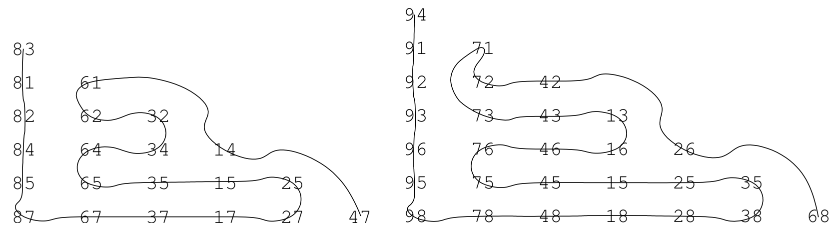


Figure 22: Table 9 on the left and Table 10 on the right

$k-1$, α_1, α_2, $\ldots$, α_{k-4}. Index the rows in order with α_2, α_3, $\ldots$, α_{k-2}. This way we get a $k-3 \times k-3$ table. Obviously, due to symmetry, only half of the table, say the lower triangle (including the diagonal) is necessary to consider. Observe, that in the beginning, the valid moves are exactly the moves between up-down or left-right neighbors in the table. However, while covering a table, new valid moves appear.

We need two different constructions to cover the tables: one kind for even indexed tables and one for odd. Instead of writing down how to cover the tables in an abstract way, let us just show the examples of Tables 7 to 10 for the case of $n = 12$ and the lexicographically first 2-stub. The examples will clearly show the pattern and how to generalize it. See Figures 21 and 22.

One can check that every step is valid inside a table. Also, we start to cover Table 7 with the set $\{2, 6, 7\}$, and this is possible, because the last element of Table 6 was $\{2, 4, 6\}$ and it is valid to move to $\{2, 4, 6, 7\}$ and then to $\{2, 6, 7\}$, because $\{2, 4, 7\}$ and $\{4, 6, 7\}$ were covered during the 2-stub. This is true in general, i.e. it is always possible to move from one table to the next one. We end Table k with the element $\{a, b, k\}$, where row k (covering the 2-stub) ended with the sets $\{a, k\}$, $\{*, k\}$, $\{b, k\}$ (lower right corner of the table). We start Table $k+1$ with the element $\{a, k, k+1\}$ (upper left corner of the table). The implied 4-set between them is $\{a, b, k, k+1\}$. The 3-sets that have to be covered to make this a valid step, are $\{a, b, k+1\}$ and $\{b, k, k+1\}$. The former was covered in the beginning of row $k+1$, which started as $\{b, k+1\}$, $\{a, k+1\}$ implying the missing set. The latter was covered between row k and row $k+1$; row k ended with $\{b, k\}$, row $k+1$ started with $\{b, k+1\}$, implying the missing set.

It remains to be shown how to traverse Table n. Remember that the table is special for two reasons: first row n was covered irregularly in the 2-stub, and second, because when we started to cover the 3-sets, and we had to "go back" to the set $\{1, 2, 3\}$, we used some entries of Table n.

The beauty of the construction is that these two forces neutralize each other. Remember, that the first three sets covered in row n were $\{3, n\}$, $\{2, n\}$ and $\{1, n\}$. Let us ignore these for a moment, and concentrate on the order the remaining sets were covered. Say this order is $\{\alpha_1, n\}$, $\ldots$, $\{\alpha_{n-5}, n\} = \{6, n\}$. Index the columns of Table n with $n-1$, α_1, $\ldots$,$\alpha_{n-5} = 6$, 1, and index the rows with α_2, $\ldots$, $\alpha_{n-5} = 6$, 1, 2, 3.

When we draw the lower triangular $n-3 \times n-3$ table now, it is not completely "accurate" for the following reasons:

- The moves between column 6 and column 1 are not valid.
- The set $\{1, 6, n\}$ is missing from the table. This is not an actual difference, because this set was covered in the beginning on the 3-stub.
- The sets $\{1, 3, n\}$, $\{2, 6, n\}$ and $\{3, 6, n\}$ are present on the table, but they are in fact covered in the beginning of the 3-stub.

We can easily take care of the last problem by dropping the corresponding elements from the table. As we mentioned, the second problem does not cause difficulty. Now look at the first problem. After we drop the elements mentioned in the third problem, the separated columns on the right are dropped, so the problem resolved itself.

Now it is really easy to traverse Table n, because we can use the exact same techniques that we used to traverse the previous tables. Let us just illustrate how to cover the table for $n = 12$ on Figure 23. The construction needs to be changed for different parity in the same way as it was for the smaller tables. We use the letters A and B as before.

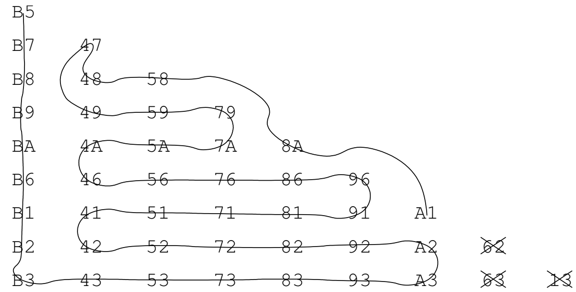


Figure 23: Table 12

4.3 The general problem

Of course the immediate question arises if this technique exhibited here is useful for solving the general problem. If nothing else, maybe for constructing a 4-stub. In our opinion, with hard work, it is probably possible to construct a 4-stub using the same ideas, but it is at least questionable if the gained insight would be enough to solve the general question.

This argument has several points that are very special for 3-stubs. The basic idea, that we cover the 2-sets "row by row" and then the 3-sets "table by table" does not work without modification. This is why we needed to do special things in the beginning of covering the 2-sets and even more special things in the beginning of covering the 3-sets. The fact that the two "tweaks" fit together perfectly, almost seems to be lucky.

We hope that this construction, or a simplification of this, can provide a general pattern, that will help answer the question if there is monotone Hamiltonian path of every subset lattice.

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