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Research Article

Time-Delay and Fractional Derivatives

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This paper proposes the calculation of fractional algorithms based on time-delay systems. The study starts by analyzing the memory properties of fractional operators and their relation with time delay. Based on the Fourier analysis an approximation of fractional derivatives through time-delayed samples is developed. Furthermore, the parameters of the proposed approximation are estimated by means of genetic algorithms. The results demonstrate the feasibility of the new perspective.

1. Introduction

Fractional calculus (FC) deals with the generalization of integrals and derivatives to a noninteger order [1–7]. In the last decades the application of FC verified a large development in the areas of physics and engineering and considerable research about a multitude of applications emerged such as, viscoelasticity, signal processing, diffusion, modeling, and control [8–17]. The area of dynamical systems and control has received a considerable attention, and recently several papers addressing evolutionary concepts and fractional algorithms can be mentioned [18, 19]. Nevertheless, the algorithms involved in the calculation of fractional derivatives require the adoption of numerical approximations [20–26], and new research directions are clearly needed.

Bearing these ideas in mind, this paper addresses the optimal system control using fractional order algorithms and is organized as follows. Section 2 introduces the calculation of fractional derivatives and formulates the problem of optimization through genetic algorithms (GAs). Section 3 presents a set of experiments that demonstrate the effectiveness of the proposed optimization strategy. Finally, Section 4 outlines the main conclusions.

2. Problem Formulation and Adopted Techniques

There are several definitions of fractional derivatives. The Riemann-Liouville, the Grünwald-Letnikov, and the Caputo definitions of a fractional derivative of a function $f(t)$ are given by

$$
\begin{gathered}
{}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n, \\
{}_aD_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{[(t-a)/h]} \gamma(\alpha,k) f(t-kh), \\
\gamma(\alpha,k) = (-1)^k \frac{\Gamma(\alpha+1)}{(\alpha-k+1)}, \\
{}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n,
\end{gathered}
\tag{2.1}
$$

where $\Gamma(\cdot)$ is the Euler's gamma function, $[x]$ means the integer part of x, and h is the step time increment.

On the other hand, it is possible to generalize several results based on transforms, yielding expressions such as the Fourier expression

$$
F\{ {}_0D_t^{\alpha} f(t)\} = (j\omega)^{\alpha} F\{f(t)\} - \sum_{k=0}^{n-1} (j\omega)^k \, {}_0D_t^{\alpha-k-1} f(0^+), \tag{2.2}
$$

where ω and F represent the Fourier variable and operator, respectively, and $j = \sqrt{-1}$.

These expressions demonstrate that fractional derivatives have memory, contrary to integer derivatives that consist in local operators. There is a long standing discussion, still going on, about the pros and cons of the different definitions. These debates are outside the scope of this paper, but, in short, while the Riemann-Liouville definition involves an initialization of fractional order, the Caputo counterpart requires integer order initial conditions which are easier to apply (often the Caputo's initial conditions are called freely as "with physical meaning"). The Grünwald-Letnikov formulation is frequently adopted in numerical algorithms because it inspires a discrete-time calculation algorithm, based on the approximation of the time increment h through the sampling period.

We verify that a fractional derivative requires an infinite number of samples capturing, therefore, all the signal history, contrary to what happens with integer order derivatives that are merely local operators [27]. This fact motivates the evaluation of calculation strategies based on delayed signal samples and leads to the study presented in this paper. In this line of thought we can think in concentrating the delayed samples into a finite number of points that somehow "average" a given set number of sampling instants (see Figure 1).

The concept of time-delayed samples for representing the signal memory can be formulated analytically as

$$
{}_aD_t^{\alpha} f(t) \approx a_0 f(t) + \sum_{k=1}^{r} a_k f(t+\tau_k), \tag{2.3}
$$

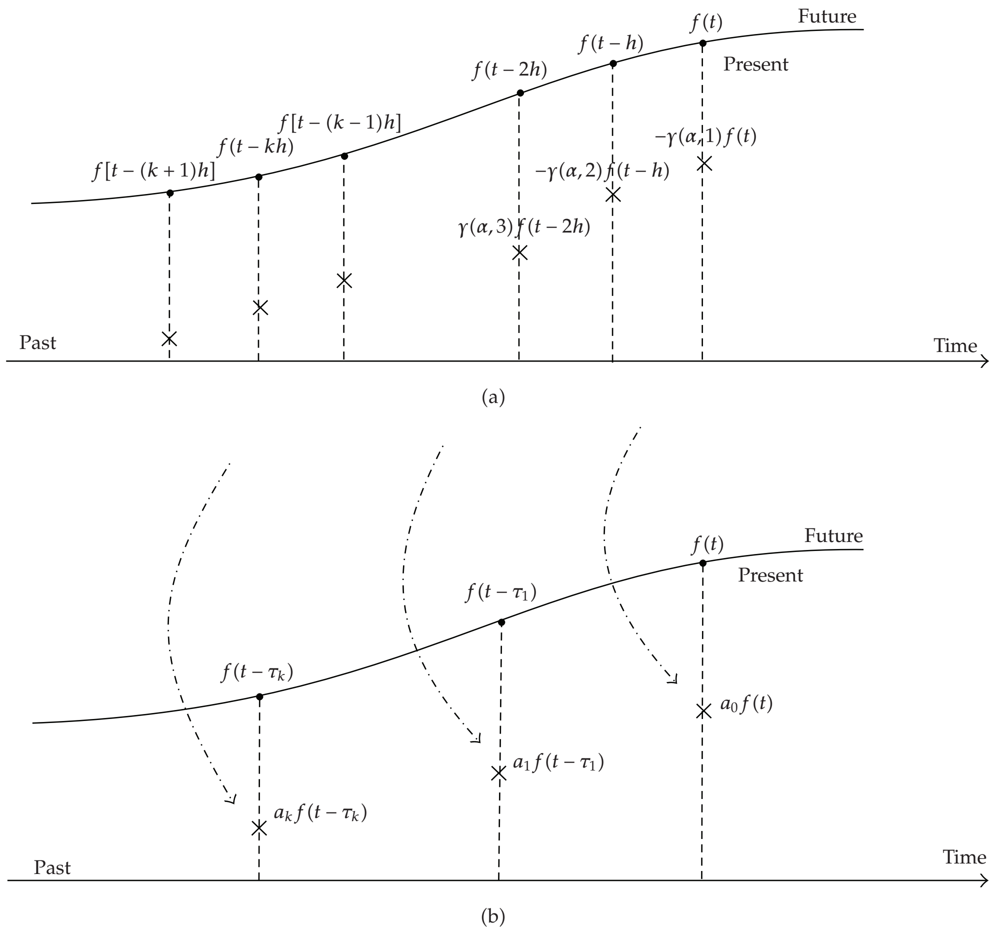

Figure 1: Conceptual diagram of the time delay perspective of the fractional derivative.

where $a_k \in \Re$ and $\tau_k \in \Re$ are weight coefficients and the corresponding delays and $r \in \aleph$ is the order of the approximation.

Before continuing we must mention that, although based on distinct premises, expression (2.3), inspired by the interpretation of fractional derivatives proposed in [27], is somehow a subset of the interesting multiscaling functional equation proposed by Nigmatullin in [28]. Besides, while in [28] we can have complex values, in the present case we are restricted to real values for the parameters. In fact, expression (2.3) adopts the well-known time-delay operator, usual in control system theory, following the Laplace expression $L\{f(t+\tau_k)\} = e^{\tau_k s}L\{f(t)\}$, where s and L represent the Laplace variable and operator, respectively.

Another aspect that deserves attention is the fact that while stability and causality may impose restrictions to the parameters in (2.3) it was decided not to impose *a priori* any restriction to the numerical values in the optimization procedure to be developed in the sequel. For example, in what concerns the delays, while it seems not feasible to "guess" the future values of the signal and only the past is available for the signal processing, it is important to analyze the values that emerge without establishing any limitation *a priori* to their values. Nevertheless, in a second phase, the stability and causality will be addressed.

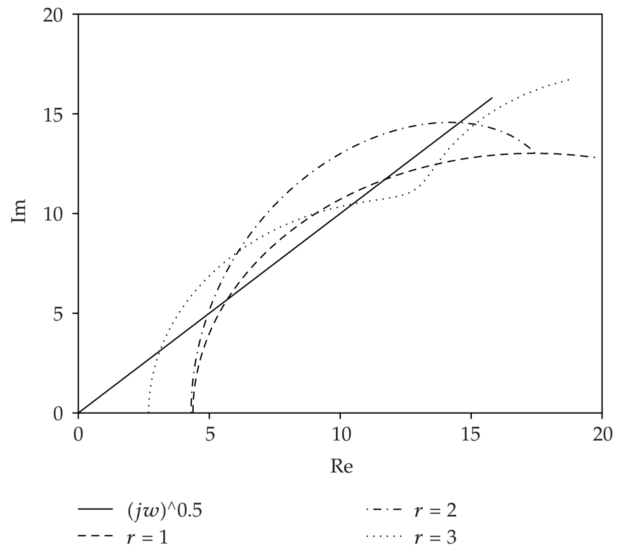

Figure 2: Polar diagram of $(j\omega)^{\alpha}$ and the approximation $a_0 + \sum_{k=1}^{r} a_k e^{j\omega\tau_k}$ for $r = \{1, 2, 3\}$, $\alpha = 0.5$, and $\omega_{\max} = 500$.

The development of an algorithm for the calculation of $\{a_0, a_k, \tau_k\}$, $k = 1, \ldots, r$, given the approximation and fractional orders r and α, respectively, can be established either in the time or the frequency domains. In this paper we adopt the Fourier expression (2.2) with null initial conditions, leading to

$$F\{{}_0D_t^{\alpha} f(t)\} = (j\omega)^{\alpha} F\{f(t)\} \approx a_0 + \sum_{k=1}^{r} a_k e^{j\omega\tau_k} F\{f(t)\}. \tag{2.4}$$

The parameters a_k and τ_k can be optimized in the perspective of the functional

$$J(r, \alpha) = \sum_{i=1}^{n} \left| (j\omega)^{\alpha} - \left(a_0 + \sum_{k=1}^{r} a_k e^{j\omega\tau_k} \right) \right|, \tag{2.5}$$

where i represents an index of the sampling frequencies ω_i within the bandwidth $\omega_{\min} \leq \omega_i \leq \omega_{\max}$ and n denotes the total number of sampling frequencies. Therefore, the quality of the approximation depends not only on the orders r and α, but also on the bandwidth $\omega_{\min} \leq \omega \leq \omega_{\max}$.

For the optimization of J in (2.5) it is adopted a genetic algorithm (GA). GAs are a class of computational techniques to find approximate solutions in optimization and search problems [29, 30]. GAs are simulated through a population of candidates of size N that evolve computationally towards better solutions. Once the genetic representation and the fitness function are defined, the GA proceeds to initialize a population randomly and then to improve them through the repetitive application of mutation, crossover, and selection operators. During the successive iterations, a part or the totality of the population is selected to breed a new generation. Individual solutions are selected through a fitness-based process, where fitter solutions (measured by a fitness function J) are usually more likely to be selected.

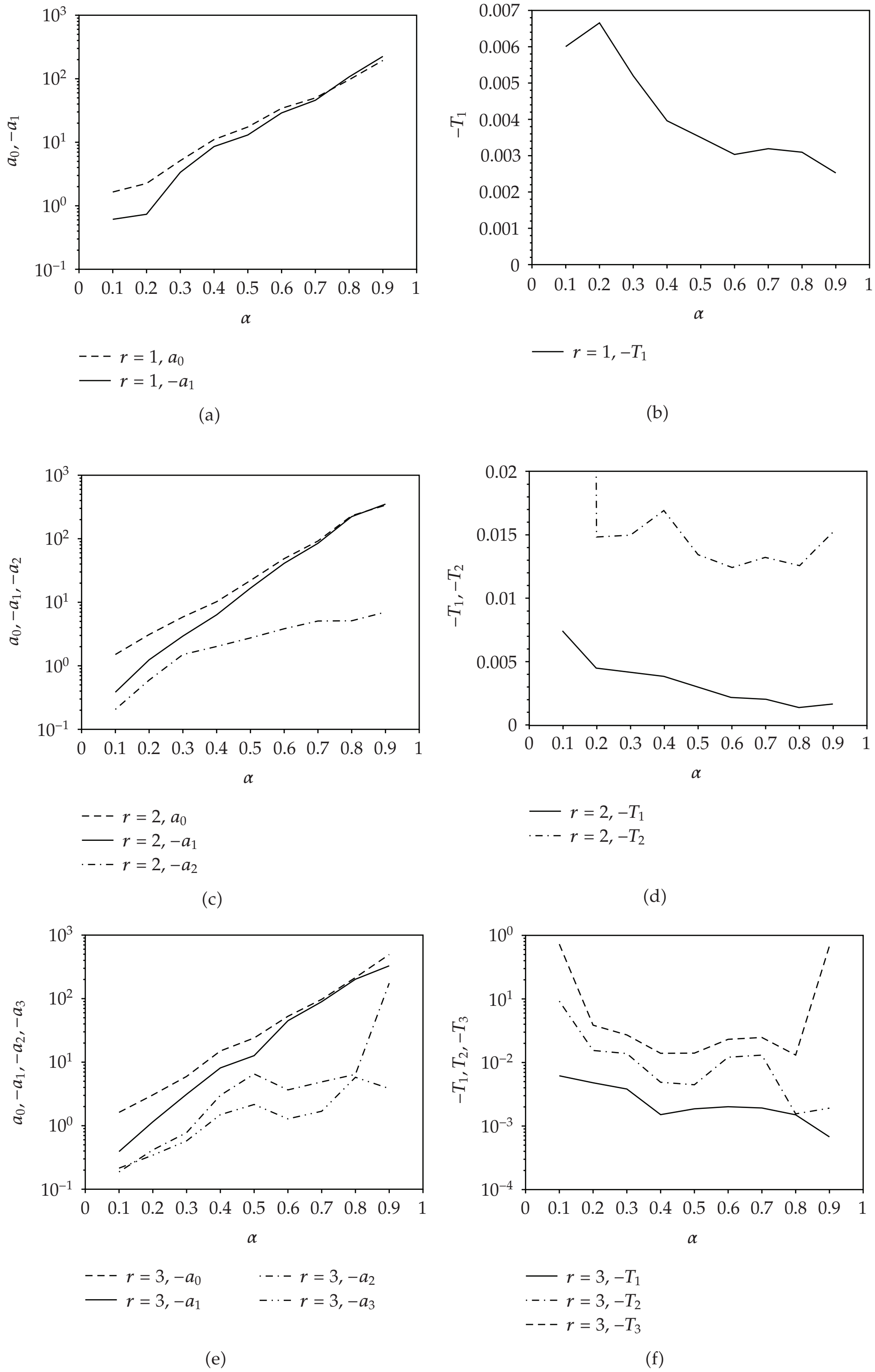

Figure 3: Evolution of the approximation parameters and fitness function versus α for $r = \{1, 2, 3\}$ and $\omega_{\max} = 500$.

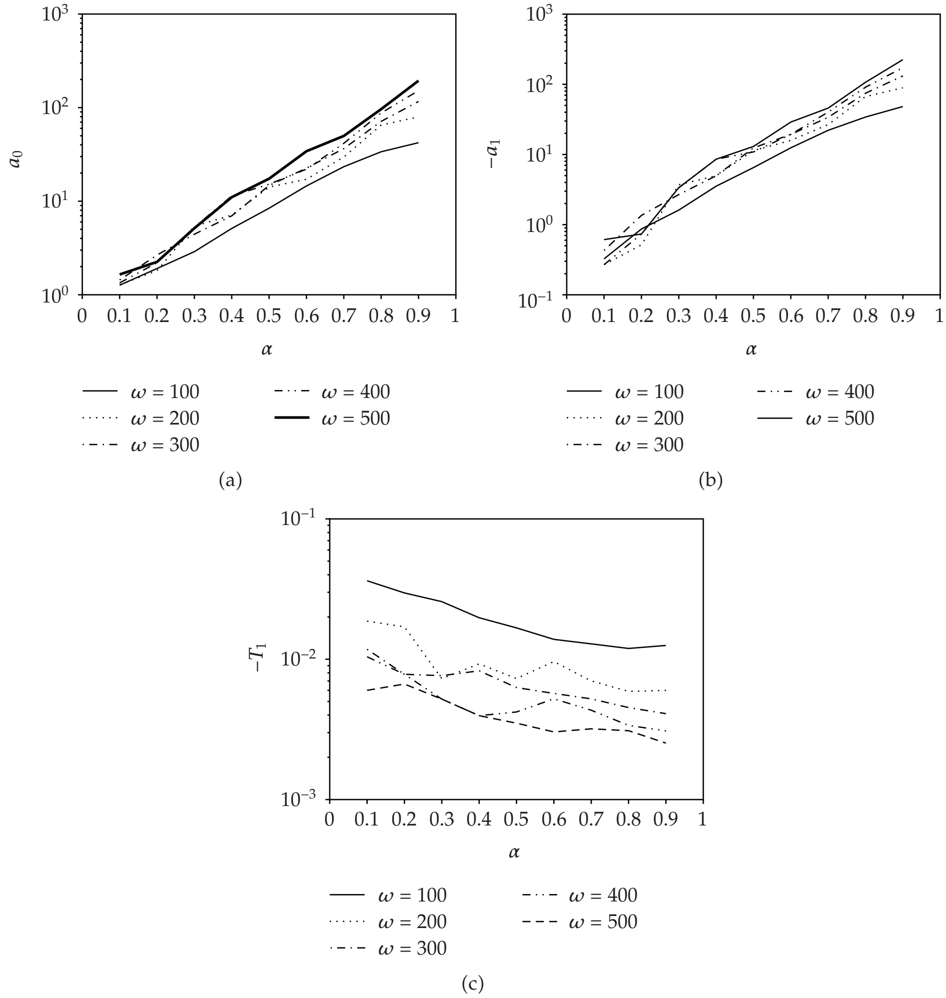

Figure 4: Comparison of the approximation parameters versus α for the bandwidths $\omega_{\max} = \{100, 200, \ldots, 500\}$ and $r = 1$.

The GA terminates when either the maximum number of generations I is produced, or a satisfactory fitness level has been reached.

The pseudocode of a standard GA is as follows.

(1) Choose the initial population

(2) Evaluate the fitness of each individual in the population

(3) Repeat

   (a) Select best-ranking individuals to reproduce
   (b) Breed new generation through crossover and mutation and give birth to offspring
   (c) Evaluate the fitness of the offspring individuals
   (d) Replace the worst ranked part of population with offspring

(4) Until termination.

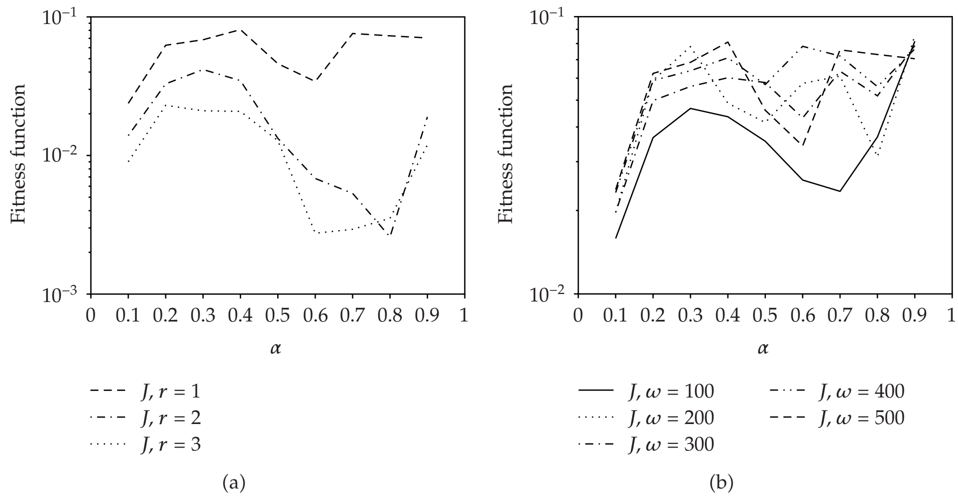

Figure 5: Fitness function versus α for (a) $r = \{1, 2, 3\}$ and $\omega_{\max} = 500$, (b) the bandwidths $\omega_{\max} = \{100, 200, \ldots, 500\}$ and $r = 1$.

A common complementary technique, often adopted to speed-up the convergence, denoted as elitism, is the process of selecting the better individuals to form the parents in the offspring generation.

We observe that we have not introduced *a priori* any restriction to the numerical values of the parameters that result during the optimization procedure. It is well known that one of the advantages of GAs over classical optimization techniques is precisely its characteristic of handling easily these situations. One technique is simply to substitute "not suitable" elements of the GA population by new ones generated randomly. Furthermore, during the generation of the GA elements it is straightforward to impose restrictions. As mentioned previously, in a first phase it is not considered any limitation in order to reveal more clearly the pattern that emerges freely with the time-delay algorithm. After having the preliminary results, in a second phase, several restrictions are considered, and the optimization GA is executed again.

3. Numerical Experiments and Results

In this section we develop a set of experiments for the analysis of the proposed concepts. Therefore, we study the case of approximation orders and fractional orders $r = \{1, 2, 3\}$ and $\alpha = \{0.1, 0.2, \ldots, 0.9\}$, respectively. In what concerns the bandwidth are considered $\omega_{\min} = 0$, and $\omega_{\max} = \{100, 200, \ldots, 500\}$ (rad/s). In all cases are adopted $n = 60$ and sampling frequencies and identical distances between consecutive measuring points along the *locus* of $(j\omega)^{\alpha}$.

Experiments demonstrated some difficulties in the GA acquiring the optimal values, being the problem harder the higher the value of r, that is, the larger the number of parameters to be estimated. Consequently, several measures to overcome that problem were envisaged, namely, a large GA population with $N = 2 \times 10^4$ elements, the crossover of all population elements and the adoption of elitism, a mutation probability of 10%, and an evolution with $I = 10^3$ iterations. Even so, it was observed that the GA tended to stabilize in suboptimal solutions and other values for the GA parameters had no significant impact.

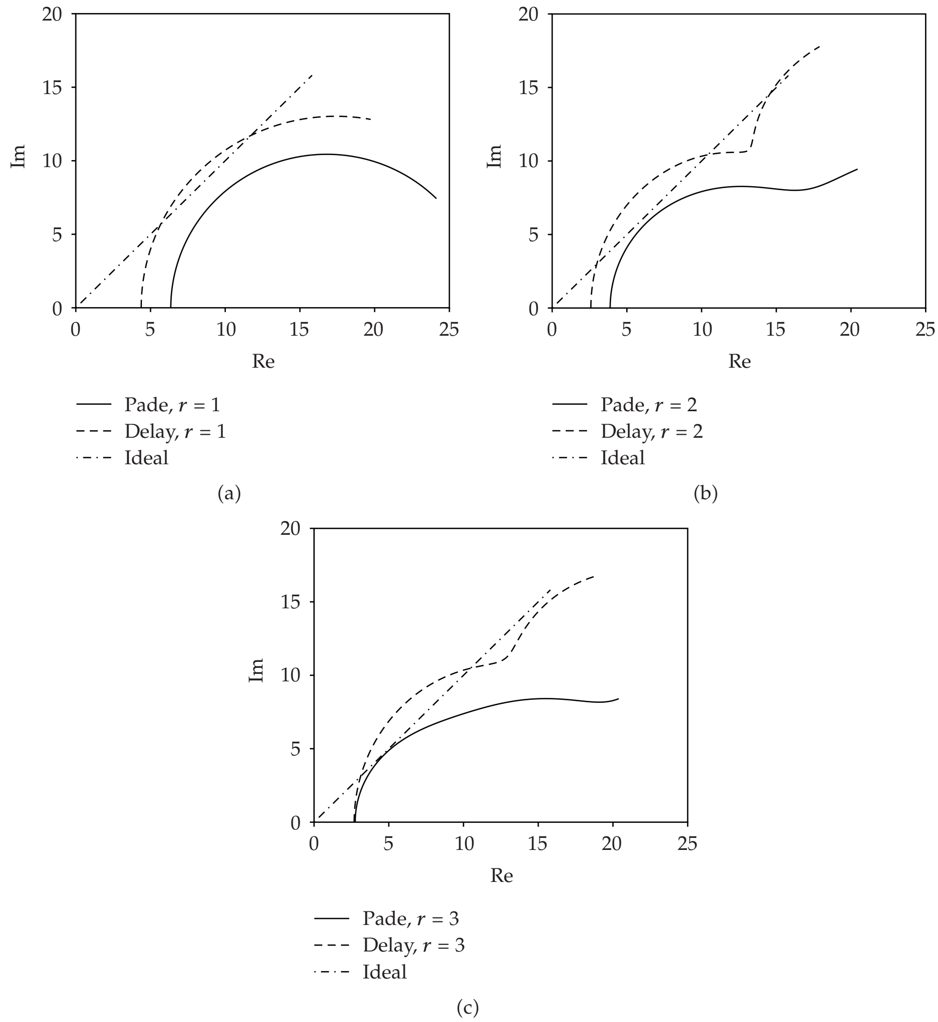

Figure 6: Polar diagrams of $(j\omega)^{\alpha}$ and the approximations (2.4) and (3.3) for $r = \{1, 2, 3\}$, $\alpha = 0.5$ and $0 \leq \omega \leq 500\,\text{rad/s}$, $h = 0.005\,\text{s}$.

Therefore, a complementary strategy was taken to prevent such behavior, by restarting the base GA population and executing new trials until getting a good solution.

It was also observed that all GA executions lead to positive values of a_0. In what concerns a_k and τ_k, $k = 1, \ldots, r$, it was verified that most experiments lead to negative values; nevertheless, in some cases, particularly for α near integer values, where the GA had more convergence difficulties, occasionally some positive values occurred. Several experiments restricting the GA to negative values proved that the fitting was possible with good accuracy, and, therefore, for avoiding scattered results with unclear meaning, those restrictions were included in the optimization algorithm.

Figure 2 shows a typical case, namely, the polar diagram of $(j\omega)^{\alpha}$ and the approximation $a_0 + \sum_{k=1}^{r} a_k e^{j\omega\tau_k}$ for $r = \{1, 2, 3\}$, $\alpha = 0.5$, and $\omega_{\max} = 500$.

Figure 3 shows the evolution of the approximation parameters and fitness function versus α, for $r = \{1, 2, 3\}$, $\omega_{\max} = 500$. Figure 4 compares the cases of increasing bandwidth $\omega_{\max} = \{100, 200, \ldots, 500\}$ for $r = 1$.

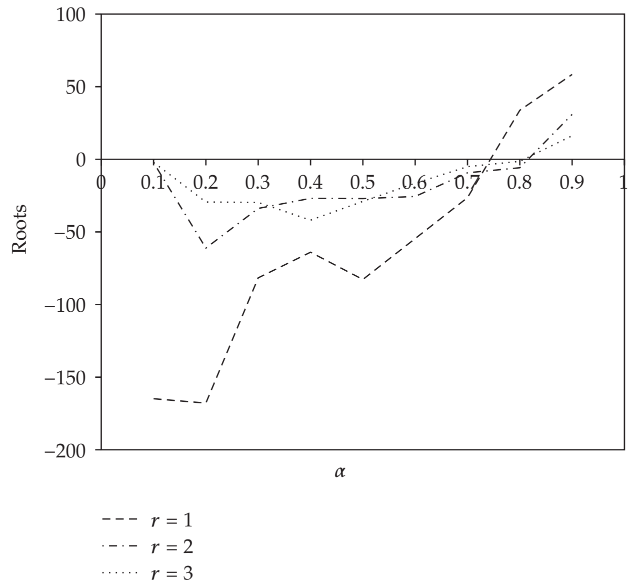

Figure 7: Roots of the characteristic equation of approximation (2.4) versus α for $r = \{1, 2, 3\}$.

Figure 5 depicts the variation of the fitness function J for different orders of approximation and for different bandwidths.

It is clear that the higher the value of r, the better the approximation, that is, the smaller the value of J. When the bandwidth increases we observe larger values of the weighting factors a_k, but the delays τ_k remain in a limited range a small values, being more close/apart to/from zero for values of α near/far the unit. Moreover, for larger bandwidths the GA has more difficulties in estimating the parameters of the approximation.

While the primary goal of this paper is to explore the relationship between the fractional derivative and the time-delay operators, it is interesting to compare the results of the present approach with those of classical approximations. In this perspective, we consider the discrete time domain and the Euler and Tustin rational expressions, $H_0(z^{-1}) = (1/h)(1 - z^{-1})$ and $H_1(z^{-1}) = (2/h)((1 - z^{-1})/(1 + z^{-1}))$, where z represents the Z transform operator and h the sampling period. These expressions are also called generating approximants of zero and first order, respectively, and their generalization to a noninteger order α yields

$$s^{\alpha} \approx \left[\frac{1}{h}\left(1 - z^{-1}\right)\right]^{\alpha} = H_0^{\alpha}\left(z^{-1}\right),$$
$$s^{\alpha} \approx \left(\frac{2}{h}\frac{1 - z^{-1}}{1 + z^{-1}}\right)^{\alpha} = H_1^{\alpha}\left(z^{-1}\right). \tag{3.1}$$

Weighting $H_0^{\alpha}(z^{-1})$ and $H_1^{\alpha}(z^{-1})$ by the factors p and $1 - p$ leads to the average

$$H_{av}\left[z^{-1}; (p, \alpha)\right] = pH_0^{\alpha}\left(z^{-1}\right) + (1 - p)H_1^{\alpha}\left(z^{-1}\right). \tag{3.2}$$

The so-called Al-Alaoui operator corresponds to an interpolation of $H_0^\alpha(z^{-1})$ and $H_1^\alpha(z^{-1})$ with weighting factor $p = 3/4$ [31–33]. Often it is adopted a Padé expansion of order $r \in \aleph$ in the neighborhood of $z = 0$, leading to a rational fraction of the type

$$H_k\left(z^{-1}\right) = \frac{\sum_{i=0}^{r} a_i z^{-i}}{\sum_{i=0}^{r} b_i z^{-i}}, \quad a_i, b_i \in \Re. \tag{3.3}$$

Figure 6 compares the frequency response of the proposed algorithm (2.4) and the fraction (3.3), with $p = 3/4$ and $h = 0.005\,\text{s}$, for $0 \le \omega \le 500\,\text{rad/s}$ and the orders $r = \{1, 2, 3\}$, $\alpha = 0.5$.

It is clear that expression (2.4) leads to a superior approximation. Furthermore, although not particularly important with present day computational resources, expression (2.4) poses a calculation load which is inferior to the one of (3.3). In fact, since in real time the delay consists simply in a memory shift, we have r versus $2r$ sums and r versus $2r + 1$ multiplications for (2.4) and (3.3), respectively.

The stability of the resulting expression is also important. Figure 7 depicts the roots of the characteristic equation of approximation (2.4) versus α for $r = \{1, 2, 3\}$. We verify that we may have stability problems near integer values of α.

In conclusion, while the aim of this paper was to explore the relationship between the fractional operator and the time delay, it was verified that the proposed algorithm can be applied to successfully approximate fractional expressions.

4. Conclusions

The recent advances in FC point towards important developments in the application of this mathematical concept. During the last years several algorithms for the calculation of fractional derivatives were proposed, but the fact is that the results are still far from the desirable. In this paper, a new method, based on the intrinsic properties of fractional systems, that is, inspired by the memory effect of the fractional operator, was introduced. The optimization scheme for the calculation of fractional approximation adopted a genetic algorithm, leading to near-optimal solutions and to meaningful results. The conclusions demonstrate not only the goodness of the proposed strategy, but point also towards further studies in its generalization to other classes of fractional dynamical systems and to the evaluation of time-based techniques.

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