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BIPEDAL LOCOMOTION: A FRACTIONAL CPG FOR GENERATING PATTERNS

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Abstract: There has been an increase interest in the study of animal locomotion. Many models for the generation of locomotion patterns of different animals, such as centipedes, millipedes, quadrupeds, hexapods, bipeds, have been proposed. The main goal is the understanding of the neural bases that are behind animal locomotion. In vertebrates, goal-directed locomotion is a complex task, involving the central pattern generators located somewhere in the spinal cord, the brainstem command systems for locomotion, the control systems for steering and control of body orientation, and the neural structures responsible for the selection of motor primitives.
In this paper, we focus in the neural networks that send signals to the muscle groups in each joint, the so-called central pattern generators (CPGs). We consider a fractional version of a CPG model for locomotion in bipeds. A fractional derivative $D^{\alpha} f(x)$, with α non-integer, is a generalization of the concept of an integer derivative, where $\alpha = 1$. The integer CPG model has been proposed by Golubitsky, Stewart, Buono and Collins, and studied later by Pinto and Golubitsky. It is a four cells model, where each cell is modelled by a system of ordinary differential equations. The coupling between the cells allows two independent permutations, and, as so, the system has D_2 symmetry. We consider $0 < \alpha \le 1$ and we compute, for each value of α, the amplitude and the period of the periodic solutions identified with two legs' patterns in bipeds. We find that the amplitude and the period increase as α varies from zero up to one.

1. Introduction

Animal locomotion is a complex mechanism that involves not only the neural networks located in the spinal cord (CPGs), that are capable of producing the rhythms associated to different animal gaits, but but also the brainstem command systems, the steering and posture control systems, and in a top layer, the structures that decide which motor pattern is to be activated at each moment of time (Grillner *et al*, 2008).

In this study, we focus mainly on the rhythmic generators – the CPGs. Mathematically, CPGs consist of networks of identical systems of ordinary differential equations, where the individual systems model (collections of) neurons (or cells) (Collins & Stewart, 1993a; Collins & Stewart, 1993b; Collins & Stewart, 1994; Kopell & Ermentrout, 1988; Golubitsky et al, 1998; Golubitsky et al, 1999). In the field of robotics, there has been an increase use of systems of coupled oscillators, inspired in animal CPGs, to control locomotion in robots (Taga *et al*, 1991; Liu *et al*, 2008; Baydin, 2008; Righetti & Ijspeert, 2008).. The interesting properties of these dynamical systems, such as limit cycle behaviour, phase locking modes, robustness against small perturbations, smooth online modulation of trajectories, by changes in parameter values of the equations, and, most important, their ability to produce sustained rhythmic activation patterns even when isolated from external stimuli, make them an attractive option to model the control of legged robot locomotion for bipedal, quadrupedal and other designs. ((Marder & Bucher, 2001; Matos *et al*, 2009, Righetti *et al*, 2006).

Golubitsky, Stewart, Buono and Collins (Golubitsky *et al*, 1998; Golubitsky *et al*, 1999) propose models for locomotion CPGs for $2n$ -legged animals, for all values of $n \in \aleph$. These networks consist of $4n$ coupled identical cells, arranged in two cycles of $2n$ cells, that generate the rhythms seen in the animal gaits. These phase relations are explained by the bilateral symmetry of animals and the cyclic translational symmetry presented in the model. Similar network architectures also generate locomotion patterns of legless animals, such as worms, snakes and lampreys, where the term "leg" should be interpreted as "muscular unit". These new structures for CPG models are derived assuming that each joint receives signals from two neurons. This reasoning draws an analogy with muscular action in real joints, since two muscle groups, flexors and extensors, control most joints. Thus, abstractly, CPG models control muscle groups rather than legs and minimal CPG networks should have two cells per leg.

The network architecture *LEG* for the central pattern generator for bipedal legs rhythms studied here is shown in Fig.1.
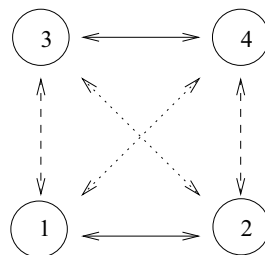

Fig. 1. CPG network *LEG* for the control of biped legs. Cells 1 and 3 send signals to the left leg, cells 2 and 4 send signals to the right leg.

The CPG network *LEG* consists of four coupled identical systems of ordinary differential equations. Each system models a cell (neuron). In the graph of Fig. 1, the nodes represent the cells and the arrows represent the couplings. The existence of three different arrows means three distinct coupling strengths. This network is capable of producing periodic solution types identified with biped locomotion rhythms, namely, *walk*, *run*, *two-legged jump*, *two-legged hop*, *hesitation-walk, asymmetric hop, skip, gallop, and one-legged hop* (Pinto *et al,* 2006).

In this paper we consider a fractional version of the CPG *LEG* represented in Fig.1. We vary the order of α and we analyze the amplitude and period changes of two periodic solutions produced by the four-cell CPG model and identified with legs' rhythms in bipeds in the *run* and in the *two-legged jump*.

2. Fractional calculus - summary

Fractional calculus (FC) is a generalization of the ordinary integer differentiation and integration to an arbitrary order (Oldham & Spanier, 1974; Samko et al, 1993, Miller & Ross, 1993). The subject was initiated in 1695 by Leibniz that sent a letter to L'Hospital with the question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?". In the last two decades we witnessed an increasing interest in the FC and relevant applications emerged in the areas of physics and engineering (Outstaloup, 1991; Mainardi, 1996, Machado , 1997; Nigmatullin, 2006; Podlubny, 1999; Tenreiro Machado, 2001; Chen & Moore, 2002; Baleanu, 2009; Tenreiro Machado, 2009).

There are several definitions of fractional derivatives, being three of the most important the Riemann - Liouville, the Grunwald - Letnikov, and the Caputo given by:

$$_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \, n-1 < \alpha < n \tag{1}$$

$$_aD_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh) \tag{2}$$

$$_aD_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \, n-1 < \alpha < n \tag{3}$$

where $\Gamma(\ )$ is the Euler's gamma function, $[x]$ means the integer part of x, and h is the step time increment.

It is also possible to generalize several results based on transforms, yielding expressions such as the Laplace expression:

$$L\left\{_0 D_t^\alpha f(t)\right\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^k \, _0 D_t^{\alpha-k-1} f\left(0^+\right)$$ (4)

where s and L represent the Laplace variable and operator, respectively.

These definitions demonstrate that fractional derivatives capture the history of the variable, or, by other words, have memory, contrary to integer derivatives, that are local operators.

The Grunwald - Letnikov formulation inspires the numerical calculation of the fractional derivative based on the approximation of the time increment h through the sampling period T and the series truncation at the rth term. This method is often denoted as Power Series Expansion (PSE) yielding the equation in the z – domain:

$$Z\{D^\alpha x(t)\} \approx \left[\frac{1}{T^\alpha} \sum_{k=0}^{r} \frac{(-1)^k \Gamma(\alpha+1)}{k! \, \Gamma(\alpha-k+1)} z^{-k}\right] X(z)$$ (5)

where $X(z) = Z\{x(t)\}$ and z and Z represent the z-transform variable and operator, respectively.

In fact, expression (2) represents the Euler (or first backward difference) approximation in the $s \to z$ discretization scheme, being the Tustin approximation another possibility. The Euler and Tustin rational expressions, $H_0\left(z^{-1}\right) = \frac{1}{T}\left(1 - z^{-1}\right)$ and $H_1\left(z^{-1}\right) = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$ respectively, are often called generating approximants of zero and first order, respectively. Therefore, the generalization of these conversion methods leads to the non-integer order results:

$$s^\alpha \approx \left[\frac{1}{T}\left(1 - z^{-1}\right)\right]^\alpha = H_0^\alpha\left(z^{-1}\right)$$ (6.a)

$$s^\alpha \approx \left(\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}\right)^\alpha = H_1^\alpha\left(z^{-1}\right)$$ (6.b)

We can obtain a family of fractional differentiators generated by $H_0^\alpha\left(z^{-1}\right)$ and $H_1^\alpha\left(z^{-1}\right)$ weighted by the factors p and $1 - p$, yielding:

$$s^\alpha \approx p H_0^\alpha\left(z^{-1}\right) + (1-p) H_1^\alpha\left(z^{-1}\right)$$ (7)

In order to get a rational expression, the final approximation corresponds to a PSE or a rational fraction expansion. This approach is often denoted by Continued Fraction Expansion (CFE) of order $k \in \aleph$, based on a Padé expansion in the neighborhood of $z^{-1} = 0$, yielding:

$$H_k\left(z^{-1}\right) = \frac{\sum_{i=0}^{k} a_i z^{-i}}{\sum_{i=0}^{k} b_i z^{-i}}, \quad a_i, b_i \in \mathfrak{R} \tag{8}$$

Since one parameter is linearly dependent, usually it is established that $b_0 = 1$.

3. CPG model for bipeds - review

In this chapter we review the work of Pinto and Golubitsky (Pinto & Golubitsky, 2006) for the CPG model for legs rhythms in bipeds. We state the general class of differential equations corresponding to *LEG* and we review the so-called H/K theory (Golubitsky & Stewart, 2002) that helps to identify periodic solutions produced by CPG *LEG* with known biped locomotion patterns.

3.1 .CPG *LEG* – equations and symmetries

The class of ordinary differential equations (ODEs) governing the CPG model in Fig.1 is the following:

$$\begin{aligned}
\dot{x}_1 &= F(x_1, x_2, x_3, x_4) \\
\dot{x}_2 &= F(x_2, x_1, x_4, x_3) \\
\dot{x}_3 &= F(x_3, x_4, x_1, x_2) \\
\dot{x}_4 &= F(x_4, x_3, x_2, x_1)
\end{aligned} \tag{9}$$

where $x_i \in \mathfrak{R}^k$ is the cell $i = 1, 2, 3, 4$ variables, $k \in \mathbb{N}$ is the dimension of the cells' internal dynamics, and $F : \left(\mathfrak{R}^k\right)^4 \to \mathfrak{R}^k$ is an arbitrary mapping. F is the same for all cells, since all cells are assumed to be identical.

The coupled cell system in (1) allows two independent transpositions, namely, $\rho = (12)(34)$, that switches muscle groups between legs, and $\tau = (13)(24)$, that switches muscle groups in each leg. These two independent permutations generate the symmetry group of order 4, denoted as D_2 and given by:

$$D_2 = Z_2(\rho) \times Z_2(\tau) \tag{10}$$

This CPG model *LEG* is capable of producing eight periodic solutions corresponding to the bipedal gaits of *two-legged hop*, *two-legged jump*, *walk*, *run*, *one-legged hop*, *hesitation walk*, *gallop*, *skip*, and *asymmetric hop*. See Pinto and Golubitsky (Pinto & Golubitsky, 2006), for the reasoning of this identification.

3.2. The H/K theory

The identification of the periodic solutions of the network model *LEG* with biped leg rhythms is done using symmetry arguments. The H/K theorem (Golubitsky & Stewart, 2002) uses the symmetry group of a network to enumerate those pairs of spatiotemporal symmetry subgroups H and K that can correspond to periodic solutions of differential equations associated to the network. These solutions are then identified with known biped locomotion patterns.

Let $x(t)$ be a periodic solution of a coupled cell system with symmetry group Γ. Define $K \subseteq \Gamma$ as the subgroup of all spatial symmetries and $H \subseteq \Gamma$ as the subgroup of all spatiotemporal symmetries. Mathematically, we have:

$$K = \left\{ \gamma \in \Gamma : \gamma x(t) = x(t), \forall t \right\}$$
$$H = \left\{ \gamma \in \Gamma : \gamma \left\{ x(t) \right\} = \left\{ x(t) \right\}, \forall t \right\}$$

(12)

In what follows, we discuss how symmetries of periodic solutions (H and K), in the CPG model *LEG*, can correspond to the biped gaits of *run*. A similar reasoning applies to the other bipedal gaits.

Run is a gait in which the two legs move half period out-of-phase. Moreover, muscle groups in each joint move in unison (Mann, 1982; Mann et al, 1986). This can be explained by the *run* symmetry group pair $\left(H, K \right) = (D_2, \tau)$. The permutation τ is a *spatial* symmetry for the *run*. This implies that muscles inside each leg receive the same signal, that is, cells 1 and 3 receive the same signal, and analogously for cells 2 and 4. As D_2 is a spatiotemporal symmetry for the run, this forces the two distinct signals to be phase shifted by one half period. Thus, cells in the left leg joints and cells in the right leg joint receive the same signal but with a phase shift of one half period.

For H and K to correspond to symmetries of periodic solutions of CPG *LEG*, there are some algebraic properties to be satisfied, that simplify for this system, namely, H/K must be a cyclic group (Pinto & Golubitsky, 2006, and references therein).

To conclude, we note that the H/K theory may be used to infer the CPG structure, for locomotion of $2n$-legged animals, from the desired types of periodic solutions. Moreover, Buono & Golubitsky (Buono & Golubitsky, 2001) have used this mathematical tool to prove that there is only one CPG model with eight cells capable of producing periodic solution types identified with the quadruped walk, trot and pace.

3.3. Results and discussion

We simulate the fractional version of the coupled cell system (9), given by:

$$D^{\alpha} x_1 = F(x_1, x_2, x_3, x_4)$$
$$D^{\alpha} x_2 = F(x_2, x_1, x_4, x_3)$$
$$D^{\alpha} x_3 = F(x_3, x_4, x_1, x_2)$$
$$D^{\alpha} x_4 = F(x_4, x_3, x_2, x_1)$$
(13)

where D^{α}, with α non-integer, is a generalization of the concept of an integer derivative, where $\alpha = 1$.

We consider the Morris-Lecar equations (Morris & Lecar, 1981) as internal cell dynamics. The coupling is linearly diffusive. The nondimensionalized Morris-Lecar equations (Rinzel & Ermentrout, 1989) are a system of two ordinary differential equations given by:

$$\dot{v} = -g_{Ca} m(v)(v-1) - g_1(v - v_1) - g_k w(v - v_k) + i = f(v, w)$$
$$\dot{w} = \phi \tau(v)(n(v) - w) = g(v, w)$$
(14)

where $m(v) = \dfrac{1}{2}\left(1 + \tanh\left(\dfrac{v - v_1}{v_2}\right)\right)$, $n(v) = \dfrac{1}{2}\left(1 + \tanh\left(\dfrac{v - v_3}{v_4}\right)\right)$, $\tau(v) = \cosh\left(\dfrac{v - v_3}{2 v_4}\right)$.

We adopt the PSE method for the approximation of the fractional derivative in the discrete time numerical integration. However, several experiments demonstrated that it was required a slight adaption to the standard approach based on a simple truncation of the series. In fact, since our objective is to generate limit cycles, the truncation corresponds to a diminishing of the gain (Tenreiro Machado, 2009) and, consequently, leads to difficulties in the promotion of periodic orbits. Therefore, in order to overcome this limitation, we decided to include a gain adjustment factor corresponding to the sum of the truncated series coefficients. In this line of thought, since the sum of the coefficients from $k = 1$ up to $k = \infty$ must be 1.0 (Tenreiro Machado, 2009), the fractional derivative approximation adopted in the sequel is:

$$Z\{D^{\alpha} x(t)\} \approx \left(\sum_{k=1}^{r} \frac{(-1)^k \Gamma(\alpha+1)}{k! \, \Gamma(\alpha-k+1)} \right)^{-1} \left[\frac{1}{T^{\alpha}} \sum_{k=0}^{r} \frac{(-1)^k \Gamma(\alpha+1)}{k! \, \Gamma(\alpha-k+1)} z^{-k} \right] X(z)$$
(15)

The dynamics governing the fractional CPG model *LEG* is given by:

$$x_1(k+1) = H(x_1(k)) + \alpha^{\Delta t}\left(f(x_1(k),y_1(k)) - k_{21}(x_1(k)-x_2(k)) - k_{11}(x_1(k)-x_3(k)) - k_{31}(x_1(k)-x_4(k))\right)$$

$$y_1(k+1) = y_1(k) + \alpha^{\Delta t} g(x_1(k),y_1(k))$$

$$x_2(k+1) = H(x_2(k)) + \alpha^{\Delta t}\left(f(x_2(k),y_2(k)) - k_{22}(x_2(k)-x_1(k)) - k_{11}(x_2(k)-x_4(k)) - k_{31}(x_2(k)-x_3(k))\right)$$

$$y_2(k+1) = y_2(k) + \alpha^{\Delta t} g(x_2(k),y_2(k))$$

$$x_3(k+1) = H(x_3(k)) + \alpha^{\Delta t}\left(f(x_3(k),y_3(k)) - k_{21}(x_3(k)-x_4(k)) - k_{12}(x_3(k)-x_1(k)) - k_{32}(x_3(k)-x_2(k))\right) \tag{16}$$

$$y_3(k+1) = y_3(k) + \alpha^{\Delta t} g(x_3(k),y_3(k))$$

$$x_4(k+1) = H(x_4(k)) + \alpha^{\Delta t}\left(f(x_4(k),y_4(k)) - k_{22}(x_4(k)-x_3(k)) - k_{12}(x_4(k)-x_2(k)) - k_{32}(x_4(k)-x_1(k))\right)$$

$$y_4(k+1) = y_4(k) + \alpha^{\Delta t} g(x_4(k),y_4(k))$$

where $k_{ij} \in \Re$ are the coupling constants and Δt is the time increment.

The parameter values of the Morris-Lecar equations, in the numerical simulations, are chosen to be $v_1 = 0.1$, $v_2 = 0.4$, $v_3 = 0.3$, $v_4 = 0.2$, $g_l = 0.6$, $g_k = 1.8$, $v_l = -1.8$, $v_k = 0.8$ and $i = 1$.

Table 1. Initial conditions for the bipedal gaits of *two-legged jump*, and *run*, where $(x_i(k), y_i(k))$ represents the initial condition of cell i at sample k.

Gait	Initial conditions
two-legged jump	$x_1(1)=0.2911; x_1(2)=0.2886; x_1(3)=0.2861; x_1(4)=0.2835; x_1(5)=0.2810;$ $y_1(1)=0.4319; y_1(2)=0.4326; y_1(3)=0.4332; y_1(4)=0.4337; y_1(5)=0.4340;$ $x_2(1)=0.2911; x_2(2)=0.2886; x_2(3)=0.2861; x_2(4)=0.2835; x_2(5)=0.2810;$ $y_2(1)=0.4319; y_2(2)=0.4326; y_2(3)=0.4332; y_2(4)=0.4337; y_2(5)=0.4340;$ $x_3(1)=0.0470; x_3(2)=0.0477; x_3(3)=0.0484; x_3(4)=0.0493; x_3(5)=0.0502;$ $y_3(1)=0.1497; y_3(2)=0.1482; y_3(3)=0.1468; y_3(4)=0.1453; y_3(5)=0.1440;$ $x_4(1)=0.0470; x_4(2)=0.0477; x_4(3)=0.0484; x_4(4)=0.0493; x_4(5)=0.0502;$ $y_4(1)=0.1497; y_4(2)=0.1482; y_4(3)=0.1468; y_4(4)=0.1453; y_4(5)=0.1440;$
Run	$x_1(1)=0.2911; x_1(2)=0.2886; x_1(3)=0.2861; x_1(4)=0.2835; x_1(5)=0.2810;$ $y_1(1)=0.4319; y_1(2)=0.4326; y_1(3)=0.4332; y_1(4)=0.4337; y_1(5)=0.4340;$ $x_2(1)=0.0470; x_2(2)=0.0477; x_2(3)=0.0484; x_2(4)=0.0493; x_2(5)=0.0502;$ $y_2(1)=0.1497; y_2(2)=0.1482; y_2(3)=0.1468; y_2(4)=0.1453; y_2(5)=0.1440;$ $x_3(1)=0.2911; x_3(2)=0.2886; x_3(3)=0.2861; x_3(4)=0.2835; x_3(5)=0.2810;$ $y_3(1)=0.4319; y_3(2)=0.4326; y_3(3)=0.4332; y_3(4)=0.4337; y_3(5)=0.4340;$ $x_4(1)=0.0470; x_4(2)=0.0477; x_4(3)=0.0484; x_4(4)=0.0493; x_4(5)=0.0502;$ $y_4(1)=0.1497; y_4(2)=0.1482; y_4(3)=0.1468; y_4(4)=0.1453; y_4(5)=0.1440;$

In Tables 1 and 2, we show initial conditions and parameter values for the periodic orbits, identified with the biped gaits of *two-legged jump* (Figure 2), and *run* (Figure 3). Each simulation was executed until a stable periodic solution was found. Moreover, it was considered $\alpha \in]0,1]$ and, during

the numerical experiments, were evaluated cases in steps of $\Delta\alpha = 0.01$. We computed the amplitude A and the period T of the solutions of system (16), for each value leading to Figures 4-5. We find that, both for two-legged jump (in red) and the run (in blue), the amplitude and period increase as α goes from near 0 to values close to 1. For A we observe a maximum at $\alpha = 0.95$ while for T we have a monotonous variation with α. A remark is that there is a slight increase in the values of the amplitude and the period for $\alpha = 1$. This is a numerical effect due to the series truncation in the approximation of the fractional derivative.

Table 2. Parameter values for the bipedal gaits of *two-legged jump* and *run,* where Δt is the time increment, and k_{ij} are the coupling constants. For more information, see text.

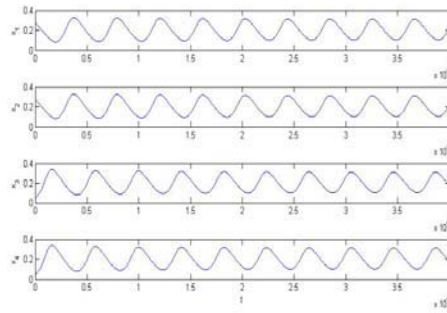| Gait	Δt	ϕ	g_{Ca}	k_{11}	k_{12}	k_{21}	k_{22}	k_{31}	k_{32}
two-legged jump	0.0005	1.0	1.8	0.1	0.1	−0.5	−0.5	−0.8	−0.8
Run	0.0005	1.0	1.8	0.1	0.1	0.0	0.0	−0.7	−0.7

Fig. 2. Periodic solution of the CPG network *LEG* identified with the *two-legged jump*, for $\alpha = 0.90$.
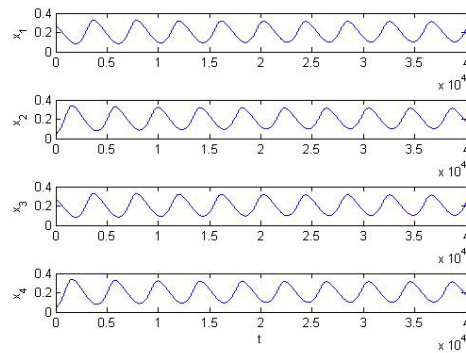

Fig. 3. Periodic solution of the CPG network *LEG* identified with the *run*, for $\alpha = 0.90$.
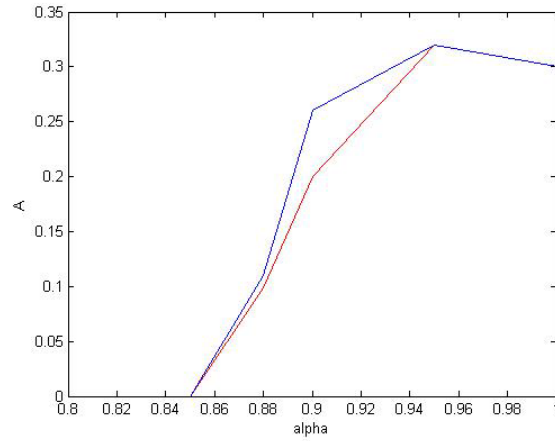
Fig. 4. Amplitude (A) of the periodic solutions produced by the CPG network *LEG* identified with the

biped gaits of *two-legged jump* (red) and *run* (blue)*, for different values of* α .
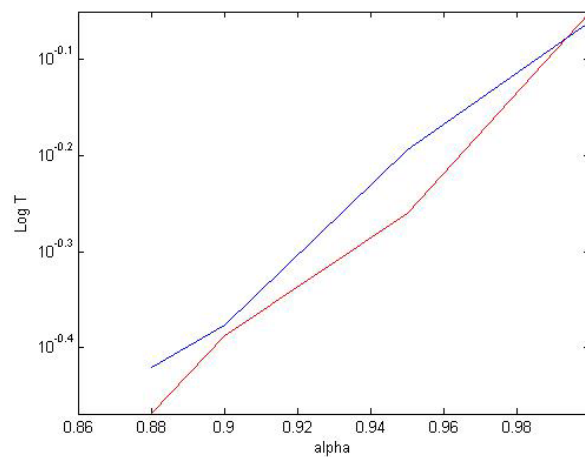

Fig. 5. Period (T) of the periodic solutions produced by the CPG network *LEG* identified with the

biped gaits of *two-legged jump* (red) and *run* (blue)*, for different values of* α .

As we can observe from Figures 2-5, α behaves like a bifurcation parameter, that is, as we vary α the dynamical behaviour of the system changes. This parameter, in particular, simplifies the work of the researcher. In fact, we do not need to know deeply the parameters of the equations, that model the dynamics of each cell, in order to increase the frequency or the amplitude of a given periodic solution, we just have to vary the value of α.

4 Conclusions and future work

We studied a fractional version of a CPG network model for legs rhythms in bipeds. We analysed the amplitude and the period values of two periodic solutions, identified with two biped locomotion patterns (*two-legged jump* and *run*), for different values of α. We found that the amplitude and the period values increase as α goes from values near zero up to values close to one. In future work, we intend to use a Continued Fraction Expansion, based on a Padé expansion, to approximate the fractional derivative values and we intend to study more periodic solutions identified with other biped locomotion patterns.

5. Acknowledgments

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