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Weakly spectrally complete pair of matrices

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Let A and B be $n \times n$ matrices over an algebraically closed field F. Let $c_1, \dots, c_n$ be elements of F such that $\det(AB) = c_1 \dots c_n$ and $\#\{i \in \{1, \dots, n\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\}$. We give necessary and sufficient condition for the existence of matrices A' and B' similar to A and B, respectively, such that $A'B'$ has eigenvalues $c_1, \dots, c_n$.

Keywords: eigenvalues; invariant polynomials; factorization of matrices

AMS Subject Classifications: 15A18; 15A23

Let F be an algebraically closed field and $A, B \in F^{n\times n}$, where $n \geq 2$.

In this paper, we study the possible eigenvalues of the product $A'B'$, where $A', B' \in F^{n\times n}$ are matrices similar to A, B, respectively. If $c_1, \dots, c_n \in F$ are the eigenvalues of $A'B'$ then there are two conditions that the eigenvalues must satisfy:

$$\det(AB) = c_1 \dots c_n, \tag{1}$$

$$\#\{i \in \{1, \dots, n\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\}. \tag{2}$$

The pair (A, B) is *spectrally complete*, if for every sequence $c_1, \dots, c_n \in F$ such that (1) is satisfied, there exist matrices $A', B' \in F^{n\times n}$ similar to A, B, respectively, such that $A'B'$ has eigenvalues $c_1, \dots, c_n$.

A complete description of the spectrally complete pair of matrices was given in [1], and previously, was given in [2] for the nonsingular case. The concept of spectral completeness was introduced in [3] in order to study the possible eigenvalues of the sum of matrices.

The pair (A, B) is said to be *weakly spectrally complete* if, for every sequence $c_1, \dots, c_n \in F$ such that (1) and (2) are satisfied, there exist matrices A', B' similar to A, B, respectively, such that $A'B'$ has eigenvalues $c_1, \dots, c_n$.

Note that there exist A', B' similar to A, B, respectively, such that $A'B'$ has eigenvalues $c_1, \dots, c_n$ if and only if there exists A'' similar to A such that $A''B$ has eigenvalues $c_1, \dots, c_n$ if and only if there exists B'' similar to B such that AB'' has eigenvalues $c_1, \dots, c_n$.

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Given a monic polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, we denote by $C(f)$ the companion matrix of f:

$$C(f) = \begin{bmatrix} 0 & 1 & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \in F^{n\times n}.$$

We denote by $i(A)$ the number of nonconstant invariant polynomials of A. We make the convention that the invariant polynomials are always monic. If $\alpha_1 | \dots | \alpha_n$ are the invariant polynomials of A, then A is similar to $C(\alpha_{n-i(A)+1}) \oplus \cdots \oplus C(\alpha_n)$.

We say that $\lambda \in F$ is a *primary eigenvalue* of A if λ is a eigenvalue of $\alpha_{n-i(A)+1}$. Note that if λ is a primary eigenvalue of A, then $\operatorname{rank}(A - \lambda I_n) = n - i(A)$.

If $C = [c_{i,j}] \in F^{n\times n}$ is a matrix such that $c_{i,j} = 0$ if $j > i+1$, we denote by $\chi(C)$ the number of indices $i \in \{1, \dots, n-1\}$ such that $c_{i,i+1} \neq 0$. We have $i(C) \leq n - \chi(C)$.

The next theorem is our main theorem:

Theorem 1 *Let $\alpha_1 | \dots | \alpha_n$ and $\beta_1 | \dots | \beta_n$ be the invariant polynomials of A and B, respectively. The pair (A, B) is weakly spectrally complete if and only if the following are satisfied:*

(1.1) *If $i(A) + i(B) > n$ and $\alpha_{n-i(A)+1}(x) = x - \lambda$, with $\lambda \in F\backslash\{0\}$, then*

$$\beta_1(x) \dots \beta_{i(A)}(x) = x^{i(A)+i(B)-n};$$

(1.2) *If $i(A) + i(B) > n$ and $\beta_{n-i(B)+1}(x) = x - \mu$, with $\mu \in F\backslash\{0\}$, then*

$$\alpha_1(x) \dots \alpha_{i(B)}(x) = x^{i(A)+i(B)-n};$$

(1.3) *At least one of the following conditions holds:*

- *$n = 2$,*
- *$\deg(\alpha_n) \neq 2$,*
- *$\deg(\beta_n) \neq 2$,*
- *$i(A) \leq i(B)$ and 0 is a primary eigenvalue of B,*
- *$i(B) \leq i(A)$ and 0 is a primary eigenvalue of A.*

Lemma 2 *If the pair (A, B) is weakly spectrally complete, then* (1.1) *is satisfied.*

Proof Suppose that (A, B) is weakly spectrally complete, $i(A) + i(B) > n$ and $\alpha_{n-i(A)+1}(x) = x - \lambda$, with $\lambda \in F\backslash\{0\}$. If A and B are nonsingular then for every sequence $c_1, \dots, c_n \in F$ such that $\det(AB) = c_1 \dots c_n$, there exist matrices $A', B' \in F^{n\times n}$ similar to A, B, respectively, such that $A'B'$ has eigenvalues $c_1, \dots, c_n$ and then the pair (A, B) is spectrally complete. By Theorem 1 of [2], we have $i(A) + i(B) \leq n$, which is impossible. Then one of the matrices A, B is singular and there exists a matrix $B' \in F^{n\times n}$ similar to B such that AB' has all its eigenvalues equal to 0. Let $\gamma_1(x) | \dots | \gamma_n(x)$ be the invariant polynomials of AB'. Then

$$\gamma_1(x) \dots \gamma_n(x) = x^n. \tag{3}$$

We have $AB' = \lambda B' + (A - \lambda I_n)B'$. If $\beta_1(x)|\dots|\beta_n(x)$ are the invariant polynomials of B then $\beta_1(\lambda^{-1}x)|\dots|\beta_n(\lambda^{-1}x)$ are the invariant polynomials of $\lambda B'$. As λ is a primary eigenvalue of A, we have $\operatorname{rank}((A - \lambda I_n)B') \leq n - i(A)$, and by [4, Theorem 2], we conclude that

$$\beta_j(\lambda^{-1}x)|\gamma_{j+n-i(A)}(x), \qquad j \in \{1, \dots, i(A)\}. \tag{4}$$

Using (3) and (4), the invariant polynomials $\beta_{n-i(B)+1}(x), \dots, \beta_{i(A)}(x)$ must be powers of x and $\operatorname{rank}(B) = n - i(B) < i(A) \leq \operatorname{rank}(A)$.

Let $c_1 = \dots = c_{n-i(B)} = 1$ and $c_{i(B)} = \dots = c_n = 0$. There exists a matrix $B'' \in F^{n\times n}$ similar to B such that AB'' has eigenvalues $c_1, \dots, c_n$. Let $\delta_1(x)|\dots|\delta_n(x)$ be the invariant polynomials of AB''. As in the previous argument, we have

$$\beta_j(\lambda^{-1}x)|\delta_{j+n-i(A)}(x), \quad j \in \{1, \dots, i(A)\}.$$

Note that

$$\delta_1(x)\dots\delta_n(x) = x^{i(B)}(x-1)^{n-i(B)} \tag{5}$$

and $\operatorname{rank}(AB'') \leq \operatorname{rank}(B'') = n - i(B)$, so $\delta_{n-i(B)+1}(0) = \dots = \delta_n(0) = 0$. Then

$$\delta_k(x) = x(x-1)^{l_k}, \quad k \in \{n - i(B) + 1, \dots, n\},$$

for some $l_k \in \mathbb{N}_0$. Therefore,

$$\beta_{n-i(B)+1}(x) = \dots = \beta_{i(A)}(x) = x$$

and

$$\beta_1(x)\dots\beta_{i(A)}(x) = x^{i(A)+i(B)-n}.$$

□

LEMMA 3 *If the pair* (A, B) *is weakly spectrally complete then* (1.3) *is satisfied.*

Proof Suppose that the pair (A, B) is weakly spectrally complete and $n \neq 2$, $\deg(\alpha_n) = \deg(\beta_n) = 2$. Then A and B are similar to matrices of the form

$$A' = \begin{bmatrix} \lambda I_{i(A)} & * \\ 0 & \nu I_{n-i(A)} \end{bmatrix} \text{ and } B' = \begin{bmatrix} \mu I_{i(B)} & * \\ 0 & \epsilon I_{n-i(B)} \end{bmatrix},$$

respectively, where λ, ν are the roots of α_n and μ, ϵ are the roots of β_n.

Suppose that $i(A) \leq i(B)$ as the complementary case is analogous. We shall say that a sequence $c_1, \dots, c_n$ of elements of F are *admissible* if there exist matrices A', B' similar to A, B, respectively, such that $A'B'$ has eigenvalues $c_1, \dots, c_n$.

Let $c_1, \dots, c_n \in F$ be any admissible sequence. Using the arguments presented in the proof of Theorem 1 of [2], we deduce that there exists a permutation π of $\{1, \dots, n\}$ such that

$$c_{\pi(2i-1)}c_{\pi(2i)} = \lambda\nu\mu\epsilon, \qquad 1 \leq i \leq n - i(B) \tag{6}$$

$$c_{\pi(j)} = \lambda\mu, \qquad 2(n - i(B)) < j \leq n + i(A) - i(B) \tag{7}$$

$$c_{\pi(j)} = \nu\mu, \qquad n + i(A) - i(B) < j \leq n. \tag{8}$$

If A and B are nonsingular, we can find a sequence $c_1, \dots, c_n \in F$ such that $\det(AB) = c_1 \dots c_n$ but the equalities (6)–(8) are not satisfied.

Suppose that at least one of the matrices A, B is singular. As the pair (A, B) is weakly spectrally complete, the sequence of n zeros is admissible and should satisfy the equalities (6)–(8). Then $\lambda = \nu = 0$ or $\mu = 0$. If $\lambda = \nu = 0$, then the sequence of n zeros is the only admissible sequence, which contradicts the assumption that the pair (A, B) is weakly spectrally complete, $A \neq 0$ and $B \neq 0$. Therefore, $\mu = 0$ and 0 is a primary eigenvalue of B. □

Using the definition of weakly spectrally complete pair, Lemma 11 of [5] can be stated as follows:

LEMMA 4 *If one of the matrices A, B is singular and the other is nonderogatory, then the pair (A, B) is weakly spectrally complete.*

LEMMA 5 [1, Lemma 4] *If* $\min\{\text{rank}(A), \text{rank}(B)\} \geq n - 1$*, one of the matrices A, B is nonderogatory and the other is nonscalar, then the pair (A, B) is spectrally complete.*

According to the two previous Lemmas, we have:

LEMMA 6 *If one of the matrices A, B is nonderogatory and the other is nonscalar, then the pair (A, B) is weakly spectrally complete.*

LEMMA 7 *If $i(A) + i(B) \leq n$ and, either $n = 2$ or at least one of the polynomials α_n, β_n has degree different from 2, then (A, B) is weakly spectrally complete.*

Proof This proof is by induction on n. If $\min\{\text{rank}(A), \text{rank}(B)\} \geq n-1$, then, according to [1, Theorem 1], the pair (A, B) is spectrally complete and then is weakly spectrally complete.

Suppose that $\min\{\text{rank}(A), \text{rank}(B)\} < n - 1$. Suppose, without loss of generality [2, Lemma 1], that $\text{rank}(A) \leq \text{rank}(B)$. If B is nonderogatory the result follows from Lemma 4. In particular, Lemma 4 covers the case $n \leq 3$.

Suppose that $n \geq 4$ and B is derogatory. Let $c_1, \ldots, c_n$ be elements of F such that $\det(AB) = c_1 \ldots c_n$ and $\#\{i \in \{1, \ldots, n\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\}$ in order to prove that there exist matrices $A', B' \in F^{n\times n}$ similar to A, B, respectively, such that $A'B'$ has eigenvalues $c_1, \ldots, c_n$. Suppose, without loss of generality, that $c_{n-1} = c_n = 0$. If there exists $i \in \{\{1, \ldots, n-2\} : c_i \neq 0\}$, suppose, without loss of generality, that $c_1 \neq 0$.

Case 1. Suppose that $c_1 \neq 0$. The matrix A is similar to the direct sum of the companion matrices of its nonconstant invariant polynomials $K = C(\alpha_n) \oplus \cdots \oplus C(\alpha_{n-i(A)+1})$.

- If $\deg(\alpha_n) \geq 3$, then, according to [1, Lemma 5], K is similar to a matrix of the form

$$K' = \begin{bmatrix} * & * & 1 \\ * & K_0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $K_0 \in F^{(n-2)\times(n-2)}$ is a direct sum of companion matrices, $\chi(K_0) = \chi(K) - 1$ and $\det(K_0) = 0$. Moreover, if $i(A) \leq n - 3$ (i.e. $\chi(K) \geq 3$), then K_0 has been chosen so that at least one of the companion matrices appearing in K_0 is of size $u \times u$, with $u \geq 3$ and then the minimum polynomial of K_0 has degree greater than 2;

- If $\deg(\alpha_n) = 2$, then $C(\alpha_n)$ is similar to a matrix of the form

$$\begin{bmatrix} * & 1 \\ 0 & 0 \end{bmatrix}$$

and K is similar to a matrix of the form

$$K' = \begin{bmatrix} * & 0 & 1 \\ 0 & K_0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $K_0 = C(\alpha_{n-i(A)+1}) \oplus \cdots \oplus C(\alpha_{n-1})$. Note that $\det K_0 = 0$ and $\chi(K_0) = \chi(K) - 1$.

Analogously, the matrix B is similar to the direct sum of the companion matrices of its nonconstant invariant polynomials $L = C(\beta_n) \oplus \cdots \oplus C(\beta_{n-i(B)+1})$.

- If $\deg(\beta_n) \geq 3$, then, according to a variant of [1, Lemma 5] or a variant of [2, Lemma 4], L is similar to a matrix of the form

$$L' = \begin{bmatrix} 0 & 0 & * \\ 0 & L_0 & * \\ c_1 & * & * \end{bmatrix},$$

where $L_0 \in F^{(n-2)\times(n-2)}$ is a direct sum of companion matrices, $\det(K_0) = \det(L_2 \oplus \cdots \oplus L_s)$, and $\chi(L_0) = \chi(L) - 1$. Moreover, if $i(B) \leq n-3$ (i.e. $\chi(L) \geq 3$), then L_0 has been chosen so that at least one of the companion matrices appearing in L_0 is of size $u \times u$, with $u \geq 3$ and then the minimum polynomial of L_0 has degree greater than 2;
- If $\deg(\beta_n) = 2$, then $C(\beta_n)$ is similar to a matrix of the form

$$\begin{bmatrix} 0 & * \\ c_1 & * \end{bmatrix}$$

and L is similar to a matrix of the form

$$L' = \begin{bmatrix} 0 & 0 & * \\ 0 & L_0 & 0 \\ c_1 & 0 & * \end{bmatrix},$$

where $L_0 = C(\beta_{n-i(B)-1}) \oplus \cdots \oplus C(\beta_{n-1})$. Note that $\chi(L_0) = \chi(L) - 1$.

We have $\det(K_0L_0) = 0 = c_2 \ldots c_{n-1}, \#\{i \in \{2, \ldots, n-1\} : c_i \neq 0\} \leq \min\{\text{rank}(A), \text{rank}(B)\} - 1 = \min\{\text{rank}(K_0), \text{rank}(L_0)\}$ and $i(K_0) + i(L_0) \leq (n - 2 - \chi(K_0)) + (n - 2 - \chi(L_0)) = 2n - \chi(K) - \chi(L) - 2 = i(A) + i(B) - 2 \leq n - 2$. Now, we shall prove that either $n = 4$ or at least one of the minimum polynomial of the matrices K_0, L_0 has degree greater than 2.

- If $\deg(\alpha_n) \geq 3$ and $i(A) \leq n-3$, then the minimum polynomial of the matrix K_0 has degree greater than 2;
- If $\deg(\beta_n) \geq 3$ and $i(B) \leq n-3$, then the minimum polynomial of the matrix L_0 has degree greater than 2;
- If $\deg(\alpha_n) = 2$ and $i(B) > n-3$, then $(n/2) + (n-2) \leq i(A) + i(B) \leq n$ and therefore $n = 4$;
- If $\deg(\beta_n) = 2$ and $i(A) > n-3$, then with similar arguments to the previous case, we conclude that $n = 4$.

By the induction assumption, there exist nonsingular matrices $X_0, Y_0 \in F^{(n-2)\times(n-2)}$ such that $X_0K_0X_0^{-1}Y_0L_0Y_0^{-1}$ has eigenvalues $c_2, \ldots, c_{n-1}$. Let $X = [1] \oplus X_0 \oplus [1]$ and $Y = [1] \oplus Y_0 \oplus [1]$. The matrix $X^{-1}K'XY^{-1}L'Y$ has eigenvalues $c_1, \ldots, c_n$.

Case 2. Suppose that $c_1 = 0$. Then $c_1 = \cdots = c_n = 0$. Let $p = \min\{j \in \{n - i(A)+1, \ldots, n-1\} : \alpha_j(0) = 0\}$. Let $\alpha'_{p-1}(x) = \alpha_p(x)/x$ and $\alpha'_j = \alpha_{j+1}$, for every $j \in \{1, \ldots, n-1\}$ and $j \neq p-1$.

The matrix A is similar to a matrix of the form

$$A' = \begin{bmatrix} A_0 & * \\ 0 & 0 \end{bmatrix},$$

where A_0 has invariant polynomials $\alpha'_1 | \ldots | \alpha'_{n-1}$ and $\det(A_0) = 0$.

Subcase 2.1 Suppose that $i(A)+i(B) < n$ or $\deg(\beta_{n-i(B)+1}) = 1$. Let μ be a primary eigenvalue of B. Let $\beta'_{n-i(B)+1}(x) = \beta_{n-i(B)+1}(x)/(x-\mu)$. The matrix B is similar to a matrix of the form

$$B' = \begin{bmatrix} B_0 & * \\ 0 & \mu \end{bmatrix},$$

where

$$\begin{aligned} B_0 &= C(\beta'_{n-i(B)+1}) \oplus C(\beta_{n-i(B)+2}) \oplus \cdots \oplus C(\beta_n), && \text{if } \deg(\beta_{n-i(B)+1}) \geq 2, \\ B_0 &= C(\beta_{n-i(B)+2}) \oplus \cdots \oplus C(\beta_n), && \text{if } \deg(\beta_{n-i(B)+1}) = 1. \end{aligned}$$

We have $i(A_0) + i(B_0) \leq n-1$ and at least one of the minimum polynomials of A_0, B_0 has degree greater than 2. According to the induction assumption, (A_0, B_0) is spectrally complete and it is easy to conclude that (A, B) is also weakly spectrally complete.

Subcase 2.2 Suppose that $i(A) + i(B) = n$ and $\deg(\beta_{n-i(B)+1}) \geq 2$. Let $d = \deg(\beta_{n-i(B)+1})$. Analogously to the subcase 2.2.2 of the proof of Theorem 1 of [1], we conclude that

$$\#\{j \in \{1, \ldots, n\} : \deg(\alpha_j) = 1\} \geq d-1.$$

Then $\alpha_{n-i(A)+1}(x) = \cdots = \alpha_{n-i(A)+d-1}(x) = x - \lambda$, where λ is a primary eigenvalue of A. If $\lambda = 0$, then $p = n - i(A) + 1$ and $i(A_0) = i(A) - 1$. Let B' be the matrix similar to B as in the previous subcase. We have $i(A_0) + i(B_0) = n-1$ and α_n, β_n are the minimum polynomials of A', B'. According to the induction assumption, there exist $X_0, Y_0 \in F^{(n-1)\times(n-1)}$ such that $X_0A_0X_0^{-1}Y_0B_0Y_0^{-1}$ has eigenvalues $c_1, \ldots, c_{n-1}$. The matrix $(X_0 \oplus [1])A'(X_0 \oplus [1])^{-1}(Y_0 \oplus [1])B'(Y_0 \oplus [1])^{-1}$ has eigenvalues $c_1, \ldots, c_n$.

Suppose that $\lambda \neq 0$. Let $\alpha''_{p-d}(x) = \alpha_p(x)/x$ and $\alpha''_j = \alpha'_{p+d}$, for every $j \in \{1, \ldots, n-d\}$ and $j \neq p-d$. The matrix A' is permutation similar to a matrix of the form

$$\begin{bmatrix} D & * \\ 0 & K_0 \end{bmatrix},$$

where $D = I_{d-1} \oplus [0]$ and $K_0 \in F^{(n-d)\times(n-d)}$ has invariant polynomials $\alpha''_1|\dots|\alpha''_{n-d}$. The matrix B is similar to

$$C(\beta_{n-i(B)+1}) \oplus L_0, \quad \text{where} \quad L_0 = C(\beta_{n-i(B)+2}) \oplus \cdots \oplus C(\beta_n).$$

We have $i(K_0) + i(L_0) = (i(A) - d + 1) + (i(B) - 1) = n - d$ and α_n, β_n are the minimum polynomials of A', B'. Then, we conclude that (K_0, L_0) is weakly spectrally complete. By Lemma 4 the pair $(D, C(\beta_{n-i(B)+1}))$ is also weakly spectrally complete. It is easy to complete the proof. □

Lemma 8 *If* (1.1) *and* (1.2) *are satisfied and at least one of the polynomials α_n, β_n has degree different from 2, then the pair (A, B) is weakly spectrally complete.*

Proof By induction on n. The proof has already been done when $i(A) + i(B) \le n$. Suppose that $i(A) + i(B) > n$. Suppose, without loss of generality [2, Lemma 1], that $i(A) \ge i(B)$. Then $\deg(\alpha_{n-i(A)+1}) = 1$. Let $p = \#\{j \in \{1,\dots,n\} : \deg(\alpha_j) = 1\}$ and $d = \deg(\beta_{n-i(B)+1})$. In order to obtain a contradiction, assume that $p < d$. Then

$$i(A) \le p + \frac{n-p}{2}, \qquad\qquad i(B) \le \frac{n}{d} \le \frac{n}{p+1}.$$

From

$$n + 1 \le i(A) + i(B) \le p + \frac{n-p}{2} + \frac{n}{p+1},$$

it follows that $0 \le h(p)$, where $h(p) = p^2 - (n+1)p + n - 2$, which is impossible because $h(1)$ and $h(n)$ are negative numbers. Therefore $p \ge d$. Let λ be the primary eigenvalue of A. The matrices A, B are, respectively, similar to the matrices

$$\begin{aligned} A' &= \lambda I_d \oplus K_0, && \text{where} \quad K_0 = C(\alpha_{n-i(A)+d+1}) \oplus \cdots \oplus C(\alpha_n),\\ B' &= C(\beta_{n-i(B)+1}) \oplus L_0, && \text{where} \quad L_0 = C(\beta_{n-i(B)+2}) \oplus \cdots \oplus C(\beta_n). \end{aligned}$$

Let $\alpha'_1|\dots|\alpha'_{n-d}$ and $\beta'_1|\dots|\beta'_{n-d}$ be the invariant polynomials of the matrices K_0 and L_0, respectively. Note that $i(K_0) = i(A) - d$ and $i(L_0) = i(B) - 1$.

Case 1. Suppose that $\lambda = 0$. Then $\operatorname{rank}(A) = n - i(A)$. If $p = n$, then $A = 0$ and the result is trivial.

Suppose that $p < n$. If $d = 1$ and $C(\beta_{n-i(B)+1})$ is singular, then $\operatorname{rank}(L_0) = \operatorname{rank}(B) = n - i(B) \ge n - i(A) = \operatorname{rank}(A) = \operatorname{rank}(K_0)$. If $d > 1$ or $C(\beta_{n-i(B)+1})$ is nonsingular, then $\operatorname{rank}(L_0) \ge i(L_0) = i(B) - 1 \ge n - i(A) = \operatorname{rank}(A) = \operatorname{rank}(K_0)$ and $\operatorname{rank}(B) \ge i(B) > n - i(A) = \operatorname{rank}(A)$.

Let $c_1,\dots,c_n \in F$ be such that $\#\{i \in \{1,\dots,n\} : c_i \ne 0\} \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\} = \operatorname{rank}(A) = n - i(A)$. Suppose without loss of generality, that $c_1 = \cdots = c_{i(A)} = 0$.

If $\beta'_{(n-d)-i(L_0)+1}(x) = x - \mu$, with $\mu \in F\backslash\{0\}$, then, as $\beta'_{(n-d)-i(L_0)+1}(x) = \beta_{n-i(B)+2}(x)$, we have $\beta_{n-i(B)+1}(x) = \beta_{n-i(B)+2}(x) = x - \mu$. By (1.2), we have

$$\alpha_1(x)\dots\alpha_{i(B)}(x) = x^{i(A)+i(B)-n}$$

and then

$$\alpha'_1(x)\dots\alpha'_{i(L_0)}(x) = \frac{\alpha_1(x)\dots\alpha_{i(B)}(x)}{x} = x^{i(A)+i(B)-n-1} = x^{i(K_0)+i(L_0)-(n-1)}.$$

Note that $\mathrm{rank}(L_0) \geq \mathrm{rank}(K_0) = n - i(A)$ and at least one of the polynomials $\alpha'_{n-d} = \alpha_n$ and $\beta'_{n-d} = \beta_n$ has degree different from 2. According to the induction assumption, there exist $X_0, Y_0 \in F^{(n-d)\times(n-d)}$ such that $X_0^{-1}K_0X_0Y_0^{-1}L_0Y_0$ has eigenvalues $c_{d+1}, \ldots, c_n$. Consider the matrices $X = I_d \oplus X_0$ and $Y = I_d \oplus Y_0$. The matrix $X^{-1}A'XY^{-1}B'Y$ has eigenvalues $c_1, \ldots, c_n$.

Case 2. Suppose that $\lambda \neq 0$. By (1.1), we have

$$\beta_1(x) \ldots \beta_{i(A)}(x) = x^{i(A)+i(B)-1}$$

which implies that

$$\beta_{n-i(B)+1}(x) = \cdots = \beta_{i(A)}(x) = x.$$

Note that $d = 1$ and $\mathrm{rank}(B) = n - i(B) < i(A) \leq \mathrm{rank}(A)$. Let $c_1, \ldots, c_n \in F$ be such that $\#\{i \in \{1, \ldots, n\} : c_i \neq 0\} \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\} = \mathrm{rank}(B) = n - i(B)$. Suppose without loss of generality, that $c_1 = \cdots = c_{i(B)} = 0$.

If $\deg(\alpha_{n-i(A)+2}) = 1$, then

$$\beta'_1(x) \ldots \beta'_{i(K_0)}(x) = \frac{\beta_1(x) \ldots \beta_{i(A)}(x)}{x} = x^{i(A)+i(B)-n-1} = x^{i(K_0)+i(L_0)-(n-1)}.$$

Note that $\mathrm{rank}(L_0) = \mathrm{rank}(B) < \mathrm{rank}(A) = \mathrm{rank}(K_0)+1$ and least one of the polynomials $\alpha'_{n-1} = \alpha_n$ and $\beta'_{n-1} = \beta_n$ has degree different from 2. According to the induction assumption, there exist $X_0, Y_0 \in F^{(n-1)\times(n-1)}$ such that $X_0^{-1}K_0X_0Y_0^{-1}L_0Y_0$ has eigenvalues $c_2, \ldots, c_n$. Consider the matrices $X = [1] \oplus X_0$ and $Y = [1] \oplus Y_0$. The matrix $X^{-1}A'XY^{-1}B'Y$ has eigenvalues $c_1, \ldots, c_n$. □

LEMMA 9 *If $n = 2 = \deg(\alpha_2) = \deg(\beta_2)$, then the pair (A, B) is weakly spectrally complete.*

Proof Follows from Lemma 6. □

LEMMA 10 *If $\deg(\alpha_n) = \deg(\beta_n) = 2$, $i(A) \leq i(B)$ and 0 is a primary eigenvalue of B, then the pair (A, B) is weakly spectrally complete.*

Proof Let λ, ν be the roots of α_n and λ a primary eigenvalue of A. Let $0, \epsilon$ be the roots of β_n. The matrix A is similar to

$$A' = \lambda I_{2i(A)-n} \oplus \bigoplus_{i=1}^{n-i(A)} C, \quad \text{where} \quad C = \begin{bmatrix} \lambda & 1 \\ 0 & \nu \end{bmatrix},$$

and B is similar to

$$B' = 0_{2i(B)-n} \oplus \bigoplus_{i=1}^{n-i(B)} D, \quad \text{where} \quad D = \begin{bmatrix} 0 & 1 \\ 0 & \epsilon \end{bmatrix}.$$

Note that $\mathrm{rank}(B) = n - i(B) \leq n - i(A) \leq \mathrm{rank}(A)$. Let $c_1, \ldots, c_n \in F$ be such that $\#\{i \in \{1, \ldots, n\} : c_i \neq 0\} \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$. Suppose, without loss of generality, that $c_{n-i(B)+1} = \cdots = c_n = 0$. According to the previous lemma, for every $j \in \{1, \ldots, n - i(B)\}$, there exists $D_j \in F^{2\times 2}$ similar to D such that CD_j has eigenvalues

c_j, 0. Then, B' is similar to $B'' = 0_{2i(B)-n} \oplus D_1 \oplus \cdots \oplus D_{n-i(B)}$ and $A'B''$ has eigenvalues $c_1, \ldots, c_n$. □

Proof of Theorem 1 The necessity follows from Lemmas 2 and 3. The sufficiency follows from Lemmas 8–10. □

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