An extension of estimation of domain of attraction for fractional order linear system subject to saturation control

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ABSTRACT

This paper employs the Lyapunov direct method for the stability analysis of frac- tional order linear systems subject to input saturation. A new stability condition based on saturation function is adopted for estimating the domain of attraction via ellipsoid approach. To further improve this estimation, the auxiliary feedback is also supported by the concept of stability region. The advantages of the proposed method are twofold: (1) it is straightforward to handle the problem both in analysis and de- sign because of using Lyapunov method, (2) the estimation leads to less conservative results. A numerical example illustrates the feasibility of the proposed method.

Keywords:

Fractional-order systems Stability, Domain of attraction Input saturation Ellipsoid approach

1. Introduction

Saturation is a ubiquitous phenomenon in physical systems that plays an important role in mathematics and engineering. During the last decades this topic was studied in the scope of Integer Order (IO) sys- tems [1,2]. Hu et al. [1] derived a condition stability for IO linear system subject to actuator saturation in terms of an auxiliary feedback using the ellipsoid approach. However, it is still an open problem of Fractional Order (FO) systems. In spite of the interest in FO dynamical system in modeling and control [3–6], only a few papers were devoted to saturation nonlinearity [7,8]. Lim et al. [7] obtained the sufficient stability based on the solution of the fractional linear equation. They adopted the Gronwall–Bellman lemma and the property of sector bounded saturation in the general case with $0 < \alpha < 2$, where α represents the fractional order. In [8], the stability of FO saturation system is addressed by means of a Lyapunov function using Riemann–Liouville definition. In [9] and references therein, it has been shown that the fractional derivative of Lyapunov function (${}^C_0D^\alpha_t V$) is a finite series and that there exists a boundedness on ${}^C_0D^\alpha_t V$.

The contribution of this paper is to analyze the stability of FO linear system with saturation nonlinearity via the relation between the Riemann–Liouville and the Caputo definitions. The Lyapunov direct method is employed where the estimated region of attraction is obtained through the ellipsoid approach based on the boundedness of ${}^{C}_{0}D^{\alpha}_{t}V$. An auxiliary feedback is also utilized to improve the estimation of domain of attraction. Having these idea in mind the paper is organized as follows. Section 2 presents the fundamental concepts. Section 3 describes the problem and formulates the stability analysis. Section 4 estimates the domain of attraction by using auxiliary feedback. Section 5 presents simulation results. Finally, Section 6 outlines the main conclusions.

2. Fundamental concepts

There are several definitions of FO derivatives being well-known the Riemann–Liouville and Caputo for- mulations [10]. The physical interpretations of fractional derivative are given in [11]. The operators ${}^{C}D^{\alpha}_{t}$ and ${}^{R}D^{\alpha}_{t}$ denote the Caputo and Riemann–Liouville fractional derivatives, respectively.

Remark 1 (*[12]*). There is the following relation between ${}^{C}D^{\alpha}_{t}$ and ${}^{R}D^{\alpha}_{t}$.

$$ {}^{R}D^{\alpha}_{t} f(t) = {}^{C}D^{\alpha}_{t} f(t) + \sum_{k=1}^{m} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{k}(0). \tag{1}$$

Remark 2 (*[12]*). Using Riemann–Liouville definition, FO derivative of positive constant $a > 0$ is

$$ {}^{R}D^{\alpha}_{t}(a) = \frac{a t^{-\alpha}}{\Gamma(1-\alpha)}. \tag{2}$$

In the rest of the paper, we refer to x instead of $x(t)$ to simplify the notation.

Remark 3 (*[9]*). According to Leibniz's rule of differentiation in FO system, the αth order time derivative of $h(x) = x^{T}x$ can be extended as

$$ D^{\alpha}_{t} h(x) = (D^{\alpha}_{t} x)^{T} x + x^{T}(D^{\alpha}_{t} x) + 2\gamma, \tag{3}$$

where

$$ \gamma = \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha)[D^{k}_{t} x]^{T}[D^{\alpha-k}_{t} x]}{\Gamma(1+k)\Gamma(1-k+\alpha)}. \tag{4}$$

From [9], γ is bounded as follows.

$$ \|\gamma\| \le \sigma \|x\|, \qquad \sigma > 0. \tag{5}$$

Consider the following Lyapunov function:

$$ V = x^{T}Px \tag{6}$$

where P is a positive definite matrix. According to Remarks 1 and 3, we can easily conclude

$$ {}^{C}D^{\alpha}_{t} V = [{}^{R}D^{\alpha}_{t} x]^{T} Px + x^{T} P[{}^{R}D^{\alpha}_{t} x] - {}^{R}D^{\alpha}_{t}[x(0)^{T} Px(0)] + \gamma, \tag{7}$$

where

$$ \|\gamma\| \le \beta \|x\|, \qquad \beta > 0. \tag{8}$$

with $\beta = p \times \sigma$ and p is maximum eigenvalue of P.

Lemma 1 ([12]). *For each vector T and Y, there is a positive scalar $\varepsilon > 0$ that the following inequality is satisfied:*

$$T^T Y \le \varepsilon T^T T + \varepsilon^{-1} Y^T Y. \tag{9}$$

Theorem 1 ([13]). *For fractional linear system $D^{\alpha} x = Ax$, if and only if there exists a positive definite matrix P such that*

$$(-(-A)^{\frac{1}{2-\alpha}})^T P + P(-(-A)^{\frac{1}{2-\alpha}}) < 0. \tag{10}$$

Then the linear system is $t^{-\alpha}$ asymptotically stable.

Theorem 2 ([14]). *Let $x = 0$ is an equilibrium point for nonautonomous ${}^{C}D^{\alpha} x = f(x, t)$ and $\aleph \subset R^n$ be a domain containing $x = 0$. Let $V(t,x) : [0,\infty) \times \aleph \to R$ be a continuously differentiable function such that*

$$W_1(x) \le V(t, x) \le W_2(x), \tag{11}$$

$${}^{C}D_t^{\alpha} V(t,x) \le -W_3(x), \qquad \forall t \ge 0,\ \forall x \in \aleph,\ 0 < \alpha < 1, \tag{12}$$

where $W_1(x), W_2(x)$ and $W_3(x)$ are continuous positive definite functions on $\aleph$. Then, $x = 0$ is uniformly asymptotically stable.

3. Stability analysis

Consider the system with $0 < \alpha < 1$ described by

$${}^{R}D_t^{\alpha} x = Ax + Bsat(u), \qquad u = Kx, \{-u_0 \le u \le u_0\}, \tag{13}$$

where $x \in R^n$ and $u \in R^m$ are the state vector and control input vector, respectively, $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $K \in R^{m\times n}$ are constant matrices and $u_0 > 0$ is the bound of control. Moreover, $sat(.) : R^m \to R^m$, is a saturation function such that $sat(u) = [sat(u_1)\ \ sat(u_2)\ \ \ldots\ \ sat(u_m)]^T$ where $sat(u_i) = sign(u_i)\min(\|u_i\|, 1)$. Then the corresponding closed-loop system is

$${}^{R}D_t^{\alpha} x = Ax + Bsat(Kx). \tag{14}$$

By defining $\psi = \{x \,|\, |Kx| \le u_{0i}\}, i = 1, \ldots, m$, the aim is to find the stability condition in two cases.

Case I: If $u_{0i} \le 1,\ \ i = 1, \ldots, m$

In this case, we have $sat(Kx) = Kx$. Thus, the system can be represented by the FO linear system as follows.

$${}^{R}D_t^{\alpha} x = (A + BK)x. \tag{15}$$

Lemma 2. *The system* (15) *is stable if and only if there exist a positive definite matrix P such that*

$$(-(-(A+BK))^{\frac{1}{2-\alpha}})^T P + P(-(-(A+BK))^{\frac{1}{2-\alpha}}) < 0. \tag{16}$$

Proof. This Lemma is simply concluded from Theorem 1 by substituting $(A+BK)$ instead of A.

Remark 4. Lemma 2 can be used as a criterion to investigate whether the linear system is in stability region or not.

Case II: If $u_{0i} > 1$, $i = 1, \ldots, m$

In this case, we consider the general form of FO linear system subject to saturation nonlinearity. The system (14) can be rewritten as

$$ {}^{R}D_t^{\alpha} x = A_{cl}x + B\phi(x), \tag{17} $$

where $A_{cl} = A + BK$ and $\phi(x) = sat(Kx) - Kx$.

Lemma 3. *The following inequality can be explored from the special property of saturation function.*

$$ \|\phi(x_1) - \phi(x_2)\| \le \|K(x_1 - x_2)\|. \tag{18} $$

Proof. It can be easily seen that for saturation function if $u_{0i} \le 1$, then $\|\phi(x_1) - \phi(x_2)\| \le \|K(x_1 - x_2)\|$. Also, if $u_{0i} > 1$, then $sat(Kx) = \pm 1 \Rightarrow \|\phi(x_1) - \phi(x_2)\| \le \|K(x_1 - x_2)\|$.

In the following, the stability condition is given for equilibrium of the system (17).

Theorem 3. *Let* $x = 0$ *is the equilibrium point of the system* (17). *Then,* $x = 0$ *is uniformly asymptotically stable if*

$$ A_{cl}^T P + PA_{cl} + PBB^T P\varphi_1 + \varphi_2 (eK)^2 I < 0, \tag{19} $$

where $\varphi_1 = \varepsilon_1 + \varepsilon_2 > 0$, $\varphi_2 = \varepsilon_1^{-1} + \varepsilon_2^{-1} > 0$, $\varepsilon_1, \varepsilon_2 > 0$, *and* eK *is the maximum of absolute of the eigenvalue of matrix* $K^T K$, *and* I *is the identical matrix with appropriate dimension.*

Proof. If the conditions (11) and (12) given in Theorem 2 are satisfied, then the proof is completed. Consider Lyapunov function (6). Assume that there exists this function satisfying $W_1(x) \le V(t, x) \le W_2(x)$. It then follows that the condition (11) is satisfied. Afterward, we must establish a condition to satisfy inequality (12). To do this, by taking FO derivative from (6) and substituting system (17), we have

$$ {}^{C}D_t^{\alpha} V = (A_{cl}x + B\phi(x))^T Px + x^T P(A_{cl}x + B\phi(x)) - {}^{R}D_t^{\alpha}(x(0)^T Px(0)) + \gamma, \tag{20} $$

in which Eq. (8) is satisfied. By simplifying (20), it leads to

$$ {}^{C}D_t^{\alpha} V = x^T A_{cl}^T P x + \phi^T(x) B^T Px + x^T PA_{cl}x + x^T PB\phi(x) - {}^{R}D_t^{\alpha}(x(0)^T Px(0)) + \gamma. \tag{21} $$

Using (2), since $2\frac{t^{\alpha}}{\Gamma(1-\alpha)}\|x(0)\| \ge 0$ we have

$$ {}^{C}D_t^{\alpha} V = x^T(A_{cl}^T P + PA_{cl})x + \phi^T(x) B^T Px + x^T PB\phi(x) + \gamma. \tag{22} $$

By considering Lemma 1, it yields

$$ \begin{aligned} {}^{C}D_t^{\alpha} V &= x^T(A_{cl}^T P + PA_{cl})x + \varepsilon_1 x^T PBB^T Px + \varepsilon_1^{-1}\phi^T(x)\phi(x) + \varepsilon_2 x^T PBB^T Px + \varepsilon_2^{-1}\phi^T(x)\phi(x) + \gamma \\ &= x^T(A_{cl}^T P + PA_{cl})x + (\varepsilon_1 + \varepsilon_2)x^T PBB^T Px + (\varepsilon_1^{-1} + \varepsilon_2^{-1})\phi^T(x)\phi(x) + \gamma. \end{aligned} \tag{23} $$

From (8) and (18), it is straightforward to see that

$$ {}^{C}D_t^{\alpha} V = x^T(A_{cl}^T P + PA_{cl})x + (\varepsilon_1 + \varepsilon_2)x^T PBB^T Px + (\varepsilon_1^{-1} + \varepsilon_2^{-1})x^T K^T Kx + \beta\|x\|. \tag{24} $$

If $W_3(x) = \varepsilon_3\|x\|^2$, $\varepsilon_3 > 0$, then we get

$$ {}^{C}D_t^{\alpha} V = x^T(A_{cl}^T P + PA_{cl})x + (\varepsilon_1 + \varepsilon_2)x^T PBB^T Px + (\varepsilon_1^{-1} + \varepsilon_2^{-1})x^T K^T Kx + \beta\|x\| \le -\varepsilon_3\|x\|^2. \tag{25} $$

From there, it can be rewritten as follows:

$$ x^T(A_{cl}^T P + PA_{cl})x + (\varepsilon_1 + \varepsilon_2)x^T PBB^T Px + (\varepsilon_1^{-1} + \varepsilon_2^{-1})x^T K^T Kx \le -\varepsilon_3\|x\|^2 - \beta\|x\| \le -\varepsilon_3\|x\|^2. \tag{26} $$

Therefore, inequality (19) is satisfied. This completes the proof.

As it is evident above, Theorem 3 illustrates the sufficient stability condition based on special property of saturation function. Moreover, if $P_1 = P^{-1}$ is multiplied in both direction of (19), then the stability condition can be written in the form of Linear Matrix Inequality (LMI). Hence, it is very easy to handle the problem in hand in both analysis and design.

4. Extension of the estimation of domain of attraction

In this section, we extend the estimation of domain of attraction by using the special property of saturation nonlinearity as well as an auxiliary feedback $C \in R^{m\times n}$ which is always in the linear region of saturation function. This means that $L : \{x \mid x \in \Re(P,\rho), |Cx| \le 1\}$ where $\Re(P,\rho) = \{x \in R^n, x^T Px \le \rho\}$ is a domain of attraction. Consider vector $s \in R^m$ where $s \in \theta$ and $\theta = \{s \in R^m : s = 0 \;\; or \;\; s = 1\}$; we can define the following matrix.

$$Co(s,K,C) = \begin{bmatrix} (1-s_1)k_1 + s_1c_1 \\ \vdots \\ (1-s_m)k_m + s_mc_m, \end{bmatrix} \tag{27}$$

where k_i and c_i are the ith row of K and C, respectively. In the following, we have a condition for an ellipsoid to be inside the domain of attraction.

Theorem 4. *Consider the ellipsoid $\Re(P,1)$. If there exists an auxiliary feedback $C \in R^{m\times n}$ such that*

$$A^T P + PA + PBCo(sK,C) + Co(s,K,C)^T B^T P < 0, \qquad s \in \theta,\; i = [1,m], \tag{28}$$

then, $\Re(P,1)$ is contractively invariant set.

Proof. Consider the Lyapunov function (6). By taking the FO derivative from (6) and substituting in system (14), we have

$$^{C}D_t^{\alpha} V = (Ax + Bsat(Kx))^T Px + x^T P(Ax + Bsat(Kx)) \;{}^{R}\_\; D_t^{\alpha}(x(0)^T Px(0)) + \gamma. \tag{29}$$

By simplifying (29) and considering (2), it yields

$$^{C}D_t^{\alpha} V \le (A^T P + PA)x + [sat(Kx)]^T B^T Px + x^T PB\,[sat(Kx)] + \gamma. \tag{30}$$

From [1], we have $x^T Pb_i sat(k_i x) \le x^T Pb_i \max(k_i x, c_i x) \quad \forall x \in \Re(P,\rho), i = [1,m]$.

Thus, the following result can be obtained:

$$^{C}D_t^{\alpha} V \le x^T (A^T P + PA)x + \sum_{i=1}^{m} x^T Pb_i((1-s_i)k_i + s_ic_i)x + x^T((1-s_i)k_i + s_ic_i)^T b_i^T Px + \gamma. \tag{31}$$

Using (8) and (27), it concludes

$$^{C}D_t^{\alpha} V \le x^T (A^T P + PA + PBCo(s,K,C) + [Co(s,K,C)]^T B^T P)x + \beta \|x\|. \tag{32}$$

Choosing W_3 as mentioned previously, we have

$$^{C}D_t^{\alpha} V \le x^T (A^T P + PA + PBCo(s,K,C) + [Co(s,K,C)]^T B^T P)x + \beta \|x\| \le -\varepsilon_3 \|x\|^2, \tag{33}$$

$$x^T (A^T P + PA + PBCo(s,K,C) + [Co(s,K,C)]^T B^T P)x \le -\varepsilon_3 \|x\|^2 - \beta\|x\| \le -\varepsilon_3 \|x\|^2. \tag{34}$$

If (28) is satisfied, then it follows that $\Re(P,1)$ is contractively invariant set and the proof is completed.

Let $X_r = \{x \in R^n, x^T Rx < 1\}$ be a prescribed bounded convex set. For a set $G \subset R^n$, we can define $v(G) := \sup(v > 0, vX_r \subset G)$. If $v(G) > 1$, then $X_r \subset G$ [1]. According to the results given in Theorem 4 and the initial X_r, we propose the following optimization problem, such that vX_r is maximized:

$$
\begin{aligned}
& \sup_{P>0,\ \rho=1,\ C}(v) \\
& st \begin{cases} \text{(a) } vX_r \subset \Re(P,\rho) \\ \text{(b) } A^T P + PA + PBCo(sK,C) + Co(s,K,C)^T B^T P < 0 \\ \text{(c) } \Re(P,\rho) \subset L. \end{cases}
\end{aligned}
\tag{35}
$$

In the optimization problem mentioned above, the constraint (b) is the stability condition given in (28) and (c) is the constraint on auxiliary feedback based on [1]. The first and third constraints can be converted to LMI. The constraint (a) is equal to $\begin{bmatrix} \varpi R & I \\ I & P_1 \end{bmatrix} \geq 0$, where $P_1 = P^{-1}$ and $\varpi = \frac{1}{v^2}$. Regarding (c), it can be rewritten as $\begin{bmatrix} 1 & n_i \\ n_i^T & P_1 \end{bmatrix} \geq 0$, where $N = CP_1$ and n_i is the ith row of N [1,2]. From there, (35) can be simplified as

$$
\begin{aligned}
& \inf_{P_1 > 0,\ N}(\varpi) \\
& st \begin{cases} \text{(a) } \begin{bmatrix} \varpi R & I \\ I & P_1 \end{bmatrix} \geq 0 \\ \text{(b) } P_1 A^T + AP_1 + BCo(s,K,C)P_1 + P_1 Co(s,K,C)^T B^T < 0 \\ \text{(c) } \begin{bmatrix} 1 & n_i \\ n_i^T & P_1 \end{bmatrix} \geq 0. \end{cases}
\end{aligned}
\tag{36}
$$

Remark 5. There are 2^m matrix inequalities in constraint (b) for all $s_i \in \theta$, $i = [1, m]$.

Remark 6. To achieve estimation of domain of attraction X_r, we update the algorithm given in [8] by considering the stability condition proposed in this paper, namely expression (19).

Step 1: Compute K such that matrix A_{cl} is in stability region by Lemma 1. Step 2: Compute P such that the inequality (19) satisfies.

Step 3: Compute $\rho = \min \frac{u_{0i}^2}{k_i P k_i^T}$.

This algorithm offers a controller that the linear system is in stability region. Based on this, we have an estimation of domain of attraction as $\Re(P,\rho)$.

Remark 7. In practice, the parameters of the system are not precisely known, or may deviate from nominal values; therefore, the control action may be sub-optimal [15,16]. Based on this, future works in this area is to develop the proposed approach by considering parametric uncertainties.

5. Simulation results

In order to demonstrate the performance of proposed control method, we compare it with a previous method proposed in [7]. To this goal, the unstable system adopted in [7] is used in the sequel. Considering system (13) with $A = \begin{bmatrix} 0.1 & -3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, $\alpha = 0.8$, $u_{0i} = 15$ and $i = 1, 2$. To perform a fair comparison, we opt $K = \begin{bmatrix} -1 & -3 \\ 2 & -1 \end{bmatrix}$, since the linear system is stable. Based on the algorithm given

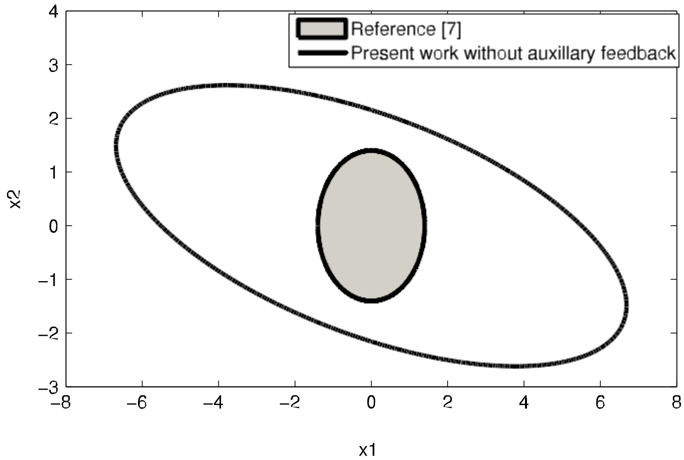

Fig. 1. Comparison results of the region of domain of attraction.

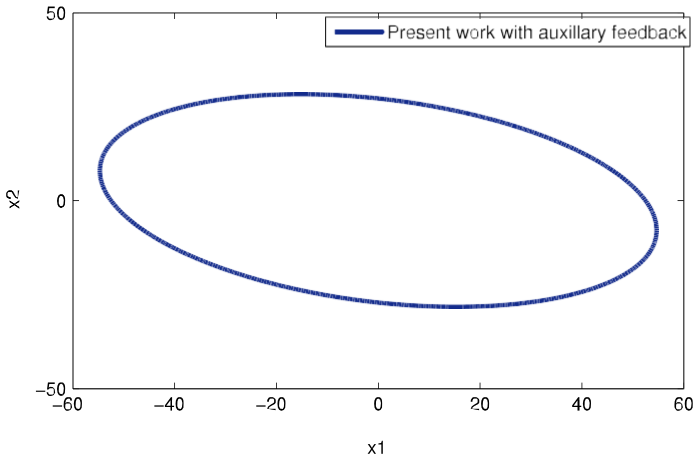

Fig. 2. Estimated stable region obtained by the present work with auxiliary feedback.

in Section 4, we can obtain the primary estimation of domain of attraction as follows. By choosing $P = \begin{bmatrix} 1.6979 & 2.4594 \\ 2.4594 & 11.0999 \end{bmatrix}$, $\phi_1 = 5.5056 \times 10^{-4}$ and $\phi_2 = 0.0194$, it leads to $\rho = 51.4794$. Fig. 1 shows the corresponding results where we verify the superiority of the new method. In addition, the procedure of auxiliary feedback design is given as follows. By using the previous estimation of domain of attraction as a primary estimation, $X_r = x^T P^* x$, $P^* = P_1^{-1} = R$, we can solve the problem (36). Using LMI, it yields: $P = \begin{bmatrix} 0.0004 & 0.0004 \\ 0.0004 & 0.0015 \end{bmatrix}$, $N = \begin{bmatrix} -0.3125 & 0.0079 \\ -0.3125 & 0.0079 \end{bmatrix} \times 10^{-14}$, $\varpi = 2.3305 \times 10^{-4}$ and $C = \begin{bmatrix} -0.8876 & 0.0224 \\ 0.0144 & -0.0004 \end{bmatrix} \times 10^{-11}$.

Fig. 2 depicts the result of region of domain of attraction with auxiliary feedback. It is apparent that the estimation of domain of attraction has been extended by the present work with auxiliary feedback. Referring to Figs. 1 and 2, it can be seen the effectiveness of the present work without/with auxiliary feedback. The corresponding phase portrait of the proposed control method, with and without auxiliary feedback, is illustrated in Figs. 3 and 4.

6. Conclusions

In this paper, we studied the stability analysis of linear systems with fractional-order belonging to $0 < \alpha < 1$ subject to saturation nonlinearity, by means of the Lyapunov direct method. Sufficient stability

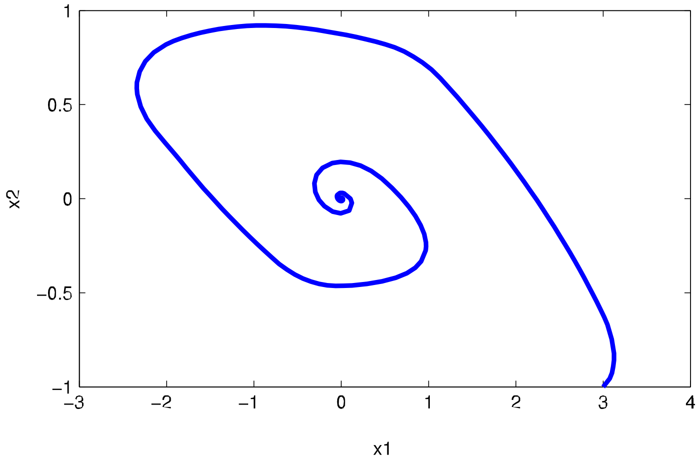

Fig. 3. Phase portrait of without auxiliary feedback.

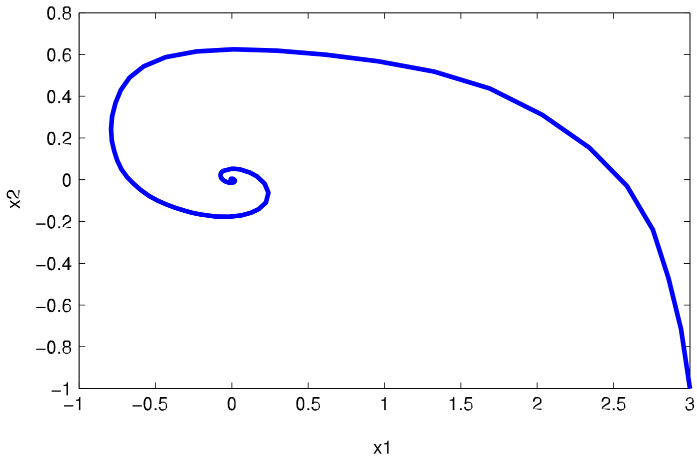

Fig. 4. Phase portrait of the system with auxiliary feedback.

condition was provided based on the properties of saturation function that can be used to the estimation of domain of attraction based on ellipsoid approach. We showed that the ellipsoid approach is a successful tool, both for FO and IO systems. The auxiliary feedback was employed in a second phase to extend the primary estimation. An illustrative example was provided to demonstrate the applicability of the proposed approach. Future work in this area will include the effect of parametric uncertainty upon the stability.

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