The effect of fractional order in variable structure control

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ABSTRACT

This paper studies fractional variable structure controllers. Two cases are considered namely, the sliding reference model and the control action, that are generalized from integer into fractional orders. The test bed consists in a mechanical manipulator and the effect of the fractional approach upon the system performance is evaluated. The results show that fractional dynamics, both in the switching surface and the control law are important design algorithms in variable structure controllers.

Keywords

1. Introduction

Variable structure systems (VSS) were proposed during the seventies [1] revealing good feasibility and high robustness. Variable structure controllers (VSC) are particularly suited for plants exhibiting complex dynamics, where standard algorithms fail to control the system. Mechanical manipulators have non-linear dynamics that require sophisticated control algorithms. In the application of VSC for mechanical manipulators [2,3], the most frequently used strategy is to consider that each link mimics a first order linear decoupled law. The resulting trajectories reveal two distinct parts, namely the reaching phase and the sliding mode. During the reaching phase, the robot evolves towards the reference law, but there is no guarantee of convergence. On the other hand, during the sliding phase the system follows the reference dynamics, but considerable chattering may occur. The chattering is due to the switching of the control effort, that imposes a considerable stress over the actuator and may even excite resonant modes of the mechanical structure. Therefore, the chattering is a result of the high frequency switching control action that, in order to force the system to follow the reference dynamics, imposes high amplitude signals. To avoid that problem several schemes were proposed such as smoothing the VSC, by transforming the 'on–off' into a 'saturation', or by reducing the VSC component by including additional adaptive or feedforward control terms [4]. Realizing that first order reference laws may be not well adapted to the system dynamics, another proposed approach [5] consists in adopting a second order reference law. The performance is superior but the algorithm adopts second order derivatives which requires either adequate sensors, or real time signal differentiation. This paper revisits this problem by taking advantage of the generalization provided by fractional calculus (FC) [6–10].

The adoption of FC concepts in VSCs has been addresses during the last years [11–16]. Since we can have fractional derivatives of any order the question is to find how the continuous variation of the order either in the reference model, or in the control law, affect the intrinsic VSC switching activity.

Having these ideas in mind, this paper studies the application of fractional derivatives in VSC and is organized as follows. Section 2 introduces the fundamental concepts about manipulator dynamics, VSC and FC. Section 3 develops numerical experiments with a simple mechanical manipulator when varying the fractional order of the sliding surface. Section 4 repeats the analysis when varying the fractional order of the control law. Finally, Section 5 outlines the main conclusions.
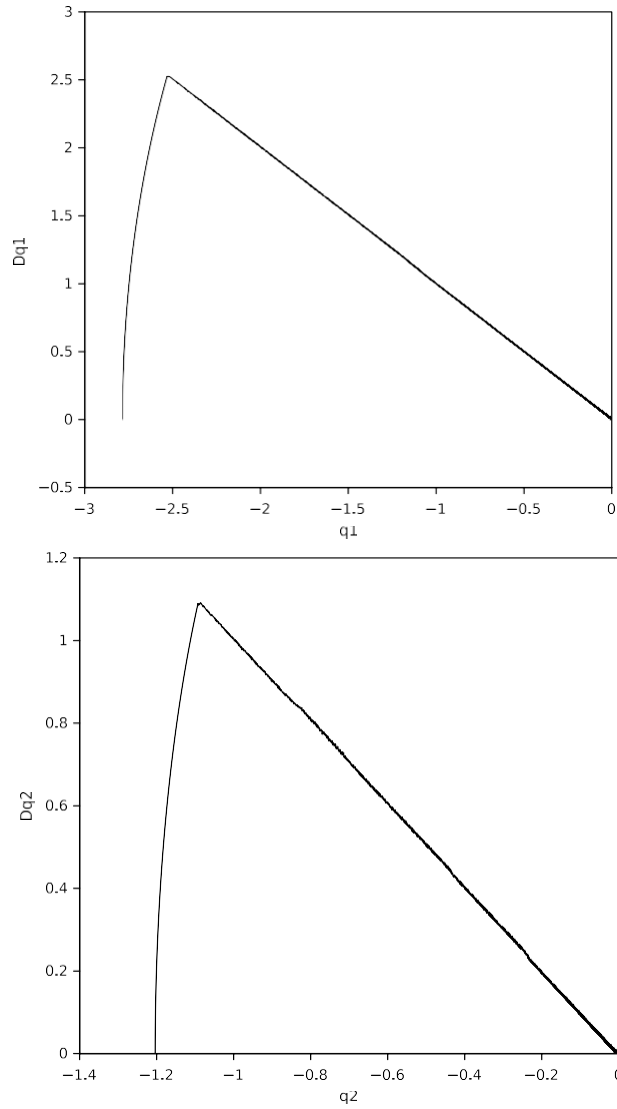
Fig. 1. Phase plane trajectories of the robot joints for $a = 0.5$ when $\delta_i = 10^{-4}$.

2. Preliminaries

We consider a mechanical manipulator with n degrees of freedom described by the dynamical equations [17]:

$$T = J(q)\ddot{q} + C(q, \dot{q}) + G(q) \tag{1}$$

where $J(q)$ is the $n \times n$ inertial matrix, $C(q, \dot{q})$ and $G(q)$ represent the $n \times 1$ vectors of Coriolis/centripetal and gravitational torques, and q, $\dot{q}$ and $\ddot{q}$ are the $n \times 1$ vectors of joint positions, velocities and accelerations, respectively.

When adopting VSC each link is constrained to follow a first order reference law ($i = 1, \ldots, n$):

$$\sigma_i = \dot{e}_i + \lambda_i e_i = 0 \tag{2}$$
$$e_i = q_{di} - q_d \tag{3}$$

where $\{q_{di}, \dot{q}_{di}\}$ and $\{\dot{q}_i, q_i\}$ denote the desired and actual positions and velocities for the ith joint of the robot, respectively, σ_i is a switching variable and e_i is the position error. The characteristic equation $s + \lambda_i = 0$ has a real eigenvalue that determines the dynamics of the sliding phase.

The controller implements a set of decision equations that produce a control action $u(t)$, forcing the manipulator to match the reference model (2). Usually the VSC obeys a switching law of the type:
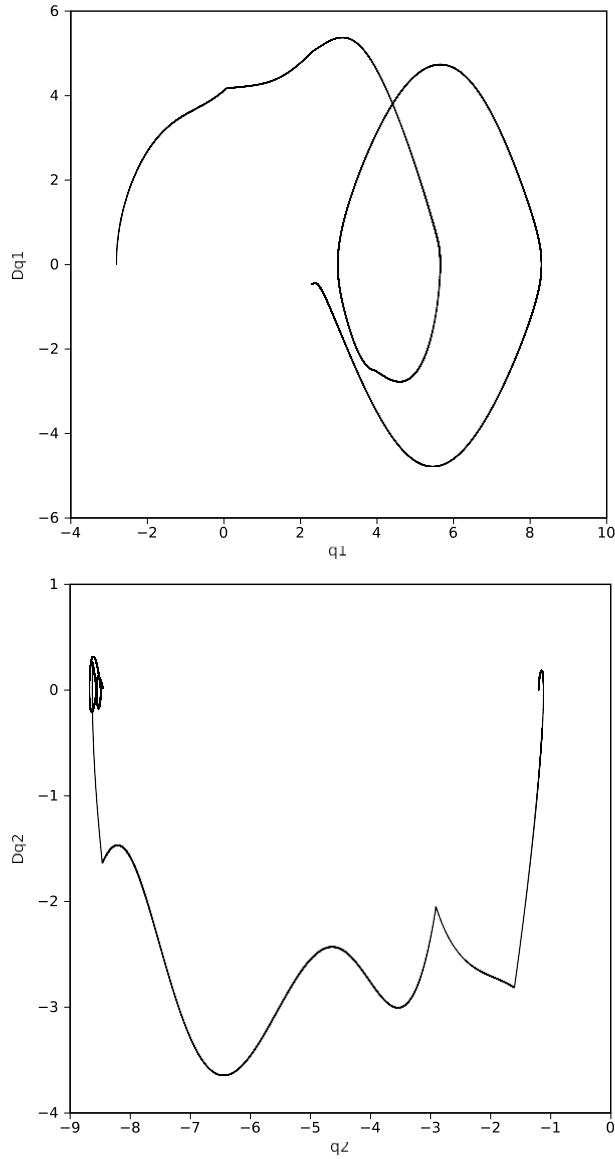
$$u(t) = u[\text{sgn}(\sigma)] \tag{4}$$

Fig. 2. Phase plane trajectories of the robot joints for $a = 1.0$ when $\delta_i = 10^{-4}$.

where sgn $(\cdot)$ represents the sign function. If the VSC satisfies the condition $(i = 1, \ldots, n)$:

$$\sigma_i \dot{\sigma}_i < 0 \tag{5}$$

then the asymptotic convergence is guaranteed. In [18], it was concluded that this algorithm imposes conflicting requirements because first order dynamics (2) has discontinuous trajectories in the phase plane, while robots have inertias that impose time continuity both in the positions and velocities. The first order discontinuous trajectories demand infinite joint torques during transients, that lead to actuator stress. Consequently, the actual reaching phase is not instantaneous and, due to its sensitivity to perturbations, convergence is not certain. In order to overcome this limitation, a second order reference law was proposed [5]:

$$\sigma_i = \ddot{e}_i + 2\zeta_i \omega_{ni} \dot{e}_i + \omega_{ni} e_i = 0 \tag{6}$$

where ζ_i is the damping coefficient and ω_{ni} is the undamped natural frequency. If the model $s^2 + 2\zeta_i \omega_{ni} s + \omega_{ni}^2 = (s + \lambda_{i1})(s + \lambda_{i2})$ has two roots, then expression (6) implements an over-damped or a critically damped reference model.

For a given initial condition in the phase plane, while the first order model (2) leads to a single trajectory, the second order model (6) produces an infinite number of continuous trajectories. It was concluded that the reaching phase and the chattering phenomena were avoided but, on the other hand, emerged the necessity of a second order derivative with the associated problems of requiring extra sensors.
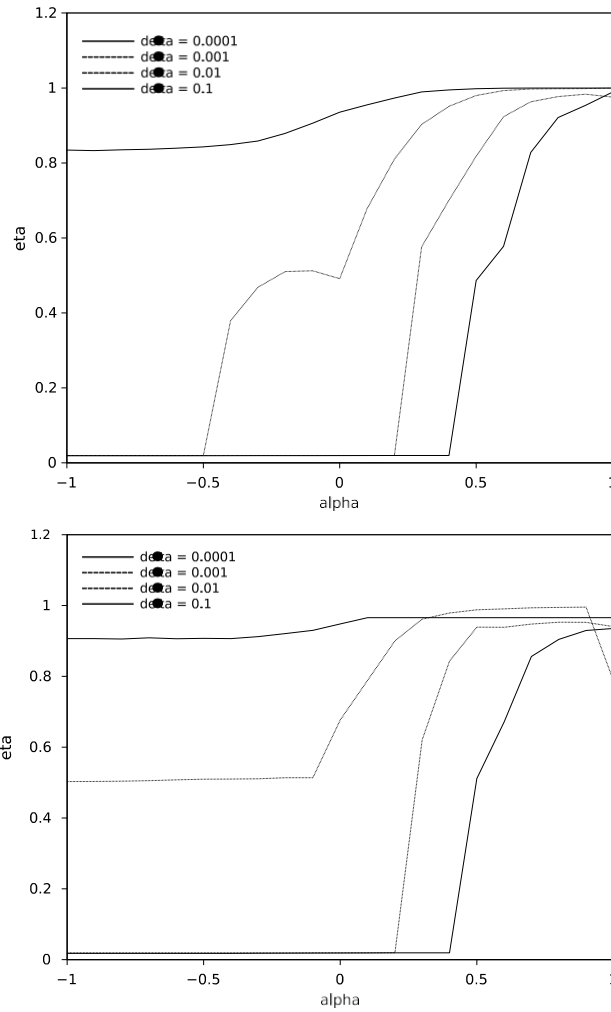
Fig. 3. Switching control activity: variation of η_i versus a for $\delta_i = \left[10^{-4}, \, 10^{-3}, \, 10^{-2}, \, 10^{-1} \right]$.

Later, in [19] the idea of introducing an integral action in the sliding reference law was explored, leading to the expression:

$$\sigma_i = \dot{e}_i + 2\zeta_i\omega_{ni}e_i + \omega_{ni}^2 \int e_i dt = 0 \qquad (7)$$

yielding the reference model $\frac{s^2 + 2\zeta_i\omega_{ni}s + \omega_{ni}^2}{s}$.

In this paper, this problem is revisited by taking advantage of the continuous variation provided by FC. We recall that the a order fractional derivative $D^a x(t)$ of the signal $x(t)$ is no longer restricted to integer values. Therefore, in this paper, we reformulate the reference model (2) as:

$$\sigma_i = D^{a+1}e_i + \lambda_i^a D^1 e_i + \lambda_i D^a e_i + \lambda_i^{a+1} = 0. \qquad (8)$$

The model becomes $(s + \lambda_i) \left[s^a + \lambda_i^a \right]$ and the case of $-1 \le a \le 1$ is explored. The design of a fractional reference model and, consequently, of the switching law, is not trivial since we have now additional freedom, but a limited know-how about the resulting dynamics. Eq. (8) is an ''interpolation'' between models (6) and (7), for $\zeta_i = 1.0$, that proved to lead to interesting results, but other reference models, namely with a different number of eigenvalues, can be explored in the future. Another aspect that deserves discussion is the tuning of the control algorithm versus the system stability. Often are adopted mathematical concepts, such as Lyapunov stability criteria, but the fact is that user gets a limited intuition into the overall performance. On the other hand, tools such as root locus, or frequency response, are no longer adequate since we are handling highly nonlinear systems. In spite of this limitation, for each link of the robot we can interpret heuristically, in the root locus, reference model (8) as two zeros ($a > 0$), or two zeros and one pole ($a < 0$), the robot dynamics (between input torque and output position) as two poles, and the other coupling effects as perturbations. The position of the two robot poles vary significantly [17] and the VSC adjusts the gain in real-time so that the global dynamics is close to the reference model.
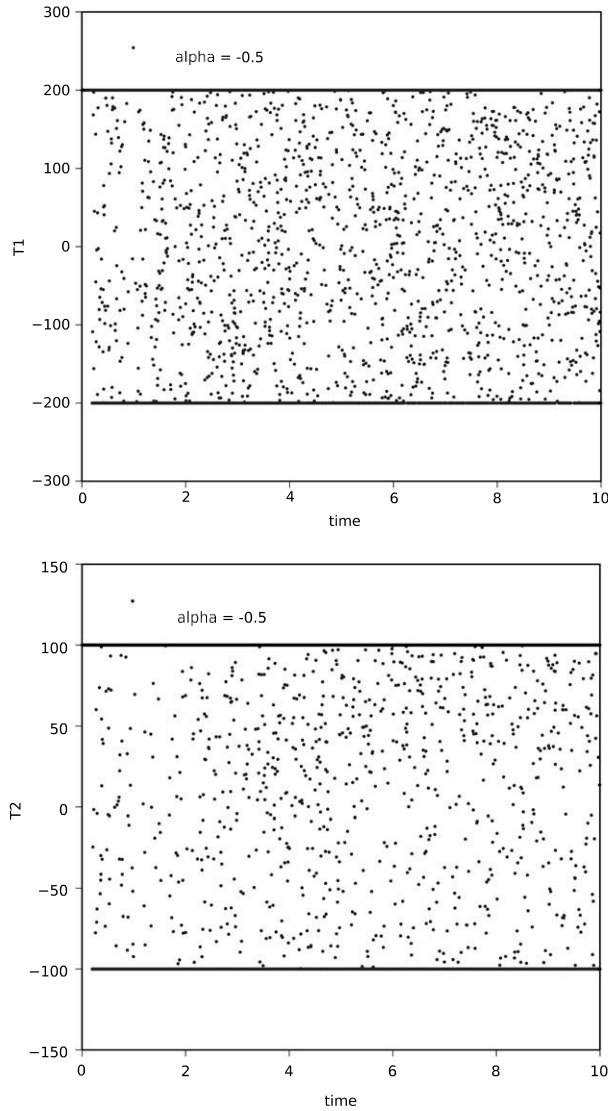
Fig. 4. Control torques versus time for $a = -0.5$ when $\delta_i = 10^{-4}$.

Therefore, the positioning and the number of the zeros and poles of the reference model must be compatible with the rest of the dynamical system, avoiding either an under-, or an over-, compensation, that would lead to poor results. In this line of thought, expression (8) together with the robot dynamics seem to establish a good balance between total number of poles and zeros providing, therefore, an intuitive prototype reference model and will be followed in the sequel.

For the purpose of implementation the fractional derivatives and integrals is followed the Grünwald–Letnikov definition:

$$
{}_aD_t^\alpha x(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} x(t - kh) \tag{9}
$$

where $\Gamma(\cdot)$ is Euler's gamma function, $[x]$ means the integer part of x, and h is the step time increment.

For obtaining the discrete time algorithm, that is, for converting expressions from continuous to discrete time, the approximation is often considered:

$$
Z\{D^\alpha x(t)\} \approx \left[\frac{1}{T_s}(1 - z^{-1})\right]^\alpha X(z^{-1}) \tag{10}
$$

where z and T_s represent the Z-transform variable and controller sampling period, respectively. This expression corresponds to the generalization of the Euler backward operator with the infinitesimal time increment h replaced by T_s.
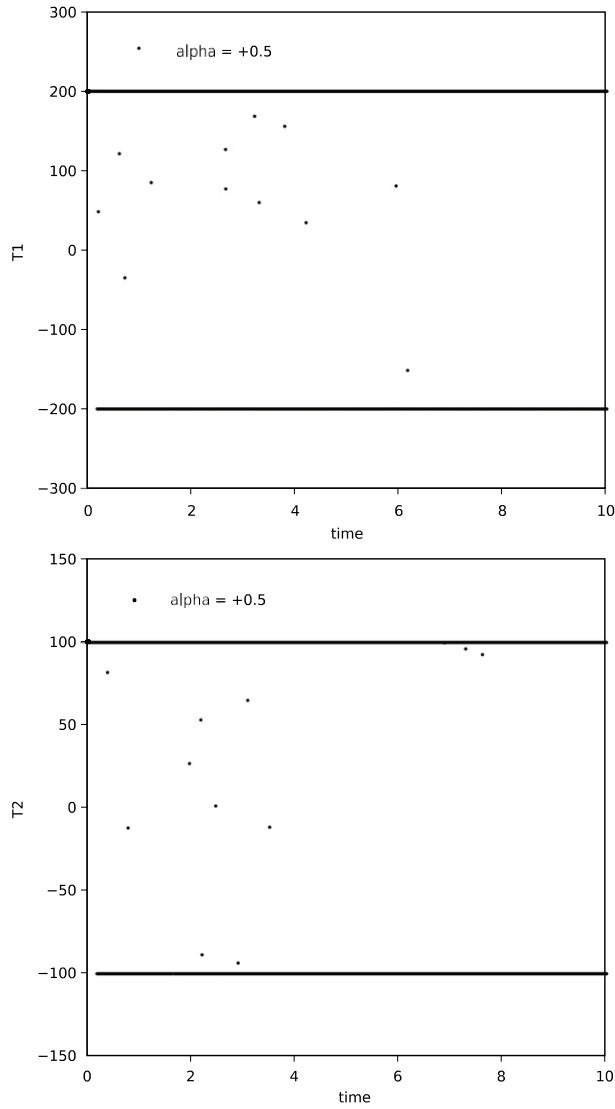
Fig. 5. Control torques versus time for $\alpha = +0.5$ when $\delta_i = 10^{-4}$.

In order to obtain rational expressions, Taylor or Padé expansions of order r are usually adopted, in the neighbourhood of $z = 0$. In the present study, a simple Taylor series expansion is adopted:

$$Z_r^{Taylor}\left[D^\alpha x(t)\right] \approx \frac{1}{T_s^\alpha}\left[1 - \alpha z^{-1} - \frac{\alpha(1-\alpha)}{2}z^{-2} - \frac{\alpha(1-\alpha)(2-\alpha)}{6}z^{-3} - \cdots\right]X\left(z^{-1}\right).\tag{11}$$

3. Integer variable structure control and fractional sliding mode

In this section, a simple manipulator with two rotational degrees of freedom ($n = 2$) is adopted and numerical values identical to those adopted in [2,3] are considered. Therefore, the robot dynamics is given by:)

$$J(q) = \begin{bmatrix} 15.75 + 10.0\cos(q_2) & 4.0 + 5.0\cos(q_2) \\ 4.0 + 5.0\cos(q_2) & 9.0 \end{bmatrix}.\tag{12}$$

$$C(q,\dot{q}) = \begin{bmatrix} -(5.0\dot{q}_2 + 10.0\dot{q}_1)\sin(q_2)\dot{q}_2 \\ 5.0\sin(q_2)\dot{q}_1^2 \end{bmatrix}.\tag{13}$$

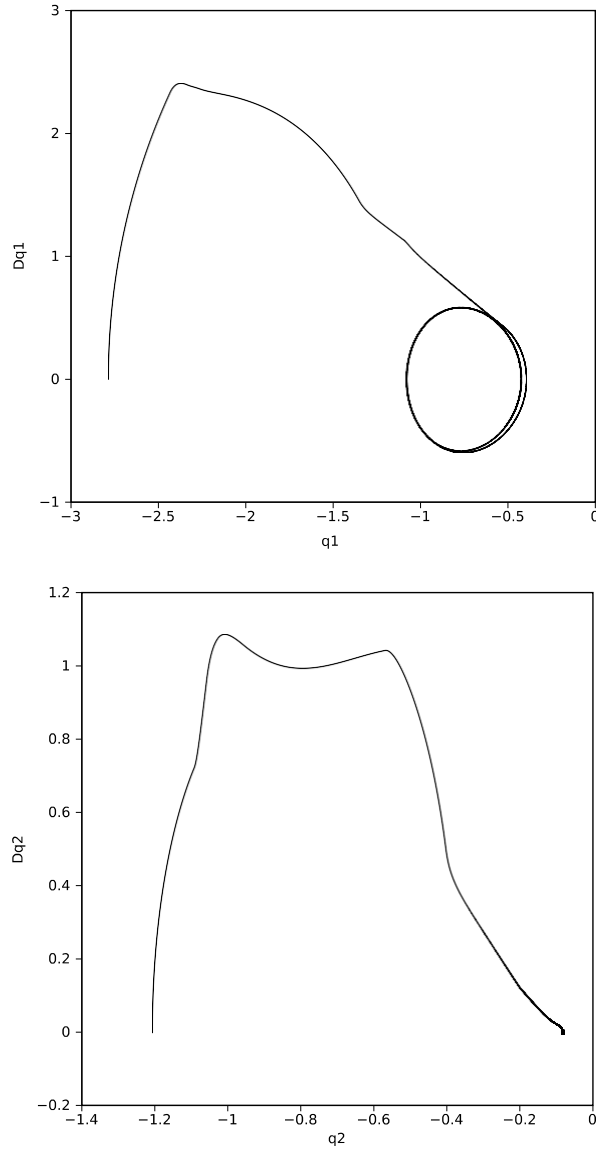$$G(q) = \begin{bmatrix} 66.15\cos(q_1) + 49.0\cos(q_1 + q_2) \\ 49.0\cos(q_1 + q_2) \end{bmatrix}.\tag{14}$$

Fig. 6. Phase plane trajectories of the robot joints for $a = -0.1$ when $\delta_i = 10^{-1}$.

We start by considering that the VSC torque is given by a simple proportional and saturation function:

$$T_i^{VSS} = \begin{cases} +D_i, & \sigma_i > \delta_i \\ \dfrac{\sigma_i}{\delta_i} D_i, & -\delta_i \leq \sigma_i \leq \delta_i \\ -D_i, & \sigma_i < -\delta_i \end{cases} \qquad (15)$$

$$T_i = T_i^{VSS} \qquad (16)$$

where δ_i defines the width of the proportional band and D_i and the maximum torque   amplitude.

In the following experiments, a standard test consisting of moving the manipulator from the initial state [2,3] is considered:

$$[q_1(0), q_2(0)]^T = [-2.784, -1.204]^T . \qquad (17)$$

$$[\dot{q}_1(0), \dot{q}_2(0)]^T = [0, 0]^T . \qquad (18)$$

To the final state:

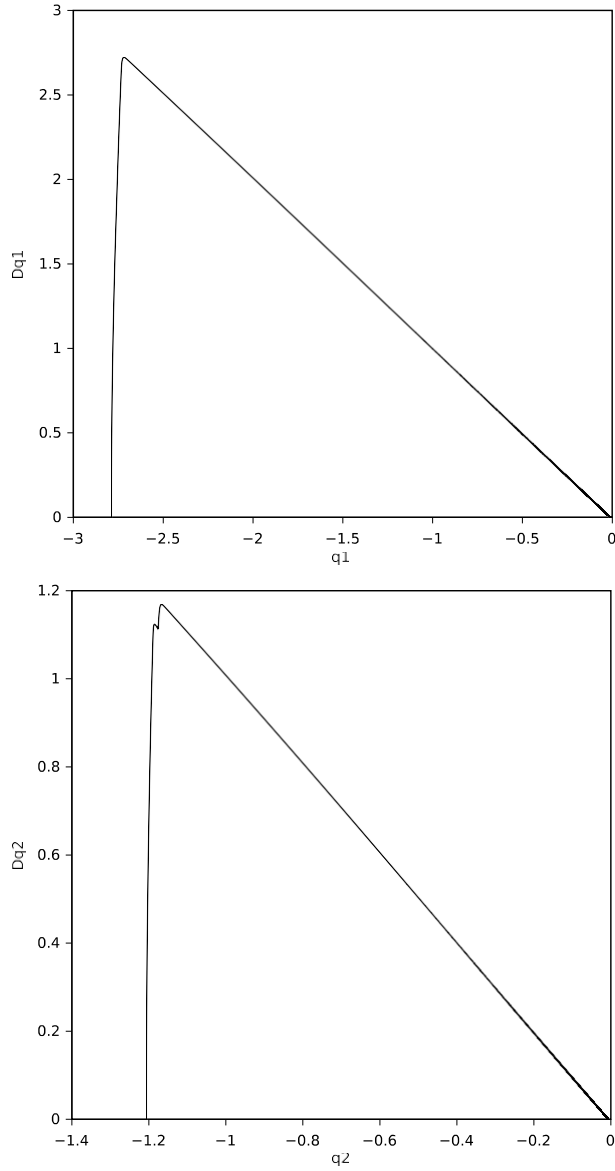$$[q_1(\infty), q_2(\infty)]^T = [0, 0]^T . \qquad (19)$$

Fig. 7. Phase plane trajectories of the robot joints for $a = 0.2$ when $\delta_i = 10^{-1}$.

$$[\dot{q}_1(\infty), \dot{q}_2(\infty)]^T = [0, 0]^T. \tag{20}$$

The chattering, visible in the sliding phase, is characteristic of VSC action. To measure this phenomenon, the index of control switching ($i = 1, 2$) is defined [20]:

$$\eta_i = \frac{N_i^p}{N_i^p + N_i^s} \tag{21}$$

where N_i^p and N_i^s are the number of samples of torque T_i that fit into the proportional and saturation bands of (15), respectively. Therefore, it is of relevance to analyse the variation of η with a and δ as a symptom of the VSC activity for compelling the non-linear system to follow the reference dynamics.

For the fractional derivative approximation, the Taylor series expansion (11) with $r = 10$ and $T_s = 10^{-4}$ [21] is considered. In the reference model $\lambda_i = 1.0$, ($i = 1, 2$) is considered, and in the control action $D_1 = 200$, $D_2 = 100$ is adopted. The experiments consist of varying the fractional order in the interval $-1 \leq a \leq 1$ and the width of the proportional band $\delta_i = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$, to evaluate their effect upon the system performance and the chattering dynamics.
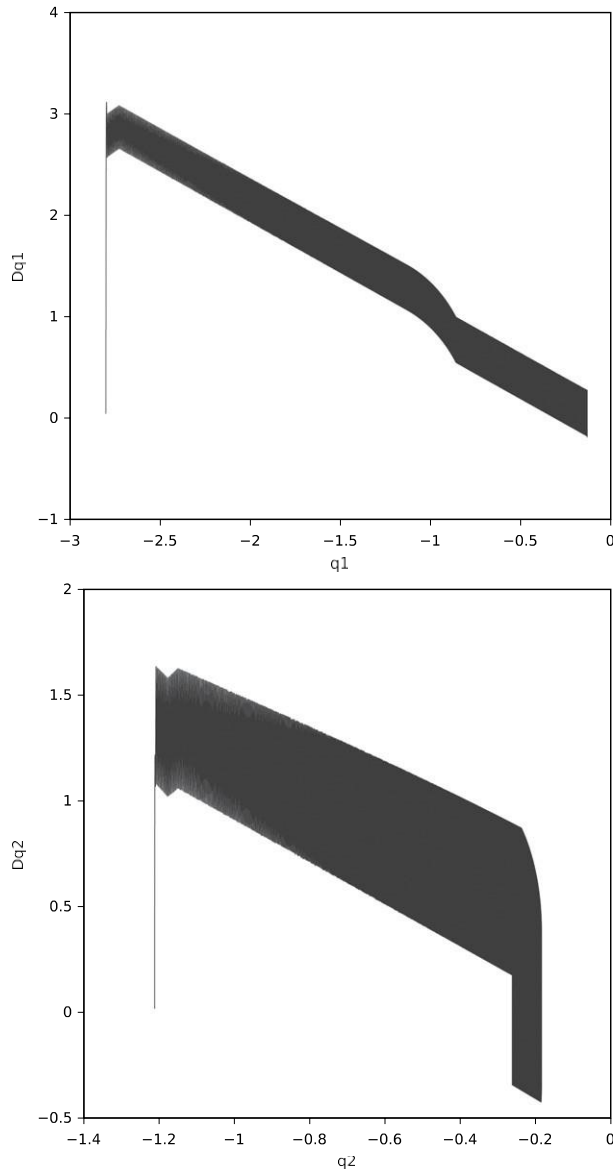
Fig. 8. Phase plane trajectories of the robot joints for $a = 0.6$ when $\delta_i = 10^{-1}$.

Fig. 1 depicts the phase plane trajectories for $a = 0.5$ when $\delta_i = 10^{-4}$. Analysing the different combinations of values of the parameters, we verify a good match against the reference dynamics, with exception of the neighbourhood of $a = 1.0$ (Fig. 2) where unstable responses occur.

Fig. 3 shows the evolution of η_i, ($i = 1, 2$), versus a for $\delta_i = \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}\}$. For example, Figs. 4 and 5 show the torque time evolution for $a = 0.5$ and $a = 0.5$ when $\delta_i = 10^{-4}$. As expected, we verify that the chattering activity increases with the diminishing of the proportional band. Furthermore, we conclude that the occurrence of control switching is higher the larger the value of a. This is due to the higher dynamical requirements posed by faster reference dynamics (i.e., larger a) that required a larger control action. In fact, observing the phase plane portraits for all tested cases (i.e., combinations of values of a when δ_i) it is concluded that high values of η occur when the VSC has convergence difficulties.

4. Fractional variable structure control and integer sliding mode

In this section, the effect of having a fractional order control law is explored. Again, this possibility emerges from [5] where the substitution of the classical VSS proportional-like algorithm by a proportional and integral controller having a VSS action in the integral component was proposed in order to adapt its strength in real-time. Therefore, it is considered the effect of having a fractional derivative in series with the VSS switching algorithm. Therefore, the switching line is (2) and
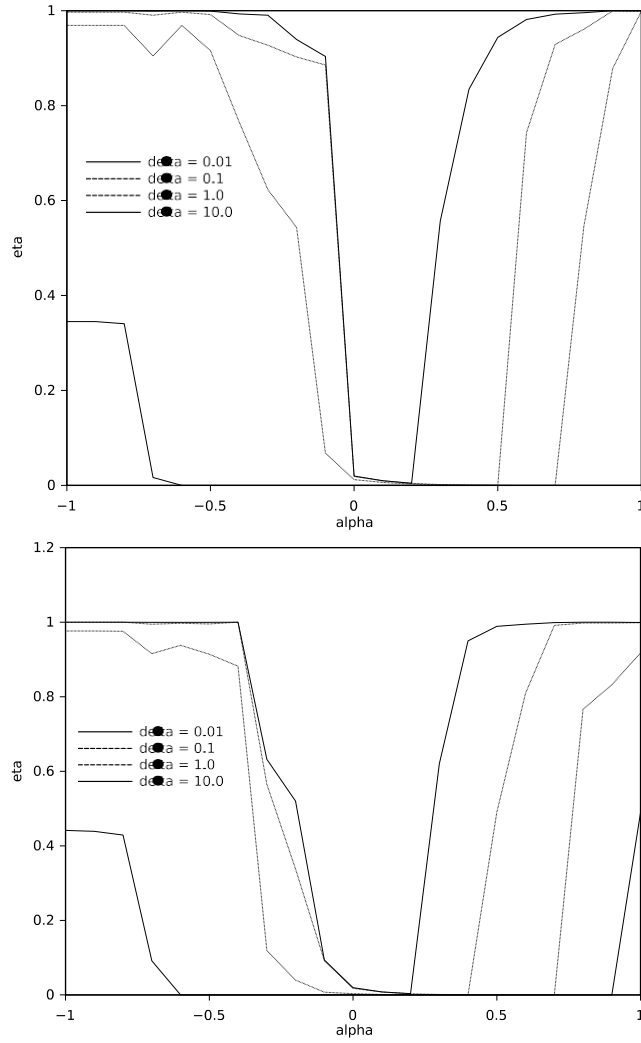
Fig. 9. Switching control activity: variation of η_i versus a for $\delta_i = \left[10^{-2},\ 10^{-1},\ 10^{0},\ 10^{1} \right]$.

the controller is given by:

$$T_i^{VSS} = \begin{cases} +D_i, & \sigma_i > \delta_i \\ \dfrac{\sigma_i}{\delta_i} D_i, & -\delta_i \le \sigma_i \le \delta_i \\ -D_i, & \sigma_i < -\delta_i. \end{cases} \tag{22}$$

$$T_i = D^\alpha \left[T_i^{VSS}(t) \right]. \tag{23}$$

We analyse the variation of η with a and δ while the controller and robot parameters remain identical to the previous ones (i.e., $r = 10$, $T_s = 10^{-4}$, $\lambda_i = 1.0$, $i = 1,\ 2$, and $D_1 = 200$, $D_2 = 100$) with exception of δ_i. For example, Figs. 6–8 depict the phase-plane trajectories for $\delta_i = 10^{-1}$ and $a = \{-0.1,\ 0.2,\ 0.6\}$, respectively. We observe good results for $a = 0.2$, but convergence difficulties and significant chartering phenomena for $a = -0.1$ and $a = 0.6$, respectively.

Fig. 9 shows the evolution of η_i, $(i = 1,\ 2)$, versus a for $\delta_i = \left[10^{-2},\ 10^{-1},\ 10^{0},\ 10^{1} \right]$, $\lambda_i = 1.0$, $(i = 1,\ 2)$. We observe that the chattering activity increases with the diminishing of the proportional band. Furthermore, we conclude that the occurrence of control switching is smaller in the centre of the range of values of a, while near the extreme cases of integer order integral (at the left) or integer order derivative (at the right) the controller experiments difficulties.

In conclusion, we demonstrated that FC is a tool that leads to an extra degree of freedom when tuning variable structure systems, both at the sliding surface and control law, that may lead to superior dynamical performances.

5. Conclusions

In this paper, the application of FC concepts was studied in the analysis of VSCs. These algorithms are a class of non-linear controllers that has been intensively studied during the past decades and, therefore, it is important to generalize the results for FC concepts. The test bed consisted in a two degrees of freedom manipulator under the action of two alternative control strategies, namely a fractional sliding surface, and a fractional control law. The dynamical performance was monitored via the phase plane and the control action was accessed by means of the switching activity during time. The experiments revealed that the variation of the fractional order represents an useful tool for the continuous tuning and adjustment of the closed loop system performance. The extra freedom of design constituted by the fractional order leads to new possibilities for the design of the reference model and the control law, that must be further explored before advancing to industrial cases. These results encourage further research on embedding FC into VSC structures and, by consequence, to the adoption of FC-VSCs in real-world applications.

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