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Controllability results for impulsive mixed-type functional integro-differential evolution equations with nonlocal conditions

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Abstract

In this paper, we establish the controllability for a class of abstract impulsive mixed-type functional integro-differential equations with finite delay in a Banach space. Some sufficient conditions for controllability are obtained by using the Mönch fixed point theorem via measures of noncompactness and semigroup theory. Particularly, we do not assume the compactness of the evolution system. An example is given to illustrate the effectiveness of our results.

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1 Introduction

In recent years, the theory of impulsive differential equations has provided a natural framework for mathematical modeling of many real world phenomena, namely in control, biological and medical domains. In these models, the investigated simulating processes and phenomena are subjected to certain perturbations whose duration is negligible in comparison with the total duration of the process. Such perturbations can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. These processes tend to be more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state. For more details on this theory and its applications, we refer to the monographs of Bainov and Simeonov [1], Lakshmikantham *et al.* [2] and Samoilenko and Perestyuk [3] and the papers of [4–12].

On the other hand, the concept of controllability is of great importance in mathematical control theory. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to the final state, where the initial and final states may vary over the entire space. Many authors have studied the controllability of nonlinear systems with and without impulses; see, for instance, [13–18]. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic systems [14, 16, 19–24], and the nonlocal initial condition, in many cases, has a much better effect in applications than the traditional initial condition. As remarked by Byszewski and Lakshmikantham (see [25, 26]), the nonlocal initial value problems can be more useful than the standard initial value problems to describe many physical phenomena.

The study of Volterra-Fredholm integro-differential equations plays an important role in abstract formulation of many initial, boundary value problems of perturbed differential partial integro-differential equations. Recently, many authors studied mixed type integro-differential systems without (or with) delay conditions [27–31]. In [16] the controllability of impulsive functional differential systems with nonlocal conditions was studied by using the measures of noncompactness and the Mon̈ch fixed point theorem, and some sufficient conditions for controllability were established. Here, without assuming the compactness of the evolution system, [29] establishes the existence, uniqueness and continuous dependence of mild solutions for nonlinear mixed type integro-differential equations with finite delay and nonlocal conditions. The results are obtained by using the Banach fixed point theorem and semigroup theory.

More recently, Shengli Xie [31] derived the existence of mild solutions for the nonlinear mixed-type integro-differential functional evolution equations with nonlocal conditions, and the results were achieved by using the Mon̈ch fixed point theorem and fixed point theory. Here some restricted conditions on *a priori* estimates and measures of noncompactness estimation were not used even if the generator $A = 0$.

To the best of our knowledge, up to now no work has reported on controllability of an impulsive mixed Volterra-Fredholm functional integro-differential evolution differential system with finite delay, and nonlocal conditions has been an untreated topic in the literature, and this fact is the main aim of the present work.

This paper is motivated by the recent works [16, 29, 31] and its main purpose is to establish sufficient conditions for the controllability of the impulsive mixed-type functional integro-differential system with finite delay and nonlocal conditions of the form

$$x'(t) = A(t)x(t) + f\left(t, x_t, \int_0^t h(t,s,x_s)\,ds, \int_0^b k(t,s,x_s)\,ds\right) + (Bu)(t),$$
$$t \in J = [0,b], t \neq t_i, i = 1,2,\ldots,s, \tag{1.1}$$

$$\Delta x|_{t=t_i} = I_i(x_{t_i}), \quad i = 1,2,\ldots,s, \tag{1.2}$$

$$x_0 = \phi + g(x), \quad t \in [-r,0], \tag{1.3}$$

where $A(t)$ is a family of linear operators which generates an evolution system $\{U(t,s) : 0 \leq s \leq t \leq b\}$. The state variable $x(\cdot)$ takes the values in the real Banach space X with the norm $\|\cdot\|$. The control function $u(\cdot)$ is given in $L^2(J,V)$, a Banach space of admissible control functions with V as a Banach space, and thereby $T = \{(t,s) : 0 \leq s \leq t \leq b\}$. B is a bounded linear operator from V into X. The nonlinear operators $h : T \times \mathcal{D} \to X$, $k : T \times \mathcal{D} \to X$ and $f : J \times \mathcal{D} \times X \times X \to X$ are continuous, where $\mathcal{D} = \{\psi : [-r,0] \to X : \psi(t)$ is continuous everywhere except for a finite number of points t_i at which $\psi(t_i^+)$ and $\psi(t_i^-)$ exist and $\psi(t_i) = \psi(t_i^-)\}$; $I_i : \mathcal{D} \to X$, $i = 1,2,\ldots,s$, are impulsive functions, $0 < t_1 < t_2 < \cdots < t_s < t_{s+1} = b$, $\Delta\xi(t_i)$ is the jump of a function ξ at t_i, defined by $\Delta\xi(t_i) = \xi(t_i^+) - \xi(t_i^-)$.

For any function $x \in \mathcal{PC}$ and any $t \in J$, x_t denotes the function in $\mathcal{D}$ defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-r,0],$$

where $\mathcal{PC}$ is defined in Section 2. Here $x_t(\cdot)$ represents the history of the state from the time $t - r$ up to the present time t.

Our work is organized as follows. In the next section, fundamental notions and facts related to MNC are recalled. Section 3 is devoted to analyzing controllability results of the problem (1.1)-(1.3). Section 4 contains an illustrative example.

2 Preliminaries

In this section, we recalled some fundamental definitions and lemmas which are required to demonstrate our main results (see [20–24, 32–35]).

Let $L^1([0,b],X)$ be the space of X-valued Bochner integrable functions on $[0,b]$ with the norm $\|f\|_{L^1} = \int_0^b \|f(t)\|\,dt$. In order to define the solution of the problem (1.1)-(1.3), we consider the following space: $\mathcal{PC}([-r,b],X) = \{x : [-r,b] \to X$ such that $x(\cdot)$ is continuous except for a finite number of points t_i at which $x(t_i^+)$ and $x(t_i^-)$ exist and $x(t_i) = x(t_i^-)\}$.

It is easy to verify that $\mathcal{PC}([-r,b],X)$ is a Banach space with the norm

$$\|x\|_{\mathcal{PC}} = \sup\{\|x(t)\| : t \in [-r,b]\}.$$

For our convenience, let $\mathcal{PC} = \mathcal{PC}([-r,b],X)$ and $J_0 = [0,t_1]$; $J_i = (t_i,t_{i+1}]$, $i = 1,2,\ldots,s$.

Definition 2.1 Let E^+ be a positive cone of an order Banach space $(E,\leq)$. A function Φ defined on the set of all bounded subsets of the Banach space X with values in E^+ is called a measure of noncompactness (MNC) on X if $\Phi(\overline{co}\Omega) = \Phi(\Omega)$ for all bounded subsets $\Omega \subseteq X$, where $\overline{co}\Omega$ stands for the closed convex hull of Ω.

The MNC Φ is said to be

(1) Monotone if for all bounded subsets Ω_1, Ω_2 of X we have $(\Omega_1 \subseteq \Omega_2) \Rightarrow (\Phi(\Omega_1) \leq \Phi(\Omega_2))$;
(2) Nonsingular if $\Phi(\{a\} \cup \Omega) = \Phi(\Omega)$ for every $a \in X$, $\Omega \subset X$;
(3) Regular if $\Phi(\Omega) = 0$ if and only if Ω is relatively compact in X.

One of the many examples of MNC is the noncompactness measure of Hausdorff β defined on each bounded subset Ω of X by

$$\beta(\Omega) = \inf\{\epsilon > 0; \Omega \text{ can be covered by a finite number of balls of radii smaller than } \epsilon\}.$$

It is well known that MNC β verifies the above properties and other properties; see [32, 33] for all bounded subsets Ω, Ω_1, Ω_2 of X,

(4) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$, where $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$;
(5) $\beta(\Omega_1 \cup \Omega_2) \leq \max\{\beta(\Omega_1), \beta(\Omega_2)\}$;
(6) $\beta(\lambda\Omega) \leq |\lambda|\beta(\Omega)$ for any $\lambda \in \mathbb{R}$;
(7) If the map $Q : D(Q) \subseteq X \to Z$ is Lipschitz continuous with a constant k, then $\beta_Z(Q\Omega) \leq k\beta(\Omega)$ for any bounded subset $\Omega \subseteq D(Q)$, where Z is a Banach space.

Definition 2.2 A two-parameter family of bounded linear operators $U(t,s)$, $0 \leq s \leq t \leq b$, on X is called an evolution system if the following two conditions are satisfied:

(i) $U(s,s) = I$, $U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq b$;
(ii) $(t,s) \to U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq b$.

Since the evolution system $U(t,s)$ is strongly continuous on the compact operator set $J \times J$, there exists $M_1 > 0$ such that $\|U(t,s)\| \leq M_1$ for any $(t,s) \in J \times J$. More details about the evolution system can be found in Pazy [34].

Definition 2.3 A function $x(\cdot) \in \mathcal{PC}$ is said to be a mild solution of the system (1.1)-(1.3) if $x(t) = \phi(t) + g(x)(t)$ on $[-r,0]$, $\Delta x|_{t=t_i} = I_i(x_{t_i})$, $i = 1,2,\ldots,s$, the restriction of $x(\cdot)$ to the interval J_i $(i = 1,2,\ldots,s)$ is continuous and the following integral equation is satisfied.

$$\begin{aligned} x(t) &= U(t,0)\big[\phi(0) + gx(0)\big] \\ &\quad + \int_0^t U(t,s)\left[Bu(s) + f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau, \int_0^b k(s,\tau,x_\tau)\,d\tau\right)\right] ds \\ &\quad + \sum_{0<t_i<t} U(t,t_i) I_i(x_{t_i}), \quad t \in J. \end{aligned}$$

Definition 2.4 The system (1.1)-(1.3) is said to be nonlocally controllable on the interval J if, for every initial function $\phi \in \mathcal{D}$ and $x_1 \leq X$, there exists a control $u \in L^2(J,V)$ such that the mild solution $x(\cdot)$ of (1.1)-(1.3) satisfies $x(b) = x_1$.

Definition 2.5 A countable set $\{f_n\}_{n=1}^\infty \subset L^1([0,b],X)$ is said to be semicompact if the sequence $\{f_n\}_{n=1}^\infty$ is relatively compact in X for almost all $t \in [0,b]$, and if there is a function $\mu \in L^1([0,b],\mathbb{R}^+)$ satisfying $\sup_{n\geq 1} \|f_n(t)\| \leq \mu(t)$ for a.e. $t \in [0,b]$.

Lemma 2.1 (See [32]) *If $W \subset C([a,b],X)$ is bounded and equicontinuous, then $\beta(W(t))$ is continuous for $t \in [a,b]$ and*

$$\beta(W) = \sup\big\{\beta\big(W(t)\big), t \in [a,b]\big\}, \quad \textit{where } W(t) = \big\{x(t) : x \in W\big\} \subseteq X.$$

Lemma 2.2 (See [12]) *If $W \subset \mathcal{PC}([a,b],X)$ is bounded and piecewise equicontinuous on $[a,b]$, then $\beta(W(t))$ is piecewise continuous for $t \in [a,b]$ and*

$$\beta(W) = \sup\big\{\beta\big(W(t)\big), t \in [a,b]\big\}.$$

Lemma 2.3 (See [19]) *Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $L^1([0,b],\mathbb{R}^+)$. Assume that there exist $\mu, \eta \in L^1([0,b],\mathbb{R}^+)$ satisfying $\sup_{n\geq 1}\|f_n(t)\| \leq \mu(t)$ and $\beta(\{f_n(t)\}_{n=1}^\infty) \leq \eta(t)$ a.e. $t \in [0,b]$, then for all $t \in [0,b]$, we have*

$$\beta\left(\left\{\int_0^t U(t,s) f_n(s)\,ds : n \geq 1\right\}\right) \leq 2M_1 \int_0^t \eta(s)\,ds.$$

Lemma 2.4 (See [19]) *Let $(Gf)(t) = \int_0^t U(t,s)f(s)\,ds$. If $\{f_n\}_{n=1}^\infty \subset L^1([0,b],X)$ is semicompact, then the set $\{Gf_n\}_{n=1}^\infty$ is relatively compact in $C([0,b],X)$. Moreover, if $f_n \rightharpoonup f_0$, then for all $t \in [0,b]$,*

$$(Gf_n)(t) \to (Gf_0)(t) \quad \textit{as } n \to \infty.$$

The following fixed-point theorem, a nonlinear alternative of Mońch type, plays a key role in our proof of controllability of the system (1.1)-(1.3).

Lemma 2.5 (See [35, Theorem 2.2]) *Let D be a closed convex subset of a Banach space X and $0 \in D$. Assume that $F : D \to X$ is a continuous map which satisfies Mońch's condition, that is, ($M \subseteq D$ is countable, $M \subseteq \overline{co}(\{0\} \cup F(M)) \Rightarrow \overline{M}$ is compact). Then F has a fixed point in D.*

3 Controllability results

In this section, we present and demonstrate the controllability results for the problem (1.1)-(1.3). In order to demonstrate the main theorem of this section, we list the following hypotheses.

(H1) $A(t)$ is a family of linear operators, $A(t) : D(A) \to X$, $D(A)$ not depending on t and a dense subset of X, generating an equicontinuous evolution system $\{U(t,s) : 0 \le s \le t \le b\}$, *i.e.*, $(t,s) \to \{U(t,s)x : x \in B\}$ is equicontinous for $t > 0$ and for all bounded subsets B and $M_1 = \sup\{\|U(t,s)\| : (t,s) \in T\}$.

(H2) The function $f : J \times \mathcal{D} \times X \times X \to X$ satisfies the following:

(i) For $t \in J$, the function $f(t,\cdot,\cdot,\cdot) : \mathcal{D} \times X \to X$ is continuous, and for all $(\phi, x) \in \mathcal{D} \times X$, the function $f(\cdot,\phi,x,y) : J \to X$ is strongly measurable.

(ii) For every positive integer k_1, there exists $\alpha_{k_1} \in L^1([0,b];\mathbb{R}^+)$ such that

$$\sup_{\|\phi\|_{\mathcal{D}} \le k_1} \|f(t,\phi)\| \le \alpha_{k_1}(t) \quad \text{for a.e. } t \in J,$$

and

$$\lim_{r\to\infty} \inf \int_0^b \frac{\alpha_{k_1}(t)}{k_1}\, dt = \sigma < \infty.$$

(iii) There exists an integrable function $\eta : [0,b] \to [0,\infty)$ such that

$$\beta\bigl(f(t,D,A,B)\bigr) \le \eta(t)\Bigl[\sup_{-r\le\theta\le 0} \beta\bigl(D(\theta)\bigr) + \beta(A) + \beta(B)\Bigr]$$
$$\text{for a.e. } t \in J \text{ and } D \subset \mathcal{D}, A, B \subset X,$$

where $D(\theta) = \{v(\theta) : v \in D\}$.

(H3) The function $h : T \times \mathcal{D} \to X$ satisfies the following:

(i) For each $(t,s) \in T$, the function $h(t,s,\cdot) : \mathcal{D} \to X$ is continuous, and for each $x \in \mathcal{D}$, the function $h(\cdot,\cdot,x) : T \to X$ is strongly measurable.

(ii) There exists a function $m \in L^1(T,\mathbb{R}^+)$ such that

$$\|h(t,s,x_s)\| \le m(t,s)\|x_s\|_{\mathcal{D}}.$$

(iii) There exists an integrable function $\zeta : T \to [0,\infty)$ such that

$$\beta\bigl(h(t,s,H)\bigr) \le \zeta(t,s) \sup_{-r\le\theta\le 0} H(\theta) \quad \text{for a.e } t \in J$$

and $H \subset \mathcal{D}$, where $H(\theta) = \{w(\theta) : w \in H\}$ and β is the Hausdorff MNC.

For convenience, let us take $L_0 = \max \int_0^t m(t,s)\, ds$ and $\zeta^* = \max \int_0^s \zeta(t,s)\, ds$.

(H4) The function $k : T \times \mathcal{D} \to X$ satisfies the following:

(i) For each $(t,s) \in T$, the function $k(t,s,\cdot) : \mathcal{D} \to X$ is continuous, and for each $x \in \mathcal{D}$, the function $k(\cdot,\cdot,x) : T \to X$ is strongly measurable.

(ii) There exists a function $m \in L^1(T, \mathbb{R}^+)$ such that

$$\left\| k(t,s,x_s) \right\| \le m^\star(t,s) \|x_s\|_{\mathcal{D}}.$$

(iii) There exists an integrable function $\gamma : T \to [0,\infty)$ such that

$$\beta\bigl(k(t,s,H)\bigr) \le \gamma(t,s) \sup_{-r\le\theta\le 0} H(\theta) \quad \text{for a.e. } t \in J$$

and $H \subset \mathcal{D}$, where $H(\theta) = \{w(\theta) : w \in H\}$.

For convenience, let us take $L_1 = \max \int_0^t m^\star(t,s)\,ds$ and $\gamma^* = \max \int_0^s \gamma(t,s)\,ds$.

(H5) $g : \mathcal{PC}([0,b] : X) \to X$ is a continuous compact operator such that

$$\lim_{\|y\|_{PC}\to\infty} \frac{\|g(y)\|}{\|y\|_{PC}} = 0.$$

(H6) The linear operator $W : L^2(J,V) \to X$ is defined by

$$W = \int_0^b U(t,s)Bu(s)\,ds \quad \text{such that}$$

(i) W has an invertible operator W^{-1} which takes values in $L^2(J,V)/\ker W$, and there exist positive constants M_2 and M_3 such that

$$\|B\| \le M_2, \qquad \left\| W^{-1} \right\| \le M_3.$$

(ii) There is $K_W \in L^1(J, \mathbb{R}^+)$ such that, for every bounded set $Q \subset X$,

$$\beta\bigl(W^{-1}Q\bigr)(t) \le K_W(t)\beta(Q).$$

(H7) $I_i : \mathcal{D} \to X$, $i = 1,2,\ldots,s$, is a continuous operator such that

(i) There are nondecreasing functions $L_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\left\| I_i(x) \right\| \le L_i\bigl(\|x\|_{\mathcal{D}}\bigr), \quad i = 1,2,\ldots,s, x \in \mathcal{D},$$

and

$$\lim_{\rho\to\infty} \inf \frac{L_i(\rho)}{\rho} = \lambda_i < \infty, \quad i = 1,2,\ldots,s.$$

(ii) There exist constants $K_i \ge 0$ such that

$$\beta\bigl(I_i(S)\bigr) \le K_i \sup_{-r\le\theta\le 0} \beta\bigl(S(\theta)\bigr), \quad i = 1,2,\ldots,s,$$

for every bounded subset S of $\mathcal{D}$.

(H8) The following estimation holds true:

$$N = \Bigg[\big(M_1 + 2M_1^2 M_2 \|K_W\|_{L^1}\big) \sum_{i=1}^{s} K_i \\ + \big[1 + 2(\zeta^* + \gamma^*)\big]\big(2M_1 + 4M_1^2 M_2 \|K_W\|_{L^1}\big) \|\eta\|_{L^1} \Bigg] < 1.$$

Theorem 3.1 *Assume that the hypotheses* (H1)-(H8) *are satisfied. Then the impulsive differential system* (1.1)-(1.3) *is controllable on J provided that*

$$M_1\big(1 + M_1 M_2 M_3 b^{\frac{1}{2}}\big)\big[\sigma(1 + L_0 + L_1)\big] + \sum_{i=1}^{s} \lambda_i < 1. \tag{3.1}$$

Proof Using the hypothesis (H6)(i), for every $x \in \mathcal{PC}([-r,b],X)$, define the control

$$u_x(t) = W^{-1}\Bigg[x_1 - U(b,0)\varphi(0) - \int_0^b U(b,s) \\ \times f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau, \int_0^b k(s,\tau,x_\tau)\,d\tau\right) ds - \sum_{0<t_i<b} U(b,t_i) I_i(x_{t_i})\Bigg](t).$$

We shall now show that when using this control, the operator defined by

$$(Fx)(t) = \begin{cases} \phi(t), & t \in [-r,0], \\ U(t,0)[\phi(0) + gx(0)] + \int_0^t U(t,s) \\ \quad \times [f(s, x_s, \int_0^t h(t,\tau,x_\tau)\,d\tau, \int_0^b k(s,\tau,x_\tau)\,d\tau)\,ds + (Bu_x)(s)]\,ds \\ \quad + \sum_{0<t_i<t} U(t,t_i) I_i(x_{t_i}), & t \in J, \end{cases}$$

has a fixed point. This fixed point is then a solution of (1.1)-(1.3). Clearly, $x(b) = (Fx)(b) = x_1$, which implies the system (1.1)-(1.3) is controllable. We rewrite the problem (1.1)-(1.3) as follows.

For $\phi \in \mathcal{D}$, we define $\hat{\phi} \in \mathcal{PC}$ by

$$\hat{\phi}(t) = \begin{cases} U(t,0)[\phi(0) + gx(0)], & t \in J, \\ \phi(t), & t \in [-r,0]. \end{cases}$$

Then $\hat{\phi} \in \mathcal{PC}$. Let $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-r,b]$. It is easy to see that y satisfies $y_0 = 0$ and

$$y(t) = \int_0^t U(t,s) \\ \times \left[f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau, \int_0^s k(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau\right) + Bu_y(s)\right] ds \\ + \sum_{0<t_i<t} U(t,t_i) I_i(y_{t_i} + \hat{\phi}_{t_i}),$$

where

$$u_y(s) = W^{-1}\bigg[x_1 - U(b,0)\big[\phi(0) + gx(0)\big] - \int_0^b U(b,s)f\bigg(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau, \int_0^b k(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau\bigg)\,ds - \sum_{i=1}^{s} U(b,t_i)I_i(y_{t_i} + \hat{\phi}_{t_i})\bigg](s)$$

if and only if x satisfies

$$x(t) = U(t,0)\big[\phi(0) + gx(0)\big] + \int_0^t U(t,s)\bigg[f\bigg(s, x_s, \int_0^s h(s,\tau,x_\tau)\,d\tau, \int_0^b k(s,\tau,x_\tau)\,d\tau\bigg) + Bu_x(s)\bigg]\,ds + \sum_{0<t_i<t} U(t,t_i)I_i(x_{t_i}),$$

and $x(t) = \phi(t) + gx(t)$, $t \in [-r,0]$. Define $\mathcal{PC}_0 = \{y \in \mathcal{PC} : y_0 = 0\}$. Let $G : \mathcal{PC}_0 \to \mathcal{PC}_0$ be an operator defined by

$$(Gy)(t) = \begin{cases} 0, \quad t \in [-r,0], \\ \int_0^t U(t,s)[f(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau, \\ \quad \int_0^b k(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau) + Bu_y(s)]\,ds \\ \quad + \sum_{0<t_i<t} U(t,t_i)I_i(y_{t_i} + \hat{\phi}_{t_i}), \quad t \in J. \end{cases} \tag{3.2}$$

Obviously, the operator F has a fixed point is equivalent to G has one. So, it turns out to prove G has a fixed point. Let $G = G_1 + G_2$, where

$$(G_1y)(t) = \sum_{0<t_i<t} U(t,t_i)I_i(y_{t_i} + \hat{\phi}_{t_i}), \tag{3.3}$$

$$(G_2y)(t) = \int_0^t U(t,s)\bigg[f\bigg(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau, \int_0^b k(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau\bigg) + Bu_y(s)\bigg]\,ds. \tag{3.4}$$

Step 1: There exists a positive number $q \geq 1$ such that $G(B_q) \subseteq B_q$, where $B_q = \{y \in \mathcal{PC}_0 : \|y\|_{\mathcal{PC}} \leq q\}$.

Suppose the contrary. Then for each positive integer q, there exists a function $y^q(\cdot) \in B_q$ but $G(y^q) \notin B_q$, *i.e.*, $\|G(y^q)(t)\| > q$ for some $t \in J$.

We have from (H1)-(H7)

$$\begin{aligned} q &< \big\|(Gy^q)(t)\big\| \\ &\leq M_1 \int_0^b \bigg\|f\bigg(s, y^q_s + \hat{\phi}_s, \int_0^s h\big(s,\tau,y^q_\tau + \hat{\phi}_\tau\big)\,d\tau, \int_0^b k\big(s,\tau,y^q_\tau + \hat{\phi}_\tau\big)\,d\tau\bigg) + Bu_{y^q}(s)\bigg\|\,ds \\ &\quad + M_1 \sum_{i=1}^{s} L_i\big(\big\|y^q_{t_i} + \hat{\phi}_{t_i}\big\|_{\mathcal{D}}\big). \end{aligned}$$

Since

$$\int_0^t \left\| f\left(s, y_s^q + \hat{\phi}_s, \int_0^s h\left(s,\tau, y_\tau^q + \hat{\phi}_\tau\right) d\tau, \int_0^b k\left(s,\tau, y_\tau^q + \hat{\phi}_\tau\right) d\tau\right) \right\| ds \le \int_0^b \alpha_{q^*}(s)\, ds,$$

where, $q^* = (1+L_0)q'$ and $q' = q + \|\hat{\phi}\|_{\mathcal{PC}}$, we have

$$q \le M_1 \int_0^b \alpha_{q^*}(s)\, ds + M_1 M_2 b^{\frac{1}{2}} \|u_{y^q}\|_{L^2} + M_1 \sum_{i=1}^s L_i\left(q'\right), \tag{3.5}$$

where

$$\|u_{y^q}\|_{L^2} \le M_3 \left[\|x_1\| + M_1 \|\phi\|_{\mathcal{D}} + M_1 \int_0^b \alpha_{q^*}(s)\, ds + M_1 \sum_{i=1}^s L_i\left(q'\right) \right]. \tag{3.6}$$

Hence by (3.5)

$$\begin{aligned} q &< M_1 \int_0^b \alpha_{q^*}(s)\, ds + M_1 M_2 b^{\frac{1}{2}} M_3 \left[\|x_1\| + M_1 \|\phi\|_{\mathcal{D}} + M_1 \int_0^b \alpha_{q^*}(s)\, ds + M_1 \sum_{i=1}^s L_i\left(q'\right) \right] \\ &\quad + M_1 \sum_{i=1}^s L_i\left(q'\right) \\ &\le \left(1 + M_1 M_2 M_3 b^{\frac{1}{2}}\right) M_1 \left[\int_0^b \alpha_{q^*}(s)\, ds + \sum_{i=1}^s L_i\left(q'\right) \right] + M, \end{aligned}$$

where $M = M_1 M_2 M_3 b^{\frac{1}{2}} (\|x_1\| + M_1 \|\phi\|_{\mathcal{D}})$ is independent of q and $q' = q + \|\hat{\phi}\|_{\mathcal{PC}}$.

Dividing both sides by q and noting that $q' = q + \|\hat{\phi}\|_{\mathcal{PC}} \to \infty$ as $q \to \infty$, we obtain

$$\begin{aligned} \lim_{q\to+\infty} \inf\left(\frac{\int_0^b \alpha_{q^*}(s)\, ds}{q} \right) &= \lim_{q\to+\infty} \inf\left(\frac{\int_0^b \alpha_{q^*}(s)\, ds}{q^*} \cdot \frac{q^*}{q} \right) = \sigma(1 + L_0 + L_1), \\ \lim_{q\to+\infty} \inf\left(\frac{\sum_{i=1}^s L_i(q')}{q} \right) &= \lim_{q\to+\infty} \inf\left(\frac{\sum_{i=1}^s L_i(q')}{q'} \cdot \frac{q'}{q} \right) = \sum_{i=1}^s \lambda_i. \end{aligned}$$

Thus we have

$$1 \le M_1\left(1 + M_1 M_2 M_3 b^{\frac{1}{2}}\right)\left(\sigma(1 + L_0 + L_1) + \sum_{i=1}^s \lambda_i\right).$$

This contradicts (3.1). Hence, for some positive number q, $G(B_q) \subseteq B_q$.

Step 2: $G : \mathcal{PC}_0 \to \mathcal{PC}_0$ is continuous.

Let $\{y^{(n)}(t)\}_{n=1}^\infty \subseteq \mathcal{PC}_0$ with $y^{(n)} \to y$ in $\mathcal{PC}_0$. Then there is a number $q > 0$ such that $\|y^{(n)}(t)\| \le q$ for all n and $t \in J$, so $y^{(n)} \in B_q$ and $y \in B_q$.

From (H2) and (H5) we have

(i)

$$f\left(t, y_t^{(n)} + \hat{\phi}_t, \int_0^t h(t,\tau, y_\tau^{(n)} + \hat{\phi}_t)\,d\tau, \int_0^b k(t,\tau, y_\tau^{(n)} + \hat{\phi}_t)\,d\tau\right) \to f\left(t, y_t + \hat{\phi}_t, \int_0^t h(t,\tau, y_t + \hat{\phi}_t)\,d\tau, \int_0^b k(t,\tau, y_t + \hat{\phi}_t)\,d\tau\right)$$

and

$$\left\| f\left(t, y_t^{(n)} + \hat{\phi}_t, \int_0^t h(t,\tau, y_\tau^{(n)} + \hat{\phi}_t)\,d\tau, \int_0^b k(t,\tau, y_\tau^{(n)} + \hat{\phi}_t)\,d\tau\right) - f\left(t, y_t + \hat{\phi}_t, \int_0^t h(t,\tau, y_t + \hat{\phi}_t)\,d\tau, \int_0^b k(t,\tau, y_\tau^{(n)} + \hat{\phi}_t)\,d\tau\right)\right\| \le 2\alpha_{q^*}(t).$$

(ii) $I_i(y_{t_i}^{(n)} + \hat{\phi}_{t_i}) \to I_i(y_{t_i} + \hat{\phi}_{t_i})$, $i = 1, 2, \ldots, s$.

Then we have

$$\|G_1 y^{(n)} - G_1 y\|_{\mathcal{PC}} \le \sum_{i=1}^{s} \|I_i(y_{t_i}^{(n)} + \hat{\phi}_{t_i}) - I_i(y_{t_i} + \hat{\phi}_{t_i})\| \tag{3.7}$$

and

$$\begin{aligned} \|G_2 y^{(n)} - G_2 y\|_{\mathcal{PC}} &\le M_1 \int_0^b \left\| f\left(s, y_s^{(n)} + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau, \int_0^b k(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau\right) \right. \\ &\quad \left. - f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau + \hat{\phi}_\tau)\,d\tau, \int_0^b k(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau\right)\right\| ds \\ &\quad + M_1 M_2 \int_0^b \|u_{y^{(n)}}(s) - u_y(s)\|\,ds \\ &\le M_1 \int_0^b \left\| f\left(s, y_s^{(n)} + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau, \int_0^b k(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau\right) \right. \\ &\quad \left. - f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau + \hat{\phi}_\tau)\,d\tau, \int_0^b k(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau\right)\right\| ds \\ &\quad + M_1 M_2 b^{\frac{1}{2}} \|u_y^{(n)} - u_y\|_{L^2}, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \|u_y^{(n)} - u_y\|_{L^2} &\le M_3 \left[M_1 \int_0^b \left\| f\left(s, y_s^{(n)} + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau, \int_0^b k(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau\right) \right.\right. \\ &\quad \left. - f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau + \hat{\phi}_\tau)\,d\tau, \int_0^b k(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau)\,d\tau\right)\right\| ds \\ &\quad \left. + M_1 \sum_{i=1}^{s} \|I_i(y_{t_i}^{(n)} + \hat{\phi}_{t_i}) - I_i(y_{t_i} + \hat{\phi}_{t_i})\| \right]. \end{aligned} \tag{3.9}$$

Observing (3.7)-(3.9), by the dominated convergence theorem, we have that

$$\|Gy^{(n)} - Gy\|_{\mathcal{PC}} \le \|G_1y^{(n)} - Gy\|_{\mathcal{PC}} + \|G_2y^{(n)} - G_2y\|_{\mathcal{PC}} \to 0 \quad \text{as } n \to +\infty.$$

That is, G is continuous.

Step 3: G is equicontinuous on every J_i, $i = 1, 2, \ldots, s$. That is, $G(B_q)$ is piecewise equicontinuous on J.

Indeed, for $t_1, t_2 \in J_i$, $t_1 < t_2$ and $y \in B_q$, we deduce that

$$\begin{aligned}
&\|(Gy)(t_2) - (Gy)(t_1)\| \\
&\quad\le \int_0^{t_1} \Big\| \big[U(t_2,s) - U(t_1,s)\big] \\
&\qquad \times f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau, \int_0^b k\big(s,\tau,y_\tau^{(n)} + \hat{\phi}_\tau\big)\,d\tau\right) + Bu_y(s)\Big\|\,ds \\
&\qquad + \int_{t_1}^{t_2} \Big\| U(t_2,s) f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)\,d\tau,\right. \\
&\qquad \left.\int_0^b k\big(s,\tau,y_\tau^{(n)} + \hat{\phi}_\tau\big)\,d\tau\right) + Bu_y(s)\Big\|\,ds \\
&\quad\le \int_0^{t_1} \|U(t_2,s) - U(t_1,s)\| \alpha_{q^*}(s)\,ds + \int_0^{t_1} \|U(t_2,s) - U(t_1,s)\| \\
&\qquad \times M_2M_3\left[\|x_1\| + M_1\|\phi(0)\| + M_1\int_0^b \alpha_{q^*}\,ds + M_1\sum_{i=1}^s L_i\big(q'\big)\right] ds \\
&\qquad + \int_{t_1}^{t_2} \|U(t_2,s)\| \alpha_{q^*}(s)\,ds + \int_{t_1}^{t_2} \|U(t_2,s)\| \\
&\qquad \times M_2M_3\left[\|x_1\| + M_1\|\phi(0)\| + M_1\int_0^b \alpha_{q^*}\,ds + M_1\sum_{i=1}^s L_i\big(q'\big)\right] ds.
\end{aligned} \tag{3.10}$$

By the equicontinuity of $U(\cdot,s)$ and the absolute continuity of the Lebesgue integral, we can see that the right-hand side of (3.10) tends to zero and is independent of y as $t_2 \to t_1$. Hence $G(B_q)$ is equicontinuous on J_i ($i = 1, 2, \ldots, s$).

Step 4: Moňch's condition holds.

Suppose $W \subseteq B_q$ is countable and $W \subseteq \overline{co}(\{0\} \cup G(W))$. We shall show that $\beta(W) = 0$, where β is the Hausdorff MNC.

Without loss of generality, we may assume that $W = \{y^{(n)}\}_{n=1}^{\infty}$. Since G maps B_q into an equicontinuous family, $G(W)$ is equicontinuous on J_i. Hence $W \subseteq \overline{co}(\{0\} \cup G(W))$ is also equicontinuous on every J_i.

By (H7)(ii) we have

$$\begin{aligned}
&\beta\big(\{G_1y^{(n)}(t)\}_{n=1}^{\infty}\big) \\
&\quad = \beta\left(\left\{\sum_{0<t_i<t} U(t,t_i)I_i\big(y_{t_i}^{(n)} + \hat{\phi}_{t_i}\big)\right\}_{n=1}^{\infty}\right) \\
&\quad \le M_1 \sum_{i=1}^s \beta\big(\{I_i\big(y_{t_i}^{(n)} + \hat{\phi}_{t_i}\big)\}_{n=1}^{\infty}\big)
\end{aligned}$$

$$
\begin{aligned}
&\le M_1 \sum_{i=1}^{s} K_i \sup_{-r\le\theta\le 0} \beta\big(\big\{y^{(n)}(t_i+\theta)+\hat{\phi}(t_i+\theta)\big\}_{n=1}^{\infty}\big) \\
&\le M_1 \sum_{i=1}^{s} K_i \sup_{0\le\tau_i\le t_i} \beta\big(\big\{y^{(n)}(\tau_i)\big\}_{n=1}^{\infty}\big).
\end{aligned} \tag{3.11}
$$

By Lemma 2.3 and from (H3)(iii), (H4)(iii), (H6)(ii) and (H7)(ii), we have that

$$
\begin{aligned}
&\beta_V\big(\big\{u_{y^{(n)}}(s)\big\}_{n=1}^{\infty}\big) \\
&\le K_W(s)\Bigg[\beta\Bigg(\Bigg\{\int_0^b U(b,s)f\Bigg(s, y_s^{(n)}+\hat{\phi}_s, \int_0^s h\big(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau\big)\,d\tau, \\
&\qquad \int_0^b k\big(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau\big)\,d\tau\Bigg)\,ds\Bigg\}_{n=1}^{\infty}\Bigg) + \beta\Bigg(\Bigg\{\sum_{i=1}^{s} U(b,t_i)I_i\big(y_{t_i}^{(n)}+\hat{\phi}_{t_i}\big)\Bigg\}_{n=1}^{\infty}\Bigg)\Bigg] \\
&\le K_W(s)\Bigg[2M_1\int_0^b \eta(s)\Big[\sup_{-r\le\theta\le 0}\beta\big(\big\{y^{(n)}(s+\theta)+\hat{\phi}(s+\theta)\big\}_{n=1}^{\infty}\big) \\
&\qquad + \beta\Bigg(\Bigg\{\int_0^s h\big(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau\big)\,d\tau, \int_0^b k\big(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau\big)\,d\tau\Bigg\}_{n=1}^{\infty}\Bigg)\Bigg]\,ds \\
&\qquad + M_1\sum_{i=1}^{s} K_i \sup_{-r\le\theta\le 0}\beta\big(\big\{y^{(n)}(t_i+\theta)+\hat{\phi}(t_i+\theta)\big\}_{n=1}^{\infty}\big)\Bigg] \\
&\le K_W(s)\Bigg[2M_1\int_0^b \eta(s)\sup_{0\le\tau\le s}\beta\big(\big\{y^{(n)}(\tau)\big\}_{n=1}^{\infty}\big)\,ds \\
&\qquad + 2M_1\int_0^b \eta(s)\beta\Bigg(\Bigg\{\int_0^s h\big(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau\big)\,d\tau\Bigg\}_{n=1}^{\infty}\Bigg)\,ds \\
&\qquad + 2M_1\int_0^b \eta(s)\beta\Bigg(\Bigg\{\int_0^s k\big(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau\big)\,d\tau\Bigg\}_{n=1}^{\infty}\Bigg)\,ds \\
&\qquad + M_1\sum_{i=1}^{s} K_i \sup_{0\le\tau_i\le t_i}\beta\big(\big\{y^{(n)}(\tau_i)\big\}_{n=1}^{\infty}\big)\Bigg] \\
&\le K_W(s)\Bigg[2M_1\int_0^b \eta(s)\sup_{0\le\tau\le s}\beta\big(\big\{y^{(n)}(\tau)\big\}_{n=1}^{\infty}\big)\,ds \\
&\qquad + 4M_1\int_0^b \eta(s)\big(\zeta^*+\gamma^*\big)\sup_{0\le\mu\le\tau}\beta\big(\big\{y^{(n)}(\mu)\big\}_{n=1}^{\infty}\big)\,ds \\
&\qquad + M_1\sum_{i=1}^{s} K_i \sup_{0\le\tau_i\le t_i}\beta\big(\big\{y^{(n)}(\tau_i)\big\}_{n=1}^{\infty}\big)\Bigg].
\end{aligned} \tag{3.12}
$$

This implies that

$$
\begin{aligned}
&\beta\big(\big\{G_2y^{(n)}(t)\big\}_{n=1}^{\infty}\big) \\
&\le \beta\Bigg(\Bigg\{\int_0^t U(t,s)f\Bigg(s, y_s^{(n)}+\hat{\phi}_s, \int_0^s h\big(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau\big)\,d\tau, \\
&\qquad \int_0^b k\big(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau\big)\,d\tau\Bigg)\,ds\Bigg\}_{n=1}^{\infty}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \beta\left(\left\{\int_0^t U(t,s)Bu_{y^{(n)}}(s)\,ds\right\}_{n=1}^{\infty}\right) \\
&\leq 2M_1 \int_0^b \eta(s) \sup_{0\leq\tau\leq s} \beta\big(\big\{y^{(n)}(\tau)\big\}_{n=1}^{\infty}\big)\,ds \\
&\quad + 4M_1 \int_0^b \eta(s)\big(\zeta^* + \gamma^*\big) \sup_{0\leq\mu\leq\tau} \beta\big(\big\{y^{(n)}(\mu)\big\}_{n=1}^{\infty}\big)\,ds \\
&\quad + 2M_1M_2 \int_0^b \beta_V\big(\big\{u_{y^{(n)}}(s)\big\}_{n=1}^{\infty}\big)\,ds \\
&\leq \big[1 + 2\big(\zeta^* + \gamma^*\big)\big]2M_1 \int_0^b \eta(s) \sup_{0\leq\tau\leq s} \beta\big(\big\{y^{(n)}(\tau)\big\}_{n=1}^{\infty}\big)\,ds \\
&\quad + \big[1 + 2\big(\zeta^* + \gamma^*\big)\big]4M_1^2M_2\left(\int_0^b K_W(s)\,ds\right)\left(\int_0^b \eta(s) \sup_{0\leq\tau\leq s} \beta\big(\big\{y^{(n)}(\tau)\big\}_{n=1}^{\infty}\big)\right) \\
&\quad + 2M_1^2M_2 \sum_{i=1}^{s} K_i \sup_{0\leq\tau_i\leq t_i} \beta\big(\big\{y^{(n)}(\tau_i)\big\}_{n=1}^{\infty}\big).
\end{aligned}
\tag{3.13}
$$

From (3.11) and (3.13) we obtain that

$$
\begin{aligned}
&\beta\big(\big\{Gy^{(n)}(t)\big\}_{n=1}^{\infty}\big) \\
&\leq \beta\big(\big\{G_1y^{(n)}(t)\big\}_{n=1}^{\infty}\big) + \beta\big(\big\{G_2y^{(n)}(t)\big\}_{n=1}^{\infty}\big) \\
&\leq M_1 \sum_{i=1}^{s} K_i \sup_{0\leq\tau_i\leq t_i} \beta\big(\big\{y^{(n)}(\tau_i)\big\}_{n=1}^{\infty}\big) \\
&\quad + \left(\big[1 + 2\big(\zeta^* + \gamma^*\big)\big]2M_1 + 4M_1^2M_2 \int_0^b K_W(s)\,ds\right) \\
&\quad \times \int_0^b \eta(s) \sup_{0\leq\tau\leq s} \beta\big(\big\{y^{(n)}(\tau)\big\}_{n=1}^{\infty}\big)\,ds \\
&\quad + 2M_1^2M_2 \int_0^b K_W(s)\,ds\left(\sum_{i=1}^{s} K_i \sup_{0\leq\tau_i\leq t_i} \beta\big(\big\{y^{(n)}(\tau_i)\big\}_{n=1}^{\infty}\big)\right)
\end{aligned}
\tag{3.14}
$$

for each $t \in J$.

Since W and $G(W)$ are equicontinuous on every J_i, according to Lemma 2.2, the inequality (3.14) implies that

$$
\begin{aligned}
&\beta\big(\big\{Gy^{(n)}\big\}_{n=1}^{\infty}\big) \\
&\leq \left[M_1 \sum_{i=1}^{s} K_i + \big[1 + \big(\zeta^* + \gamma^*\big)\big]\big(2M_1 + 4M_1^2M_2\|K_W\|_{L^1}\big)\|\eta\|_{L^1}\right]\beta\big(\big\{y^{(n)}\big\}_{n=1}^{\infty}\big) \\
&\quad + \left[2M_1^2M_2\|K_W\|_{L^1} \sum_{i=1}^{s} K_i\right]\beta\big(\big\{y^{(n)}\big\}_{n=1}^{\infty}\big) \\
&= \left[\big(M_1 + 2M_1^2M_2\|K_W\|_{L^1}\big) \sum_{i=1}^{s} K_i\right. \\
&\quad \left. + \big[1 + 2\big(\zeta^* + \gamma^*\big)\big]\big(2M_1 + 4M_1^2M_2\|K_W\|_{L^1}\big)\|\eta\|_{L^1}\right]
\end{aligned}
$$

$$\times \beta\big(\{y^{(n)}\}_{n=1}^{\infty}\big)$$
$$= N\beta\big(\{y^{(n)}\}_{n=1}^{\infty}\big).$$

That is, $\beta(GW) \leq N\beta(W)$, where N is defined in (H8). Thus, from Mońch's condition, we get that

$$\beta(W) \leq \beta\big(\overline{co}\big(\{0\} \cup G(W)\big) = \beta\big(G(W)\big) \leq N\beta(W)$$

since $N < 1$, which implies that $\beta(W) = 0$. So, we have that W is relatively compact in $\mathcal{PC}_0$. In the view of Lemma 2.5, *i.e.*, Mońch's fixed point theorem, we conclude that G has a fixed point y in W. Then $x = y + \hat{\phi}$ is a fixed point of F in $\mathcal{PC}$, and thus the system (1.1)-(1.3) is nonlocally controllable on the interval $[0, b]$. This completes the proof. □

Here we must remark that the conditions (H1)-(H8) given above are at least sufficient, because it is an open problem to prove that they are also necessary or to find an example which points out clearly that the mentioned conditions are not necessary to get the main result proved in this section.

4 An example

Consider the partial functional integro-differential systems with impulsive conditions of the form

$$\begin{aligned}\frac{\partial}{\partial t}z(t,\xi) = {}& \frac{\partial}{\partial \xi}z(t,\xi) + m(\xi)u(t,\xi)\\ & + F\left(t, z(t-r,\xi), \int_0^t k_1\big(t, w(x,\xi-r)\big)\,ds, \int_0^b h_1\big(t, w(x,\xi-r)\big)\,ds\right)\\ & \text{for } \xi \in [0,\pi], t \in [0,b], t \neq t_i, i = 1,2,\ldots,s, \end{aligned} \tag{4.1}$$

$$z(t,0) = 0, \quad t \in [0,b], \tag{4.2}$$

$$z\big(t_i^+,\xi\big) - z\big(t_i^-,\xi\big) = I_i\big(z\big(t_i^-,\xi\big)\big), \quad \xi \in (0,\pi], i = 1,2,\ldots,s, \tag{4.3}$$

$$z_0(\xi) = \varphi(t,\xi) + \int_0^b h(s)\log\big(1 + \big|x(\theta,\xi)\big|\big)\,ds, \quad t \in [-r,0], \xi \in [0,\pi], \tag{4.4}$$

where $r > 0$, $I_i > 0$, $i = 1,2,\ldots,s$, $\varphi \in \mathcal{D} = \{\psi : [-r,b] \times [0,\pi] \to \mathbb{R}$, ψ is continuous everywhere except for a countable number of points at which $\psi(s^-)$, $\psi(s^+)$ exist with $\psi(s^-) = \psi(s)\}$, $0 = t_0 < t_1 < t_2 < \cdots < t_{s+1} = b$, $z(t_i^+) = \lim_{(h,\xi)\to(0^+,\xi)} z(t_i + h,\xi)$, $z(t_i^-) = \lim_{(h,\xi)\to(0^-,\xi)} z(t_i + h,\xi)$, $B : X \to X$.

Let $X = L^2[0,\pi]$ and $A(t) \equiv A : X \to X$ be defined by $Aw = w'$ with the domain $D(A) = \{w \in X : w$ is absolutely continuous $w' \in X$, $w(\xi) = w(0) = 0\}$. It is well known that A is an infinitesimal generator of a semigroup $T(t)$ defined by $T(t)w(s) = w(t+s)$ for each $w \in X$. $T(t)$ is not a compact semigroup on X and $\beta(T(t)D) \leq \beta(D)$, where β is the Hausdorff MNC. We also define the bounded linear control operator $B : X \to X$ by

$$(Bu)(\xi) = m(\xi)u(\xi) \quad \text{for almost every } \xi \in [0,\pi].$$

We assume that

(1) $f : [0,b] \times X \times X \times X \to X$ is a continuous function defined by

$$f(t,x,k_1,h_1)(\xi) = F\big(t,x(\xi,t),k_1(\xi,t),h_1(\xi,t)\big),$$
$$k_1(\xi,t) = \int_0^t k_1\big(t,w(x,\xi-r)\big)\,ds,$$
$$h_1(\xi,t) = \int_0^b h_1\big(t,w(x,\xi-r)\big)\,ds.$$

We take $F(t,x(\xi,t),k_1(\xi,t),h_1(\xi,t)) = C_0\sin(x(\xi))$, C_0 is a constant. F is Lipschitz continuous for the second variable. Then f satisfies the hypotheses (H2) and (H3) of Section 3.

(2) $I_i : X \to X$ is a continuous function for each $i = 1,2,\ldots,s$ defined by

$$I_i(x)(\xi) = I_i\big(x(\xi)\big).$$

We take $I_i(x)(\xi) = \int_{[0,\pi]} \rho_i(\xi,y)\cos^2(x(y))\,dy$, $x \in X$, $\phi_i \in C([0,\pi] \times [0,\pi], R)$, for each $i = 1,2,\ldots,s$. Then I_i is compact and satisfies the hypothesis (H6)(i).

(3) $g : \mathcal{PC}([0,b] : X) \to X$ is a continuous function defined by

$$g(\varphi)(\xi) = \int_0^b h(s)\log\big(1 + \big|\varphi(s)(\xi)\big|\big)\,ds, \quad \varphi \in PC\big([0,b] : X\big)$$

with $\varphi(s)(\xi) = z(s,\xi)$. Then g is a compact operator and satisfies the hypothesis (H5).

Therefore, the above partial differential system (4.1)-(4.4) can be written to the abstract form (1.1)-(1.3) and all conditions of Theorem 3.1 are satisfied. We can conclude that the system (4.1)-(4.4) is nonlocally controllable on the interval J.

Conclusions

In the current paper, we are focused on finding some sufficient conditions to establish controllability results for a class of impulsive mixed-type functional integro-differential equations with finite delay. The proof of the main theorem is based on the application of the Moňch fixed point theorem with a noncompact condition of the evolution system. An example is also included to illustrate the technique.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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