Two new power indices based on winning coalitions

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Abstract

Deegan and Packel (1979) and Holler (1982) proposed two power indices for simple games: the Deegan– Packel index and the Public Good Index. In the definition of these indices, only minimal winning coalitions are taken into account. Using similar arguments, we define two new power indices. These new indices are defined taking into account only those winning coalitions that do not contain null players. The results obtained with the different power indices are compared by means of two real-world examples taken from the political field.

Keywords

Power indices; simple games; decision-making processes

1. Introduction

In the field of political science, the study of the *a priori* distribution of power in a voting body has a main role. A particular class of Transferable Utility (TU) games, the simple games, can be used to model the decision-making process. Different power indices have been suggested in order to assess the *a priori* distribution of power among the players. A power index gives a measure of the ability that players have to transform a losing coalition into a winning one. We arise far from consensus over the issue of choice of an appropriate power index in a given context, and several power indices are employed.

The main power indices of the literature include the Shapley– Shubik index [7], the Banzhaf index [2], the Deegan– Packel index [3] and the Public Good Index [4]. The first two power indices are based on vulnerable winning coalition. A winning coalition is vulnerable when it has at least one member whose removal would cause the resulting coalition to be a losing coalition. An agent is considered critical when his elimination from a winning coalition turns this coalition into a losing coalition. In Banzhaf's model, the power of an agent is proportional to the number of coalitions in which he is critical.

A minimal winning coalition is one such that all its members are critical. According to Deegan and Packel [3], only minimal winning coalitions should be considered in establishing the power of a voter. They assume that all minimal winning coalitions are equiprobable and all the voters in a minimal winning coalition divide the spoils equally. With this assumption, they define the Deegan– Packel index. Holler [4] proposes that only minimal winning coalitions should be considered when it comes to measuring power and the outcome is a public good and he defines the Public Good Index. The Public Good Index is determined by the number of minimal winning coalitions containing the voter divided by the sum of such numbers across all the voters.

Several desirable properties have been introduced in the context of power indices. In this paper, some of these properties will be mentioned, as well as some characterizations of the main power indices according to them. We define two modifications of the Deegan–Packel index and two modifications of the Public Good Index. Although it will not be discussed which index is most appropriate, two real-world examples taken from the political field are used to compare the results obtained with different power indices. We should mention that issues related to those of this paper have been studied by Pinto [6].

2. Simple games

In this section, we recall the main notions related with characteristic function games and simple games. *A characteristic function game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and v, the characteristic function, is a real function on $2^N = \{S : S \subseteq N\}$ with $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a coalition. Shorthand notation will be used and will be written $S \cup i$ for the set $S \cup \{i\}$ and $S \backslash i$ for the set $S \backslash \{i\}$.

A *null player* in a game (N, v) is a player $i \in N$ such that $v(S \cup i) = v(S)$ for all $S \subseteq N \backslash i$. Two players $i, j \in N$ are *symmetric* in a game (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$.

An important subclass of characteristic function games is the class of simple games. A *simple game* is a characteristic function game (N, v) such that:

- $v(S) = 1$ or $v(S) = 0$, for every $S \subseteq N$;
- v is a monotone function, that is, $v(S) \leq v(T)$, for every $S \subseteq T \subseteq N$; and
- $v(N) = 1$.

$SI(N)$ denotes the set of simple games with player set N. In a simple game (N, v), a coalition $S \subseteq N$ is *winning* if $v(S) = 1$, and S is *losing* if $v(S) = 0$. $W(v)$ denotes the set of winning coalitions of the game (N, v) and by $W_i(v)$ the subset of $W(v)$ formed by coalitions $S \subseteq N$ such that $i \in S$. A winning coalition $S \subseteq N$ is a *minimal winning* coalition if every proper subset of S is a losing coalition, that is, S is a minimal winning coalition in (N, v) if $v(S) = 1$ and $v(T) = 0$ for any $T \subset S$. $M(v)$ denotes the set of minimal winning coalitions of the game (N, v) and by $M_i(v)$ the subset of $M(v)$ formed by coalitions $S \subseteq N$ such that $i \in S$.

A simple game (N, v) is called a weighted voting game (briefly, weighted game) if there exist natural integers $w_1, \ldots, w_n$ such that every coalition $S, S \in W(v)$ if and only if the sum of the w_i's, $i \in S$, is at least equal to some preset quota q.

Given $S \subseteq N, (N, u_S)$ denotes the *unanimity game* of the coalition S, i.e. $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ otherwise. Note that $|M(u_S)| = 1$.

A winning coalition $S \subseteq N$ is a *quasi-minimal winning* coalition if there does not exist a null player $i \in S$. $WNP(v)$ denotes the set of quasi-minimal winning coalitions of the simple game (N, v) and by $WNP_i(v)$ the subset of $WNP(v)$ formed by coalitions $S \subseteq N$ such that $i \in S$. For every simple game (N, v),

$$M(v) \subseteq WNP(v) \subseteq W(v).$$

Given a simple game (N, v), a *swing* for a player $i \in N$ is a coalition $S \subseteq N$ such that $S \backslash i$ is a losing coalition and S is a winning one. $\eta_i(v)$ denotes the set of swings for player $i \in N$. A winning coalition $S \subseteq N$ is a minimal winning coalition if and only if $S \in \eta_i(v)$ for every $i \in S$.

Given a family of simple games $H \subseteq SI(N)$, a *power index* on H is a function f, which assigns to a simple game $(N, v) \in H$ a vector

$$(f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^n,$$

where the real number $f_i(N, v)$ is the 'power' of the player i in the game (N, v) according to f. The power index of a simple game can be interpreted as a measure of the ability of the different players to turn a losing coalition into a winning one. It is useful to single out a list of three desirable properties of power indices.

- a solution f satisfies the *null player* property if $f_i(N, v) = 0$ for every $(N, v) \in H$ and every null player $i \in N$;
- a solution f is *symmetric* if $f_i(N, v) = f_j(N, v)$ for every $(N, v) \in H$ and for every pair of symmetric players $i, j \in N$; and
- a solution f is *efficient* if $\sum_{i \in N} f_i(N, v) = 1$ for every $(N, v) \in H$.

Young [8] proposed the strong monotonicity property. A solution f satisfies *strong monotonicity* if $f_i(N, v) \geq f_i(N, w)$ for every pair of games $(N, v), (N, w) \in H$ and for all $i \in N$ such that $v(S \cup i) - v(S) \geq w(S \cup i) - w(S)$ for all $S \subseteq N \backslash i$.

In Lorenzo-Freire et al. [5], a new characterization of Deegan–Packel index is obtained with a monotonicity property. In Alonso-Meijide et al. [1], a new characterization of the Public Good Index by means of a property of monotonicity with a similar flavour is provided and the other indices are revisited comparing new and old characterizations.

3. Two new power indices

In this section, we define two new power indices. Our first intention was to consider as a measure of a player's power, two indices similar to those proposed by Holler and Deegan and Packel but using the set of winning coalitions instead of the set of minimal winning coalitions, that is,

$$f_i(N, v) = \frac{|W_i(v)|}{\sum_{j \in N} |W_j(v)|} \quad \text{or} \quad g_i(N, v) = \frac{1}{|W(v)|} \sum_{S \in W_i(v)} \frac{1}{|S|}.$$

Hence, from the inclusions in the formula of the set of winning coalitions instead of the set of minimal winning coalitions, it can be deduced that these new power indices involve more amount of information. As the cardinality of the set $W_i(v)$ and $W_j(v)$ coincide if players i and j are symmetric, then these new two indices satisfy the symmetry property. By construction, the two indices satisfy efficiency. The main problem of the previous indices is that they do not satisfy the null player property, because the set $W_i(v)$ is not empty for a null player i.

Then, we consider a modification of the index f, defined as follows

$$f_i'(N, v) = \frac{|WNP_i(v)|}{\sum_{j \in N} |WNP_j(v)|}.$$

This index satisfies efficiency and symmetry as the index f. As the set $WNP_i(v)$ is empty if i is a null player, then f' satisfies the null player property, too. In the definition of this index, we use an argument similar to that employed to Holler to define the Public Good Index. Using a parallel argument to that used by Deegan and Packel, we could define an index similar to Deegan– Packel index and it is given by the formula

$$g_i'(N, v) = \frac{1}{|WNP(v)|} \sum_{S \in WNP_i(v)} \frac{1}{|S|},$$

as the power index f^0, g^0 satisfies efficiency, null player and symmetry.

3.1 Example: The Portuguese Parliament

In this subsection, we apply the previous indices to the Portuguese Parliament (IX Legislature 2002):

Parties	Members
PPD/PSD (player 1)	105
PS (player 2)	96
CDS/PP (player 3)	14
PCP (player 4)	10
BE (player 5)	3
PEV (player 6)	2

This Parliament can be seen as a weighted game, where the weights of each player coincides with his number of members. The Portuguese Parliament is constituted by 230 members and the quota is equal to 116. We have a total of 64 coalitions. First, we list the 33 losing coalitions (each player is identified with the previous numbers): Y, 1, 2, 3, 4, 5, 6, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56, 156, 235, 236, 245, 246, 256, 345, 346, 356, 456, 2356, 2456, 3456.

The 31 winning coalitions are: 12, 13, 123, 124, 125, 126, 134, 135, 136, 145, 146, 234, 1234, 1235, 1236, 1245, 1246, 1256, 2345, 2346, 1345, 1346, 1356, 1456, 12345, 12346, 12356, 12456, 13456, 23456, *N*.

The five minimal winning coalitions are: 12, 13, 145, 146, 234.

It is easy to prove that a player is null if and only if he does not belong to none of the minimal winning coalitions. Then, in this case, the set of null players is empty. Then, f coincides with f^0, and g coincides with g^0, because the sets $W_i$ðvÞ and WNP_iðvÞ are equal, for every player i.

Player 1 belongs to two minimal winning coalitions of size 2 and to two minimal winning coalitions of size 3. Players 2 and 3 belong to one minimal wining coalition of size 2 and to one minimal winning coalition of size 3. Player 4 belongs to three minimal winning coalitions of size 3. Finally, players 5 and 6 belong to one minimal winning coalition of size 3.

The Deegan– Packel index is equal to (0.333; 0.167; 0.167; 0.2; 0.067; 0.067).

The Public Good Index is equal to (0.307; 0.153; 0.153; 0.231; 0.076; 0.076).

Player 1 belongs to two winning coalitions of size 2, to nine winning coalitions of size 3, to 10 winning coalitions of size 4, to five winning coalitions of size 5 and to a winning coalition of size 6. Players 2 and 3 belong to a winning coalition of size 2, to 5 winning coalition of size 3, to eight winning coalitions of size 4, to 5 winning coalitions of size 5 and to a winning coalition of size 6. Player 4 belongs to five winning coalition of size 3, to eight winning coalitions of size 4, to five winning coalitions of size 5 and to a winning coalition of size 6. Finally, players 5 and 6 belong to three winning coalition of size 3, to seven winning coalitions of size 4, to five winning coalitions of size 5 and to a winning coalition of size 6.

The indices f and f^0 are equal to (0.228; 0.169; 0.169; 0.161; 0.136; 0.136).

The indices g and g^0 are equal to (0.215; 0.172; 0.172; 0.156; 0.126; 0.126).

3.2 *Example: The Council of the European Union 1958–1972*

In this subsection, we apply the previous indices to the Council of the European Union (1958 – 1972):

Countries	Votes
France (player 1)	4
Germany (player 2)	4
Italy (player 3)	4
Belgium (player 4)	2
Netherlands (player 5)	2
Luxembourg (player 6)	1

As the previous example, the Council of European Union can be seen as a weighted game. In this case, the total of votes is 17 and the quota is equal to 12. There are 14 winning coalitions: 123, 1245, 1345, 2345, 1236, 23456, 12456, 13456, 1234, 1235, 12346, 12356, 12345, N.

In this case, there are four minimal winning coalitions: 123, 1245, 1345, 2345. Luxembourg is a null player because it does not belong to any minimal winning coalition.

There are seven quasi-winning coalitions: 123, 1245, 1345, 2345, 1234, 1235, 12345. Then, f does not coincide with f^0, and g does not coincide with g^0.

Taking into account that France, Germany and Italy have four votes, they are symmetric players, Belgium and Netherlands are symmetric, too. In this case, as the non-null players belong to the same number of minimal winning coalitions, the Public Good Index of these countries coincide.

The Deegan– Packel index is equal to (0.208; 0.208; 0.208; 0.188; 0.188; 0).

The Public Good Index is equal to (0.2; 0.2; 0.2; 0.2; 0.2; 0).

The index f is equal to (0.190; 0.190; 0.190; 0.159; 0.159; 0.111).

The index g is equal to (0.196; 0.196; 0.196; 0.155; 0.155; 0.101).

The index f^0 is equal to (0.214; 0.214; 0.214; 0.179; 0.179; 0).

The index g^0 is equal to (0.219; 0.219; 0.219; 0.171; 0.171; 0).

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