Preservice teachers' knowledge on elementary geometry concepts

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Abstract

This text is based on research, which is still in progress, whose main objective is to identify and understand what the main difficulties of future mathematics teachers of basic education are, regarding their knowledge in geometry in the context of the curricular unit of Geometry during their undergraduate degree. We chose a qualitative approach in the form of case study, in which data collection was done through observation, interviews, a diverse set of tasks, a diagnostic test and other documents. This paper focuses on the test given to prospective teachers at the beginning of the course. The preliminary analysis of the data points to a weak performance of future teachers in the test issues addressing elementary knowledge of Geometry.

Key-words: elementary concepts of geometry, initial training for teachers, geometrical knowledge

Introduction

The constant acknowledgement of the lack of basic structuring concepts among the students who apply, and have been applying since 2007/08, to the bachelor degrees of Basic Education (LEB), opened the way to the intention of learning how teaching mathematics could lead, not only to learning mathematics, but also to learning about mathematics. We aim at understanding how to develop, among the students of LEB, solid training in mathematics and didactics, as well as a more positive attitude towards mathematics and geometry skills. Concerning the importance of liking mathematics and enjoying teaching it, Braumman (2004) says that social influences and, mainly, the influence of teacher, are crucial. He also mentions that it is important for the teacher of the 1st cycle to like Mathematics and to transmit that liking, since it is impossible to feign a liking which doesn’t exist (Braumman, 2004). Therefore, we hope that these students, in their teaching activity, will able to awaken their joy for mathematics and, consequently, make them more skilled in mathematics.

The different curricular recommendations and orientations, at a national and international level (e.g. PMEB, DGIDC, NCTM), regarding geometry for the 1st and 2nd cycles, assume that future teachers are proficient in these matters. Therefore, we decided to search, identify and understand how future teachers relate with each other regarding their perceptions and knowledge of the contents concerning geometry in a common classroom environment.

Teachers' initial training

Teachers' training has been a research field, especially since the 90s. Since then, the research concerning initial training for teachers has been huge, and there's a considerable amount of research in relation to the knowledge developed by future teachers in order to teach (Ponte & Chapman, 2008). Many of these students do not consider “the mathematical knowledge they are developing in terms of the development of subject knowledge for teaching” (Oliveira & Hannula, 2008, p. 16). Several teachers' trainers (e. g. Ball, Bass, Sleep & Thames, 2007; Bullough & Gittlin, 2001; Korthagen, Kessels, Koster, Lagerwerf & Wubbels, 2001, Loughran, 2006; Ma, 2009; Segal, 2002; Shulman, 1986) mention questions regarding the teacher and teachers' training. However, for several reasons, these contributions have failed in providing an answer for some dilemmas that still persist today in training teachers.

One of the goals of initial training is to develop practical knowledge and skills in teachers so that they, not only reproduce them but also, so that their practice is more dynamic, interactive and reflexive (Vale, 2002). This idea is sustained by Shulman (1986) when, regarding teachers' training, he mentions that education researchers' task is to understand the phenomena behind educations, to learn how to improve its implementation and to discover ways to prepare and train educators and teachers. Several researches highlight the importance of providing the teachers, during their training, with experiences which increase their mathematical knowledge and their knowledge about mathematics (e.g. Ball et al., 2007, Ma, 2009). However, the development of the necessary knowledge to fulfil the teaching profession comprehends different components which, during the last few years, have been described in several ways, not moving away too much from the teacher's knowledge model of Shulman (1986). In the decade of 1960, research shows that knowledge and pedagogy are in-dissociable parts of comprehension (Shulman, 1986).

Nowadays, it's consensual that, in order to teach mathematics, it's necessary to develop not only mathematical knowledge and knowledge about mathematics, but also knowledge on how to teach, considering both didactics and pedagogy. Ma (2009) says that “a limited understanding of the subject confines the teacher’s capacity to promote conceptual understanding amongst students” (p. 83). She also says that among teachers “pedagogical knowledge may not compensate for the ignorance of the concept” (Ma, 2009, p. 135). Wu (1999) states, particularly, that the teacher cannot teach what s/he doesn't know. In order to be a good professional, capable of teaching maths, it's crucial to deeply know mathematics and therefore, as maintained by Hill, Sleep, Lewis and Ball (2007), it's crucial to have the ability of putting to work the strategies which are capable of making the students learn. Also Ponte and Chapman (2008) mention that in order to teach properly, the teacher must know the contents of what s/he is teaching, the students, the context and the teaching techniques. Furthermore, teachers teach what they are.

This makes us analyse whether we have been taking the necessary steps concerning the initial training of our basic education teachers. In Ponte's opinion (2006), there are many critics concerning teachers' training and, within our society, we can perceive a lack of confidence concerning the quality of the initial teachers' training. There are even some who consider that everything that is done in this domain only increase the problems that education faces. However, the current model of teachers' training, according to Bologna, deeply changed the weight of mathematics curricular units. A lot of the basic education teachers' training courses of the different variations comprehended (except the mathematics and sciences variations) only 120 hours of mathematics, which is quite insufficient on order to overcome the weaknesses faced by teacher candidates' basic and secondary education. However, the model that follows the Bologna process has yet another problem as it provides a wide basic training, allowing candidates with different backgrounds to have access to the training of future basic education teachers. A lot of these candidates studied humanities and, therefore, have little preparation concerning mathematics, or a small success rate in mathematical courses.

Learning mathematics “is like a multi-storey building. The foundations may be invisible from the upper floors but they are the ones which hold the whole building” (Ma, 2009, p. 205). As trainers, it's quite difficult to diagnose which foundations the students are missing. Students are afraid to show their scientific weaknesses thinking that with more knowledge they will be able to overcome them. It's the same as if we try to add one or two floors more to a building designed only to have two floors, without strengthening its foundations, which would undoubtedly lead to its collapse. Therefore, it's crucial that future teachers know the basic concepts (the foundations) very well in order to understand other more complex concepts (the upper floors) or else they should take down the whole building and build it up again.

There is a goal that seems to be consensual within teachers' training which is "to develop the reflective capacity of future teachers so as to contribute to their formation as responsible professionals who are autonomous, ethically challenging, and able to effectively reflect on their teaching practice” (Oliveira & Cyrino, 2011, p. 111).
We can say, then, that besides other skills, a good teacher should not only be passionate about what s/he teaches, but also have a mathematical and didactic knowledge which allows him/her to identify: what s/he can teach, how, and what the student is capable of learning.

Teaching and learning geometry

Our teaching system allows the student to progress without having succeeded in mathematics, without having absorbed basic and structuring concepts, particularly concerning geometry. Previous orientations concerning school mathematics weren't very concerned with geometry. His relevance was recovered during the recent PMEB (Mathematics program for basic education). This is ascertained by Veloso (2008) when declaring: “How is it possible to spend 9 years looking at cylinders and cones without once imagining to cut them and see the plane?!!” (p. 19). Therefore, the hardships which geometry is facing are quite predictable. The PMEB assumes that, throughout the three study cycles, teaching and learning mathematics goes through a development process based on four fundamental foundations: the work with numbers and operations, the development of geometrical thinking, algebraic thinking and the work with data. Once some of the relevance that geometry had thirty years ago was recovered, it seemed pertinent to direct our research towards the study of the acquisition of geometrical basic concepts of future teachers for the 1st and 2nd study cycles.

The study of geometry is crucial for the mathematical education of our youth. During the 1st and 2nd study cycles, children start to develop the cognitive structure which enables rational thinking within a linear and deductive thinking system. As suggested by the Principles and Standards for School Mathematics, during the first years, children should start building mathematical arguments which are inductive about ideas and geometric connections (NCTM, 2000). The development of mathematical arguments enables the transition from informal to a more formal thinking method, which stresses mathematical reasoning, including the inductive and deductive processes, the formulation and reasoning of conjectures and the classification and definition of geometrical objects. Concerning geometry, the Mathematics program for basic education particularly stresses the visualisation and comprehension of properties of geometrical figures, understanding how important these are for the development of the student's spatial awareness and also introduces the study of geometrical transformations from the first years, which is progressively widened and more deeply analysed during the more advanced years.

For several years, mathematics educators have been studying the Van Hiele levels (e. g. Burger & Shaughnessy, 1986; Gutiérrez, Jaime & Fortuny, 1991; Jaime & Gutiérrez, 1994; Saads & Davis, 1997) and space visualisation skills (Arcavi, 2003; Battista, 2007; Battista & Clements, 2002; Saads & Davis, 1997). According to Battista (2007) it's important to develop, within the child, the skill to "see", analyse and think about the spatial objects and their images. Also according to Vale and Barbosa (2009) “to see” is a very important component of generalization and it should be explored from early years. Concerning the role and importance of visual representation, Arcavi (2003) defines “visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings” (p. 217). Besides visualisation, geometrical skills involve two other important cognitive processes: construction and reasoning (Duval, 1998). Reasoning is strengthened by the means of the relations which are established when looking for geometrical objects in certain conditions.

However, mathematics in general and geometry in particular, do not accept the lack of basic concepts where other more complex ones are based. Geometry is like a net of interconnected thoughts and concepts and of representation systems used to conceptualize and understand physical and imagined spatial environments (Battista, 2007). If there is a broken cycle, we must understand exactly what went wrong. This idea, supported by van Hiele's theory, finds its way as a reference for teaching geometry. Therefore, it is important to understand the constructive, global and progressive process of Hiele's theory for teaching and learning geometry. This theory presupposes the existence of five sequential levels for the development of geometrical thought. These levels get progressively more complex and the student's evolution throughout the levels is determined by the teaching methods. Van Hiele also considers that the teacher has a crucial role within the process of teaching and learning of his/her students. The teacher must define the adequate tasks and activities which are able to lead the students to reach further levels of thought. In order to assess the level of development of the students, the teacher needs a tool which allows him/her to assess whether the student has progressed and how so. According to van Hiele's theory, the progression in these levels happens as students develop their geometrical maturity. Geometrical thought is developed, gradually, starting by recognising figures and going on to its differentiation up to the emergence of deductive reasoning.

The development of geometrical thought is an important auxiliary to solving problems in students' daily lives. However, the acquisition of these ideas depends greatly upon the teacher and his/her knowledge according to Gomes (2003) when stating that the teacher's knowledge of the contents is crucial to the students' learning process, and Jones (2000) when referring that the success of teaching geometry depends upon the teacher's knowledge and teaching methods.

Taking into account the considerations made up to now, we decided to direct this communication towards the analysis of the Test as a means of diagnosing and characterising the students' geometrical knowledge at the beginning of their study of geometry.

The study

The on-going study is developed within the context of a second year class of LEB, at the beginning of the curricular unit of geometry of the second semester, taught by a teacher who is the first author of this paper. Our main goal is to identify and understand the main difficulties of the students regarding geometry. Based on the previous knowledge the students acquire in basic and secondary education, we intend to identify possible weaknesses so that we can understand, throughout the curricular unit, how this knowledge progresses. Therefore, in the first class of the curricular unit, we gave a test to the twenty four students in the class that we chose to the research.

This study took place in a classroom environment where the participants were the class students, the teacher and the researcher who had a role of non-participant observer. The selection of this class among four classes was based, mainly, on criteria of good informer students and availability.

Results and discussion

As was already mentioned, we will analyse some of the answers obtained in five of the Test's answers. However, before that, in order to have a global idea of the class, we will start by contextualising the Test and analysing the results obtained by the class.

During the first class of geometry curricular unit, we gave the Test to the class which was going to be the object of the research. This Test was created based on adapted questions taken from national tests, assessment tests of the 1st, 2nd and 3rd cycles, as well as from international tests, TIMSS, PISA and van Hiele's test. The Test has twenty five questions and while creating it, we took into account, not only the specific knowledge of some geometry topics (65% of the questions were about plane geometry and 35% about space geometry), but also transversal skills: solving problems, communication and reasoning.

Table 1 sums up the results of the class (%) spread by the transversal knowledge and skills. In 1032 possible points only 347 we obtained, that is, 33.6% of correct answers.

KNOWLEDGE AND SKILLS			
Knowledge and understanding of mathematical concepts and knowledge	Reasoning	Communication	Solving problems
34%	35%	33%	31%

Table 1 – Percentage of the class' results divided by knowledge and skills

Despite the way in which questions were grouped by knowledge and skills being debatable, we take it as a reference to assess the level of some basic geometry knowledge.

This first element that characterises the class proves our idea of the insufficient basic geometry knowledge. The results show a low level of basic knowledge acquisition. None of the transversal skills and knowledge set by the PMEB reached even a 36% success rate. There were only 34% correct answers in knowledge and understanding of concepts and mathematical knowledge. Solving problems proved to be the weakest point for these students, with only 31% correct answers, followed by question involving communication with a 33% success rate. Concerning questions involving reasoning, only 35% of the students answered adequately.

We will now analyse the class performance in five of the twenty five questions. We have selected at least one question per knowledge and skill.

Question 3 – "Explain why the following statement is true: *A right triangle cannot be equilateral.*"
This question is part of the reasoning category. It concerns plane geometry concepts related to the triangle. It demands knowledge about the internal angles of a triangle and also about the classification of triangles by internal angles and by the relative lengths of the sides. Given the needed knowledge, despite being basic, we didn't expect good outcomes by the students.

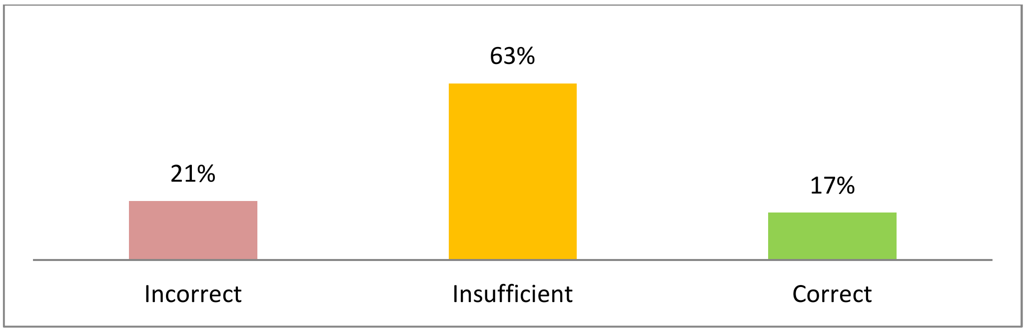


Figure 1: Student outcomes to question 3.

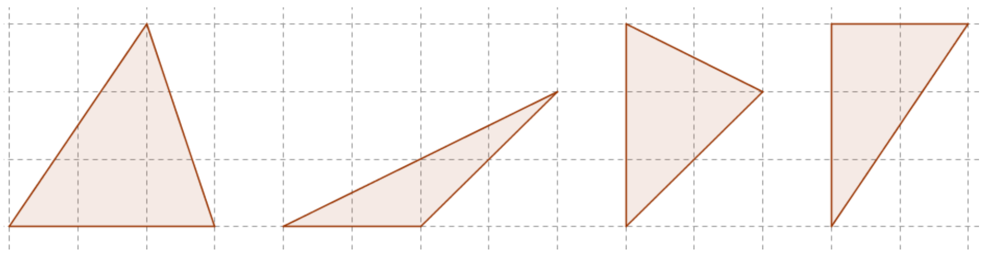
17% of correct answers and 63% give an insufficient explanation. The mistakes in some answers are interesting. A student said: "A right triangle has a 90^0 angle; an equilateral triangle has two equal sides and one that is different so it cannot have a 90^0 angle." The student mistakes the notion of equilateral triangle by the notion of isosceles triangles. Another wrote: "Because an equilateral triangle has all angles with an amplitude of 45^0

and therefore, if this is a right triangle, this is, with an angle of 90^0, it is not possible that it is simultaneously an equilateral triangle”. The student doesn't realise that $45^0 \text{x} 3 \neq 180^0$.

With this answer, we don't know whether the student is aware that the sum of the internal angles of a triangle equals 180^0.

Concerning the results obtained in this question, our expectations were confirmed.

Question 4 – “For each of the triangles draw, on the figure, a height.”



This question falls into the category of knowledge and understanding of concepts and mathematical knowledge. Within plane geometry, concerning the triangle, this question concerns basic knowledge of one of its elements: the height. We expected a high percentage of correct answers.

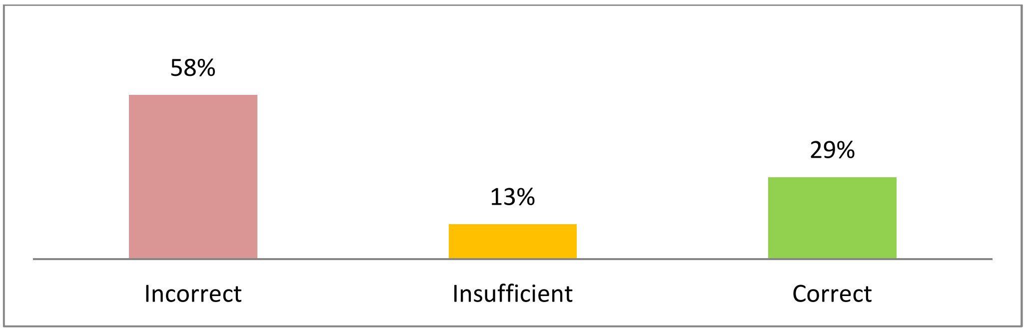


Figure 2: Student outcomes to question 4.

However, according to Figure 2, only 29% of the students were able to answer correctly. This was a low outcome considering what we expected: the knowledge needed - the height of a triangle - is basic. In figure 3 we typified an answer of these students.

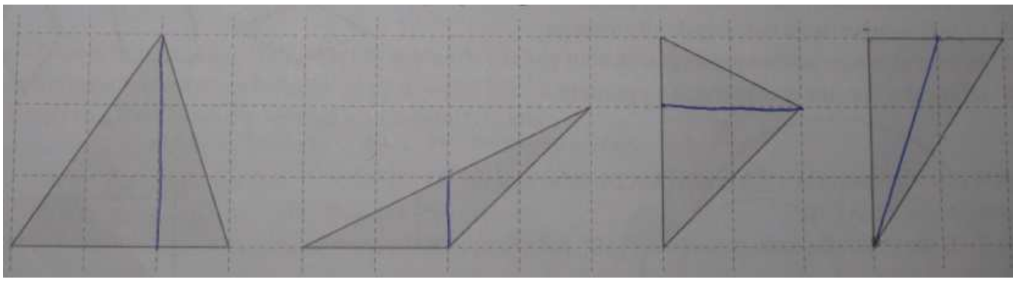
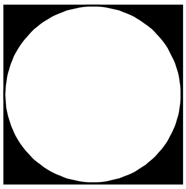


Figure 3: One of the responses to question 4.

Question 6 – "The sweaters of the participants in a handball tournament will have the drawing showed in the figure. Cátia will call Mr. Tomás. She needs to describe the drawing in order for him to do it. Put yourself in Cátia's role and describe the drawing for Mr. Tomás.



The question was adapted from the mathematics assessment test for the 1st cycle of basic education in 2008 and falls into the communication category. We expected a good performance by the students in this question given that it concerns plane geometry and involves the circle, the square and the notion of in-circle.

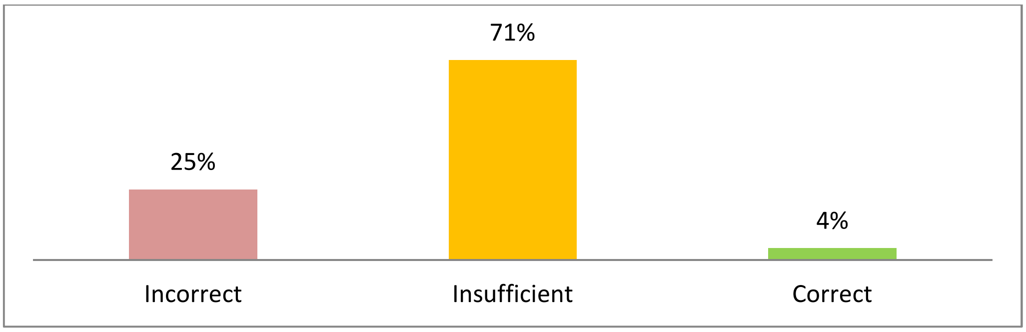
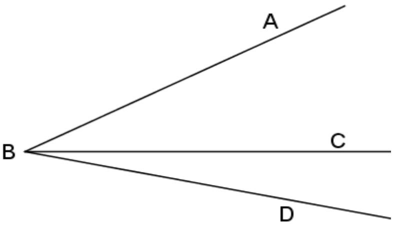


Figure 4: Student outcomes to question 6.

4% of correct answers as only one student answers correctly. 71% of the students gave and insufficient answer and 25% didn't answer. These results were surprising, as well as some of the answers. One of the students wrote: "… a black background square superposed by a white background circumference". Another student said: "The figure represents a square and inside it there a solid figure, in this case, the circle" And another wrote: "A black square with a white circumference. The diameter of this circumference should be half of square's measurement (or the perimeter of the circumference should touch all the sides of the square)".

The difficulty showed in mathematical communication, the confusion and lack of basic concepts showed by the answers of these students to answer a simple question is a factor that should worry all of us, educators.

Question 11 – "How many angles can you identify in this figure? Mark them clearly."



This question falls into the category of knowledge and understanding of concepts and mathematical knowledge. Within plane geometry, it concerns the notion of angle. This being a question that demanded not only the notion of angle, but also the understanding of the concepts of complementary and supplementary angles, we didn't expect a good outcome for this question.

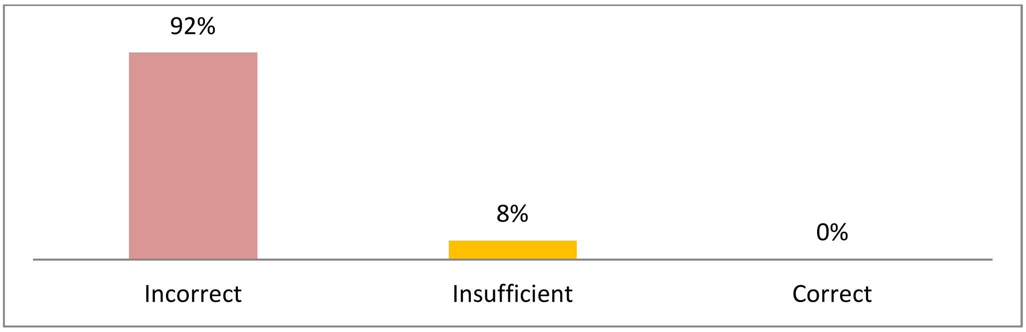
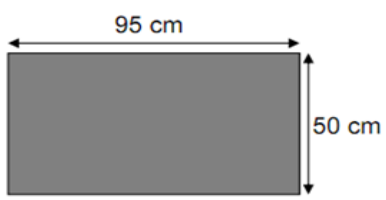


Figure 5: Student outcomes to question 11.

Despite the fact that we didn't expect a good outcome, figure 5 shows the lack of a single correct answer (eight angles) and this was a surprise. Only one students was able to identify six angles and another one four angles. Of the remaining, one student did not answer, three students identified only two angles and eleven students identified three angles. As educators, we should be worried by the low performance of students in question involving visualisation.

Question 12 – "Ana put twelve photos, without superposition, in a rectangular card with the dimensions marked on the figure. Each photograph has the form of a rectangle 20cm long and 15cm wide. What is the area of the card that hasn't been taken by the photos?



The question was taken from the mathematics assessment test for the 1st cycle of basic education in 2010 and falls into the problem solving category. This is a plane geometry problem involving the area of the rectangle where the student can sketch the steps that need to be taken to solve it. Given that the concepts involved in this question are basic, we expected that a good part of the students was going to be able to solve this problem. However, none of the students was able to solve the question correctly.

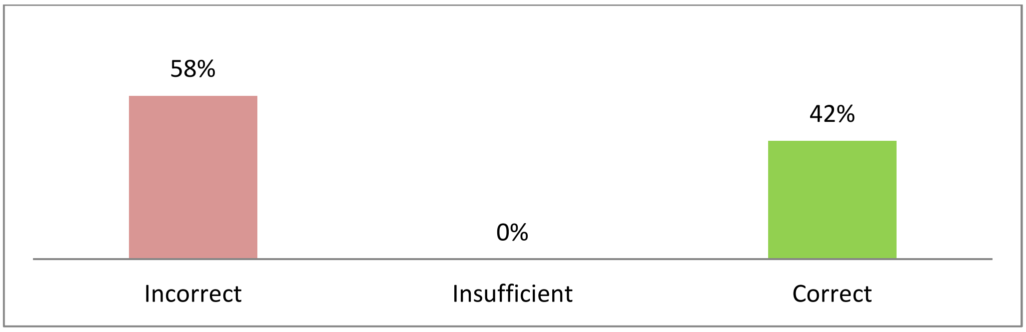
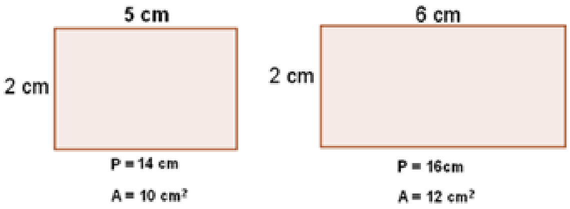


Figure 6: Student outcomes to question 12.

The analysis of the graph of figure 6 shows that over half of the students weren't able to think of a correct strategy to solve a simple problem of subtraction of areas.



Question 13 – "In a class of the 5th grade, a student got in the classroom and said to the teacher: *Teacher, I've found a new rule: In any figure, if we increase the perimeter, the area will also increase. I brought an example to prove it is true.* Put yourself on the teacher's shoes. How would you comment on the conjecture of the student?"

This question was adapted from Ma, 2009, and falls into the category of reasoning and, within plane geometry, it concerns the relation between the perimeter and the area of a rectangle. Given that this is a conjecture that is apparently obvious, we had a low expectation, but we were far from imagining that there wouldn't be any correct answers, as figure 7 shows.

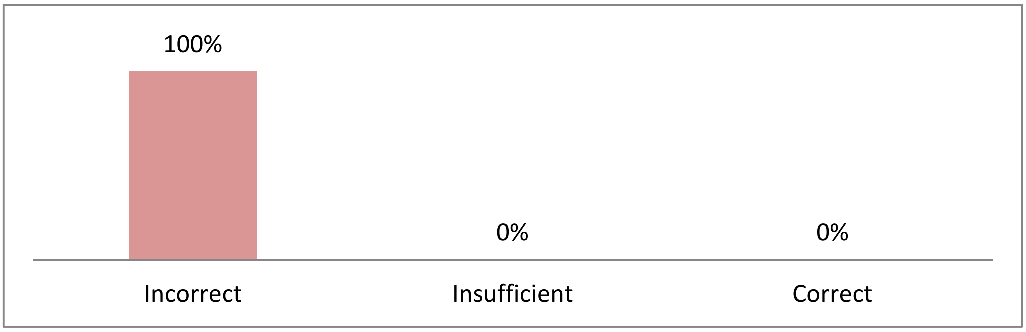


Figure 7: Student outcomes to question 13.

The students confirmed the conjecture, that is not always true, and their answer can be typified in one of the students' answers: “Very good, I see that you now understand that if the perimeter increases that means that the length of sides also increases implying, naturally, that the area also increases, given that it also depends of the length of the sides".

It is easy to answer this kind of question incorrectly for this is a new formulation for these students. It demands more than the simple geometry knowledge. It involves didactic knowledge and the knowledge of assessment of oral presentation at a retroaction level. The students don't usually reflect on their learning or question themselves on what the teacher is teaching. And mathematics needs a lot of "curiosity".

Final considerations

These results confirm one of the pre-suppositions of this study which is the weak preparation of future teachers and the results are according to the results obtained by the students on the different levels of their basic education. And these are the students who will be the future teachers of these levels of education. Therefore, it is important that we, as mathematical trainers and educators, give special attention to the geometry topic, identifying possible weaknesses in the knowledge of future teachers in order to make initial training overcome those same deficiencies in a timely manner.

These results, which identify some of the flaws on the geometrical knowledge of these students, in accordance to the results obtained in some studies concerning initial training (e. g. Gomes, 2003), are the starting point for the widened study, of which this presentation is part, and may lead to a set of strategies and recommendations for the designing of the curricular programme for geometry.

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