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Model Risk in the Pricing of Exotic Options

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Abstract

The growth experimented in recent years in both the variety and volume of structured products implies that banks and other financial institutions have become increasingly exposed to model risk. In this article we focus on the model risk associated with the local volatility (LV) model and with the Variance Gamma (VG) model. The results show that the LV model performs better than the VG model in terms of its ability to match the market prices of European options. Nevertheless, both models are subject to significant pricing errors when compared with the stochastic volatility framework.

Keywords: Model risk, exotic options, local volatility, stochastic volatility, Variance Gamma process, path dependence.

JEL: G12, G13.

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1 Introduction

In recent years there has been a remarkable growth of structured products with embedded exotic options. In this sense, the European Commission[1] stated that the use of derivatives has grown exponentially over the last decade, with over-the-counter transactions being the main contributor to this growth. At the end of December 2009, the size of the over-the-counter derivatives market by notional value equaled approximately $615 trillion, a 12% increase with respect to the end of 2008. These exotic options can be quite model dependent and, as a consequence, the financial institutions that commercialize these structured products are exposed to the existence of model risk. That is, they are exposed to the risk that arises from the utilization of an inadequate model. In fact, as Derman and Wilmott (2009) pointed out, one of the reasons of the financial crisis that started in August 2007 with the collapse in subprime mortgages was the use of inadequate models to valuate complex credit derivatives. In this sense, the Basel Committee on Banking Supervision[2] has recently stated that it will supplement the risk-based capital requirements with a leverage ratio that intends to introduce additional safeguards against model risk and measurement error.

Since the stock market crash on October 1987, equity options markets have been characterized by a persistent negative implied volatility skew. There are a number of models that have been proposed to deal with this stylized fact. Stochastic volatility models leave the constant instantaneous volatility assumption of the Black-Scholes (1973) model and assume that volatility follows a stochastic process possibly correlated with the process for the stock price. Within this group are Hull and White (1987) and Heston (1993) stochastic volatility models. Merton (1976), among others, incorporates the possibility of jumps in the stochastic process for the underlying asset price. Local volatility (LV) models postulate that the instantaneous volatility (called local volatility) is a deterministic function of the underlying asset price and time. Within this group we have the works of Dupire (1994), Derman and Kani (1994) Rubinstein (1994), Andersen and Brotherton-Ratcliffe (1998) or Brown and Randall (1999). Wilmott (2006) indicates that the local volatility and stochastic volatility models are the most widely used models by financial institutions to price exotic options.

In absence of jumps all these models give continuous paths and have an infinite first variation, that is, an infinite amount of up and down moves. But, as pointed

[1] See European Commission Press Releases, April 1, 2010, IP/10/1125. Available at http://europa.eu/rapid/pressReleasesAction.do?reference=IP/10/1125.

[2] See Basel Committee on Banking Supervision (2010).

out by Joshi (2003), stock prices move in little jumps rather than continuously and the total amount of up and down moves is finite rather than infinite. The Variance Gamma (VG) process, introduced by Madan et al. (1998), attempts to address these features. Under this process each unit of calendar time can be viewed as having a time length given by an independent random variable that is gamma distributed.

But as pointed out by Carr et al. (2007), the VG process is not very popular between practitioners. In this sense, this article has two main objectives. The first one is to investigate if the VG process can be an alternative to the models widely used by practitioners to account for the existence of volatility skew. To this end, we compare the ability of the model proposed by Madan et al. (1998) and the ability of the LV model to replicate the market price of vanilla options. The second objective is related to the measurement of model risk in the pricing of exotic options. Hull and Sou (2002) studied the model risk arising from the pricing of compound options as well as barrier options. To this end, they assumed that market prices are governed by a plausible no-arbitrage stochastic volatility model (the "true model"). They then determined the parameters of the true model by fitting it to representative market data. Finally, they compared the pricing and hedging performance of the model being tested with the pricing and hedging performance of the true model for the exotic option.

Note that the key point of the model risk is that different models can yield the same price for European options but, at the same time, very different prices for exotic options depending on their assumptions corresponding to the evolution of the underlying asset price and its volatility. Hull and Sou (2002) concluded that the model risk associated with the LV model is a function of the degree of path dependence in the exotic option being priced. They defined the degree of path dependence as the number of times that the asset price must be observed to calculate the payoff. The higher the degree of path dependence, the worse the LV model is expected to perform. In this article we extend the important work of Hull and Suo (2002) to the model risk arising from the pricing of cliquet options. These options are very common in structured products in Europe and are quite sensitive to the forward skew. Since barrier options are also usually embedded in structured products, we also consider the model risk associated with these exotic options. The results of our study show that, although the path dependency can play an important role in determining the magnitude of the model risk associated with the LV model, other factors are more determinant. Namely, the dependence of the exotic option on the forward skew, as well as the volga[3] sensitivity of the exotic option being priced.

[3]The volga represents the second derivative of the option price with respect to the volatility. It

The rest of the paper proceeds as follows. Section 2 presents the models considered in the study and offers the calibration results of the different models to the information provided by European options. In section 3 we describe the exotic options considered to measure the existence of model risk, as well as the sensitivities of these options to market parameters. Section 4 displays the empirical results associated with the pricing tests. Finally, section 5 offers concluding remarks.

2 Volatility skew and calibration to market data

In the article we follow the methodology introduced by Hull and Suo (2002) to measure the model risk embedded in the pricing of exotic options. To this end, we mimic the way in which practitioners price these options. They typically use a model to price a particular exotic option in terms of the observed market prices at a particular time. In this sense, they calibrate the model parameters to the market prices of vanilla instruments at a point in time and use the model parameters to price exotic options at the same time. Following Hull and Suo (2002), we assume that market prices are governed by a stochastic volatility model. In particular, we consider the Heston (1993) model and we determine the model parameters fitting it to representative market data. The main reason for this choice is that this model is one of the most popular models within the class of stochastic volatility models.

This section compares the ability of the LV model and the VG model to recover the market prices of vanilla options generated by the Heston (1993) model. To this end, we now present the main features of the models considered.

2.1 Models specifications

2.1.1 The local volatility model

The local volatility model was introduced by Dupire (1994), Derman and Kani (1994) and Rubinstein (1994). Let S_t denote the spot price of the underlying asset at time $t \in [0, \Upsilon]$, where Υ is some arbitrarily distant horizon. For simplicity, we assume that the continuously compounded risk-free rate r and dividend yield q are constant. Let us consider the probability measure Q defined on a probability space $(\Omega, \mathcal{F}, Q)$ such that asset prices expressed in terms of the current account are martingales. This probability measure is denoted as the risk-neutral measure. Under the local volatility model the asset price evolution is governed by the following stochastic

can be seen as the gamma of the option with respect to volatility.

differential equation, under the risk-neutral probability measure Q:

$$\frac{dS_t}{S_t} = (r-q)\,dt + \sigma(t,S_t)\,dW_t^Q \tag{1}$$

where W_t^Q is a Wiener process under Q and the instantaneous volatility, denoted local volatility, $\sigma(t,S_t)$ is a function of time and asset price. Dupire (1994) shows that the following relationship holds between time $t=0$ European call option prices of strike K and maturity T, $C_0(K,T)$, and the one dimensional local volatility function:

$$\sigma(T,S_T=K) = \sqrt{2\left[\frac{\frac{\partial C_0(K,T)}{\partial T} + K\frac{\partial C_0(K,T)}{\partial K}(r-q) + qC_0(K,T)}{K^2\frac{\partial^2 C_0(K,T)}{\partial K^2}}\right]} \tag{2}$$

Equation (2) shows that it is possible to recover the local volatility function using the market prices of vanilla options.

Let $D(S_t,t)$ denote the time t price of a derivative, which may be path-dependent, on an asset whose time t price is given by S_t. We assume that the asset price follows the risk-neutral process of equation (1). Replication arguments show that the derivative satisfies the Kolmogorov backward equation:

$$\frac{\partial D(S_t,t)}{\partial t} + (r-q)\,S_t\frac{\partial D(S_t,t)}{\partial S_t} + \frac{1}{2}\sigma^2(t,S_t)\,S_t^2\frac{\partial^2 D(S_t,t)}{\partial S_t^2} - rD(S_t,t) = 0 \tag{3}$$

where $\sigma(t,S_t)$ is given by equation (2). Therefore options can be priced through a Monte Carlo simulation, based on the asset price dynamics of equation (1) or through a finite-difference scheme, based on equation (3). This approach is particularly useful for instruments with early-exercise features.

2.1.2 The Heston model

The Heston (1993) model postulates that the risk-neutral evolutions of the asset price S_t and its instantaneous variance v_t under Q are governed by the following stochastic differential equations:

$$\frac{dS_t}{S_t} = (r-q)\,dt + \sqrt{v_t}dW_{S,t}^Q \tag{4}$$

$$dv_t = \kappa(\overline{v} - v_t)\,dt + \zeta\sqrt{v_t}dW_{v,t}^Q \tag{5}$$

where $W^Q_{S,t}$ and $W^Q_{v,t}$ are Wiener processes under the probability measure Q such that:

$$dW^Q_{S,t} dW^Q_{v,t} = \rho dt$$

with ρ being the instantaneous correlation, which is assumed to be constant. The parameter $\overline{v}$ represents the long-term mean corresponding to the instantaneous variance, κ denotes the speed of mean reversion and, finally, ζ represents the volatility of the variance.

Let $D(S_T)$ denote the terminal payoff corresponding to the derivative asset on S. Its time t value, denoted by $D(S_t, v_t, t)$, verifies the following partial differential equation:

$$\begin{aligned} rD_t = \frac{\partial D_t}{\partial t} + (r-q)\, S_t \frac{\partial D_t}{\partial S_t} + \kappa\,(\overline{v} - v_t)\, \frac{\partial D_t}{\partial v_t} + \\ \frac{1}{2}\frac{\partial^2 D_t}{\partial S_t^2} v_t S_t^2 + \frac{1}{2}\frac{\partial^2 D_t}{\partial v_t^2} v_t \zeta^2 + \frac{\partial^2 D_t}{\partial S_t \partial v_t} v_t S_t \rho \zeta \end{aligned} \tag{6}$$

Note that the partial differential equation corresponding to a derivative asset under the Heston (1993) model includes additional terms that were not present in the Kolmogorov backward equation associated with the LV model. In particular, $\frac{\partial D_t}{\partial v_t}$ is related to the vega of the option, the convexity factor with respect to volatility $\frac{\partial^2 D_t}{\partial v_t^2}$ has to do with the volga of the derivative asset. Finally, the cross-convexity term $\frac{\partial^2 D_t}{\partial S_t \partial v_t}$ is associated with the vanna of the option. The use of valuation models that do not consider these effects to price options which exhibit sensitivity to them can lead to valuation differences, as we will see in section 4.

One advantage of the Heston (1993) model is that under the assumptions of the model it is possible to obtain semi-analytic formulas to price European options. Let us consider a European call option with strike price K and maturity T and let $P(0,T)$ denote is the time $t = 0$ price of a zero coupon bond with maturity T. It is possible to express the payoff of the option at maturity as:

$$(S_T - K)^+ = (S_T - K)\, \mathbf{1}_{(S_T > K)}$$

where $\mathbf{1}_{(S_T > K)}$ is the Heaviside step function or unit step function. Hence, the time $t = 0$ price of a European call with strike $K$and maturity T, under the risk-neutral probability measure Q, can be expressed as:

$$
\begin{aligned}
C_0(K,T) &= P(0,T)\left(E_Q\left[S_T\mathbf{1}_{(S_T>K)}\right]-KE_Q\left[\mathbf{1}_{(S_T>K)}\right]\right)\\
C_0(K,T) &= e^{-qT}S_0E_Q\left[\frac{S_T}{F_{0,T}}\mathbf{1}_{(S_T>K)}\right]-P(0,T)KE_Q\left[\mathbf{1}_{(S_T>K)}\right]\\
C_0(K,T) &= e^{-qT}S_0P_1-P(0,T)KP_2 \qquad (7)
\end{aligned}
$$

where $F_{0,T}{=}S_0e^{(r-q)T}$ is the time $t=0$ value of a forward contract on the underlying asset with maturity T. P_2 represents the probability, under Q, that at the maturity of the option the asset price is above the strike price. The functions P_j, for $j=1,2$, can be obtained from the inverse Fourier transform:

$$P_j=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}Re\left[\frac{e^{-iz\ln(K)}f_j}{iz}\right]dz \qquad (8)$$

where $i=\sqrt{-1}$ and where f_j, for $j=1,2$, are given by:

$$
\begin{aligned}
f_j &= e^{C_j+D_jv_0+iz\ln(S_0)}\\
C_j &= (r-q)izT-2\frac{\kappa\overline{v}}{\zeta^2}\ln\left(\frac{1-g_je^{Td_j}}{1-g_j}\right)\\
&\quad +\frac{\kappa\overline{v}}{\zeta^2}(b_j-\rho\zeta iz+d_j)T
\end{aligned}
$$

$$
\begin{aligned}
D_j &= \frac{b_j-\rho\zeta iz+d_j}{\zeta^2}\left[\frac{1-e^{Td_j}}{1-g_je^{Td_j}}\right]\\
g_j &= \frac{b_j-\rho\zeta iz+d_j}{b_j-\rho\zeta iz-d_j}\\
dj &= \left[(\rho\zeta iz-b_j)^2-\zeta^2\left(2u_jiz-z^2\right)\right]^{\frac{1}{2}}
\end{aligned}
$$

with $u_1=\frac{1}{2}$, $u_2=-\frac{1}{2}$, $b_1=\kappa-\rho\zeta$ and $b_2=\kappa$. Equation (7) allows us calibrating the parameters of the model to the market prices of vanilla options. Once this is done, it is possible to price exotic options by Monte Carlo simulation, based on the dynamics of equations (4) and (5), or through a finite-difference scheme, based on equation (6).

2.1.3 The Variance Gamma model

The Variance Gamma model for financial asset returns was introduced in the symmetric case by Madan and Seneta (1990) and later extended to incorporate skewness by Madan et al. (1998). In this article we consider the version of Madan et al. (1998) that allows for the existence of implied volatility skew and we denote it as the Variance Gamma (VG) model.

The idea behind the VG model is to consider volatility as a measure of the sensitivity of a stock with respect to the arrival of information. Since the amount of information arriving is stochastic, it has to be described by a random process. In this case, each unit of calendar time can be interpreted as having an economically relevant time length that is given by an independent random variable which has a gamma distribution. In particular, the VG process $X(t;\sigma,\nu,\theta)$ is defined in terms of a Brownian motion with drift and a gamma process V:

$$X(t;\sigma,\nu,\theta) = \theta V_t + \sigma Z_t \sqrt{V_t}$$

where Z is a standard normal variable independent of the gamma variable V, which has mean t and variance νt. Conditional on the gamma time change V the VG variable, over an interval of length t, is normally distributed:

$$X(t;\sigma,\nu,\theta)\,|V_t \sim N\left(\theta V_t, \sigma^2 V_t\right)$$

Note that the realizations of V drive both volatility, via $\sigma\sqrt{V_t}$, and skewness, via θV_t. The parameter ν affects the kurtosis of the distribution. The risk neutral process corresponding to the underlying asset under the VG model is given by:

$$S_t = S_0 e^{(r-q+\omega)t + X(t;\sigma,\nu,\theta)} \tag{9}$$

where $\omega = \frac{1}{\nu}\ln\left[1-\theta\nu-\frac{\sigma^2\nu}{2}\right]$. Madan et al. (1998) showed that the European call option price on a stock whose risk neutral dynamics is given by equation (9) can be expressed as follows:

$$\begin{aligned} C_0(K,T) &= S_0 e^{-qT}\Psi\left[d\sqrt{\frac{1-c_1}{\nu}},(\alpha+s)\sqrt{\frac{\nu}{1-c_1}},\frac{T}{\nu}\right] \\ &\quad -Ke^{-rT}\Psi\left[d\sqrt{\frac{1-c_2}{\nu}},\alpha s\sqrt{\frac{\nu}{1-c_2}},\frac{T}{\nu}\right] \end{aligned} \tag{10}$$

where:

$$\begin{aligned} d &= \frac{1}{s}\left[\ln\left(\frac{S_0 e^{(r-q)T}}{K} + \frac{T}{\nu}\ln\left(\frac{1-c_1}{1-c_2}\right)\right)\right] \\ \alpha &= \xi s, \end{aligned}$$

with $\xi = -\frac{\theta}{\sigma^2}$, $s = \frac{\sigma}{\sqrt{1+\left(\frac{\theta}{\sigma}\right)^2\frac{\nu}{2}}}$, $c_1 = \frac{\nu(\alpha+s)^2}{2}$, $c_2 = \frac{\nu\alpha^2}{2}$ and where the function Ψ is defined by:

$$\Psi(a,b,\gamma) = \frac{c^{\gamma+\frac{1}{2}} e^{\mathrm{sign}(a)c}(1+u)^\gamma}{\sqrt{2\pi}\Gamma(\gamma)\gamma} K_{\gamma+\frac{1}{2}}(c)\Lambda\left[\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\mathrm{sign}(a)c(1+u)\right]$$
$$-\mathrm{sign}(a)\frac{c^{\gamma+\frac{1}{2}} e^{\mathrm{sign}(a)c}(1+u)^{1+\gamma}}{\sqrt{2\pi}\Gamma(\gamma)(1+\gamma)} K_{\gamma-\frac{1}{2}}(c)\Lambda\left[1+\gamma, 1-\gamma, 2+\gamma; \frac{1+u}{2}, -\mathrm{sign}(a)c(1+u)\right]$$
$$+\mathrm{sign}(a)\frac{c^{\gamma+\frac{1}{2}} e^{\mathrm{sign}(a)c}(1+u)^{\gamma}}{\sqrt{2\pi}\Gamma(\gamma)\gamma} K_{\gamma-\frac{1}{2}}(c)\Lambda\left[\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\mathrm{sign}(a)c(1+u)\right]$$

where $c = |a|\sqrt{2+b^2}$, $u = \frac{b}{\sqrt{2+b^2}}$, K_γ is the modified bessel function of the second kind of order γ, $\Gamma(.)$ is the gamma function and, finally, Λ is the degenerate hypergeometric function of two variables that has the following integral representation:

$$\Lambda[\alpha,\beta,\gamma;x,y] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}e^{uy}du$$

Note that the VG process is a one dimensional time homogeneous Markov process. The paths corresponding to this process consist of a large number of small jumps and, hence, the VG model leads to an incomplete markets framework with many equivalent martingale measures.

Equation (10) allows us calibrating the parameters of the model to the market prices of vanilla options. Once this is done, it is possible to price exotic options by Monte Carlo simulation, based on the dynamics of equation (9).

2.2 Calibration of the models

As said previously, we assume that market prices are governed by the Heston (1993) stochastic volatility model and we determine the model parameters fitting it to representative market data. In particular, we consider the implied volatility surface for the Eurostoxx 50 equity index corresponding to April 1, 2010 obtained from Bloomberg. We have seven maturities and nine values of moneyness[4], ranging from

[4]The moneyness is defined as $\frac{K}{S}$.

80% to 120%. Therefore, a total of 63 points on the implied volatility surface are provided. The continuously compounded risk-free rate r is equal to 1.40% and dividend yield q is equal to 4.35%. The fitted values corresponding to the parameters of the Heston (1993) model are: $\kappa = 2.03$, $\overline{v} = 0.078$, $\zeta = 0.40$, $\rho = -0.72$ and $v_0 = 0.048$.

Table 1: Implied volatility surface for the Eurostoxx 50 equity index, corresponding to April 1, 2010, generated by the Heston model.

K/T	0.25	0.5	1	1.5	2	2.5	3
70%	30.51	29.90	29.03	28.47	28.03	27.77	27.70
80%	27.86	27.55	27.19	27.00	26.81	26.82	26.87
90%	25.23	25.29	25.49	25.64	25.77	25.95	26.09
95%	23.91	24.20	24.69	25.02	25.29	25.54	25.72
100%	22.60	23.13	23.93	24.43	24.83	25.15	25.36
105%	21.33	22.11	23.21	23.88	24.39	24.76	25.01
110%	20.14	21.16	22.53	23.36	23.96	24.38	24.68
120%	18.34	19.55	21.30	22.40	23.16	23.67	24.04
130%	17.51	18.47	20.29	21.55	22.40	23.00	23.43

Notes. K represents the strike as a percentage of the asset price. T denotes the time to maturity, expressed in years, whereas implied volatilities are expressed in percentage. The parameters used in generating this table are: $\kappa = 2.03$, $\overline{v} = 0.078$, $\zeta = 0.40$, $\rho = -0.72$, $v_0 = 0.048$, $r = 1.40\%$ and $q = 4.35\%$.

Table 1 shows the implied volatility surface generated by this parameter set, whereas figure 1 shows its graphical representation. The figure reveals the existence of negative volatility skew, which is most pronounced for near-term options. This is a common pattern of behavior that has been widely observed in other well known equity indexes such as the Standard and Poor's 500 index or the Nikkei 225 index. Note that the negative sign of the correlation coefficient is consistent with the negative implied volatility skew observed in the Eurostoxx 50 index data.

We calibrate the LV model and the VG model to the implied volatility surface of table 1 generated by the Heston (1993) stochastic volatility model. Regarding the LV model, we use the approach introduced by Marabel (2010) to calculate the local volatility. This methodology consists of smoothing the implied volatility through a flexible parametric function, which is consistent with the no-arbitrage conditions developed by Lee (2004) for the asymptotic behavior of the implied volatility at extreme strikes. The local volatility function is then calculated analytically. This approach allows obtaining smooth and stable local volatility surfaces while capturing the prices of vanilla options quite accurately. Regarding the VG model, we define a quadratic loss function based on the difference between the market price of European options generated by the Heston (1993) model and the option prices obtained under the parametric specification of equation (10). The parameters are chosen so that the error between market and model prices is as small as possible measured by the following loss function:

$$L(\Theta) = \left[\frac{1}{N_i N_j} \sum_{i=1}^{N_i} \sum_{j=1}^{N_j} \left[C(K_i, T_j) - C_\Theta (K_i, T_j) \right]^2 \right]^{1/2}$$

where Θ is the vector of parameters to be estimated, $C(K_i, T_j)$ is the market price associated with a European call, generated by the specification corresponding to the Heston (1993) model of table 1, of an option with strike price K_i and maturity T_j, $C_\Theta(K_i, T_j)$ is the VG model price, N_i is the total number of strikes and, finally, N_j represents the number of maturities considered. The estimated values corresponding to the parameters of the model are: $\theta = -0.447$, $\nu = 0.244$ and $\sigma = 0.160$.

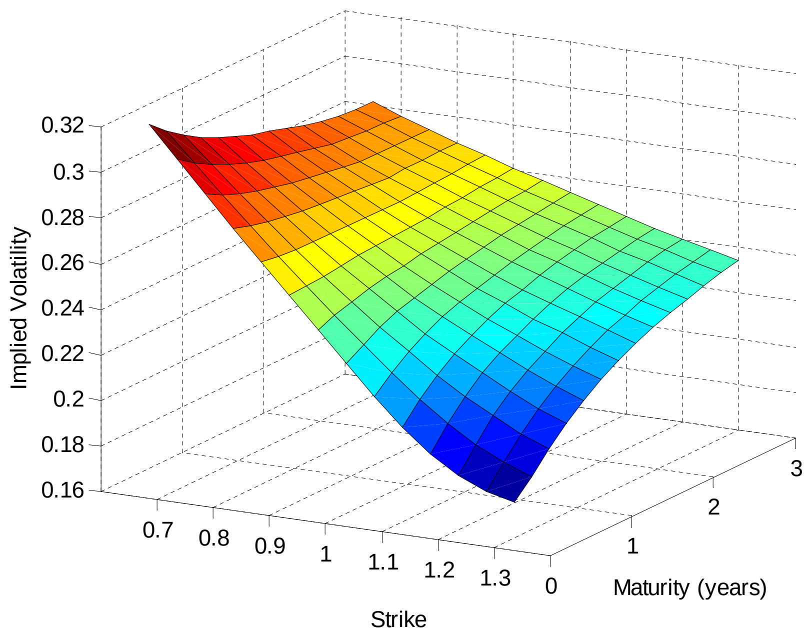

Figure 1: Implied volatility surface for the Eurostoxx 50 equity index, corresponding to April 1, 2010, generated by the Heston model. Strike prices are expressed as a percentage of the index price, whereas maturities are expressed in years. The surface is generated using the parameters of table 1.

Table 2 reports the root mean squared error (RMSE) corresponding to the pricing errors associated with the LV model and with the VG model. The table provides information about the total RMSE and the RMSE associated with at-the-money options, out-of-the-money puts and out-of-the-money calls.

Table 2: Root mean squared error (RMSE) corresponding to the pricing errors associated with the LV model and the VG model for the implied volatility surface of table 1.

	LV model	VG model
Total RMSE	0.017%	0.22%
RMSE at-the-money options	0.012%	0.19%
RMSE out-of-the-money puts	0.018%	0.27%
RMSE out-of-the-money calls	0.017%	0.14%

The results show that the LV model provides much better fit than the VG model to the market price of European options. In this sense tables 3 and 4 show, respectively, the differences between the implied volatility of table 1 and the implied volatilities generated by the LV model and the VG model. Hence, a positive number means that the implied volatility generated by the model is lower than the implied volatility of table 1.

Table 3: Differences between the implied volatilities of table 1 and the implied volatilities generated by the LV model.

K/T	0.25	0.5	1	1.5	2	2.5	3
70%	-0.67	-0.21	0.06	0.06	-0.05	-0.09	0.02
80%	-0.26	-0.03	0.06	0.03	-0.08	-0.04	0.04
90%	-0.06	0.01	0.04	0.00	-0.06	-0.01	0.04
95%	-0.06	-0.01	0.03	-0.02	-0.04	0.00	0.03
100%	-0.11	-0.04	0.02	-0.02	-0.02	0.01	0.02
105%	-0.19	-0.09	0.01	-0.01	0.00	0.01	0.01
110%	-0.24	-0.11	0.01	0.00	0.01	0.01	-0.01
120%	0.07	0.01	0.06	0.05	0.04	0.00	-0.04
130%	1.13	0.51	0.21	0.12	0.04	-0.03	-0.09

Notes. K represents the strike as a percentage of the asset price. T denotes the time to maturity, expressed in years, whereas implied volatilities are expressed in percentage.

The results show that the LV model is able to replicate quite accurately the current implied volatility surface generated by the Heston (1993) model. The major discrepancies correspond to the short term with the LV model generating a bit more implied volatility skew. On the other hand, the VG exhibits huge differences in the short term since the model generates implied volatility skews which are much more pronounced that those corresponding to the Heston (1993) model. Conversely, in the long term the VG model generates flatter skews than those implied by the stochastic volatility framework.

Table 4: Differences between the implied volatilities of table 1 and the implied volatilities generated by the VG model.

K/T	0.25	0.5	1	1.5	2	2.5	3
70%	-8.67	-3.43	-0.67	0.15	0.44	0.63	0.87
80%	-6.37	-2.75	-0.77	-0.09	0.18	0.46	0.70
90%	-3.82	-1.98	-0.80	-0.29	0.01	0.30	0.51
95%	-2.34	-1.54	-0.79	-0.37	-0.05	0.23	0.42
100%	-0.61	-1.03	-0.75	-0.42	-0.11	0.15	0.33
105%	1.53	-0.42	-0.68	-0.45	-0.16	0.08	0.25
110%	4.02	0.31	-0.57	-0.46	-0.21	0.01	0.16
120%	3.80	2.09	-0.24	-0.42	-0.28	-0.13	0.00
130%	2.31	2.91	0.28	-0.31	-0.34	-0.25	-0.16

Notes. K represents the strike as a percentage of the asset price. T denotes the time to maturity, expressed in years, whereas implied volatilities are expressed in percentage.

Note that the existence of differences between the implied volatility surface of table 1 and the implied volatility surface generated by the VG model can lead to significant differences in the pricing of exotic options that are not necessary related to the existence of model risk. In this sense, to discriminate the part of the pricing errors that comes from the existence of model risk, we will use the following argument. Given the ability of the local volatility model to fit quite accurately different implied volatility surfaces, we will also calibrate this model to the implied volatility surface generated by the VG model. We call this specification as the local volatility calibrated to Variance Gamma (LVVG). Analogously, the specification of the local volatility model calibrated to the implied volatility surface of table 1 is denoted as local volatility calibrated to market data (LVMD). The model risk associated with the LV model, for a given exotic option, can be then calculated as the difference between the market price (the one generated using the Heston (1993) model) and the LVMD price. On the other hand, we calculate the model risk associated with the VG model as follows:

$$\begin{aligned} \textit{Model risk VG} &= \textit{LVMD price - Market Price} + \textit{VG price - LVVG price} \\ &= \textit{Model risk LV} + \textit{VG price - LVVG price} \end{aligned} \tag{11}$$

where the *VG price* is the price obtained for the exotic option with the specification of the VG model calibrated to the implied volatility surface of table 1. In this sense, if the price under the VG model is the same as the price obtained with the LV model calibrated to the implied volatility surface generated by the VG model, then we consider that the VG model and the LV model have the same model risk.

Table 5 displays the RMSE corresponding to the pricing errors associated with the LVVG specification corresponding to the implied volatility surface of table 4 for the VG model. As in the case of table 2, the table provides information about the total RMSE and the RMSE associated with at-the-money options, out-of-the-money puts and out-of-the-money calls.

Table 5: Root mean squared error (RMSE) corresponding to the pricing errors associated with the LVVG specification for the VG model implied volatility surface of table 4.

	LVVG specification
Total RMSE	0.038%
RMSE at-the-money options	0.043%
RMSE out-of-the-money puts	0.028%
RMSE out-of-the-money calls	0.047%

Note that, once again, the adjustment results are fairly good. The great ability of the LV model to replicate the current implied volatility surface is one of the main reasons of its popularity between practitioners.

3 Exotic options and model risk

This section presents the exotic options considered in this study to measure the existence of model risk and analyzes its mean features and price sensitivities. In particular, we focus on the model risk associated with cliquet options, as well as barrier options. As said previously, both types of options are quite common in structured products.

3.1 Barrier options

Barrier options have a payoff that is contingent on the underlying asset reaching some specified level before expiry, called the barrier. In particular, we will focus on the model risk associated with an up-and-out call and a down-and-out put. These options are standard European options if the barriers is not touched. On the other hand, if the underlying asset reaches the barrier, then the contract becomes worthless. The main reason for considering these two options is that their gamma changes sign depending on the evolution of the underlying asset. This feature makes these options quite sensitive to second order factors such as the existence of volatility in the volatility and, hence, complicates its valuation.

3.1.1 Up-and-out call

An up-and-out call (UOC) has a knockout barrier above the current spot price. If the barrier is at or below the strike price, the UOC is worthless and, hence, we only have to consider barriers set above the strike. Formally, the time $t=0$ price of an UOC with strike K, barrier H and maturity $t=T$, under the probability measure Q, is given by:

$$\begin{aligned} UOC_0(K,H,T) &= P(0,T)\,E_Q\left[(S_T-K)^+\mathbf{1}_{(M_T<H)}\right] \\ M_T &= \max_{0\le t\le T}(S_t) \qquad H>K \end{aligned} \tag{12}$$

Under the assumptions of the Black-Scholes (1973) model (BS model), it is possible to set the following statement, called the reflection principle[5]:
Let $A(S_t,t)$ be a solution of the BS equation

$$\frac{\partial A_t}{\partial t}+\frac{S_t^2\delta^2}{2}\frac{\partial A_t^2}{\partial S_t^2}+(r-q)\,S_t\frac{\partial A_t}{\partial S_t}=rA_t \tag{13}$$

where δ represents the implied BS volatility. If we define $\lambda=\frac{1}{2}-\frac{(r-q)\delta^2}{4}$ then for any constant H:

$$B(S_t,t)=\left(\frac{S_t}{H}\right)^{2\lambda}A\left(\frac{H^2}{S_t},t\right) \tag{14}$$

is also a solution of equation (13). The reflection principle allows us to find closed form solutions for the price and the sensitivities of barrier options in terms of the prices and sensitivities of European options. In this sense, it is possible to express the price of the UOC as follows:

$$\begin{aligned} UOC_0(K,H,T) &= C_0^{BS}(S_0,K,T)-C_0^{BS}(S_0,H,T)-(H-K)\,DC_0^{BS}(S_0,H,T) \\ &-\left(\frac{S_0}{H}\right)^{2\lambda}\left[C_0^{BS}\left(\frac{H^2}{S_0},K,T\right)-C_0^{BS}\left(\frac{H^2}{S_0},H,T\right)-(H-K)\,DC_0^{BS}\left(\frac{H^2}{S_0},H,T\right)\right] \end{aligned} \tag{15}$$

[5] For a proof of the reflection principle see Appendix A.

where $C_0^{BS}(S_0, K, T)$ is the time $t = 0$ price of a European call with strike price K and maturity $t = T$, when the price of the underlying asset is equal to S_0, under the assumptions of the BS model. The value of this option is given by:

$$C_0^{BS}(S_0, K, T) = P(0,T)\left[F_{0,T}\Phi(h) - K\Phi\left(h - \delta\sqrt{T}\right)\right]$$

where $\Phi(.)$ is the standard normal cumulative distribution function and where:

$$h = \frac{1}{\delta\sqrt{T}}\left[\ln\left(\frac{F_{0,T}}{K}\right) + \frac{1}{2}\delta^2 T\right]$$

On the other hand, $DC_0^{BS}(S_0, K, T)$ denotes the time $t = 0$ price of a digital call paying a monetary unit if the asset price is above the strike K at expiration of the option, under the assumptions of the BS model. The price of this option can be expressed as follows:

$$DC_0^{BS}(S_0, K, T) = P(0,T)\Phi\left(h - \delta\sqrt{T}\right)$$

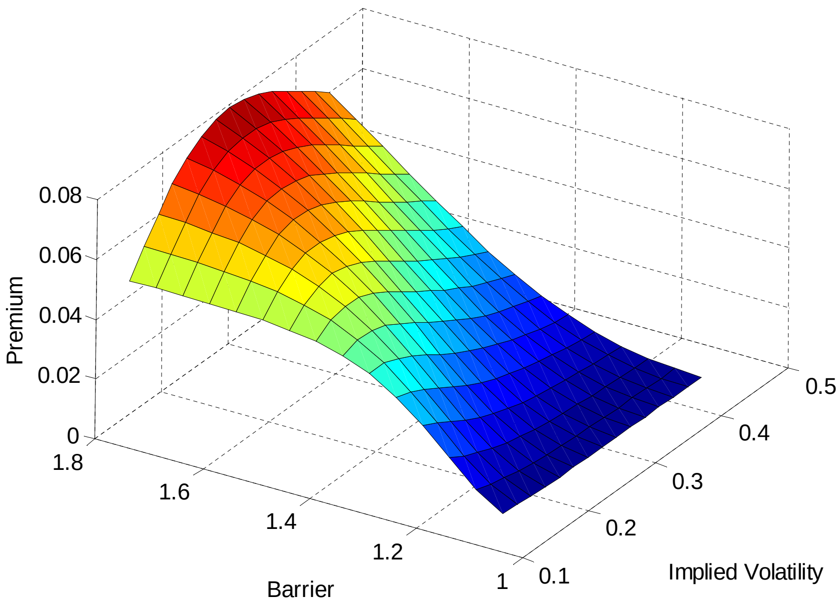

Figure 2: Prices of an UOC with strike at-the-money and maturity equal to one year, calculated using equation (15) with the following parameters values: $r = 1.40\%$ and $q = 4.35\%$. The barrier levels and the option prices are expressed as a percentage of the asset price.

From expression (15) we have that when the spot price coincides with the barrier level, the value of the replicating portfolio equals zero and, therefore, it can be canceled without cost.

Note that the assumptions of the BS model lead to a flat implied volatility surface. Hence, the formula of equation (15) is not considering the existence of implied volatility skew in the valuation of barrier options. In the case of an UOC option, the lower the volatility corresponding to the barrier compared with the volatility associated with the strike, the lower the probability of reaching the barrier and, therefore, the higher the price of the UOC option. Note that the VG model, as well as the LV model takes into account the existence of volatility skew.

On the other hand, figure 2 illustrates the prices of an UOC with strike at-the-money and maturity equal to one year, under the assumptions of the BS model, for different barrier levels and implied volatilities. The figure shows that the price of the option exhibits significant convexity with respect to the level of volatility and, hence, is quite sensitive to the existence of volatility in the volatility. Therefore, to perform a correct valuation of this kind of options a stochastic volatility framework is required. This simple example shows the importance of considering the existence of second order factors such as the existence of volatility skew and the volatility of volatility in the valuation of UOC.

3.1.2 Down-and-out put

A down-and-out put (DOP) has a knockout barrier below the current spot price. If the barrier is at or above the strike price, the DOP is worthless and, hence, we only have to consider barriers set below the strike. The holder of the option has a good return if the underlying asset stays above the barrier but finishes near to it. Formally, the time $t=0$ price of an DOP with strike K, barrier H and maturity $t=T$, under the probability measure Q, is given by :

$$\begin{aligned} DOP_0\left(K,H,T\right) &= P\left(0,T\right)E_Q\left[\left(K-S_T\right)^+\mathbf{1}_{\left(N_T>H\right)}\right] \\ N_T &= \min_{0\leq t\leq T}\left(S_t\right) \qquad H<K \end{aligned} \tag{16}$$

Under the assumptions of the BS model it is possible to use the reflection principle once again to express the price of the DOP as follows:

$$\begin{aligned} DOP_0\left(K,H,T\right) &= P_0^{BS}\left(S_0,K,T\right)-P_0^{BS}\left(S_0,H,T\right)+\left(H-K\right)DP_0^{BS}\left(S_0,H,T\right) \\ &-\left(\frac{S_0}{H}\right)^{2\lambda}\left[P_0^{BS}\left(\frac{H^2}{S_0},K,T\right)-P_0^{BS}\left(\frac{H^2}{S_0},H,T\right)+\left(H-K\right)DP_0^{BS}\left(\frac{H^2}{S_0},H,T\right)\right] \end{aligned} \tag{17}$$

where $P_0^{BS}(S_0,K,T)$ is the time $t=0$ price of a European put with strike price K and maturity $t=T$, when the price of the underlying asset is equal to S_0, under the assumptions of the BS model. Analogously, $DP_0^{BS}(S_0,K,T)$ represents the time $t=0$ price of a digital put paying a monetary unit if the asset price is below the strike K at expiration of the option. It is well known that prices of these options are given, respectively, by the following expressions:

$$\begin{aligned} P_0^{BS}(S_0,K,T) &= P(0,T)\left[K\Phi\left(\delta\sqrt{T}-h\right)-F_{0,T}\Phi(-h)\right] \\ DP_0^{BS}(S_0,K,T) &= P(0,T)\Phi\left(\delta\sqrt{T}-h\right) \end{aligned}$$

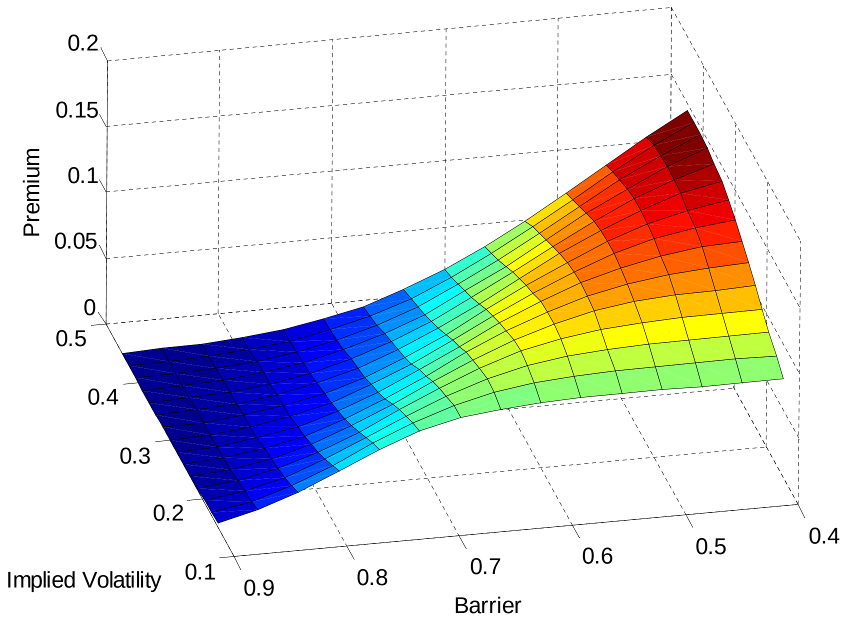

Figure 3: Prices of an DOP with strike at-the-money and maturity equal to one year, calculated using equation (17) with the following parameters values: $r=1.40\%$ and $q=4.35\%$. The barrier levels and the option prices are expressed as a percentage of the asset price.

Figure 3 exhibits the prices of a DOP with maturity within one year and strike at-the-money as a function of the barrier level and the implied volatility under the assumptions of the BS model. As in the case of the UOC, the DOP exhibits high sensitivity to the volatility of volatility and, as said previously, both options have a gamma that changes sign. But these options have different skew sensitivity. In the case of the DOP, the higher the volatility corresponding to the barrier compared with the volatility associated with the strike, the higher the probability of reaching the barrier and, therefore, the lower the price of the DOP. Therefore, the flatter the skew the higher the price of the DOP and the lower the price of the UOC.

3.2 Cliquet option with floors and caps

Cliquet options are structures where the investment horizon is divided into a series of equally spaced periods (months, quarters or years) and the underlying performance in each of these periods is used to determine the outcome of investment at the maturity of the structure. In this sense, the result is given by the sum of positive and negative performances corresponding to each period. These performances may be subject to local caps and /or local or global floors. The fact that performance is measured on each date considering the level of the underlying asset in the previous period and it is not measured with respect to its initial level, as in standard options, makes the cliquet options particularly sensitive to the future evolution of the implied volatility surface. Hence, two models that generate the same prices for European options, can lead to very different prices for cliquet options if the dynamic of the implied volatility associated with each model is different.

Formally, it is possible to express the time $t=0$ price of a cliquet option, denoted as $Cliquet_0$, with maturity $t=T$, under the risk neutral probability measure Q, as:

$$Cliquet_0 = P\left(0,T\right) E_Q \left[\max \left[\sum_{t=1}^{T} \max \left\{ \min \left(\frac{S_t}{S_{t-1}} - 1, l_c \right), l_f \right\}, G_f \right] \right] \tag{18}$$

where l_c is the local cap, l_f denotes the local floor and, finally, G_f represents the global floor. The existence of these local caps and floors makes the price of the cliquet option quite dependent on the forward implied volatility skew.

As pointed out by Gatheral (2006), cliquet options offer investors the possibility of attaining a high coupon and are very popular because, although these structures are complex from the valuation and hedging perspective, they are not so hard to explain to a retail investor.

Unlike what happens with barrier options, in the case of cliquet structures there are no analytical formulas even under the assumptions of the BS model and, therefore, even in this case we must resort to numerical methods to calculate the prices of these options.

4 Pricing tests

In this section, we compare the ability of the VG model and the LV model in the pricing of the exotic options presented in the previous section. Recall that both models account for the existence of volatility skew but none of them take into account the volatility of volatility. Moreover, since the paths corresponding to the underlying asset price under the VG model consist of a large number of small jumps, the VG model leads to an incomplete market and, therefore, to the existence of many equivalent martingale measures. To highlight the importance of the volatility skew for the correct valuation of this type of options, we also consider the prices obtained using the BS model.

4.1 Barrier options

To price the barrier options under the Heston (1993) model, the VG model and the LV model, we use Monte Carlo simulations with daily time steps and 80.000 trials and we apply the antithetic variable technique described in Boyle (1977) to reduce the variance of the estimates. For the Heston (1993) model we implement a Milstein discretization scheme as described in Gatheral (2006). In the case of the BS model, we consider the analytic expressions of equations (15) and (17).

Table 6 reports the model errors associated with the LV model, the VG model and the BS model for European barrier options with maturity equal to two years. The table shows the prices generated by the Heston (1993) model, whereas for the rest of the models it displays the pricing errors. These pricing errors are calculated using equation (11) for the VG model, as well as for the LV model, and are expressed as a percentage of the prices obtained under the Heston (1993) model. In the case of the BS model, we consider two possibilities. The price denoted *BS atm* is calculated using the implied volatility corresponding to the at-the-money strike. Conversely, the price denoted *BS barrier* exhibits the price obtained using the implied volatility associated with the barrier. In both cases the model error is calculated as the BS price minus the Heston price, expressed as a percentage of the Heston price.

Table 6: Pricing errors associated with barrier options

Panel A Up-and-out call *Model/Barrier*	120%	130%	140%	150%
Heston	0.53%	1.73%	3.45%	5.38%
LVMD	-30.19%	-20.23%	-12.17%	-8.18%
VG	48.19%	26.15%	17.53%	8.22%
Black-Scholes atm	-52.63%	-48.76%	-44.47%	-40.97%
Black-Scholes barrier	-43.45%	-36.23%	-30.19%	-27.34%

Panel B Down-and-out put *Model/Barrier*	50%	60%	70%	80%
Heston	8.35%	4.76%	1.92%	0.48%
LVMD	3.95%	6.30%	13.54%	16.67%
VG	11.38%	18.49%	28.65%	35.42%
BS atm	39.28%	43.82%	40.47%	19.30%
BS barrier	15.91%	8.16%	8.24%	-2.75%

Notes. The table shows the Heston price as a percentage of the initial asset price and the percentage error corresponding to the LV model, the VG model and the BS model. For the LV model and the VG model the percentage error is calculated using equation (11) and it is expressed as a percentage of the stochastic volatility price. The price denoted *BS atm* is calculated using the implied volatility corresponding to the at-the-money strike, whereas the price denoted *BS barrier* exhibits the price obtained using the implied volatility associated with the barrier. In both cases the model error is calculated as the BS price minus the Heston price and it is expressed as a percentage of the Heston price. Barrier levels are expressed as a percentage of the initial asset price and the maturity is two years.

The pricing errors associated with the LV model and with the BS model for the UOC are consistent with the empirical findings of Hull and Suo (2002). The results show that prices of the UOC under the LV model, as well as under the BS model are mispriced when compared with the stochastic volatility framework. The absolute magnitude of the price errors is higher for lower barrier levels. Note that the BS model does not account for the existence of volatility skew. In this case, an increase in the implied volatility means that the probability of reaching the barrier is higher and, therefore, the price of the UOC will be lower. This is the reason why the UOC is more expensive when we use the implied volatility associated with the barrier under the BS model. On the other hand, the prices of the UOC are overpriced under the VG model when compared with the stochastic volatility framework. Regarding the DOP, all the premiums are overpriced under the three models when compared with the Heston (1993) stochastic volatility model. The only exception is the *BS barrier* price for a barrier equal to 80% of the initial asset price. The reason is that, since the volatility of the barrier is higher than the at-the-money volatility and the barrier is relatively close to the strike, the probability of reaching the barrier is very high under the BS model and, therefore, the price of the option is lower than when using the stochastic volatility model. Recall that both the LV model, as well as the VG model, takes into account the existence of implied volatility skew but they do not consider the volatility of the volatility, which is particularly important for the correct valuation of barrier options. Overall, the pricing discrepancies are higher in the case of the VG model than in the case of the LV model for the barrier options considered.

4.2 Cliquet options

To price the cliquet options we use Monte Carlo simulations with weekly time steps and 80.000 trials and we apply the antithetic variable technique to reduce the variance of the estimates. For the Heston (1993) model we implement a Milstein discretization scheme, which increases the accuracy of the simulations relative to the Euler discretization scheme.

Table 7: Pricing errors associated with semiannual cliquet options

Model/Maturity	2 years	3 years
Heston	5.39%	7.84%
LVMD	-3.34%	-5.48%
VG	7.42%	7.78%
BS	-9.28%	-10.20%

Notes. For both maturities the local floor is zero and the local cap is 3%. The global floor is 2% for the option with maturity equal to two years, whereas it is 3% for the three years option. See also notes for table 6.

Table 7 shows the model errors associated with the LV model, the VG model, as well as the BS model[6] for semiannual cliquet options. Two maturities are considered: two years and three years. As in the previous case, the table shows the prices generated by the Heston (1993) model, whereas for the rest of the models it displays the percentage pricing errors. For both maturities the local cap is 3% and the local

[6]In this case, we consider the at-the-money implied volatility associated with the maturity of the option.

floor is zero. The global floor is 3% for the three years maturity and 2% for the maturity equal to two years. The results show that the priced generated by the VG model are overpriced relative to the stochastic volatility model. Conversely, the prices obtained under the BS model and under the LV model are mispriced when compared with the Heston (1993) model. Once again, the highest differences in absolute value correspond to the BS model, whereas the LV model exhibits the lowest pricing discrepancies.

Table 8: Pricing errors associated with an annual cliquet option with maturity three years

Heston	7.61%
LVMD	-3.68%
VG	1.97%
BS	-9.33%

Notes. The local floor is zero, the local cap is 6% and the global floor is 3%. See also notes for table 6.

Note that, for all the models considered, the pricing errors increase with the time to maturity. To investigate how the pricing errors depend on the frequency of the cliquet option, table 8 reports the pricing discrepancies corresponding to an annual cliquet option with three years maturity, local floor equal to zero, local cap equal to 6% and a global floor of 3%. Recall that both cliquet options with three years maturity, of tables 7 and 8, offer a maximum coupon equal to 18% and a guaranteed coupon equal to 3%. Therefore, the stochastic volatility prices associated with these options are similar and, in this sense, the percentage pricing errors are comparable. The comparison of the results of tables 7 and 8, corresponding to the options with maturity equal to three years, shows that the pricing errors increase with the frequency of the cliquet option. This fact is related to the sensitivity of this option with respect to the forward skew. Since short-term skews are steeper than the skews corresponding to longer maturities, the pricing discrepancies are higher the lower the frequency corresponding to the cliquet option. Note that, for the annual cliquet option, the lowest error in absolute value corresponds to the VG specification.

As we said in the introductory section, Hull and Suo (2002) conjectured that the model risk associated with the LV model was a function of the degree of path dependence of the exotic option being tested, defined as the number of times that the asset price has to be observed to calculate the payoff. The higher the degree of path dependence, the worse the model is expected to perform. The results corresponding to the semiannual and annual cliquet options are in line with this argument. Nevertheless, to investigate if the degree of path dependence is the main element in determining the model risk associated with a certain option, we also consider the pricing of a daily Asian option with maturity equal to two years. The time $t = 0$ price of this option can be expressed as follows:

$$AC_0(K,T) = P(0,T)\,E_Q\left[\left(\frac{1}{T}\sum_{t=1}^{T} S_t - K\right)^+\right]$$

Since, in this case, the asset price has to be observed daily to determine the payoff of the Asian call, if the degree of path dependence was the main element in determining the model risk, we should obtain higher pricing errors in the valuation of the Asian option than in the valuation of the cliquet options.

Table 9: Pricing errors associated with a daily Asian option with strike at-the-money and maturity equal to two years.

Heston	6.16%
LVMD	-0.49%
VG	0.00%
BS	6.01%

See notes for table 6.

Table 9 exhibits the the model errors associated with the LV model, the VG model, as well as the BS model for a daily Asian option with strike at-the-money and maturity equal to two years, calculated using Monte Carlo simulations with daily time steps and 80.000 trials. In this case, although the price of the underlying asset must be observed daily, there are not significant pricing errors associated with the LV model and with the VG model. The main reason is that the Asian option is not as sensitive to second order effects such as the volatility of volatility or the forward skew, that are determinant for the correct valuation of barrier and cliquet options. Therefore, we can conclude that for a given product, such as the cliquet option, the higher the degree of path dependence, the higher the model risk. But a product with higher path dependence that another product can display lower model risk if it exhibits less sensitivity to the volatility of volatility or the forward skew.

5 Conclusion

Equity options markets exhibit a persistent negative implied volatility skew. A number of models have been proposed to account for this volatility skew. Traders typically calibrate these models to plain vanilla instruments and use them to price exotic options at the same time. But, as said previously, the key point of the model risk is that different models can yield the same price for European options but, at the same time, very different prices for exotic options depending on their assumptions corresponding to the evolution of the underlying asset price and its volatility.

The growth experimented in recent years in both the variety and volume of structured products, which are trade in the over-the-counter market, implies that banks and other financial institutions have become increasingly exposed to model risk. In this article we have focused on the model risk associated with the local volatility (LV) model and with the Variance Gamma (VG) model. We have assumed that market data are generated by a stochastic volatility framework and we have analyzed the ability of these models to replicate the market prices of vanilla options and the model risk associated with each model in the valuation of barrier options, as well as cliquet options.

The results show that the LV model performs better than the VG model in terms of its ability to match the market prices of European options. This may be the reason why the LV model is more popular between practitioners. Regarding the model risk, although both models take into account the existence of skew, they do not consider properly other effects such as the volatility of volatility or the forward skew, which are quite relevant for the correct valuation of barrier options and cliquet options. Therefore, both models are subject to significant pricing errors when compared with the stochastic volatility framework. However, the LV model performs a bit better than the VG model in terms of pricing errors for the options considered in this article.

For a given product, such as the cliquet options, the model risk is an increasing function of the time to maturity, as well as of the degree of path dependence. Following Hull and Suo (2002), we define the degree of path dependence as the number of times that the asset price must be observed to calculate the payoff. Finally, the results also show that a product with higher path dependence that another product can display lower model risk if it exhibits less sensitivity to second order effects such as the volatility of volatility or the forward skew.

A The reflection principle

Let us define $E_t = \frac{H^2}{S_t}$. Differentiating equation (14) with respect to the spot price yields:

$$\frac{\partial B_t}{\partial S_t} = \left(\frac{S_t}{H}\right)^{2\lambda}\left[2\lambda\frac{A_t}{S_t} - \frac{\partial A_t}{\partial E_t}\frac{E_t}{S_t}\right] \tag{19}$$

whereas the second derivative of B_t with respect to S_t is given by:

$$\begin{aligned}\frac{\partial^2 B_t}{\partial S_t^2} &= \left(\frac{S_t}{H}\right)^{2\lambda}\left[2\lambda\left(2\lambda-1\right)\frac{A_t}{S_t^2} - 4\lambda\frac{\partial A_t}{\partial E_t}\frac{E_t}{S_t^2}\right] \\ &\quad + \left(\frac{S_t}{H}\right)^{2\lambda}\left[\frac{\partial^2 A_t}{\partial E_t^2}\frac{E_t^2}{S_t^2} + 2\frac{\partial A_t}{\partial E_t}\frac{E_t}{S_t^2}\right]\end{aligned} \tag{20}$$

Let us assume that B_t satisfies the BS differential equation (13)

$$\frac{\partial B_t}{\partial t} + \frac{S_t^2\delta^2}{2}\frac{\partial B_t^2}{\partial S_t^2} + \left(r-q\right)S_t\frac{\partial B_t}{\partial S_t} = rB_t$$

If this is true, A_t must verify the BS differential equation as well. Substituting equations (19) and (20) into the previous expression, we obtain:

$$\begin{aligned}&\frac{\delta^2 E_t}{2}\left[2\lambda\left(2\lambda-1\right)\frac{A_t}{E_t} - 4\lambda\frac{\partial A_t}{\partial E_t} + \frac{\partial^2 A_t}{\partial E_t^2}E_t + 2\frac{\partial A_t}{\partial E_t}\right] \\ &\quad + \frac{\partial A_t}{\partial t} + \left(r-q\right)\left[2\lambda A_t - \frac{\partial A_t}{\partial E_t}E_t\right] = rA_t\end{aligned}$$

Simplifying the previous equation yields:

$$\begin{aligned}&\frac{\delta^2}{2}\frac{\partial^2 A_t}{\partial E_t^2}E_t^2 + E_t\frac{\partial A_t}{\partial E_t}\left[\delta^2\left(1-2\lambda\right) - \left(r-q\right)\right] \\ &\quad + \lambda A_t\left[\delta^2\left(2\lambda-1\right) + 2\left(r-q\right)\right] + \frac{\partial A_t}{\partial t} = rA_t\end{aligned}$$

where:

$$\begin{aligned}\delta^2\left(1-2\lambda\right) - \left(r-q\right) &= \left(r-q\right) \\ \delta^2\left(2\lambda-1\right) + 2\left(r-q\right) &= 0\end{aligned}$$

Therefore, we have that:

$$\frac{\partial A_t}{\partial t} + \frac{\delta^2}{2}\frac{\partial^2 A_t}{\partial E_t^2}E_t^2 + E_t\frac{\partial A_t}{\partial E_t}\left(r-q\right) = rA_t$$

and, hence, A_t is also a solution of the BS differential equation (13).

References

[1] Andersen, L. and Brotherton-Ratcliffe, R. (1998) The equity option volatility smile: an implicit finite-difference approach. *Journal of Computational Finance*, 1, pp. 5-37.

[2] Basel Committee on Banking Supervision. (2010) Basel III: a global regulatory framework for more resilient banks and banking systems. Bank for International Settlements. Available at: http://www.bis.org/publ/bcbs189.htm.

[3] Black, F. and Scholes, M.S. (1973) The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, pp. 637-654.

[4] Boyle, P. P. (1977) Options: a Monte Carlo approach. *Journal of Financial Economics*, 4, pp. 323- 338.

[5] Brown, G. and Randall, C. (1999) If the skew fits. *Risk*, pp. 62-65.

[6] Carr, P., Hogan, A. and Stein, H. (2007) Time for a Change: The Variance Gamma Model and Option Pricing, Working Paper Quantitative Finance Bloomberg LP.

[7] Derman, E. and Kani, I. (1994) The volatility smile and its implied tree, Quantitative Strategies Research Notes, Goldman Sachs, New York.

[8] Derman, E. and Wilmott, P. (2009) Perfect models, imperfect world. *Business Week*, 12th Jan, pp. 59.

[9] Dupire, B. (1994) Pricing with a smile. *Risk*, 7, pp. 18-20.

[10] Gatheral, J. (2006) *The volatility surface. A practitioner's guide.* Hoboken, New Jersey: John Wiley & Sons.

[11] Heston, S. L. (1993) A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6, pp. 327-343.

[12] Hull, J. and Suo, W. (2002) A methodology for assessing model risk and its application to the implied volatility function model. *Journal of Financial and Quantitative Analysis*, 37, pp. 297-318.

[13] 21. Hull, J., & White, A. (1987) The pricing of options on assets with stochastic volatilities. *Journal of Financial and Quantitative Analysis*, pp. 281-300.

[14] Joshi, M.S. (2003) *The concepts and practice of mathematical finance*. Cambridge: Cambridge University Press.

[15] Lee, R. (2004) The moment formula for implied volatility at extreme strikes. *Mathematical Finance*, 14, pp. 469-480.

[16] Madan, D., Carr, P. and Chang, E. (1998) The Variance Gamma process and option pricing. *European Finance Review*, 2, pp. 79-105.

[17] Madan, D. and Seneta, E. (1990) The variance gamma (V.G.) model for share market returns, *Journal of Business*, 63 (4), pp. 511–524.

[18] Marabel, J. (2010) Fitting the skew with an analytic local volatility function. Paper presented at symposium, Finance Forum, XVIII annual meeting of the Spanish Finance Association (AEFIN), November 2010. Elche, Spain.

[19] Merton, R. C. (1976) Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, pp. 124-144.

[20] Rubinstein, M. (1994) Implied binomial trees. *Journal of Finance*, 49, pp. 771-818.

[21] Wilmott, P. (2006) *Paul Wilmott on quantitative finance*. 2nd ed. West Sussex: John Wiley & Sons.