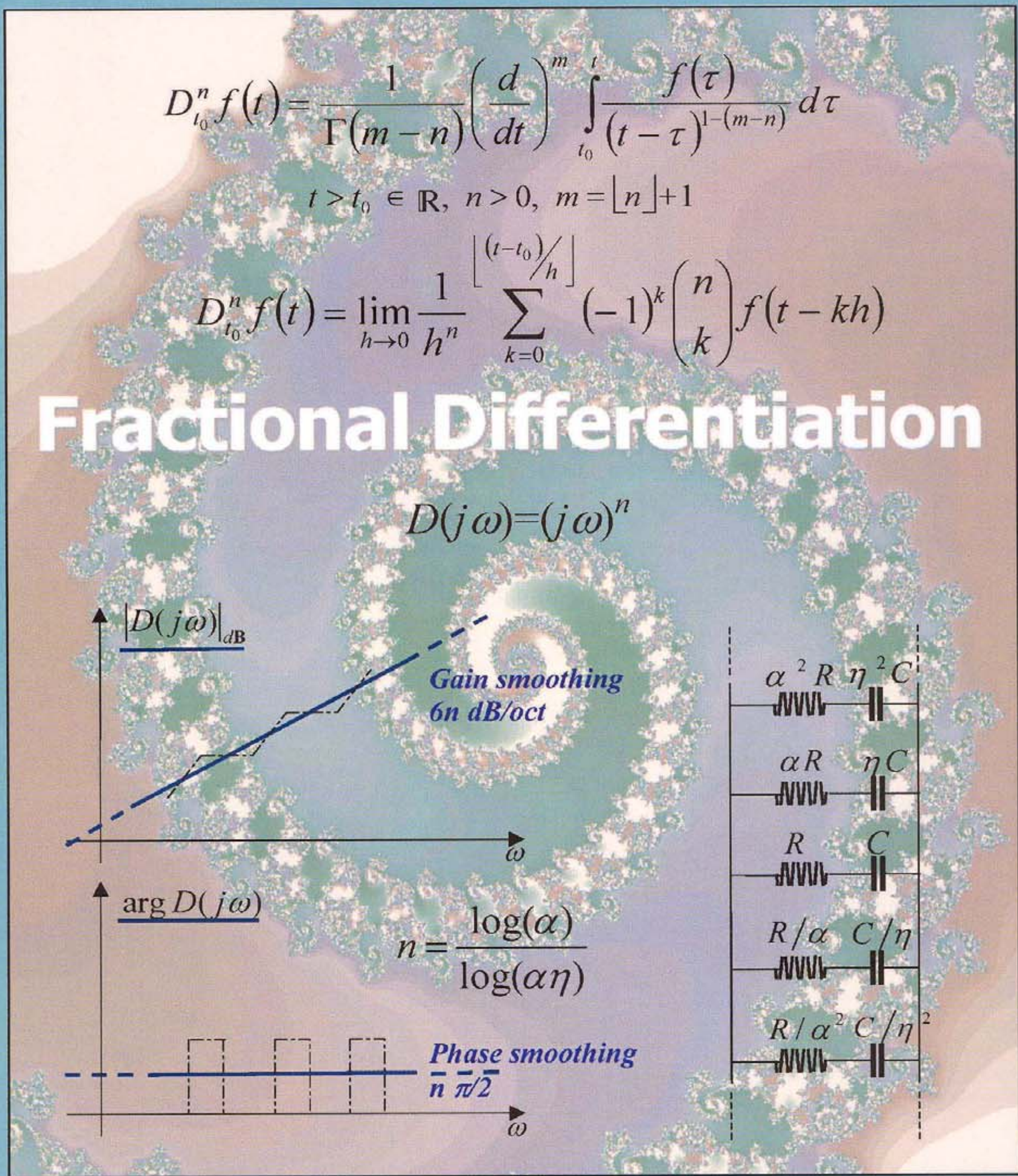
Fractional differentiation and its applications



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Design of Digital Fractional-Order Integrators and Differentiators by Least Squares

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Abstract — *In this paper we develop a method for obtaining digital rational approximations (IIR filters) to fractional-order operators of type s^α, where $\alpha \in \Re$. The proposed method is based on the least-squares (LS) minimization between the impulse responses of the digital fractional-order integrator/differentiator and of the rational-fraction approximation. The results reveal that the LS approach gives similar or superior approximations in comparison with other methods. The effectiveness of the method is demonstrated both in the time and frequency domains through an illustrative example.*

1 Introduction

In the literature we find several different definitions for the fractional-order operator $D^\alpha f(t)$, where the order α can be an arbitrary non-integer value [1, 2]. In this study we admit only values of $\alpha \in \Re$. From a control and signal processing perspective, the Grünwald-Letnikov definition [2] seems to be the most appropriate, particularly for a digital realization [3]. Furthermore, the definition poses fewer restrictions upon on the functions to which it is applied [1]. It is given by the expression:

$$D^\alpha f(t) = \lim_{h \to 0} \left\{ \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(t - kh) \right\} \tag{1}$$

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha + 1)}{\Gamma(k+1)\Gamma(\alpha - k + 1)} \tag{2}$$

where $f(t)$ is the applied function, $\Gamma(x)$ is the Gamma function and h the time increment. Note that (1) is defined by an infinite series revealing that the fractional-order operators are *global* operators and that have, implicitly, a *memory* of all past function values.

One of the mathematical tools commonly used for the analysis and synthesis of automatic control systems is the Laplace transform (LT). Fortunately, the generalization of the LT to a fractional-order is straightforward. For instance, the LT of a fractional derivative/integral of order α of the function $f(t)$, $D^\alpha[f(t)]$, under null initial conditions, is given by the simple expression:

$$L\{D^\alpha[f(t)]\} = s^\alpha F(s), \quad \alpha \in \Re \tag{3}$$

where $F(s) = L\{f(t)\}$. Note that (3) is a *direct* generalization of the classical integer-order scheme with the multiplication of the signal transform by the Laplace operator s. This means that frequency-based analysis methods have a straightforward adaptation to the fractional-order case.

The usual approach for obtaining discrete equivalents of the fractional-order operator s^α, $\alpha \in \Re$, adopts a generating function [4, 5]. By other words, given a continuous transfer function, $G(s)$, a discrete equivalent, $G(z)$, can be found by the substitution:

$$G(z) = G(s)\Big|_{s^\alpha = H^\alpha(z)} \tag{4}$$

where $H^\alpha(z)$ denotes the fractional discrete equivalent of order α of the fractional-order operator s^α, expressed as a function of the complex variable z or the shift operator z^{-1}. In these $s \rightarrow z$ conversion schemes (also called analog to digital open-loop design methods) we usually adopt either the Euler (or first backward difference) or the Tustin (or bilinear) generating functions [3]. Table 1 lists the two mentioned conversion methods that will be used in this study.

In general, the irrational functions $H^\alpha(z^{-1})$ (Table 1) are approximated either through polynomials or through rational functions ($i.e.$, the ratio of two polynomials). It is well known that rational approximations frequently converge faster than polynomial approximations and have a wider domain of convergence in the complex domain. In the work that follows, we develop rational approximations of the z variable to fractional-order operators of type s^α, $\alpha \in \Re$, which make them suited for $Z-$transform analysis and digital implementation.

Rational approximations $H_{m,n}(z^{-1})$ of m and n order to irrational transfer functions of type $H^\alpha(z^{-1})$ can be formally expressed as:

$$H^\alpha(z^{-1}) \approx \left[\frac{P_m(z^{-1})}{Q_n(z^{-1})}\right]_{m,n} = H_{m,n}(z^{-1}) \tag{5}$$

where P and Q are the polynomials of degree m and n, respectively.

Method	$H^\alpha(z^{-1})$
Euler Grünwald-Letnikov	$\left(\frac{1-z^{-1}}{T}\right)^\alpha$
Tustin	$\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha$

Table 1: $s \rightarrow z$ conversion schemes

In this paper we consider digital rational approximations of type (5) to fractional-order operators. In a first phase, we discretize the fractional-order operator s^α using one of the generating functions, yielding the irrational functions $H^\alpha(z^{-1})$, listed in Table 1, and we determine their impulse responses $h^\alpha(k)$. Then, in a second phase, we apply the least-squares (LS) minimization method to the impulse responses, $h^\alpha(k)$ and $h(k)$, of the digital fractional operator $H^\alpha(z^{-1})$ and of the digital rational approximation $H_{m,n}(z^{-1})$, respectively. We show that these new rational transfer functions of the z variable give better approximations, both in the time and frequency domains, than other approaches, namely the Padé or the continued fraction expansion (CFE) methods.

Bearing these ideas in mind, the paper is organized as follows. Section 2 derives the impulse responses of the fractional Euler/Tustin operators and section 3 gives an introduction to the signal modeling. Based on the previous results, sections 4 and 5 develop the Padé and the least-squares (LS) approximation methods, respectively. Section 6 presents an illustrative example showing the effectiveness of the proposed methods, both in the time and frequency domains. Finally, section 7 draws the main conclusions.

2 Impulse Response of Digital Fractional-Order Operators

In this section we derive the impulse responses $h^\alpha(k)$ of the fractional-order operators listed in Table 1. It is assumed that $h^\alpha(k) = 0$ for $k < 0$, $i.e.$, a causal system.

Expanding the Euler generating function $H_E^\alpha(z^{-1})$ into a power series in z^{-1}, we have:

$$\begin{aligned} H_E^\alpha(z^{-1}) &= \left[\frac{1}{T}\left(1 - z^{-1}\right)\right]^\alpha \\ &= \left(\frac{1}{T}\right)^\alpha \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} z^{-k} = \sum_{k=0}^{\infty} h_E^\alpha(k) z^{-k} \end{aligned} \tag{6}$$

where the impulse sequence $h_E^\alpha(k)$ is given by:

$$h_E^\alpha(k) = \left(\frac{1}{T}\right)^\alpha (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix}, \quad k \geq 0 \tag{7}$$

Developing a power series expansion (PSE), over the Tustin generating function $H_T^\alpha(z^{-1})$, we get:

$$\begin{aligned} H_T^\alpha(z^{-1}) &= \left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right)^\alpha \\ &= \left(\frac{2}{T}\right)^\alpha \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} \begin{pmatrix} -\alpha \\ k-j \end{pmatrix}\right] z^{-k} = \sum_{k=0}^{\infty} h_T^\alpha(k) z^{-k} \end{aligned} \tag{8}$$

where the impulse sequence $h_T^\alpha(k)$ is given by:

$$h_T^\alpha(k) = \left(\frac{2}{T}\right)^\alpha \sum_{j=0}^{k} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} \begin{pmatrix} -\alpha \\ k-j \end{pmatrix}, \quad k \geq 0 \tag{9}$$

Notice that the PSE method leads to impulse sequences of infinite duration. For a practically realizable form we need to truncate these sequences yielding approximations in the form of *finite impulse sequences* (FIR filters).

3 Signal Modeling

Consider that the impulsional response $h^{\alpha}(k)$ of the fractional-order operator is specified for $k \geq 0$. The desired rational function $H_{m,n}(z^{-1})$ that approximates the irrational transfer function $H^{\alpha}(z^{-1})$ has the form:

$$H_{m,n}\left(z^{-1}\right) = \frac{B\left(z^{-1}\right)}{A\left(z^{-1}\right)} = \frac{\sum\limits_{k=0}^{m} b_k z^{-k}}{1 + \sum\limits_{k=1}^{n} a_k z^{-k}} = \sum_{k=0}^{\infty} h\left(k\right) z^{-k} \tag{10}$$

where $h(k)$ is its impulse response. The rational approximation (10) has $m + n + 1$ parameters, namely the coefficients a_k ($k = 1, 2, \ldots, n$) and b_k ($k = 0, 1, \ldots, m$), which can be selected to minimize the sum of the squared errors:

$$J = \sum_{k=0}^{N-1} \left[h^{\alpha}\left(k\right) - h\left(k\right)\right]^2 \tag{11}$$

where N is the number of impulse values used in the summation. However, this approach leads to a nonlinear problem for the model parameters (a_k, b_k) and, consequently, the minimization of J involves the solution of a set of nonlinear equations.

If we rewrite (10) as $H_{m,n}(z^{-1})A(z^{-1}) = B(z^{-1})$, and assuming that $h^{\alpha}(k)$ is given approximately by the impulse response of $H_{m,n}(z^{-1})$, one can write the time domain equation of (10) as:

$$h^{\alpha}(k) + \sum_{l=1}^{n} a_l h^{\alpha}(k-l) = \begin{cases} b_k, & 0 \leq k \leq m \\ 0, & k > m \end{cases} \tag{12}$$

This gives a set of linear equations, which can be used in different ways to solve for the coefficients (a_k, b_k) [6, 7, 8]. Next, we consider the application of the Padé and the so-called least-squares approximation methods for the design of rational functions of type (10) to fractional-order integrators and differentiators. In our work [8] this study is extended to three linear suboptimal solutions to the problem, namely the herein presented Padé method, and the Prony and the Shanks methods.

4 Padé Approximation Method

The Padé approximation method yields an approximation that have an exactly match to $h^{\alpha}(k)$ for the first $m + n + 1$ values of k. Then, equation (12) becomes:

$$h^{\alpha}(k) + \sum_{l=1}^{n} a_l h^{\alpha}(k-l) = \begin{cases} b_k, & 0 \leq k \leq m \\ 0, & m+1 \leq k \leq m+n \end{cases} \tag{13}$$

where $h^{\alpha}(k) = 0$ for $k < 0$. All $m + n + 1$ equations (13) can be written simultaneously in the matrix form:

$$\left[\begin{array}{cccc} h^{\alpha}(0) & 0 & \cdots & 0 \\ h^{\alpha}(1) & h^{\alpha}(0) & \cdots & 0 \\ h^{\alpha}(2) & h^{\alpha}(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h^{\alpha}(m) & h^{\alpha}(m-1) & \cdots & h^{\alpha}(m-n) \\ \hline h^{\alpha}(m+1) & h^{\alpha}(m) & \cdots & h^{\alpha}(m-n+1) \\ \vdots & \vdots & \ddots & \vdots \\ h^{\alpha}(m+n) & h^{\alpha}(m+n-1) & \cdots & h^{\alpha}(m) \end{array}\right]\left[\begin{array}{c} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{array}\right]=\left[\begin{array}{c} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_m \\ \hline 0 \\ \vdots \\ 0 \end{array}\right] \tag{14}$$

In the Padé approximation method, two steps are used to solve for the coefficients a_k and b_k, first solving for the denominator coefficients a_k and then solving for the numerator coefficients b_k.

In the first step, solving for coefficients a_k, we use the last n equations of system (14), as indicated by the partitioning, which after simple manipulations, yields:

$$\left[\begin{array}{cccc} h^{\alpha}(m) & h^{\alpha}(m-1) & \cdots & h^{\alpha}(m-n+1) \\ h^{\alpha}(m+1) & h^{\alpha}(m) & \cdots & h^{\alpha}(m-n+2) \\ \vdots & \vdots & \ddots & \vdots \\ h^{\alpha}(m+n-1) & h^{\alpha}(m+n-2) & \cdots & h^{\alpha}(m) \end{array}\right]\left[\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array}\right]=-\left[\begin{array}{c} h^{\alpha}(m+1) \\ h^{\alpha}(m+2) \\ \vdots \\ h^{\alpha}(m+n) \end{array}\right] \tag{15}$$

$$\mathbf{H}_2\mathbf{a}=-\mathbf{h}_{21} \tag{16}$$

where **a** and $\mathbf{h}_{21}$ are $n \times 1$ vectors and $\mathbf{H}_2$ is an $n \times n$ nonsymmetric Toeplitz matrix. If $\mathbf{H}_2$ is nonsingular ($i.e.$, is invertible) then $\mathbf{H}_2^{-1}$ exists and the coefficients a_k are uniquely determined by:

$$\mathbf{a}=-\mathbf{H}_2^{-1}\mathbf{h}_{21} \tag{17}$$

After obtaining the coefficients a_k, the second step is to solve for the numerator coefficients b_k using the first $m+1$ equations in system (14), $i.e.$:

$$\left[\begin{array}{ccccc} h^{\alpha}(0) & 0 & 0 & \cdots & 0 \\ h^{\alpha}(1) & h^{\alpha}(0) & 0 & \cdots & 0 \\ h^{\alpha}(2) & h^{\alpha}(1) & h^{\alpha}(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h^{\alpha}(m) & h^{\alpha}(m-1) & h^{\alpha}(m-2) & \cdots & h^{\alpha}(m-n) \end{array}\right]\left[\begin{array}{c} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{array}\right]=\left[\begin{array}{c} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_m \end{array}\right] \tag{18}$$

$$\mathbf{b}=\mathbf{H}_1\bar{\mathbf{a}} \tag{19}$$

where **b** is an $(m+1)\times 1$ vector, $\bar{\mathbf{a}}=[1;\mathbf{a}]$ is an $(n+1)\times 1$ vector and $\mathbf{H}_1$ is an $(m+1)\times(n+1)$ matrix. Therefore, **b** may be found by simply multiplying $\bar{\mathbf{a}}$ by the matrix $\mathbf{H}_1$. Equivalently, the coefficients b_k may be evaluated using equation (13) as follows:

$$b_k = h^{\alpha}(k) + \sum_{l=1}^{n} a_l h^{\alpha}(k-l), \quad k = 0,\ 1,\ \ldots,\ m \tag{20}$$

In this way, we obtain a perfect match between $h(k)$ and the desired impulse response $h^{\alpha}(k)$ for the first $m+n+1$ values of the impulse sequence. The success of this method depends strongly on the number of selected model coefficients. Since the design method matches $h^{\alpha}(k)$ only up to the number of model parameters, the more complex the model, the better the approximation to $h^{\alpha}(k)$ for $0 \leq k \leq m+n$. However, in practical applications, this introduces a major limitation of the Padé method because the resulting approximation must contain a large number of poles and zeros.

It can be shown that rational approximations obtained by the CFE method are the same as those resulting by application of the Padé approximation to power series expansion ($m = n$) [9]. Nevertheless, the CFE approach is computationally less expensive than the Padé technique.

5 Least-Squares Approximation Method

In this section we develop a new approach to the problem, which is based on the standard *least-squares identification* algorithm [10].

The impulse response $h(k)$ of $H_{m,n}(z^{-1})$ to a unit sample input $\delta(k)$ corresponds to the expression:

$$h(k) + \sum_{l=1}^{n} a_l h(k-l) = \sum_{l=0}^{m} b_l \delta(k-l) \tag{21}$$

where $\delta(k-l) = 1$ (for $k = l$) and $\delta(k-l) = 0$ (for $k \neq l$) and $k = 0, 1, \ldots, N-1$ corresponding to a collect of N values from the input and output sequences.

Setting $h(k) = h^{\alpha}(k)$, expression (21) can be written in matrix form as:

$$h^{\alpha}(k) = \boldsymbol{\theta}^T \mathbf{x}(k), \quad k \geq 0 \tag{22}$$

where $\mathbf{x}(k)$ is the $(m+n+1) \times 1$ *state vector* and $\boldsymbol{\theta}$ the $(m+n+1) \times 1$ *parameter vector* defined as:

$$\mathbf{x}(k) = [-h^{\alpha}(k-1), ..., -h^{\alpha}(k-n), \delta(k), ..., \delta(k-m)]^T \tag{23}$$

$$\boldsymbol{\theta} = [a_1, a_2, \ldots, a_n, b_0, b_1, \ldots, b_m]^T \tag{24}$$

Let us introduce the matrix variables:

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}^T(0) \\ \mathbf{x}^T(1) \\ \vdots \\ \mathbf{x}^T(N-1) \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} h^{\alpha}(0) \\ h^{\alpha}(1) \\ \vdots \\ h^{\alpha}(N-1) \end{bmatrix} \tag{25}$$

If the system can be represented by equation (22) for some $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, then the vector of systems outputs becomes:

$$\mathbf{h} = \mathbf{X}\boldsymbol{\theta}^* \tag{26}$$

where $\mathbf{X}$ is an $N \times (m+n+1)$ matrix and $\mathbf{h}$ is an $N \times 1$ vector. For the construction of $\mathbf{X}$ we assume that the initial conditions of the system are zero, that is, $h^\alpha(k) = 0$ for $k < 0$.

Usually, $N >> m+n+1$ and we define the *error vector* $\mathbf{e}(\boldsymbol{\theta}) = \mathbf{h} - \mathbf{X}\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is a general parameter vector. Hence, the objective is to find an estimate $\boldsymbol{\theta}$ that minimizes:

$$J(\boldsymbol{\theta}) = \sum_{k=0}^{N-1} [e(k)]^2 = \mathbf{e}(\boldsymbol{\theta})^T \mathbf{e}(\boldsymbol{\theta}) = (\mathbf{h} - \mathbf{X}\boldsymbol{\theta})^T (\mathbf{h} - \mathbf{X}\boldsymbol{\theta}) \tag{27}$$

Solving $\partial J/\partial\boldsymbol{\theta} = 0$ we obtain the following system of *normal equations*:

$$\mathbf{X}^T\mathbf{X}\boldsymbol{\theta} = \mathbf{X}^T\mathbf{h} \tag{28}$$

If the matrix $\mathbf{X}^T\mathbf{X}$ is nonsingular, a unique solution of (28) exists and the optimum $\boldsymbol{\theta}$ is given by:

$$\boldsymbol{\theta} = \mathbf{X}^+\mathbf{h} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{h} \tag{29}$$

where $\mathbf{X}^+ = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$ is the *pseudoinverse* of $\mathbf{X}$.

It is clear from (29) that if $N = m+n+1$, the system reduces to an $N \times N$ square matrix and, consequently, the parameter vector $\boldsymbol{\theta}$ can be calculated simply by $\boldsymbol{\theta} = \mathbf{X}^{-1}\mathbf{h}$. We verify that, in this case, we get the same rational approximation as those obtained by the application of the Padé or the CFE methods.

6 Ilustrative Example

In this section we obtain rational approximation models $H_{m,n}(z^{-1})$ for the fractional-order operator s^α, with $\alpha = -1/2$, using the LS method described in the previous section. We consider the fractional Euler/Tustin operators, sampled at T = 0.01 s, $m = n = \{1, 3, 5, 7\}$, and $N = 1000$. The approximations are given in Tables 2 and 3.

For comparison purposes, we also plot the rational approximation obtained by the Padé method for $m = n = 5$, $G_{5,5}(z^{-1})$, for the Euler and Tustin operators, which are given in Table 4. We note that the sum of the coefficients of the numerator and denominator of the rational approximations $H_{m,n}(z^{-1})$ and $G_{5,5}(z^{-1})$ are approximately zero, that is:

$$\sum_{k=0}^{m} b_k = \sum_{k=0}^{n} a_k \approx 0 \tag{30}$$

Figures 1 and 2 depict the Bode diagrams and the step responses of the approximations $H_{m,n}(z^{-1})$, with $m = n = \{1, 3, 5, 7\}$ and $N = 1000$, for the Euler and the Tustin operators, respectively. Figures 3 and 4 show the results when we vary the length of the impulsional sequence $N = \{11, 100, 200, 500, 1000\}$ for a fixed order of the approximations, namely for $m = n = 5$.

It is clear that the higher the order $m = n$ (or the impulse sequence N) of the approximations the better the fitting (in a least-squares sense) both in the frequency and the step responses, of the fractional-order integrator $s^{-0.5}$. Furthermore, with the LS method we can tune the approximations for achieving better accuracy on a prescribed range of time t

(m, n)	**Rational function,** $H_{m,n}\left(z^{-1}\right)$
(1, 1)	$\dfrac{0.1-0.04397z^{-1}}{1-0.9397z^{-1}}$
(3, 3)	$\dfrac{0.1-0.161z^{-1}+0.06789z^{-2}-0.004722z^{-3}}{1-2.11z^{-1}+1.359z^{-2}-0.2479z^{-3}}$
(5, 5)	$\dfrac{0.1-0.2789z^{-1}+0.2792z^{-2}-0.1184z^{-3}+0.01861z^{-4}-0.0005166z^{-5}}{1-3.289z^{-1}+4.062z^{-2}-2.294z^{-3}+0.5644z^{-4}-0.04312z^{-5}}$
(7, 7)	$\dfrac{\begin{array}{c}0.1-0.3966z^{-1}+0.6293z^{-2}-0.5065z^{-3}+0.2155z^{-4}-0.04542z^{-5}\\+0.003798z^{-6}-0.000056456z^{-7}\end{array}}{\begin{array}{c}1-4.466z^{-1}+8.151z^{-2}-7.778z^{-3}+4.109z^{-4}-1.164z^{-5}\\+0.1544z^{-6}-0.006461z^{-7}\end{array}}$

Table 2: Approximations with LS method for the Euler operator

(m, n)	**Rational function,** $H_{m,n}\left(z^{-1}\right)$
(1, 1)	$\dfrac{0.07071+0.008843z^{-1}}{1-0.8749z^{-1}}$
(3, 3)	$\dfrac{0.07071+0.005302z^{-1}-0.05281z^{-2}-0.001747z^{-3}}{1-0.925z^{-1}-0.3218z^{-2}+0.2596z^{-3}}$
(5, 5)	$\dfrac{\begin{array}{c}0.07071+0.004728z^{-1}-0.09396z^{-2}-0.004356z^{-3}+0.02656z^{-4}\\+0.0005193z^{-5}\end{array}}{\begin{array}{c}1-0.933z^{-1}-0.8957z^{-2}+0.8007z^{-3}+0.1144z^{-4}\\-0.08461z^{-5}\end{array}}$
(7, 7)	$\dfrac{\begin{array}{c}0.07071+0.004737z^{-1}-0.1355z^{-2}-0.007165z^{-3}+0.0771z^{-4}\\+0.00279z^{-5}-0.0118z^{-6}-0.0001729z^{-7}\end{array}}{\begin{array}{c}1-0.933z^{-1}-1.483z^{-2}+1.348z^{-3}+0.5753z^{-4}-0.4936z^{-5}\\-0.04154z^{-6}+0.02785z^{-7}\end{array}}$

Table 3: Approximations with LS method for the Tustin operator

Method	**Rational function,** $G_{5,5}\left(z^{-1}\right)$
Euler	$\dfrac{0.1-0.225z^{-1}+0.175z^{-2}-0.05469z^{-3}+0.005859z^{-4} -0.00009766z^{-5}}{1-2.75z^{-1}+2.75z^{-2}-1.203z^{-3}+0.2148z^{-4} -0.01074z^{-5}}$
Tustin	$\dfrac{0.07071+0.03536z^{-1}-0.07071z^{-2}-0.02652z^{-3}+0.01326z^{-4} +0.00221z^{-5}}{1-0.5z^{-1}-z^{-2}+0.375z^{-3}+0.1875z^{-4} -0.03125z^{-5}}$

Table 4: Approximations with Padé method for $m=n=5$

(or frequency ω) in contrast with other approximations that matches only the initial-time transient corresponding to the high frequency range.

Figure 5 shows the pole-zero map of the approximations $H_{m,n}(z^{-1})$, with $m=n=\{1, 3, 5, 7\}$, for the Euler and Tustin operators. We observe that the distribution of the zeros and poles satisfies two desired properties: (i) all the poles and zeros lie inside the unit circle and (ii) they are interlaced along the segment of the real axis, corresponding to $z \in]0,\ 1[$ and $z \in]-1,\ 1[$ for the Euler and Tustin operators, respectively.

In conclusion, the proposed LS method provides causal, stable and minimum-phase rational approximations as imposed for a digital realization. Its superior nature, in comparison with the Padé and the CFE approximation methods, is illustrated in the case of typical paradigms. The results presented here seem to indicate that the LS approach is a suitable technique for obtaining discrete approximations of the fractional-order operators.

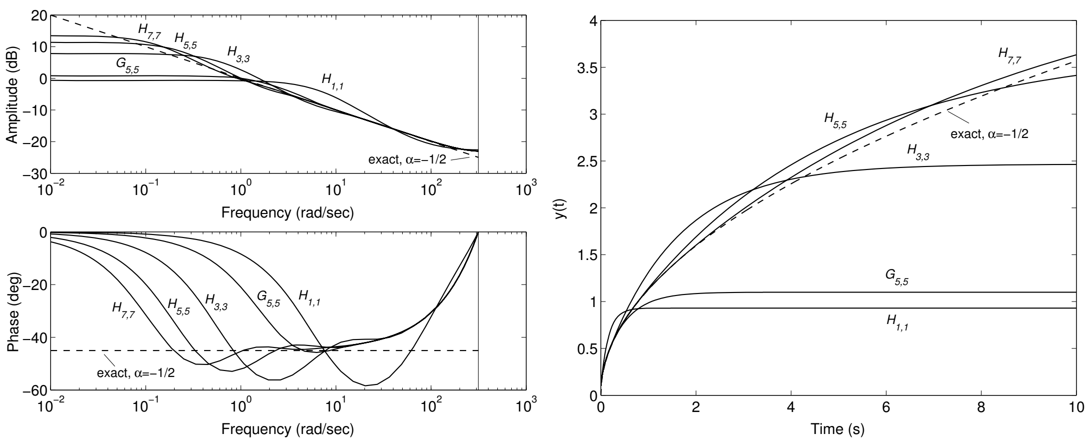

Figure 1: Bode diagrams (left) and step responses (right) of the LS approximation $H_{m,n}(z^{-1})$, $m=n=\{1,\ 3,\ 5,\ 7\}$, *vs.* the Padé approximation $G_{5,5}(z^{-1})$ for the Euler operator with $\alpha=-1/2$ and $N=1000$

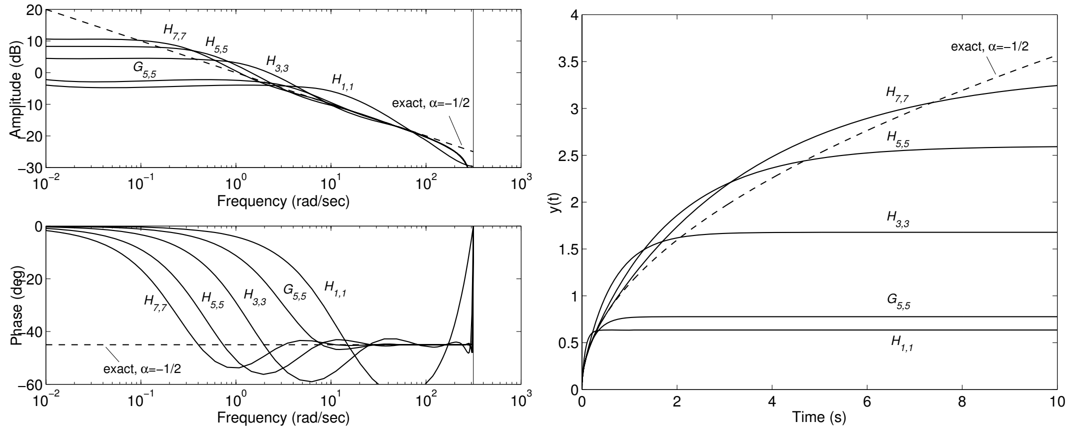

Figure 2: Bode diagrams (left) and step responses (right) of the LS approximation $H_{m,n}(z^{-1})$, $m = n = \{1, 3, 5, 7\}$, *vs.* the Padé approximation $G_{5,5}(z^{-1})$ for the Tustin operator with $\alpha = -1/2$ and $N = 1000$

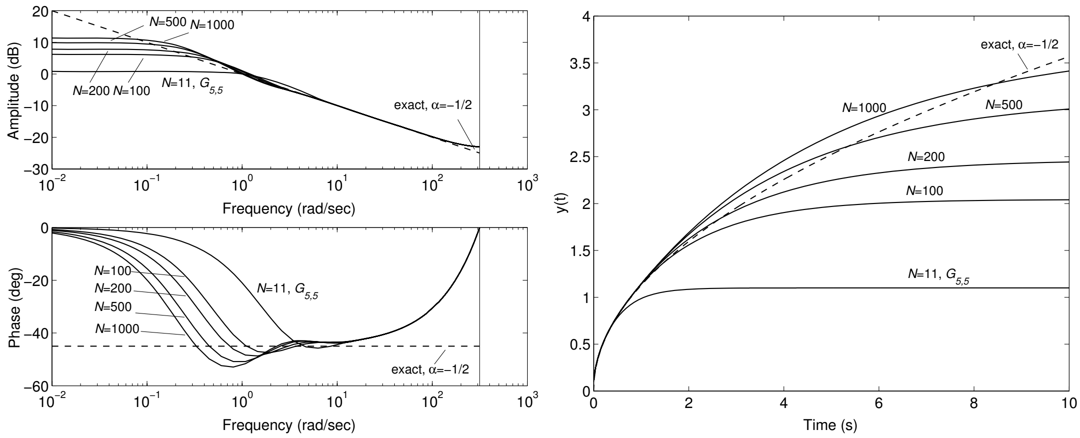

Figure 3: Bode diagrams (left) and step responses (right) of the LS approximation $H_{5,5}(z^{-1})$, *vs.* the Padé approximation $G_{5,5}(z^{-1})$ for the Euler operator with $\alpha = -1/2$ and $N = \{11, 100, 200, 500, 1000\}$

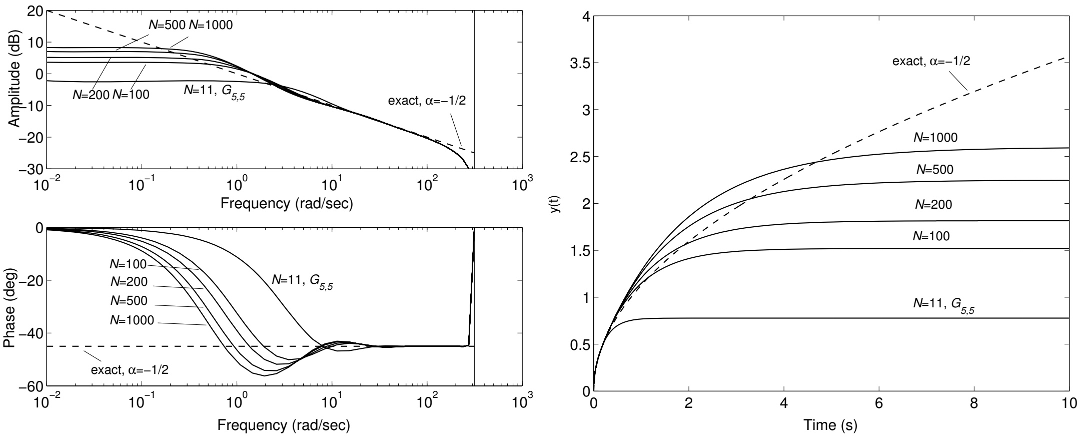

Figure 4: Bode diagrams (left) and step responses (right) of the LS approximation $H_{5,5}(z^{-1})$, *vs.* the Padé approximation $G_{5,5}(z^{-1})$ for the Tustin operator with $\alpha = -1/2$ and $N = \{11, 100, 200, 500, 1000\}$

7 Conclusions

We have described the adoption of the LS approach in the design of digital rational transfer functions that approximates fractional-order operators of type s^{α}, $\alpha \in \Re$. The method was illustrated for a fractional-order integrator ($\alpha = -1/2$), but it can be generalized to other real non-integer values. It was shown that the new discrete rational functions give better results, both in the time and frequency domains, than other approaches used for the same purpose, namely the Padé or the CFE approximations. Furthemore, the LS method yields causal, stable and minimum-phase rational transfer functions suitable for real-time implementation. In this line of thought, this paper represents a step towards the implementation of practical digital fractional-order differentiators and integrators.

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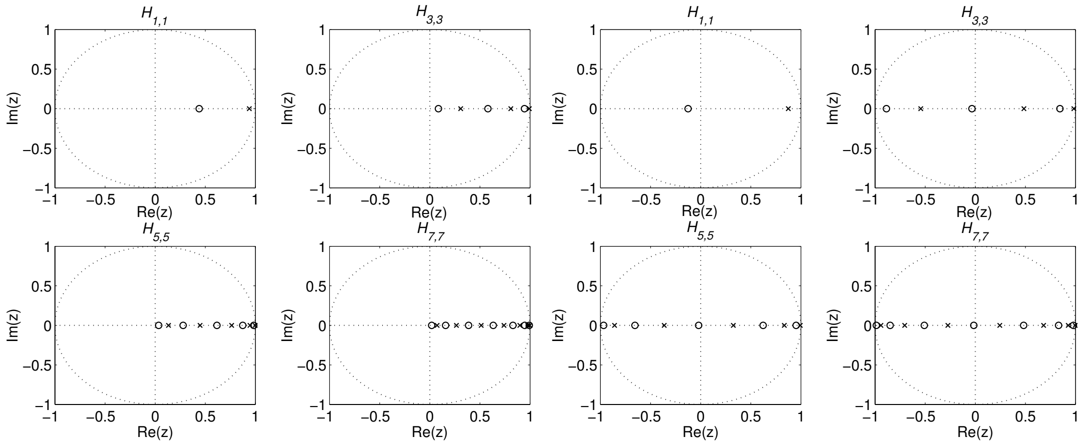

Figure 5: Pole-zero map of the LS approximation $H_{m,n}(z^{-1})$, $m = n = \{1, 3, 5, 7\}$ for the Euler (left) and Tustin (right) operators with $\alpha = -1/2$ and $N = 1000$

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