

# A fractional perspective to the bond graph modelling of world economies

J. A. Tenreiro Machado · Maria Eugénia Mata

**Abstract** Inspired in dynamic systems theory and Brewer's contributions to apply it to economics, this paper establishes a bond graph model. Two main variables, a set of inter-connectivities based on nodes and links (bonds) and a fractional order dynamical perspective, prove to be a good macro-economic representation of countries' potential performance in nowadays globalization. The estimations based on time series for 50 countries throughout the last 50 decades confirm the accuracy of the model and the importance of scale for economic performance.

**Keywords** Fractional calculus · Bond graphs · Economy · Modelling · Dynamics

## 1 Introduction

In 1695 Gottfried Leibniz conceived the derivative of order  $1/2$ . This idea leads to the area usually known as 'Fractional Calculus' (FC) and represents the general-

ization of the operations of integration and differentiation up to real or complex orders [34,46,49,52,55,57]. During the last decades researchers verified that fractional models capture memory effects of natural and artificial phenomena, while classical integer models reveal limitations when describing those properties [6,29,44,65,71]. This state of affairs has motivated an intensive research towards new areas of application of the novel concepts, and we verify a sustained growth in the number of publications involving this mathematical tool [42,43].

Bond graphs (BG) were created by Henry Paynter at MIT during the fifties [53]. Paynter noted that distinct 'domains', such as electrical, mechanical and fluid systems, were modelled by equations of the same form. These systems are nowadays commonly called 'analogous'. The BG method is a graphical modelling tool based on the energy transfer rate within the system. The constitutive elements are interconnected by 'bonds' that specify the power transfer between them. Therefore, 'power' (the time rate of energy transfer) between the distinct components is the main concept supporting the description of physical systems. Since then, the BG theory has been developed by many researchers [7,8,21,23,24,33,48,67,68,74] that extended the technique to power hydraulics, mechanics, thermo-dynamics and, more recently, to non-physical systems like micro-economy [10–12,76] and architecture [36,69,70]. The BG graphical representation and the derivation of system equations is system-

atic and can be algorithmized by existing software (e.g. 20-sim, Camp-G, ENPORT and others).

Well-succeeded countries have been studied as cases of *exceptionalism* in economic history. The UK dominance over the world in the nineteenth century was exceptional because of pioneering the steam-engine industrialization [20]. The American hegemony in the twentieth century was a case of *exceptionalism* because of exceptional resource endowments [3,54]. Other successful countries were interpreted as cases of exceptional replication opportunities [1].

According to these views, technological development did not command growth, because its adoption may be explained by the opportunity cost of labour force wages [4]. Large literature on the causes for growth and stagnation elects industrial productivity and wages as the secret for new-technology adoption [51,60]: most regions ‘could not have industrialized by following American policies since their wages were so low and their energy costs so high that the modern technology that was cost effective in Britain and the USA would not have paid in their circumstances’ [4]. At the same time contradictory interpretations are available on large territories. On the one side, economic policy is a relevant driver to growth, it is said [50]. (For example, ‘The development of Egypt and India required more draconian state intervention than a protective tariff, mass education, and infrastructure investment’ [4]). On the other side, they were victims of mother-country economies and could not develop appropriate institutional environments for economic growth, according to other views [2].

Among so large and diversified literature, scale has not deserved enough attention, although the general-equilibrium models perspective is also available [66]. This paper prefers to approach economic growth according to bond graphs modelling theories to elect a few number of variables expressing scale and observe the economic growth of 50 nations throughout the last 50 years, the period for which reliable data are available (<http://data.worldbank.org/>). *Exceptionalism* may become interpreted as a much less exceptional view.

Having these ideas in mind this paper is organized as follows. Section 2 presents the fundamental aspects of bond graph theory. Section 3 is a literature review of BG application to economic systems and a general presentation of this paper model. The discussion of results is presented in Sect. 4, and Conclusions follow up in Sect. 5.

## 2 Bond graphs: fundamentals

The fundamental idea of a BG is that power  $P$  is transmitted between connected components by a combination of ‘effort’ and ‘flow’ so that  $P(t) = e(t)f(t)$ , where  $t$  is time. In general the two variables are called ‘generalized effort’ and ‘generalized flow’. The power interconnection between elements is denoted by ‘bonds’. Table 1 presents examples of different domains and the corresponding effort and flow variables. We should note that the assignment for variables  $e$  and  $f$  can be rotated. For example, some authors switch the Force and Velocity pair in mechanical systems. The resulting changes do not preclude neither the model validity nor its generality. In some domains the product of effort and flow does not lead to power, because extending the BG method to new applications requires the generalization of the concept of power.

The effort and flow signals are carried by the single power bond and represented in Fig. 1. The direction of the half-arrow indicates the positive direction of power transport between elements A and B. Therefore, in Fig. 1, power transfer from left to right is considered positive and negative otherwise.

The fundamental variables are effort  $e$  and flow  $f$ , but we need to keep in mind the time integral of effort,  $p$ , and the time integral of flow,  $q$ . Often, the electrical domain is adopted for illustrating the BG concepts; thus, we have voltage  $v$ , current  $i$ , flux linkage  $\lambda$  and charge  $q$  as the analogous variables.

Four groups of symbols are used in BGs, namely two active elements, three one-port elements, two two-port elements and two junctions. The active ports are denoted by ‘effort source’ (SE) and ‘flow source’ (SF) represented in Table 2. In the electrical domain they are the voltage and current sources, respectively. In what concerns the one-port elements we have two energy stores and one energy dissipator, that is,  $f = \varphi^{-1}(p)$ ,  $e = \psi^{-1}(q)$  and  $e = \gamma(f)$  (or,  $f = \gamma^{-1}(e)$ ), respectively. In the electrical domain we have correspondingly the inductor  $L$ , capacitor  $C$  and resistor  $R$  with constitutive linear relations  $i = \frac{1}{L}\lambda$ ,  $v = \frac{1}{C}q$  and  $v = Ri$ , respectively. Figure 2 shows the so-called tetrahedron of states, illustrating the relationships between the state variables.

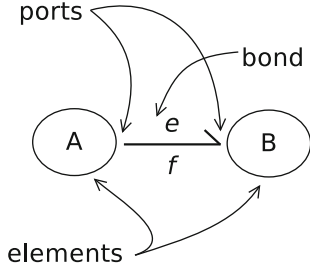
Besides the three classical one-port elements, it was recognized to be possible a fourth element denoted memristor  $M$  interconnecting  $p$  and  $q$  [16–18,30,64,75]. In fact, besides these elements it is possible to

**Table 1** Bond graph domains and variables

Domain	Effort $e$	Flow $f$	Momentum $p$	Displacement $q$	Power $P$	Energy $E$
Electric	Voltage	Current	Flux linkage	Charge	Power	Energy
	$v$	$i$	$\lambda = \int v dt$	$q = \int i dt$	$P = e \cdot i$	$E = \int P dt$
Units	[V]	[A]	[Vs]	[C]	[V A = W]	[V As = J]
Magnetic	Magnetomotive force	Magnetic flux rate	Charge linkage	Magnetic flux	Power	Energy
	$e_m$	$\dot{\phi}$	$\Gamma = \int e_m dt$	$\phi = \int \dot{\phi} dt$	$P = e_m \dot{\phi}$	$E$
	[A]	[Wbs <sup>-1</sup> ]	[C = As]	[Wb]	[A Wbs <sup>-1</sup> = W]	[A Wb = J]
Mechanics (translation)	Force	Velocity	Momentum	Displacement	Power	Energy
	$f$	$\dot{x}$	$p = \int f dt$	$x = \int \dot{x} dt$	$P = f \cdot \dot{x}$	$E$
	[N]	[ms <sup>-1</sup> ]	[Ns]	[m]	[Nms <sup>-1</sup> = W]	[V As = J]
Mechanics (rotation)	Torque	Angular velocity	Angular momentum	Angle	Power	Energy
	$\tau$	$\omega$	$h = \int \tau dt$	$\theta = \int \omega dt$	$P = \tau \cdot \omega$	$E$
	[Nm]	[rad s <sup>-1</sup> ]	[N ms]	[rad]	[W]	[J]
Hydraulic	Pressure	Volume	Pressure momentum	Volume flow rate	Power	Energy
	$p$	$q$	$\Gamma = \int p dt$	$V = \int q dt$	$P = p \cdot q$	$E$
	[N m <sup>-2</sup> ]	[m <sup>3</sup> s <sup>-1</sup> ]	[Nsm <sup>-2</sup> ]	[m <sup>3</sup> ]	[W]	[J]
Chemical	Chemical potential	Molar flow	–	Number of moles	Power	Energy
	$\mu$	$\dot{N}$		$N = \int \dot{N} dt$	$P$	$E$
	[J mole <sup>-1</sup> ]	[mole s <sup>-1</sup> ]		[mole]	[W]	[J]
Thermo-dynamics	Temperature	Entropy flow	–	Entropy	Power	Energy
	$T$	$\dot{S}$		$S = \int \dot{S} dt$	$P$	$E$
	[K]	[JK <sup>-1</sup> s <sup>-1</sup> ]		[JK <sup>-1</sup> ]	[W]	[J]
Architecture	Energy for a unit of change	Number of changes in a unit of time	Momentum	Displacement	Power	Energy
	$e$	$f$	$p_a$	$q$	$P$	$E$
	[energy/change]	[change/time]	[energy time/change]	[change]	[energy/time]	[energy]
Micro-economy	Unit price of commodity	Commodity flow rate	Economic impulse	Accumulation of orders	Money flow	Money
	$e$	$f$	$\lambda = \int e dt$	$q = \int f dt$	$\dot{M} = ef$	$M = \int \dot{M} dt$
	[\$ unit <sup>-1</sup> ]	[unit period <sup>-1</sup> ]	[\$ unit <sup>-1</sup> period]	[unit]	[\$ period <sup>-1</sup> ]	[\$]
Macro-economy	GDP per capita	Population flow	Economic impulse	Population	Money flow	Money
	$Y$	$\dot{n}$	$I$	$n$	$\dot{M} = Y\dot{n}$	$M = \int \dot{M} dt$
	[\$ capita <sup>-1</sup> ]	[capita period <sup>-1</sup> ]	[\$ capita <sup>-1</sup> period]	[capita]	[\$ period <sup>-1</sup> ]	[\$]

generalize the concepts, not only to higher order elements [31, 72], but also to fractional order inductors and capacitors [19, 26, 39, 45, 47], often called in the electrical domain as ‘fractductors’ and ‘fractances’, with

constitutive relations  $v = L \mathcal{D}_t^\beta i(t)$ ,  $0 < \beta < 1$ , and  $i = C \mathcal{D}_t^\alpha v(t)$ ,  $0 < \alpha < 1$ , respectively [32, 40], where symbol  $\mathcal{D}_t^\alpha$  denotes the fractional derivative of order  $\alpha$  with respect to time  $t$  [34, 46, 49, 55, 57].

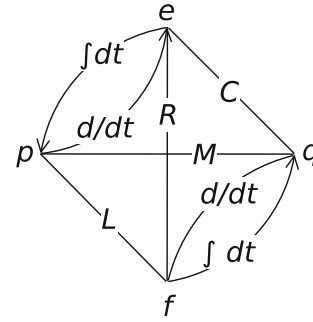


**Fig. 1** Bond between elements *A* and *B*

The BG elements are connected by means of two-port devices and junctions (see Table 2). The transformer *TF* and gyrator *GY* are two-port elements that modify the input  $\{e_1, f_1\}$  into the output  $\{e_2, f_2\}$  (or the primary into the secondary) variables, being power conservative, that is, with  $e_1 f_1 = e_2, f_2$ . The constitutive relations are given by  $\{e_1 = m e_2, f_2 = m f_1\}$  and  $\{e_1 = r f_2, e_2 = r f_1\}$ , where  $m \in \mathbb{R}^+$  and  $r \in \mathbb{R}^+$  are the transformer and gyrator modulus, respectively. On the other hand, the 1- (or effort) and 0- (or flow) junctions implement continuity constraints on the effort and flow variables, respectively. The constitutive relations correspond to  $\{e_1 + \dots + e_n = 0, f_1 = \dots = f_n\}$  for the 1-junction and  $\{e_1 = \dots = e_n, f_1 + \dots + f_n = 0\}$

for the 0-junction. In the electrical domain the 1- and 0-junctions correspond to the first (or voltage) and second (or current) Kirchoff laws, respectively.

The notion of ‘causality’ is not usually discussed during system equation derivation. However, in BG causality is an important concept and allows the input/output roles of effort and flow to be included into the graph. This is represented using the causal stroke convention. In BG notation, a causal stroke added to one end of the bond indicates that the opposite side is establishing the effort. Symmetrically, the bond side with the causal stroke defines the flow. Assigning



**Fig. 2** Tetrahedron of states  $\{e, f, p, q\}$

**Table 2** Bond graph elements and ports: symbols, causality and relations

Element	Symbol	Constitutive relations	Type
Effort source	$S_e \begin{array}{c} e \\ \leftarrow \end{array}$	$e(t)$ given, $f(t)$ arbitrary	One-port
Flow source	$S_f \begin{array}{c} \leftarrow \\ f \end{array}$	$f(t)$ given, $e(t)$ arbitrary	One-port
Effort store	$\begin{array}{c} e \\ \leftarrow \\ f \end{array} L$	$f = \frac{1}{L} p$	One-port
Flow store	$\begin{array}{c} e \\ \leftarrow \\ f \end{array} C$	$e = \frac{1}{C} q$	One-port
Dissipator	$\begin{array}{c} e \\ \leftarrow \\ f \end{array} R$	$e = R f$	One-port
	$\begin{array}{c} e \\ \leftarrow \\ f \end{array} R$	$f = \frac{1}{R} e$	
Transformer	$\begin{array}{c} e_1 \\ \leftarrow \\ f_1 \end{array} TF \begin{array}{c} e_2 \\ \leftarrow \\ f_2 \end{array}$	$e_1 = m e_2$	Two-port
	$\begin{array}{c} e_1 \\ \leftarrow \\ f_1 \end{array} TF \begin{array}{c} e_2 \\ \leftarrow \\ f_2 \end{array}$	$f_2 = m f_1$	
Gyrator	$\begin{array}{c} e_1 \\ \leftarrow \\ f_1 \end{array} GY \begin{array}{c} e_2 \\ \leftarrow \\ f_2 \end{array}$	$e_1 = r f_2$	Two-port
	$\begin{array}{c} e_1 \\ \leftarrow \\ f_1 \end{array} GY \begin{array}{c} e_2 \\ \leftarrow \\ f_2 \end{array}$	$e_2 = r f_1$	
Effort or 1-junction	$\begin{array}{c} e_1 \\ \leftarrow \\ f_1 \end{array} 1 \begin{array}{c} e_n \\ \leftarrow \\ f_n \end{array}$	$e_1 + e_2 + \dots + e_n = 0$ $f_1 = f_2 = \dots = f_n$	Junction
Flow or 0-junction	$\begin{array}{c} e_1 \\ \leftarrow \\ f_1 \end{array} 0 \begin{array}{c} e_n \\ \leftarrow \\ f_n \end{array}$	$e_1 = e_2 = \dots = e_n$ $f_1 + f_2 + \dots + f_n = 0$	Junction

**Table 3** Effort store, flow store and dissipator elements

Domain	Element type	Name	Symbol relation	Dynamic relation	Linear relation	Units
Electric	Effort store	Inductor	$L$	$i = \varphi^{-1}(\lambda)$	$i = \frac{1}{L}\lambda$	$[H = m^2 \text{ kg s}^{-2} \text{ A}^{-2}]$
	Flow store	Capacitor	$C$	$v = \psi^{-1}(q)$	$v = \frac{1}{C}q$	$[F = s^4 \text{ A}^2 \text{ m}^{-2} \text{ kg}^{-1}]$
	Dissipator	Resistor	$R$	$v = \gamma(i)$	$v = Ri$	$[\Omega = \text{kg m}^2 \text{ s}^{-3} \text{ A}^{-2}]$
Micro economy	Effort store	Inertia	$I_e$	$f = \varphi^{-1}(\lambda)$	$f = \frac{1}{I_e}\lambda$	$[\$ \text{ unit}^{-2} \text{ period}^2]$
	Flow store	Compliance	$C_e$	$e = \psi^{-1}(q)$	$e = \frac{1}{C_e}q$	$[\$^{-1} \text{ unit}^2]$
	Dissipator	Resistor	$R_e$	$e = \gamma(f)$	$e = R_e f$	$[\$ \text{ unit}^{-2} \text{ period}]$
Macro economy	Effort store	Inertia	$I_e$	$\dot{n} = \varphi^{-1}(I)$	$\dot{n} = \frac{1}{I_e}I$	$[\$ \text{ capita}^{-2} \text{ period}^2]$
	Flow store	Compliance	$C_e$	$Y = \psi^{-1}(n)$	$Y = \frac{1}{C_e}n$	$[\$^{-1} \text{ capita}^2]$
	Dissipator	Resistor	$R_e$	$Y = \gamma(\dot{n})$	$Y = R_e \dot{n}$	$[\$ \text{ capita}^{-2} \text{ period}]$

causality in a BG allows the detection of degeneracies in the system modelling. This problem occurs in situations where one or more algebraic loops exist, and the one or more state variables are linearly dependent on the rest. Causality strokes are represented in Table 2.

### 3 Bond graph modelling of economic systems

In 1975 Brewer [9] extended the BG concepts to micro-economy. Later the work was continued in [10–12]. Brewer considered effort and flow state variables to be the ‘unit price of commodity’ and ‘commodity flow rate’, respectively. Power and energy have for analogous variables ‘money flow’ and ‘money’, representing a generalization of the standard physical variables. The related dynamical concepts are shown in Tables 1 and 3.

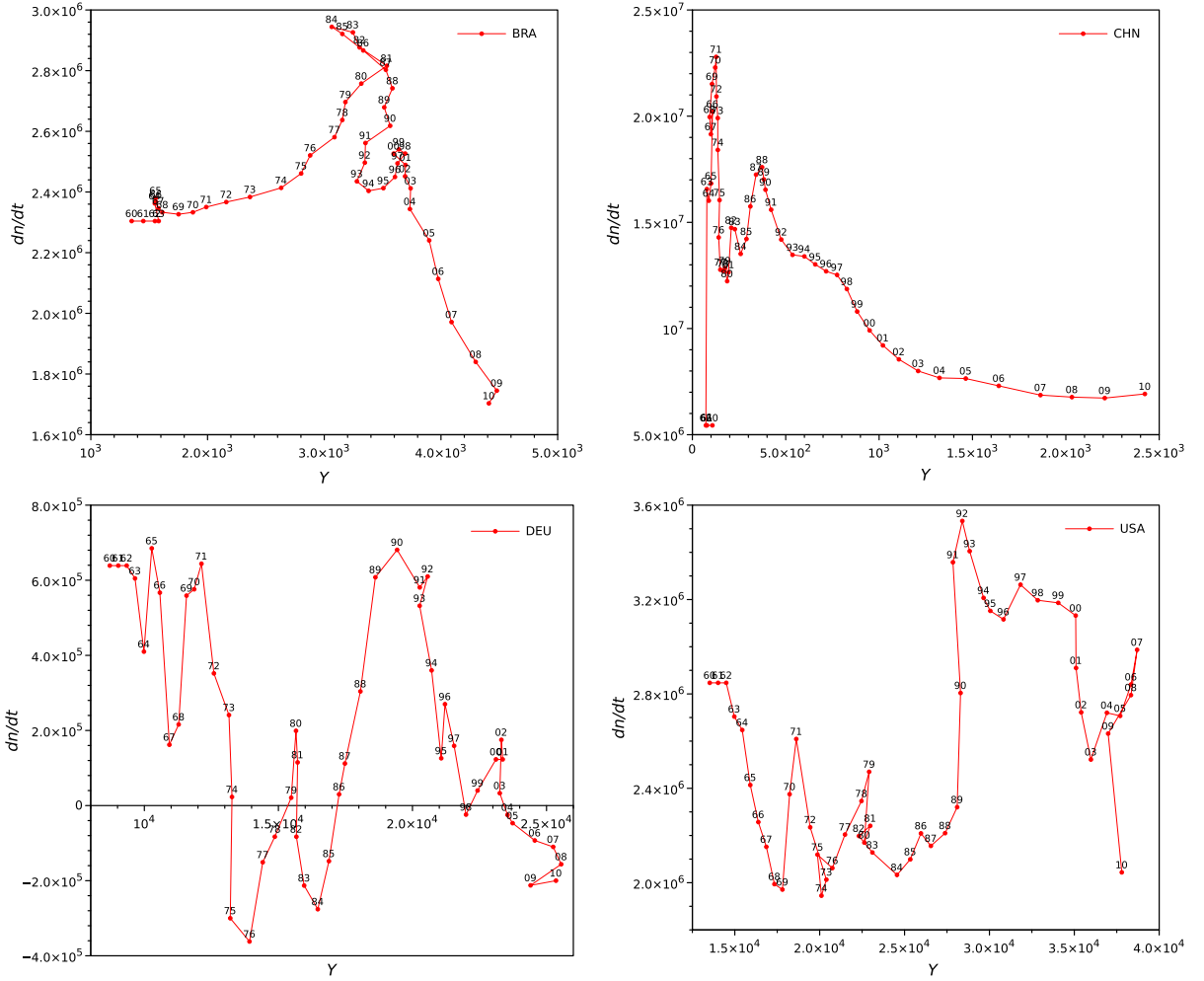
In what concerns the effort and flow junctions Brewer mentions that a ‘proper cash accounting’, known as Walras’ law to economists, leads to the requirement  $e_1 f_1 + \dots + e_n f_n = 0$ . This principle is not totally clear in macro-economy, although, at least for a given time instant, a healthy environment would lead us to such supposition. This ‘power-conservative principle’ comes from considering that different sectors of any economy are related to each other and lead to consider a system of  $n$  equations describing those relationships, using  $n$  variables. To reach a general equilibrium, a successive iteration of price adjustments is required, in order that all excess demand over supply would become zero [73]. Such a ‘power-conservative principle’ had also been established much earlier among

economists, such as Adam Smith, James Mill and Jean Baptiste Say, who considered that the production of final goods gives origin to a distribution of revenues to pay for the use of input factors during the productive process, and such revenues are available to buy those final goods, at last [58]. Say’s law currently, is coined as saying ‘Supply generates Demand’. It is only not clear for macro-economic equilibrium in monetized societies, if hoarding will occur because sub-consumption means crisis (as scarce consumption for production is equivalent to an over-production situation). However, in a dynamic perspective crises can be overcome and a new general equilibrium may be reached. In the sequel we overcome using junctions by considering a generalized one-port element.

In this work we intend to extend the concepts introduced by Brewer to macro-economy. In our formulation the effort and flow variables are represented by the ‘GDP per capita’,  $Y(t)$  and ‘Population flow’  $\dot{n}(t)$ . Therefore, power and energy are similar to Brewer proposal, namely ‘money flow’  $\dot{M}(t)$  and ‘money’  $M(t)$ . The associated dynamic variables and elements are shown in Tables 1 and 3.

Figure 3 shows, for example, the state space portrait  $\{Y, \dot{n}\}$  of BRA, CHN, DEU and USA during 1960–2010.

We observe a complex dynamical relationship between the two variables that poses problems for an accurate modelling. Brewer discussed analytically some possible non-linear relations between effort and flow in micro-economy, but in our case we follow a distinct approach for macro-economy. First we make use of real data, and we shall verify how the BG model fits in.



**Fig. 3** State space portrait  $\{Y, \dot{n}\}$  of BRA, CHN, DEU and USA during 1960–2010

Second, we consider a linear relation between effort and flow by means of a generalized fractional-order element:

$$\dot{n}(t) = C D_t^\alpha Y(t), \quad C, \alpha \in \mathfrak{R}, \quad (1)$$

Third, we allow parameters  $\{C, \alpha\}$  to be variable in time and to have both positive and negative values, so that they can adapt to the changing conditions in data.

Several issues need to be clarified before continuing. We note that the generalized one-port element (1) allows us to avoid using the interconnection of several integer order one-port elements while using only two parameters. We observe also that negative/positive values of  $C$  mean an active/passive element, while posi-

tive/negative values of  $\alpha$  represent capacitive/inductive elements (in the electrical domain). It should be highlighted that in classical terms we have  $|\alpha| \leq 1$ , but as pointed by [31, 72], values outside this range are possible. For example, the frequency-dependent negative resistor (FDNR) [14, 59, 62] implements easily the second-order constitutive equation  $i = D D_t^2 v$ , where  $D$  is a parameter, used in electronic filters.

The database for 50 countries includes GDP and population for a five decades time span, sampled with one year period (Table 4). Data were collected from the World Bank national development indicators to cover the period coming from 1960 to 2010:

- (1) GDP per capita comes from NY.GNP.PCAP.KD. It is the GNI per capita (constant 2,000 US\$). GNI per

capita is gross national income divided by midyear population. GNI (formerly GNP) is the sum of value added by all resident producers plus any product taxes (less subsidies) not included in the valuation of output plus net receipts of primary income (compensation of employees and property income) from abroad. Data are in constant 2,000 U.S. dollars (<http://data.worldbank.org/indicator/NY.GNP.PCAP.KD>).

- (2) Population density comes from EN.POP.DNST. It is the people per sq. km of land area. Population density is midyear population divided by land area in square kilometres. Population is based on the de facto definition of population, which counts all residents regardless of legal status or citizenship—except for refugees not permanently settled in the country of asylum, who are generally considered part of the population of their country of origin. Land area is a country's total area, excluding area under inland water bodies, national claims to continental shelf, and exclusive economic zones. In most cases the definition of inland water bodies includes major rivers and lakes (<http://data.worldbank.org/indicator/EN.POP.DNST>).

For implementing the fractional-order relation (1) it is adopted the Grünwald-Letnikov formulation of definition of fractional derivative:

$${}^G L \mathcal{D}_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{j=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^j \binom{\alpha}{j} f(t-jh), \quad t > a, \quad (2)$$

where  $\Gamma(\cdot)$  is Euler's gamma function,  $[x]$  means the integer part of  $x$ , and  $h$  is the step time increment.

In the numerical algorithm for calculating  $\mathcal{D}_t^\alpha$  the time increment  $h$  is approximated by the sampling period  $T_s$ , and the series is truncated at the  $r$ -th term yielding:

$$\mathcal{D}^\alpha [f(t)] \approx \frac{1}{T_s^\alpha} \sum_{k=0}^r (-1)^j \frac{\Gamma(\alpha+1)}{j! \Gamma(\alpha-j+1)} f(t-jT_s). \quad (3)$$

Other techniques adopt the Laplace or the Z transforms and lead to rational fractions [15, 38, 41, 56]. In our case, since time series have one single sample per year, it is adopted the series truncation method with  $T_s = 1$ .

For estimating the values of the parameters  $\{C, \alpha\}$  it was used a genetic algorithm (GA) due to its simplicity and robustness. In brief, the GAs are a computational technique to find exact or approximate solutions in optimization problems [25, 28, 37]. A GA consists in a population representing candidate solutions of an optimization problem that evolves (computationally) towards better solutions. The GA starts by initializing the population of solutions randomly, and improves them by means of the repetitive application of mutation, crossover and selection operators. In each iteration the fitness of every individual in the population is evaluated, and the best individuals are selected to form part of the new population. The GA terminates when either the maximum number of generations is produced, or a satisfactory fitness level has been reached.

We adopt a GA population of 400 elements, 400 iterations and a fitness function  $J$  given by

$$\dot{n}_a(k) = C \sum_{j=0}^r (-1)^j \frac{\Gamma(\alpha+1)}{j! \Gamma(\alpha-j+1)} Y(k-j) \quad (4a)$$

$$J = \sum_{k=0}^r \{\dot{n}(k) - \dot{n}_a(k)\}^2, \quad (4b)$$

where  $k$  is the discrete time index,  $\dot{n}_a$  denotes the approximation to the flux  $\dot{n}$  based on the effort  $Y$ . The upper limit  $r$  represents the total number of points involved in the fitting procedure and the memory length in (3). We decided to adopt a sliding window of one point for each calculation. This means that there is an overlap of  $r-1$  years between 2 consecutive time windows.

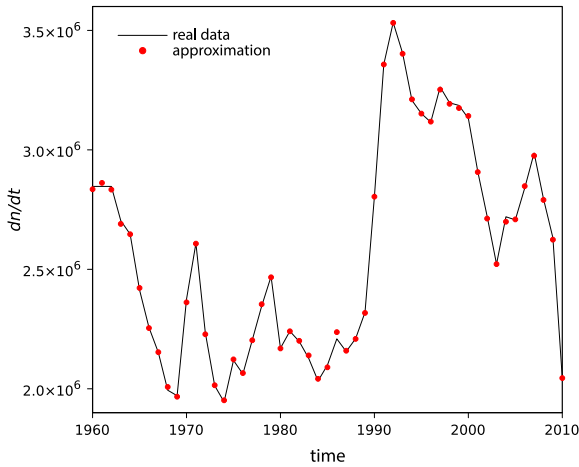
Several experiments demonstrated that the fitting between  $\dot{n}$  and  $\dot{n}_a$  was very good (see Fig. 4) at the cost of a strong variability of the parameters. For example, Fig. 5 shows the variation of  $\{C, \alpha\}$  versus time for USA, with  $r = 12$ , during 1960–2010. It was observed that the effect of  $r$  upon this variability was minor. Figure 6 depicts the contour plots of  $\{C, \alpha\}$  versus  $r$  and time for USA, where is visible the propagation of the economical variations horizontally, apparently with a minor influence of  $r$ .

Plotting the variation of the parameters in the locus  $\{C, \alpha\}$  leads to the emergence of a distinct pattern for each country. For example, Fig. 7 shows the patterns of BRA, CHN, DEU and USA in the locus  $\{C, \alpha\}$ , with  $r = 12$ , during 1960–2010.

In order to capture the signature of each country and to describe the world dynamics it was decided

**Table 4** Countries and labels

No.	Label	Country	No.	Label	Country
1	ARG	Argentina	26	IRL	Ireland
2	AUS	Australia	27	IRN	Iran
3	AUT	Austria	28	ITA	Italy
4	BEL	Belgium	29	JPN	Japan
5	BRA	Brazil	30	KOR	Korea
6	CAN	Canada	31	LUX	Luxembourg
7	CHE	Switzerland	32	MEX	Mexico
8	CHL	Chile	33	MYS	Malaysia
9	CHN	China	34	NGA	Nigeria
10	COG	Congo	35	NLD	Netherlands
11	COL	Colombia	36	NOR	Norway
12	DEU	Germany	37	NZL	New Zealand
13	DNK	Denmark	38	PAK	Pakistan
14	DZA	Algeria	39	PHL	Philippines
15	EGY	Egypt	40	PRT	Portugal
16	ESP	Spain	41	SAU	Saudi Arabia
17	FIN	Finland	42	SGP	Singapore
18	FRA	France	43	SWE	Sweden
19	GBR	United Kingdom	44	THA	Thailand
20	GRC	Greece	45	TUR	Turkey
21	HUN	Hungary	46	URY	Uruguay
22	ICE	Iceland	47	USA	United States
23	IDN	Indonesia	48	VEN	Venezuela
24	IND	India	49	ZAF	South Africa
25	ISR	Israel	50	ZWE	Zimbabwe



**Fig. 4** Variation of  $\dot{n}(t)$  and  $\dot{n}_a(t)$ , with  $r = 12$ , versus time for USA during 1960–2010

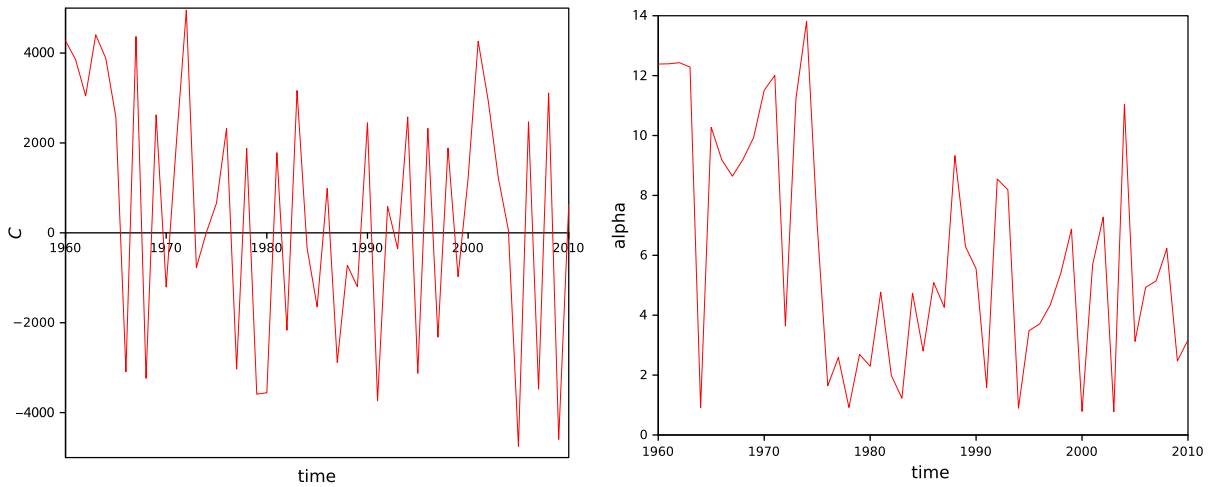
to calculate the median of the parameters  $\{C, \alpha\}$ . The median is more robust than the arithmetic average since it has a smaller sensitivity to outliers. Figure 8 shows

the 50 countries represented in the locus  $\{C, \alpha\}$ . Given the high concentration of points (countries) near the origin in Fig. 9 is represented a magnification of that area. The surface of each circle is proportional to the variability of  $\alpha$ , namely to the interquartile range of its variation along the period 1960–2010.

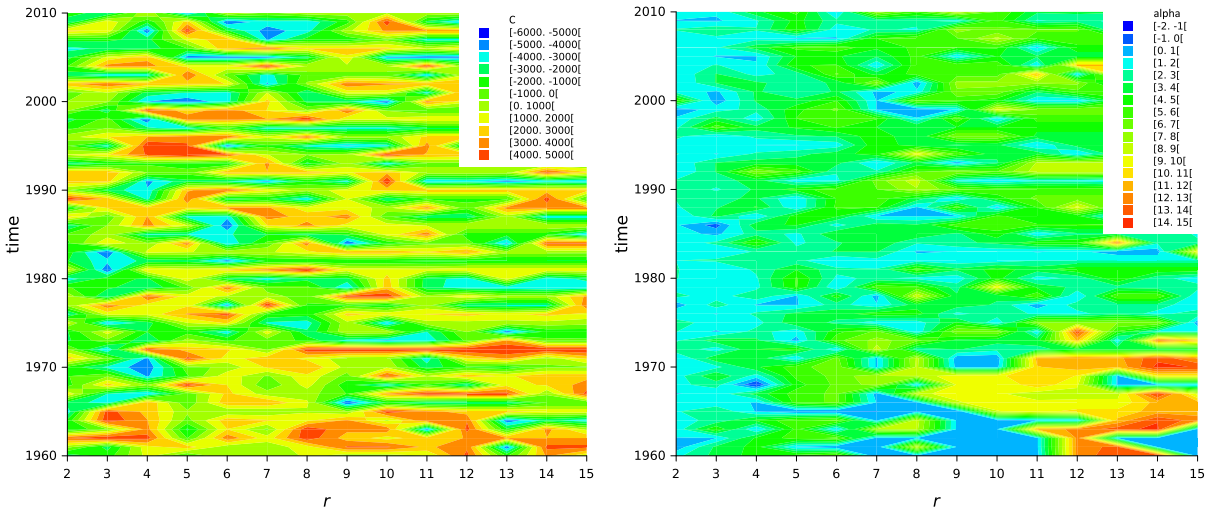
In fact, the locus  $\{C, \alpha\}$  was drawn for  $r = 2$  up to  $r = 15$ . For example, Figs. 10 and 11 depict the locus  $\{C, \alpha\}$  for  $r = 9$  and  $r = 15$ , respectively. It was observed that changing between two consecutive values of  $r$  had the effect of rotating the chart, while maintaining the general layout approximately. Therefore, points were moved from the 4th up to the 1st quadrant, being the limit value  $r = 12$ . Larger values of  $r$  had minor effect upon the chart, and it was concluded that 12 years were the adequate value for  $r$  to capture adequately the memory of the phenomenon.

We verify the emergence of a pattern, and we start by a simple analytical discussion. The elements are of the ‘capacitive type’ ( $\alpha > 0$ ) as one should expect





**Fig. 5** Variation of the parameters  $\{C, \alpha\}$ , with  $r = 12$  versus time for USA during 1960–2010



**Fig. 6** Contour plots of  $\{C, \alpha\}$  versus  $r$  and time for USA during 1960–2010

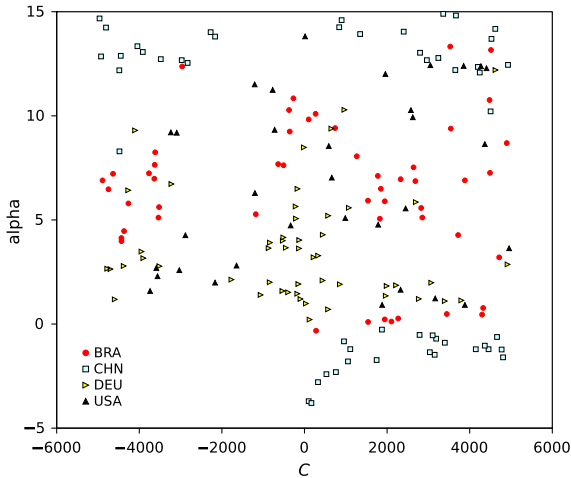
since effort  $Y(t)$  has a smooth variation, while the flux  $\dot{n}(t)$  reveals a very volatile behaviour. A central part, constituted mainly by the European countries, JPN and USA, remained invariant, but the new economies, such as BRA, CHN and IND, rotated anticlockwise while increasing  $r$ . As mentioned previously the effect stabilized for  $r = 12$ . A more detailed analysis of the results from an economical point of view is developed in the next section.

#### 4 Discussion

State space portraits are very different, as the four extreme cases depicted in Fig. 3 can show. Figure 8

is an accurate view of the last half-century economic trends in a global perspective. Its main message says that scale matters: the human creative powers matter as well as the departing GDP per capita levels. Economic development follows the traced eigen-vector. The world economic system is very sensitive to the selected vital variables representing scale, indeed.

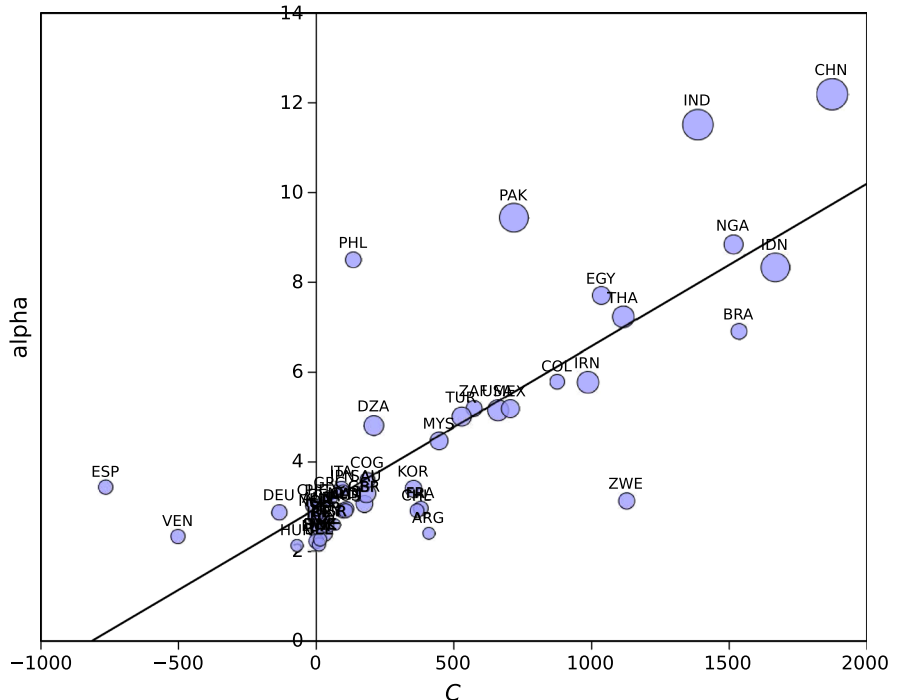
The developed countries form a large node of high-standard patterns of life and welfare in the locus  $\{C, \alpha\}$ , with  $r = 12$ . They were the large-scale economies at the departing moment. Independently from their territory dimension they were the large-scale national economies in the global scene. Several other countries are following this pattern. The promising new



**Fig. 7** Patterns of BRA, CHN, DEU and USA in the locus  $\{C, \alpha\}$ , with  $r = 12$ , during 1960–2010

performers include the large far-Asian partners (China, India, Indonesia, Thailand, Pakistan and Philippines), and the large South-American Brazil. Independently from their lateness in industrializing and modernizing they are huge at the departing moment, and this fact commands their performance in globalization.

**Fig. 8** Representation of the 50 countries in the locus  $\{C, \alpha\}$ , with  $r = 12$ , for the period 1960–2010



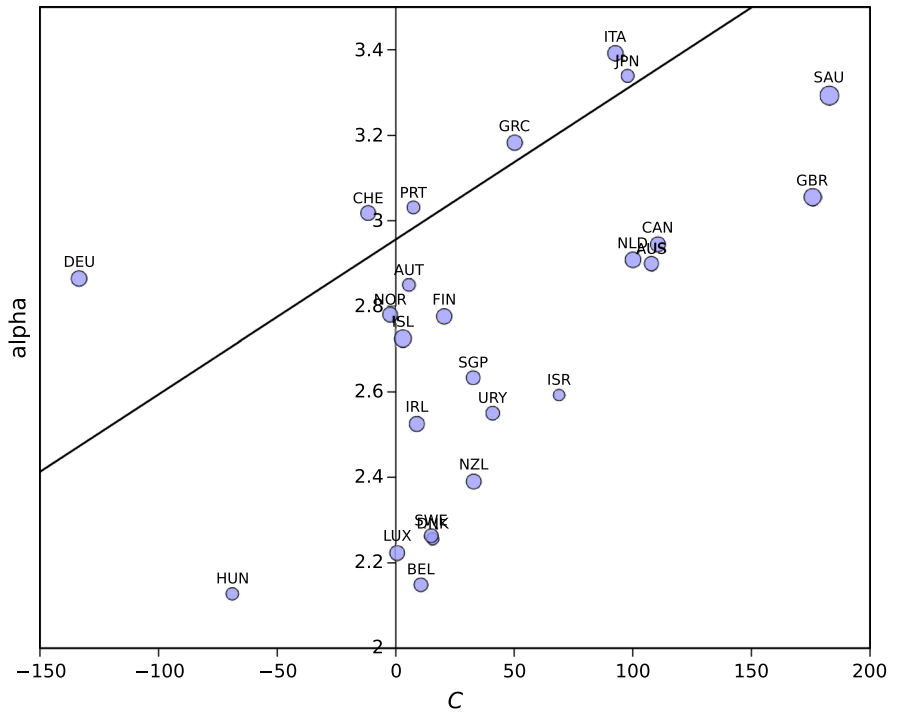
The European-neighbouring countries (Iran, Egypt, Turkey) and South–Central-American partners (such as Argentina, Colombia and Mexico) present large potential similarities for growth among them and with the Western partners, too. However, similarities among the partners belonging to the Western wealthy world are so much stronger that they all overlap. A magnified representation of the central part of the locus is required (Fig. 9). The node includes the European continent, Japan, Canada and the USA. Germany is a bit dissimilar from the other European states and is commanding them.

A remarkable feature in the universe of the countries belonging to this node has been the flourishing of democracy, while rapid economic growth provided superior competitive advantages, higher wages and mobility [13,35].

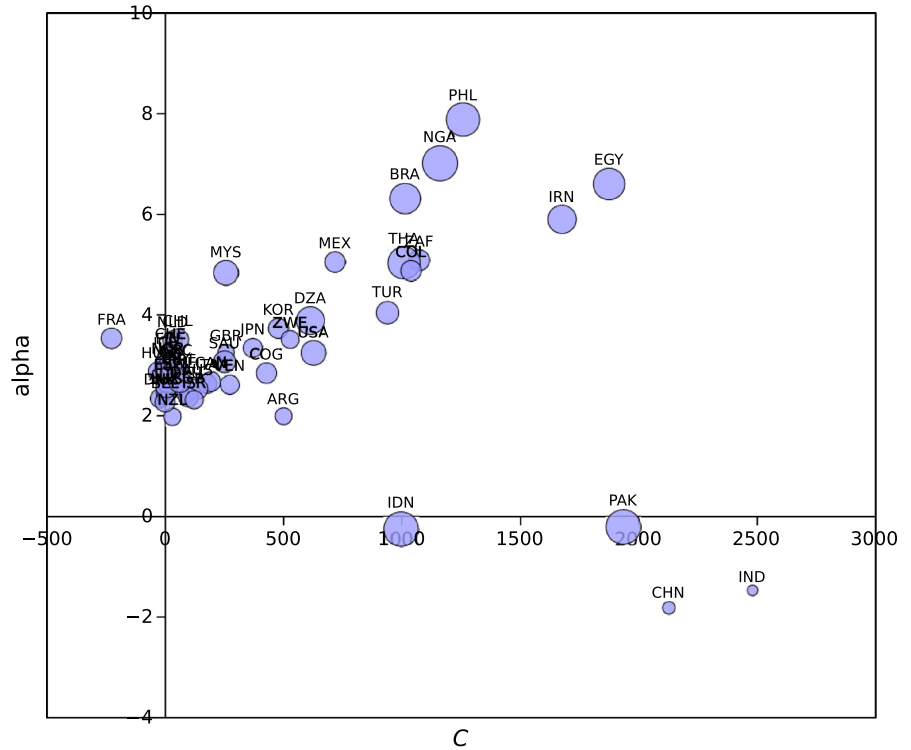
It is out of the scope of this paper to discuss if economic growth spread will bring a global democracy system or will move to the future Asian great power political regimes, as the Nobel Prize Robert Fogel suggests [22].

As the high departing levels do matter in spite of the individual space portrait differences demonstrated in Fig. 3, territory scale matters in a historical perspective,

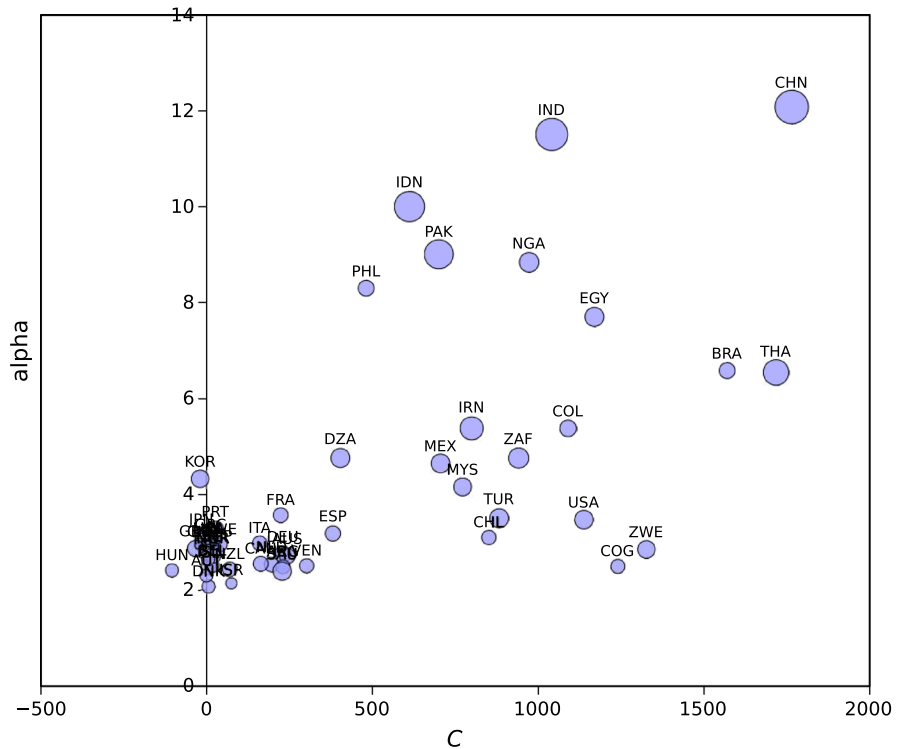
**Fig. 9** Magnified representation of the central part of the locus  $\{C, \alpha\}$ , with  $r = 12$ , for the 50 countries and the period 1960–2010



**Fig. 10** Representation of the 50 countries in the locus  $\{C, \alpha\}$ , with  $r = 9$ , for the period 1960–2010



**Fig. 11** Representation of the 50 countries in the locus  $\{C, \alpha\}$ , with  $r = 15$ , for the period 1960–2010



because it has dictated the departing levels of performance. So, the small European countries' control over large territories in the past may deserve credibility, a hypothesis to be checked from the BG perspective [2].

## 5 Conclusions

Formal modelling in economics has the special aim of fitting theories to understand a complex world, because models reflect ideas that may be 'more powerful than is commonly understood' [63].

The developed approach helps the perception of how our complex world has moved as a system. It also helps to frame a precise vision on how the future world may be like. Using simple assumptions, models may provide decisive insights into economic phenomena, and be part of the analysis for theoretical paradigms, putting in motion crucial ways for the humankind survival [5].

Models recall that day-to-day adjustments are not enough to view the whole economic picture, because constant small changes make up long-run trends for the whole pattern of changes in the global world [27]. To overlap the global economic system one may remember Shakespeare in saying that 'All the world's a stage, and all the men and women merely players' [61].

In this perspective, the paper presented a BG approach to the modelling of economies. Concepts proposed for micro-economy were generalized having in mind the macro-economy problem. In this context complex aspects such as the generalization of the principle of conservation of power and the assignment of causalities were circumvented by means of a variable fractional-order element. Several experiments analysed the influence of the memory effects and characterized the dynamical behaviour in the light of the fractional approach.

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