



Partial classification of Lorenz knots: Syllable permutations of torus knots words



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HIGHLIGHTS

- We define syllable permutations of symbolic words of Lorenz torus knots.
- We build an algorithm to construct symbolic words of satellite Lorenz knots.
- Some of the syllable permutation families of words are shown to be hyperbolic.
- Infinite families of hyperbolic Lorenz knots are generated in this way.
- The techniques used can be generalized to other families of Lorenz knots.

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ABSTRACT

We define families of aperiodic words associated to Lorenz knots that arise naturally as syllable permutations of symbolic words corresponding to torus knots. An algorithm to construct symbolic words of satellite Lorenz knots is defined. We prove, subject to the validity of a previous conjecture, that Lorenz knots coded by some of these families of words are hyperbolic, by showing that they are neither satellites nor torus knots and making use of Thurston's theorem. Infinite families of hyperbolic Lorenz knots are generated in this way, to our knowledge, for the first time. The techniques used can be generalized to study other families of Lorenz knots.

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1. Introduction

Lorenz knots

Lorenz knots are the closed (periodic) orbits in the Lorenz system [1]

$$\begin{aligned} x' &= -10x + 10y \\ y' &= 28x - y - xz \\ z' &= -\frac{8}{3}z + xy \end{aligned} \quad (1)$$

while Lorenz links are finite collections of (possibly linked) Lorenz knots.

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The systematic study of Lorenz knots and links was made possible by the introduction of the *Lorenz template* or knot-holder by Williams in [2,3]. It is a branched 2-manifold equipped with an expanding semi-flow, represented in Fig. 1. It was first conjectured by Guckenheimer and Williams and later proved through the work of Tucker [4] that every knot and link in the Lorenz system can be projected into the Lorenz template. Birman and Williams made use of this result to investigate Lorenz knots and links [5]. For a review on Lorenz knots and links, see also [6].

A $T(p, q)$ torus knot is (isotopic to) a curve on the surface of an unknotted torus T^2 that intersects a meridian p times and a longitude q times. Birman and Williams [5] proved that every torus knot is a Lorenz knot.

A satellite knot is defined as follows: take a nontrivial knot C (companion) and nontrivial knot P (pattern) contained in a solid unknotted torus T and not contained in a 3-ball in T . A satellite knot is the image of P under a homeomorphism that takes the core of T onto C .

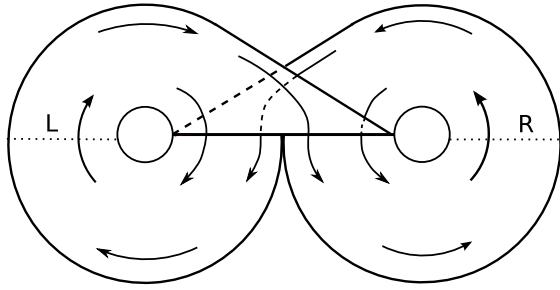


Fig. 1. The Lorenz template.

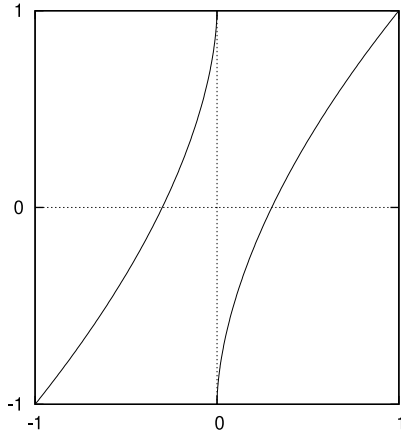


Fig. 2. Lorenz map.

A knot is hyperbolic if its complement in S^3 is a hyperbolic 3-manifold. Thurston [7] proved that a knot is hyperbolic iff it is neither a satellite knot nor a torus knot. One of the goals in the study of Lorenz knots has been their classification into *hyperbolic* and *non-hyperbolic*, possibly further distinguishing torus knots from satellites. Birman and Kofman [8] listed hyperbolic Lorenz knots taken from a list of the simplest hyperbolic knots. In a previous article we generated and tested for hyperbolicity, using the program *SnapPy*, families of Lorenz knots that are a generalization of some of those that appear in this list, which led us to conjecture that the families tested are hyperbolic [9].

The first-return map induced by the semi-flow on the *branch line* (the horizontal line in Fig. 1) is called the *Lorenz map*. If the branch line is mapped onto $[-1, 1]$, then the Lorenz map f becomes a one-dimensional map from $[-1, 1] \setminus \{0\}$ onto $[-1, 1]$, with one discontinuity at 0 and strictly increasing in each of the subintervals $[-1, 0[$ and $]0, 1]$ (Fig. 2).

Lorenz braids

If the Lorenz template is cut open along the dotted lines in Fig. 1, then each knot and link on the template can be obtained as the closure of an open braid on the cut-open template, which will be called the *Lorenz braid* associated to the knot or link [5]. These *Lorenz braids* are simple positive braids (our definition of positive crossing follows Birman and is therefore opposed to an usual convention in knot theory). Each Lorenz braid is composed of $n = p + q$ strings, where the set of p left or L strings cross over at least one (possibly all) of the q right strings, with no crossings between strings in each subset. These sets can be subdivided into subsets LL, LR, RL and RR according to the position of the startpoints and endpoints of each string. An example of a Lorenz braid is shown in Fig. 3, where we adopt the convention of drawing the overcrossing (L) strings as thicker lines than the undercrossing (R) strings. This convention will be used in other braid diagrams.

The *braid group on n strings* B_n is given by the presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i - j| \geq 2) \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (i = 1, \dots, n - 2) \end{array} \right\rangle$$

where the *generator* σ_i exchanges the endpoints of strings i and $i + 1$ with string i crossing over string $i + 1$. In particular, all Lorenz braids can be expressed as products of these generators.

Each Lorenz braid β is a simple braid, so it has an associated permutation π . This permutation has only one cycle iff it is associated to a knot, and has k cycles if it is associated to a link with k components (knots).

Symbolic dynamics for the Lorenz map

Let $f^j = f \circ f^{j-1}$ be the j th iterate of the Lorenz map f and f^0 be the identity map. We define the *itinerary* of a point x under f as the symbolic sequence $(i_f(x))_j, j = 0, 1, \dots$ where

$$(i_f(x))_j = \begin{cases} L & \text{if } f^j(x) < 0 \\ 0 & \text{if } f^j(x) = 0 \\ R & \text{if } f^j(x) > 0. \end{cases}$$

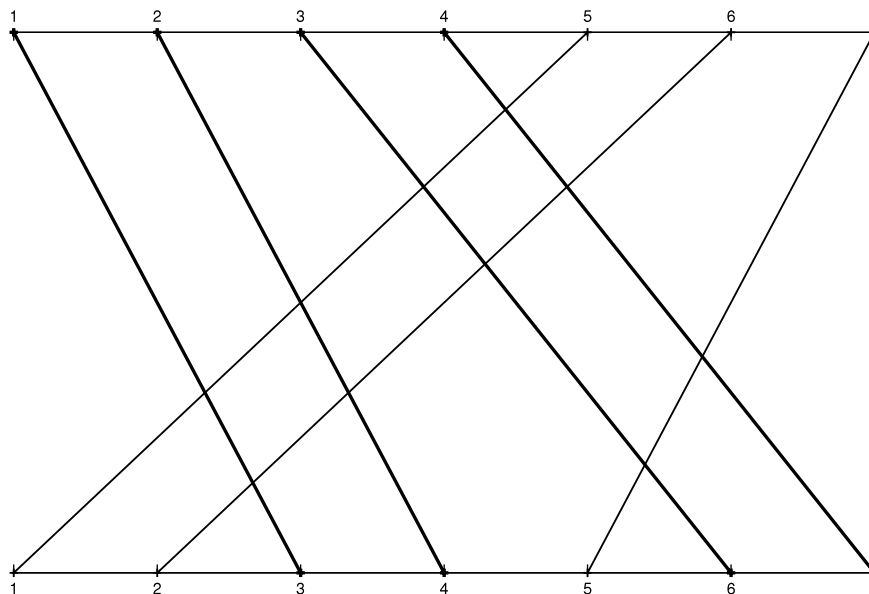


Fig. 3. A Lorenz braid.

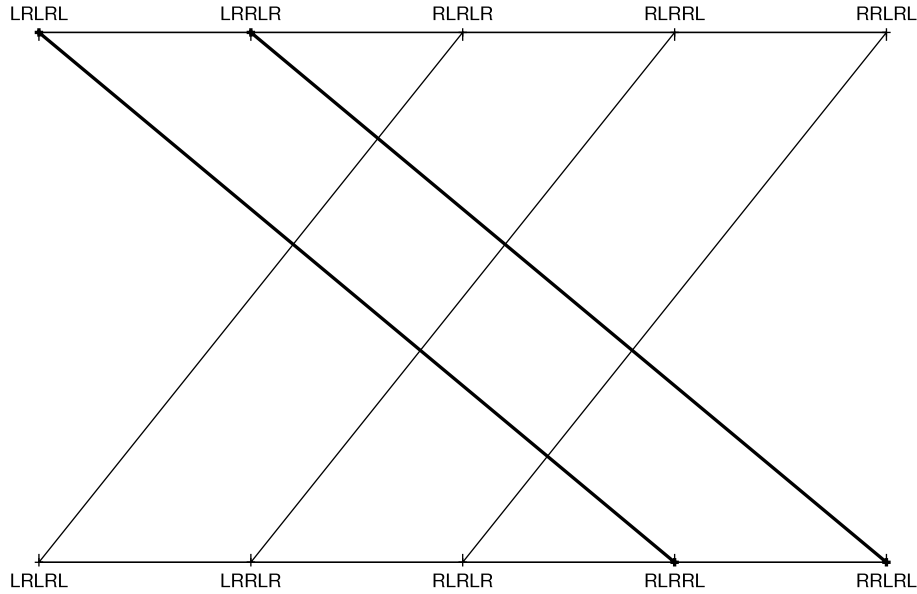


Fig. 4. Lorenz braid corresponding to $w = LRRLR$.

The itinerary of a point in $[-1, 1] \setminus \{0\}$ under the Lorenz map can either be an infinite sequence in the symbols L, R or a finite sequence in L, R terminated by a single symbol 0 (because f is undefined at $x = 0$). The length $|X|$ of a finite sequence $X = X_0 \dots X_{n-1}0$ is n , so it can be written as $X = X_0 \dots X_{|X|-1}0$. A sequence X is periodic if $X = (X_0 \dots X_{p-1})^\infty$ for some $p > 1$. If p is the least integer for which this holds, then p is the (least) period of X .

The space Σ of all finite and infinite sequences can be ordered in the lexicographic order induced by $L < 0 < R$: given $X, Y \in \Sigma$, let k be the first index such that $X_k \neq Y_k$. Then $X < Y$ if $X_k < Y_k$ and $Y < X$ otherwise.

The shift map $s : \Sigma \setminus \{0\} \rightarrow \Sigma$ is defined as usual by $s(X_0X_1 \dots) = X_1 \dots$ (it just deletes the first symbol). From the definition above, an infinite sequence X is periodic iff there is $p > 1$ such that $s^p(X) = X$. In order to simplify the notation, we will also define the shift operator on finite aperiodic words. Given a p -periodic sequence $X = (X_0X_1 \dots X_{p-1})^\infty$ where $w = X_0X_1 \dots X_{p-1}$ is a finite aperiodic word of length p , we define $s(X_0X_1 \dots X_{p-1}) = X_1 \dots X_{p-1}X_0$. Then $s(X) = (X_1 \dots X_{p-1}X_0)^\infty = (s(X_0X_1 \dots X_{p-1}))^\infty$. This shift operator defined on finite words has an inverse s^{-1} , defined by $s(X_0X_1 \dots X_{p-1}) = X_{p-1}X_0X_1 \dots X_{p-2}$. The sequence $w, s(w), \dots, s^{p-1}(w)$ will also be called the orbit of w and a word in the orbit of w will be generally called a shift of w .

A (finite or infinite) sequence X is called L -maximal if $X_0 = L$ and for $k > 0, X_k = L \Rightarrow s^k(X) \leq X$, and R -minimal if $X_0 = R$ and for $k > 0, X_k = R \Rightarrow X \geq s^k(X)$. An infinite periodic sequence $(X_0 \dots X_{n-1})^\infty$ with least period n is L -maximal (resp. R -minimal) if and only if the finite sequence $X_0 \dots X_{n-1}0$ is L -maximal (resp. R -minimal). Given an aperiodic word $w = X_0 \dots X_n$, we say that k is the maximal (resp. minimal) position in w if $s^k(w)$ is L -maximal (resp. R -minimal).

For a finite word w, n_L and n_R will denote respectively the number of L and R symbols in w , and $n = n_L + n_R$ the length of w . Each finite aperiodic word is associated to a Lorenz braid (whose closure is a Lorenz knot), which can be obtained through the following procedure: given a word w of length n , order the successive shifts $s(w), s^2(w), \dots, s^n(w) = w$ lexicographically and associate them to startpoints and endpoints in the associated Lorenz braid, with points corresponding to words starting with L lying on the left half and points corresponding to words starting with R on the right half. Each string in the braid connects the

startpoint corresponding to $s^k(w)$ to the endpoint corresponding to $s^{k+1}(w)$. Fig. 4 exemplifies this procedure for $w = LRRLR$.

Each periodic orbit of the flow has a unique corresponding orbit in the Lorenz map, which in turn corresponds to the cyclic permutation class of one aperiodic word in the symbols L, R as clearly all periodic sequences resulting from shifting a given sequence represent points in the same periodic orbit (see [5]).

The crossing number is the smallest number of crossings in any diagram of a knot K . The braid index is the smallest number of strings among braids whose closure is K .

The trip number t is the number of syllables (subwords of type L^aR^b with maximal length) in an aperiodic word. The trip number of a Lorenz link is the sum of the trip numbers of its components. Franks and Williams [10], followed by Waddington [11], proved that the braid index of a Lorenz knot is equal to its trip number. This result had previously been conjectured by Birman and Williams [5], who defined a minimal t -braid for Lorenz links [8]:

The minimal braid of a Lorenz link corresponding to a Lorenz braid β is given by

$$\Delta^2 \prod_{i=1}^{t-1} (\sigma_1 \dots \sigma_i)^{n_i} \prod_{i=t-1}^1 (\sigma_{t-1} \dots \sigma_i)^{m_{t-i}}$$

where Δ^2 is the full-twist in t strings, and, denoting by π the permutation associated with the Lorenz braid, the exponents are

$$n_i = \#\{j : \pi(j) - j = i + 1 \text{ and } \pi(j) < \pi^2(j)\}$$

$$m_i = \#\{j : j - \pi(j) = i + 1 \text{ and } \pi(j) > \pi^2(j)\}$$

Goal and plan

The purpose of this article is the classification of knots corresponding to syllable permutations of the standard word of a torus knot, that is, words composed of the same number of LR^k and LR^{k+1} syllables as the torus knot standard word, defined below, arranged in a different order.

In Section 2 we characterize the symbolic words obtained by permuting the syllables of words corresponding to torus knots and their associated Lorenz braids. Section 3 deals with braids of Lorenz satellite knots and an algorithm to find their corresponding words, also proving that the permuted words cannot be obtained this way. Finally, in Section 4 we show that in each set of syllable permutations of a torus knot word, there is at most one word that can possibly correspond to another torus knot, and prove that for some classes of these sets there is no such word.

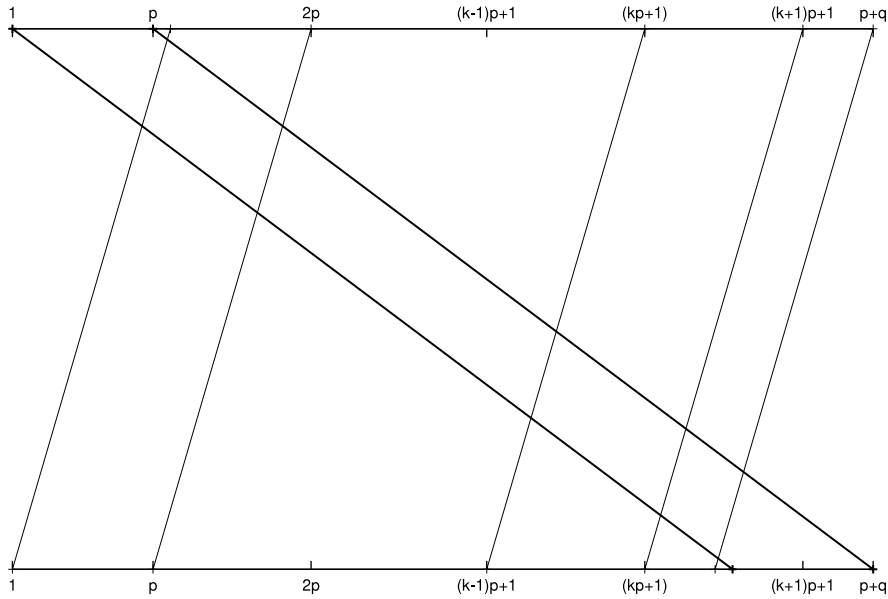


Fig. 5. Lorenz braid of $T(p, q)$ ($p < q$).

2. Torus knots and syllable permutations

As mentioned above, all torus knots are Lorenz knots. The torus knot $T(p, q)$ is the closure of a Lorenz braid in $n = p + q$ strings, with p left or L strings that cross over q right or R strings, such that each L string crosses over all the R strings. The Lorenz braid of a torus knot thus has the maximum number of crossings (pq) for a Lorenz braid with pL strings and qR strings. Since $T(q, p) = T(p, q)$ we will only consider torus knots $T(p, q)$ with $p < q$. The structure of the Lorenz braid of a torus knot $T(p, q)$ ($p < q$) is sketched in Fig. 5, where only the first and last L strings and some of the R strings are drawn. The remaining L (R) strings are parallel to the L (R) strings shown.

Lorenz knots corresponding to orbits in the Lorenz template which are represented by evenly distributed words in the alphabet $\{L, R\}$ are torus knots [5]. Also, given a torus knot $T(p, q)$ there is an evenly distributed word with $n_L = p$, $n_R = q$, that represents it. There is thus a bijection between torus knots and cyclic permutation classes of evenly distributed words. We will call the L -maximal word that represents $T(p, q)$ the *standard word* $w(p, q)$ for $T(p, q)$. These are also the words generated in the symbolic Farey tree [12]. Let $k > 0$ be the quotient of the integer division of q by p and $0 < r < p$ the remainder ($q = kp + r$). Then $w(p, q)$ has p syllables: rLR^{k+1} syllables and $p - rLR^k$ syllables, evenly distributed. This aperiodic word is unique for each p, q satisfying the conditions above.

Lemma 1. Any word resulting from permuting the syllables of the standard word $w(p, q)$ is aperiodic and therefore corresponds to a Lorenz knot with the same braid index.

Proof. Let $T(p, q)$, $p < q$ be a torus knot and w a syllable permutation of $w(p, q)$. Then $n_L(w) = p$ and $n_R(w) = q$. If there was a subword v of w such that $w = v^j$ for some $j > 1$, then we would have $p = n_L(w) = jn_L(v)$ and $q = n_R(w) = jn_R(v)$ and p, q would not be relatively prime. \square

Definition 1. We define $P(p, q)$ as the set of L -maximal words resulting from permutations of syllables of the standard word $w(p, q)$ for $T(p, q)$.

Remark 1. Any word resulting from the permutation of syllables of the standard word for $T(p, q)$ is a shift of some word in $P(p, q)$ and is therefore equivalent to that word, i.e. it corresponds to the same Lorenz knot.

Remark 2. The set of distinct permutations of p syllables, where respectively r and $p - r$ are identical, contains $\frac{p!}{r!(p-r)!}$ words. This set contains, for each L -maximal word w , p words that are shifts of w (including itself). To keep only the L -maximal words we must therefore further divide by p . Each set $P(p, q)$, $q = kp + r$ has therefore exactly $\frac{(p-1)!}{r!(p-r)!}$ words.

Remark 3. If $2 \leq p \leq 4$, then $P(p, q)$ contains only the standard word for $T(p, q)$: $P(p, q) = \{w(p, q)\}$.

Remark 4. If $q = kp + r$ and $r = 1$ or $r = p - 1$, then the only possible L -maximal word in $P(p, q)$ is $LR^{k+1}(LR^k)^{p-1}$ ($r = 1$) or $(LR^{k+1})^{p-1}LR^k$ ($r = p - 1$), so in both cases $P(p, q)$ contains only the standard word for $T(p, q)$.

In what follows, we therefore assume, whenever necessary, $p > 4$ and $1 < r < p - 1$.

Next we find bounds for the number of crossings in the Lorenz braid and in the minimal Birman–Williams (BW) braid. For a knot which is the closure of a positive braid, the genus g is related to the number of crossings c and the number of strings n by [5]

$$2g = c - n + 1. \tag{2}$$

In the symbolic words under study there are no consecutive L symbols, so for each knot in $P(p, q)$ the trip number is $t = n_L = p$. The minimal BW braid [5] is a p -string braid and the number of crossings (the length of the braid word) in the BW braid is the crossing number of the knot, because in Eq. (2), if c takes the minimum value, then n must also be minimum, as the genus g is a knot invariant.

We start by investigating the structure of the braids that correspond to the words with permuted syllables. Let w be an aperiodic word resulting from permuting the syllables of the standard word $w(p, q)$. Note that the shifts of w can be grouped and ordered lexicographically:

$$LR^kL \dots < LR^{k+1} \dots < RL \dots < R^2L \dots \\ < \dots < R^kL \dots < R^{k+1}L \dots$$

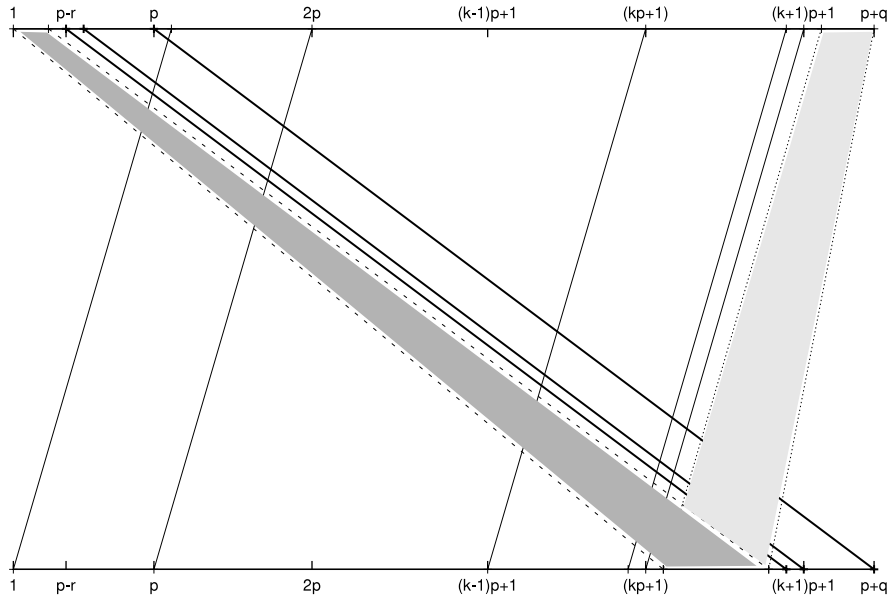


Fig. 6. Lorenz braid corresponding to a syllable permuted word.

Also, there are:

- exactly p words (and corresponding strings) with the form $LR\dots$, of which r are $LR^{k+1}\dots$ and $p-r$ are $LR^kL\dots$;
- p words with each of the forms $RL\dots, R^2L\dots, \dots, R^kL\dots$;
- r words with the form $R^{k+1}L$.

Moreover:

1. All the words of type $LR\dots$ are shifts of words of type $RL\dots$, and the shift operator preserves lexicographical order.

For $1 \leq i \leq k-1$, the words of type $R^iL\dots$ are the shifts of $R^{i+1}L\dots$ words, with order preserved.

2. The p words of type $R^kL\dots$ are shifts of the r $R^{k+1}L\dots$ and the $p-r$ $LR^kL\dots$ words, the shift operator preserving the order within each subset.

The least word with the form $R^kL\dots, w_l$, is the shift of the least $R^{k+1}L\dots$ word. To see this, assume that w_l is the shift of a word of type $LR^kL\dots$. This word has a preimage under the iterated shift operator with the form $R^kLR^kL\dots$, resulting from appending an R^kL syllable to the beginning of w_l , and therefore $R^kLR^kL\dots < w_l = R^kL\dots$, which contradicts the fact that w_l is the least $R^kL\dots$ word.

Analogously, the greatest $R^kL\dots$ word, w_g , is the shift of the greatest $LR^kL\dots$ word. Assume that $s^{-1}(w_m)$ is of type $R^{k+1}L\dots$; w_m has the form $R^kLR^{k+1}L\dots$ (otherwise it would not be the greatest $R^kL\dots$), so $s^{-1}(w_m)$ must have the form $R^{k+1}LR^{k+1}L\dots$. This word has a preimage under the shift with the form $R^kLR^{k+1}LR^{k+1}\dots$ and is greater than w_g (it has one more syllable R^{k+1} immediately following the first R^kL), which contradicts the fact that w_g is the greatest $R^kL\dots$ word.

3. The r words of type $R^{r+1}L\dots$ are the shifts of the r $LR^{k+1}\dots$ words, with order preserved.

The Lorenz braids corresponding to the permuted words thus have the structure sketched in Fig. 6. The dotted lines and enclosed shaded regions represent two sets of strings and which both have their endpoints in the range $\{kp+2, \dots, (k+1)p-1\}$, so that the strings from the first set with startpoints $1, \dots, p-r-1$ will possibly cross over the strings with startpoints $(k+1)p+2, (k+1)p+r = p+q$, preserving the order in each set of strings.

The Lorenz braids corresponding to the permuted words have a number of crossing points c satisfying

$$(p-r-1)(kp+1) + (r+1)(kp+r) \leq c \leq p(kp+r) = pq. \quad (3)$$

For the standard word $w(p, q)$, since all the pL strings cross over the $q = kp+rR$ strings, we have $c = p(kp+r)$.

Lemma 2. For the word $(LR^{k+1})^r(LR^k)^{p-r}$, the corresponding braid has the structure illustrated in Fig. 7, with exactly $(p-r-1)(kp+1) + (r+1)(kp+r)$ crossings. Since the L strings have the leftmost possible endpoints and the R strings the maximum possible endpoints, this braid has the minimum number of crossings in $P(p, q)$ and $(LR^{k+1})^r(LR^k)^{p-r}$ is the only word in $P(p, q)$ corresponding to this number of crossings.

Proof. The only endpoints that can change from one permuted braid corresponding to a word w in $P(p, q)$ are those corresponding to shifts of w with the form $R^kL\dots$. These can be ordered lexicographically as follows:

$$\begin{aligned} s(R^{k+1}(LR^k)^{p-r}(LR^{k+1})^{r-1}L) &= R^k(LR^k)^{p-r}(LR^{k+1})^{r-1}LR \\ &< s((LR^k)^{p-r}(LR^{k+1})^r) &= R^k(LR^k)^{p-r-1}(LR^{k+1})^rL \\ &< s((LR^k)^{p-r-1}(LR^{k+1})^r) \\ &= R^k(LR^k)^{p-r-2}(LR^{k+1})^rLR^kL < \dots \\ &\dots < s(LR^k(LR^{k+1})^r(LR^k)^{p-r-1}) \\ &= R^k(LR^k)(LR^{k+1})^r(LR^k)^{p-r-2}L \\ &< s(R^{k+1}(LR^{k+1})(LR^k)^{p-r}(LR^{k+1})^{r-2}L) \\ &= R^k(LR^{k+1})(LR^k)^{p-r}(LR^{k+1})^{r-2}LR < \dots \\ &\dots < s(R^{k+1}(LR^{k+1})^{r-1}(LR^k)^{p-r}L) \\ &= R^k(LR^{k+1})^{r-1}(LR^k)^{p-r}LR \\ &< s(LR^k(LR^{k+1})^r(LR^k)^{p-r-1}) \\ &= R^k(LR^{k+1})^r(LR^k)^{p-r-1}L. \end{aligned}$$

Identifying the words between brackets on the left with the startpoints of the corresponding strings and the ordered words on the right with the matching endpoints, we get the braid represented in Fig. 7. \square

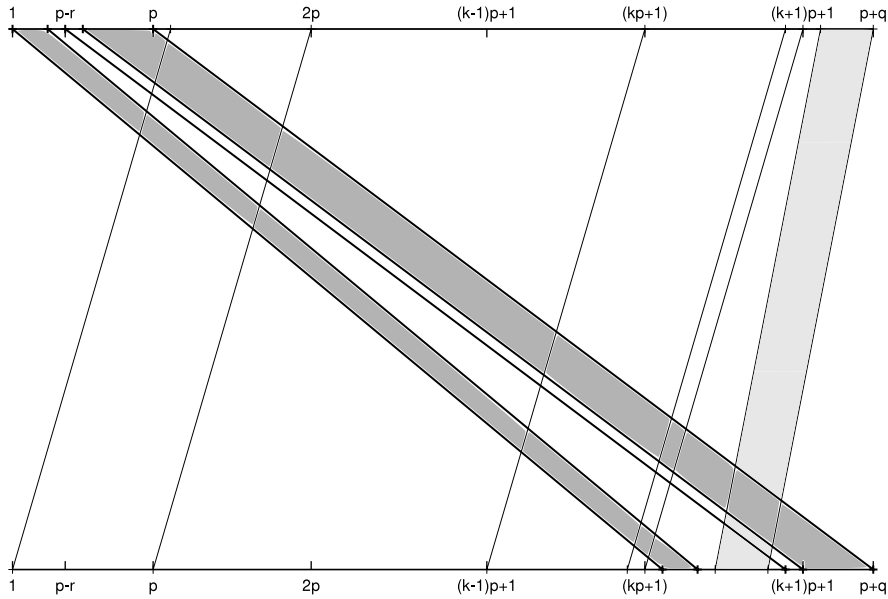


Fig. 7. Lorenz braid of $(LR^{k+1})^r(LR^k)^{p-r}$.

The Lorenz braid of $(LR^{k+1})^r(LR^k)^{p-r}$ thus has exactly $(p - r - 1)(kp + 1) + (r + 1)(kp + r)$ crossing points. This is the only braid in $P(p, q)$ with this number of crossings.

Using Eq. (2) we can now find bounds for the genus g of knots that are the closure of braids in $P(p, q)$.

$$kp(p - 1) + r(r - 1) \leq 2g \leq (p - 1)(kp + r - 1) = (p - 1)(q - 1). \quad (4)$$

Again, the minimum corresponds to $(LR^{k+1})^r(LR^k)^{p-r}$ while the maximum is the double of the torus knot genus and therefore corresponds to the standard word $w(p, q)$.

3. Lorenz satellite knots

El-Rifai studied Lorenz knots which are satellites of Lorenz knots [13,14]. His construction of a satellite knot or link can be interpreted in terms of Lorenz braids, as follows:

- Take three Lorenz braids A (left pattern), B (right pattern) and C (companion) whose closures are Lorenz knots, such that $n_R(A) = n_L(B) = k$.
- Inflate the braid C by replacing each string with k parallel strings, thus obtaining an intermediate braid corresponding to a link with k components.
- Add $n_L(A)$ vertical strings to the left and $n_R(B)$ vertical strings to the right (corresponding to identity permutations of these strings). The number of strings of the resulting braid is $n = n_L(A) + n_R(B) + kn(C)$, where $n(C)$ is the number of strings in C .
- Concatenate this braid with the n -braid obtained by putting A on the left, B on the right and extend the remaining strings in C with vertical strings, between A and B .

The procedure is illustrated in Fig. 8 for braids A, B, C corresponding to the words $w_A = LRRLR$, $w_B = LRRLRLR$ and $w_C = LRRLR$, respectively.

The construction can also be carried out with only one pattern braid A or B . In that case, take the identity k -braid for B (A) and assume $n_R(B) = 0$ ($n_L(A) = 0$) in the third step.

Remark 5. The closure of the resulting braid can be a knot or a link. The braids A and B induce a permutation of the inflated strings of C which is the product of permutations associated to A and B . If this

permutation has more than one cycle, then the resulting braid is a link. An example is given below, with $w_A = LRRLR$, $w_B = LRRLR$ and $w_C = LRRLR$ (Fig. 9).

In order to obtain the aperiodic word of the satellite knot that is the closure of the Lorenz braid constructed as above, we start by defining permutations on the L and R points, respectively, of a Lorenz braid whose closure is a Lorenz knot, or equivalently on words starting respectively with L or R in the symbolic orbit associated to the braid.

Definition 2. Let b be a Lorenz n -braid that closes to a knot, $\pi(b)$ the permutation associated to b and w the corresponding L -maximal aperiodic symbolic word. We label the n_L words which start with L in the orbit of w as l_1, \dots, l_{n_L} and the n_R words starting with R as r_1, \dots, r_{n_R} , such that, under the lexicographic order, $l_1 < \dots < l_{n_L}$ and $r_1 < \dots < r_{n_R}$.

We define the L -permutation associated to b , π_L for $i = 1, \dots, n_L$ as $\pi_L(i) = j$ if l_j is the first shifted word that starts with an L in the orbit $s(l_i), \dots, s^n(l_i)$. The R -permutation π_R is defined for $i = 1, \dots, n_R$ as $\pi_R(i) = j$ if r_j is the first shifted word that starts with an R in the orbit $s(r_i), \dots, s^n(r_i)$. Maps π_L and π_R are therefore the first return maps on the sets of L -started and R -started words, respectively.

Example. For $w = LRRLR$, the shifted words, ordered lexicographically, are $l_1 = LRLRR$, $l_2 = LRRLR$, $r_1 = RLRLR$, $r_2 = RLRLR$, $r_3 = RRLRL$, so $\pi_L(1) = 2$, $\pi_L(2) = 1$, $\pi_R(1) = 2$, $\pi_R(2) = 3$, $\pi_R(3) = 1$.

Proposition 1. The Lorenz braid constructed as above from Lorenz braids A, B and C , with $n_R(A) = n_L(B)$, is associated to a cyclic permutation and therefore closes to a knot iff $\pi_R(A)\pi_L(B)$ is cyclic (has only one cycle). If only one braid A or B is used in the construction, then the closure of the resulting Lorenz braid is always a Lorenz knot.

Proof. Let $p_A = n_L(A)$, $q_B = n_R(B)$ and $n_C = n(C)$. Let π be the permutation corresponding to the satellite braid constructed as above. For simplicity, denote $\pi_R(A)$ and $\pi_L(B)$ by π_R and π_L , respectively. The orbit of a point p under the permutation π is $(p, \pi(p), \dots, \pi^{-1}(p))$. The permutation π is cyclic iff given a point p , its orbit contains all points. We will subdivide the points in the satellite braid in the following sets:

- A contains the first $p_A + k$ points, corresponding to the original braid A ;

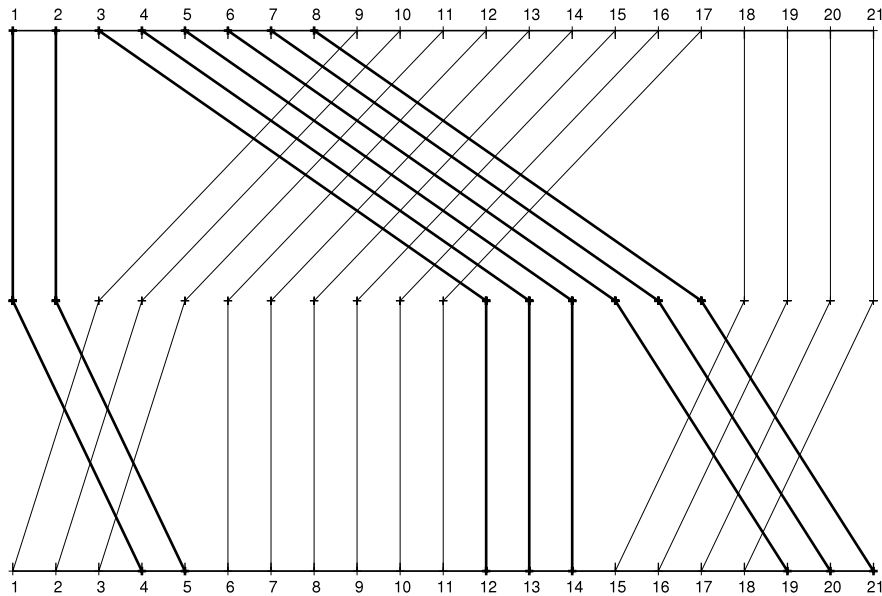


Fig. 8. Satellite for $w_A = LRRLR$, $w_B = LRRLLR$ and $w_C = LRRLR$.

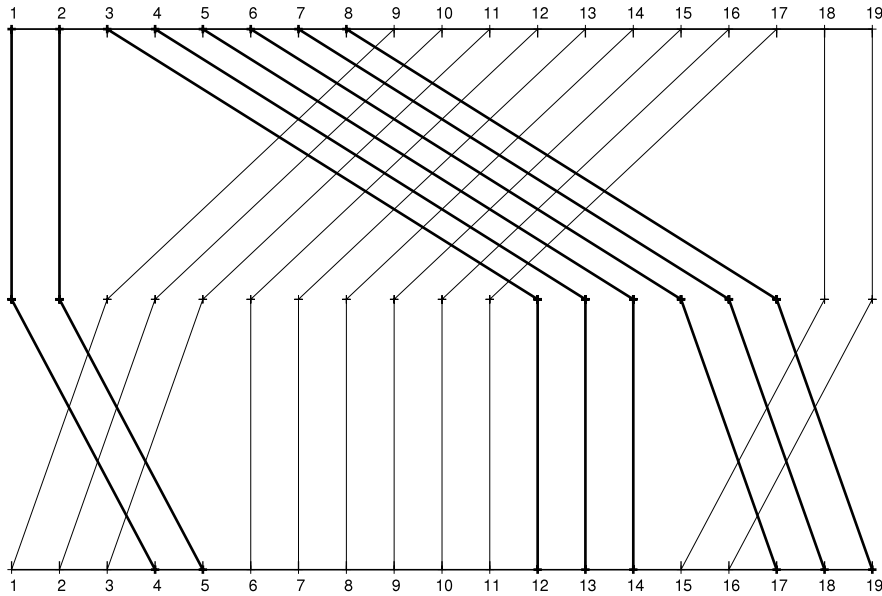


Fig. 9. Link for $w_A = LRRLR$, $w_B = LRLRL$ and $w_C = LRRLR$.

- B contains the last $q_B + k$ points, corresponding to the original braid B ;
- The remaining points (in the central part) are subdivided into k sets C_i , $i = 1, \dots, k$ according to the component of the inflated braid they belong to.

Note that:

- If $p \in C_i$, then either $\pi(p) \in C_i$ or $\pi(p) \in A$ or $\pi(p) \in B$;
- If $p \in A$ ($p \in B$) then the first point in the orbit of p not in A (B) will be in one of the sets C_i .
- If $p \in C_i$ and $\pi(p) \in A$ ($\pi(p) \in B$) then the first point in the orbit of p not contained in A (B) will be contained in a set C_j ($j \neq i$), such that $j = \pi_R(i)$ ($j = \pi_L(i)$), where π_L and π_R are the k -permutations defined above.

Consider, for example, the point p with index $p_A + 1 \in A$. The sequence of sets visited by the orbit of p is $(A, C_1, B, C_{\pi_R(1)}, A, C_{\pi_L(\pi_R(1))}, B, \dots, B, C_{\pi_L^{-1}(1)})$. The orbit of p contains all the points

iff the sequence contains each set C_i exactly twice (one immediately after A and one immediately after B). Since π_L and π_R are cyclic permutations, this is equivalent to each set C_i appearing once after A . The sequence of indices for these sets is $(1, \pi_L \pi_R(1), (\pi_L \pi_R)^2(1), \dots, (\pi_L \pi_R)^{k-1}(1))$. This sequence contains all the indices $1, \dots, k$ exactly once iff $\pi_L \pi_R$ is a cyclic permutation. Finally, $\pi_R \pi_L$ is cyclic iff $\pi_L \pi_R$ is cyclic (they are conjugate permutations), which completes the proof. \square

Given three Lorenz braids A, B and C , all of them closing to knots, let $\pi_R(A)$ be the R -permutation of A and $\pi_L(B)$ be the L -permutation of B as in Definition 2. Assume that $\pi_R(A)\pi_L(B)$ is cyclic.

For w_B let $m_R(i)$ be the number of R symbols between the first and second L symbols in l_i ($m_R(i) = 0$ whenever there are two consecutive L symbols), for $i = 1, \dots, k$. Likewise, for w_A let $m_L(i)$ be the number of L symbols between the first and second R symbols in r_i ($m_L(i) = 0$ if the i th R is immediately followed by another R),

for $i = 1, \dots, k$. For simplicity, we will write π_R for $\pi_R(A)$ and π_L for $\pi_L(B)$.

The following algorithm returns the aperiodic word $w(A, B, C)$ corresponding to the satellite braid constructed through the procedure above:

Algorithm. Input: Three aperiodic words w_A, w_B, w_C with $n_R(A) = n_L(B) = k$.

1. For $i = 1, \dots, k, w_i = w_C$.
2. $j_1 = k$.
Insert $R^{m_R(j_1)}$ in w_1 immediately after the maximal position;
 $j_2 = \pi_L(j_1)$;
Insert $L^{m_L(j_2)}$ in w_1 immediately after the minimal position.
3. For $i = 2, \dots, k$:
 $j_{2i-1} = \pi_R(j_{2i-2})$;
Insert $R^{m_R(j_{2i-1})}$ in w_i immediately after the maximal position of w_i ;
 $j_{2i} = \pi_L(j_{2i-1})$;
Insert $L^{m_L(j_{2i})}$ in w_i immediately after the minimal position of w_i .
4. $w_S = w_1 w_2 \dots w_k$
Output: Aperiodic word $w_S(A, B, C)$ of satellite in L -maximal form.

Proposition 2. *If the satellite braid constructed as above from Lorenz braids A, B and C represents a knot, then its aperiodic word is obtained by the preceding algorithm.*

Proof. First note that the satellite braid has exactly $k n(C) + n_L(A) + n_R(B)$ strings, and its corresponding word $k t(C)$ syllables, (where $n(C)$ is the number of strings in C and $t(C)$ is its trip number). Each string in C originates k strings in the satellite. To these, $n_L(A) + n_R(B)$ strings are added. The LR strings in the satellite braid result from the inflation of the LR strings in C . The $n_L(A)$ L strings of A and the $n_R(B)$ R strings of B contribute, respectively, with $n_L(A)$ LL strings and $n_R(B)$ RR strings. Therefore, the braid index of the satellite knot or link is $k t(C)$.

With the exception of the strings that result from the inflation of the first RL string and the last LR string in C (connected, respectively, to the R strings of A and the L strings of B), all strings in the inflated braid are organized in sets of beams of k parallel strings whose endpoints are the startpoints of another beam of k parallel strings. The subwords in the word of the satellite braid will be the same as in the word for C while the startpoints and endpoints of strings are within the inflated braid.

If we follow the satellite braid beginning with the maximal position of the last LR beam, corresponding to an L symbol both in C and in the satellite word, then its endpoint will be the last in B , corresponding to the endpoint of the last LR string of B . The L symbol will thus be followed by $m_R(k)$ R symbols, corresponding to RR strings in B . When an RL string of B is reached, the endpoint is the R point in the last RL beam of the satellite braid given by $j_2 = \pi_L(k)$. Since the strings in the inflated braid are parallel, the next symbols in the satellite braid are the sequence between the maxima and minimal position of C . When a position in the first RL braid is reached, $m_L(j_2)$ L symbols are added in an analogous way and the inflated braid is resumed at the $\pi_R(j_2)$ and then continued by the symbols in C until the $\pi_R(j_2)$ position in the last beam. The process continues until all the points on the satellite braid have been visited once. \square

Example. For the Lorenz braid in Fig. 8, $A = LRRLR, B = LRRLRLR$ and $C = LRRLR$ and $k = n_R(A) = n_L(B) = 3$.

In braid $B, l_1 = LRLRLRR, l_2 = LRLRRLR, l_3 = LRRLRLR$, so $m_R(1) = 1, m_R(2) = 1, m_R(3) = 2$ and, in cycle notation, $\pi_L = (123)$.

In braid $A, r_1 = RLRLR, r_2 = RLRLR, r_3 = RRLRL, m_L(1) = 1, m_L(2) = 1, m_L(3) = 0$ and $\pi_R = (123)$.

Starting with $w_1 = w_2 = w_3 = LRRLR$ and following the steps above,

- add $m_R(3) = 2$ R symbols after the maximal position of w_1 ;
- $\pi_R(3) = 1$, so add $m_L(1) = 1$ L symbol to the minimal position of w_1 ;
- $\pi_L(1) = 2$, so add $m_R(2) = 1$ R to the maximal position of w_2 ;
- $\pi_R(2) = 3$, so add $m_L(3) = 0$ L to the minimal position of w_2 ;
- $\pi_L(3) = 1$, so add $m_R(1) = 1$ R at the maximal position of w_3 ;
- $\pi_R(1) = 2$, so add $m_L(2) = 1$ L to the minimal position of w^3 .

The resulting aperiodic word

$w_S = w_1 w_2 w_3 = LRRRLLRLRRRLRLRRLLR$ corresponds to the satellite braid in Fig. 8.

Theorem 1. *No permuted words in the sets $P(p, q)$ can be obtained through the satellite braid construction defined above.*

Proof. As the words in $P(p, q)$ have no LL subwords, they would have to be obtained, by the procedure above, from the word of a companion knot C which equally has no LL subsequences, taking A as an identity braid and a Lorenz braid B . The word for C would have the structure (in L -maximal form) $LR^{b_1} \dots LR^{b_a}$ where $a = n_L(C) > 1$ (otherwise C would be trivial), $b_1 = \max\{b_i\}$ and at least one k satisfying $b_k < b_1$ (otherwise the word would be periodic). Each word in $P(p, q)$ has syllables of only two types: LR^{k+1} and LR^k . The satellite construction, in this case, only adds R symbols to the maximal positions of w_1, \dots, w_k , so if it could be obtained through this procedure then we would have $\min\{b_i\} = k$ and $b_1 = k + 1$. But after the satellite construction, the resulting word would have a first syllable with at least $b_1 + 1 = k + 2$ symbols R , and at least one syllable with k symbols R and thus will not be in any $P(p, q)$. \square

El-Rifai showed [14] that a Lorenz knot that is a satellite of a Lorenz knot can be presented as the closure of a Lorenz braid constructed as outlined above. Morton has conjectured [13,15] that all Lorenz satellite knots are cablings (satellites where the pattern is a torus knot) on Lorenz knots.

If Morton's conjecture is true, then we conclude from Theorem 1 that Lorenz knots corresponding to syllable permutations of standard torus words, that is, the knots corresponding to words in the sets $P(p, q)$ are not satellites.

4. Syllable permutations that do not correspond to torus knots

As seen above, the braid index of any braid corresponding to a word in $P(p, q)$ is given by $t = n_L = p$. So, if any of these knots were torus knots, then they would have to be of type $T(p, q')$ for some q' .

As a consequence of Remarks 3 and 4 in Section 2, we again assume $p > 4, 1 < r < p - 1$ below.

Lemma 3. *No closure of a braid corresponding to a word in $P(p, q)$ is a torus knot $T(p, q')$, for $q' > q$, and the only word in $P(p, q)$ corresponding to a braid that has $T(p, q)$ as closure is the evenly distributed word $w(p, q)$ corresponding to the standard $T(p, q)$ braid.*

Proof. If a word in $P(p, q)$ corresponds to a knot $T(p, q')$, then the genus of $T(p, q')$ must be in the range defined by Eq. (4), so in particular, $(p - 1)(q' - 1) \leq (p - 1)(q - 1) \Rightarrow q' \leq q$. The only word in $P(p, q)$ for which the genus is $(p - 1)(q - 1)$ is the word corresponding to the standard braid of $T(p, q)$. \square

Theorem 2. *For each word w in $P(p, q), 4 < p < q$, distinct from $w(p, q)$, there is at most one torus knot $T(p, q')$, $q' < q$, with the same braid index and genus as the closure of the braid corresponding to w .*

Proof. Let g be the genus of the closure of the Lorenz braid corresponding to w . As $w \in P(p, q)$, any torus knot with the same braid index must be of type $T(p, q')$ for some $q' < q$. The genus of $T(p, q')$ is given by $(p - 1)(q' - 1)$ so q' is uniquely determined by $(p - 1)(q' - 1) = 2g$. If $p - 1$ divides $2g$ then this equality determines a unique q' , otherwise there is no q' satisfying the condition. \square

Lemma 4. Let $w \in P(p, q) = P(p, kp + r)$, $p > 4$, $1 < r < p - 1$. If there is a (unique) torus knot $T(p, q') = T(p, k'p + r')$ with the same genus and braid index as the closure of the braid of w , then $k' = k$ and $1 < r' < r < p - 1$.

Proof. If $k' > k$ then $k \geq k + 1$, so $q' = k'p + r' \geq (k + 1)p + r' = kp + p + r' > kp + r = q$. But $q' < q$ (Lemma 3) so $k' \leq k$.

From the first inequality in Eq. (4), $kp(p - 1) + r(r - 1) \leq (p - 1)(k'p + r' - 1)$ or

$$((k - k')p - r' + 1)(p - 1) + r(r - 1) \leq 0.$$

If $k' < k$ then $(k - k')p \geq p$ so $(k - k')p - r' + 1 \geq p - r' + 1 \geq 0$ and the inequality above does not hold.

For $k = k'$, $q' < q \Rightarrow r' < r$. If $r' = 1$ then the same inequality gives $r(r - 1) < 0$ (impossible) which completes the proof. \square

Proposition 3. For each odd integer $p > 4$ and all integers $k > 0$, the sets $P(p, q)$, for $q = kp + 2$ and $q = (k + 1)p - 2$ contain no words corresponding to braids whose closure is a torus knot, besides $w(p, q)$.

Proof. For all odd positive integers p and all $k > 0$, $(p, kp + 2)$ and $(p, (k + 1)p - 2)$ are pairs of relatively prime integers. If $q = kp + 2$, then $r = 2$ and there is no r' satisfying the condition of Lemma 4. For $q = (k + 1)p - 2$, from the proof of Theorem 2, if there is a word in $P(p, q)$ corresponding to a braid whose closure is a torus knot of genus g , then $(p - 1)$ divides $2g$. As the maximum $2g$ in $P(p, q)$ is $(p - 1)(q - 1)$, there will be at least another (even) multiple of $(p - 1)$ in the set $\{kp(p - 1) + r(r - 1), \dots, (p - 1)(kp + r - 1)\}$ if $kp(p - 1) + r(r - 1) \leq (p - 1)(kp + r - 2)$ or

$$r(r - 1) \leq (p - 1)(r - 2).$$

This inequality above does not hold, with any $p > 4$, for $r = p - 2$, as can be seen from simple substitution, which completes the proof. \square

Proposition 4. If $p > 4$ is even and not a multiple of 3, then for any integer k the sets $P(p, kp + 3)$ and $P(p, (k + 1)p - 3)$ contain no words corresponding to braids whose closure is a torus knot. Also, if $p < 12$ and p is even, then $P(p, q)$ contains no words other than $w(p, q)$ corresponding to torus knots.

Proof. If p is even and not a multiple of 3 then $(p, kp + 3)$ and $(p, (k + 1)p - 3)$ are pairs of relatively prime integers. Also, since $p - 1$ is odd, $(p - 1)(kp + r - 1)$ is an even multiple of $p - 1$ (because it is equal to $2g$ for the torus knot $T(p, kp + r)$), so $(p - 1)(kp + r - 2)$ is odd. There will be at least another even multiple of $(p - 1)$ in the set $\{kp(p - 1) + r(r - 1), \dots, (p - 1)(kp + r - 1)\}$ if $kp(p - 1) + r(r - 1) \leq (p - 1)(kp + r - 3)$. Direct substitution shows that this inequality does not hold for $r = 3$ and $r = p - 3$ or, for any $1 < r < p - 1$, for $p < 12$ even. \square

Remark 6. If p is even, then $(p, kp + 2)$ and $(p, (k + 1)p - 2)$ are not pairs of relatively prime integers. The same happens with $(p, kp + 3)$ and $(p, (k + 1)p - 3)$ if p is a multiple of 3.

5. Conclusion

- We have defined sets $P(p, q)$ of words resulting from permuting the syllables of the standard words for torus knots $T(p, q)$, $1 < p < q$ and characterized their corresponding Lorenz braids. For $p \leq 4$ $P(p, q) = \{w(p, q)\}$. We have shown that for each word w in the sets $P(p, q)$ there is at most a torus knot $T(p, q')$ with the same braid index and genus as the knot corresponding to w .
- A procedure to obtain Lorenz satellite braids was defined, and an algorithm for finding its aperiodic word was devised. We have used this algorithm to prove that, provided Morton's conjecture is true, no set $P(p, q)$ contains words corresponding to satellite knots.
- From Theorem 1, Propositions 3 and 4 and Thurston's theorem [7] we can finally state that:
If Morton's conjecture is true, then the words in sets $P(p, kp + 2)$, $P(p, (k + 1)p - 2)$ for p odd and $P(p, kp + 3)$, $P((k + 1)p - 3)$, for p even and p not a multiple of 3, distinct from the standard $w(p, q)$ torus word, correspond to hyperbolic Lorenz knots.
- We have recently performed an extensive computational test [16], in which we computed the volumes of all knot complements corresponding to words in the (non-empty) sets $P(p, q)$ with $5 \leq p \leq 19$ and $6 \leq q \leq 100$. We found all of them to be hyperbolic, with the expected exception of the torus knots $T(p, q)$ corresponding to the standard words $w(p, q)$. This allowed us to conjecture that the results of this work can be extended to all $P(p, q)$ sets.
- We believe that the techniques used in this paper can be generalized to study other families of words and their corresponding Lorenz braids and knots and thus possibly generate other infinite families of hyperbolic Lorenz knots.

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