SOME BOUNDS OF LOSS IN CLASSICAL SEMIDETERMINISTIC IMMUNIZATION

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ABSTRACT

Since Samuelson, Redington and Fisher and Weil, duration and immunization are very important topics in bond portfolio analysis from both a theoretical and a practical point of view. Many results have been established, especially in semi-deterministic framework. As regards, however, the loss may be sustained, we do not think that the subject has been investigated enough, except for the results found in the wake of the theorem of Fong and Vasicek. In this paper we present some results relating to the limitation of the loss in the case of local immunization for multiple liabilities.

Keywords: immunization, duration, term structure, dispersion.

0. INTRODUCTION.

Since Macaulay (1938), Samuelson (1945) and Redington (1952), duration and immunization are very important topics in bond portfolio analysis from both a theoretical and a practical point of view. Both Samuelson and Redington showed, independently, that if the Macaulay's duration of assets and liabilities are equal and the balance-sheet constraint is fulfilled, a portfolio is protected against a small shift of level in a flat term structure. Fisher and Weil (1971) proved that it is possible to globally immunize a portfolio with a single liability against any level shift in the spot rate curve. Afterwards many others results are established by various authors, cfr. bibliografy. As regards, however, the loss may be sustained, we do not think that the subject has been investigated enough, except for the results found in the wake of the theorem of Fong and Vasicek (1984); cfr. also Montrucchio and Peccati (1991), Moretto (2008). In this paper we present some results relating to bounds of the loss in the case of local immunization for multiple liabilities.

1. APPROXIMATION'S FORMULAS.

Let $\delta_t(s) = \delta(t,s)$ be the force of interest at time $t \geq 0$ with maturity s such that $t \leq s \leq \omega$ and

(1.1) $\quad x = (x_1, x_2, \dots, x_n)/(t_1, t_2, \dots, t_n)$ and $y = (y_1, y_2, \dots, y_n)/(t, t_2, ., t_n)$

the assets and liabilities cash flows, where t_k is the maturity of x_k (y_k), $k = 1, .., n$ and $\omega \geq t_n - t \geq \cdots \geq t_1 - t$. Moreover, we denote by

(1.2) $\quad v(t,s) = \exp\left(-\int_t^s \delta_t(u)du\right), \qquad \delta_t = \delta_t(s), s \in [0,\omega],$

the structure of the prices of the unitary zero coupon bond with maturity s, $t \leq s \leq \omega$, that represents the financial market at time t and we consider the distribution of the times to maturity of the assets and liabilities cash flows $\tau_{x,t}$ $[\tau_{y,t}]$ or

(1.3)
$$\tau_{x,t}[\tau_{y,t}] = \begin{cases} t_1 - t & p_{x,1}(\delta_t)\ [p_{y,1}(\delta_t)] \\ t_2 - t & p_{x,2}(\delta_t)\ [p_{y,2}(\delta_t)] \\ \dots\dots & \dots\dots\dots\dots \\ t_n - t & p_{x,n}(\delta_t)\ [p_{y,n}(\delta_t)] \end{cases}$$

where

(1.4)
$$p_{x,k}(\delta_t) = \frac{x_k v(t,t_k)}{\sum_{h=1}^n x_h v(t,t_h)} \quad \left[p_{y,k}(\delta_t) = \frac{y_k v(t,t_k)}{\sum_{h=1}^n y v(t,t_h)} \right], \ k = 1, \dots\dots, n.$$

We put, also, for $k = 0, \dots\dots, m$

(1.5) $A_k = \sum_{h=1}^{n} p_{x,h}(\delta_t)\ (t_h - t)^k,\ B_k = \sum_{h=1}^{n} p_{y,h}(\delta_t)\ (t_h - t)^k\ ,\ S_k = A_k - B_k$

and we note that A_k *and* B_k are the durations of various orders of cash flows x and y, while S_k is the difference of k-order durations. If the yield curve at time t^+, immediately after t, assumes the form:

(1.6) $$\delta(t^+, s) = \delta(t, s) + Y, \quad Y \in \boldsymbol{R}$$

where $\boldsymbol{R}$ is set of real numbers, the net post-shift value is

(1.7) $$W(Y) = \sum_{h=1}^{n} (x_h - y_h) v(t, t_{h)}\, e^{-Y(t_h - t)}$$

and its derivates are:

(1.8) $$W^{(k)}(Y) = (-1)^k \sum_{h=1}^{n} (x_h - y_h) v(t, t_h)(t_h - t)^k\, e^{-Y(t_h - t)}$$

therefore at $Y = 0$ we have

(1.9) $$W^{(k)}(0) = (-1)^k (A_0 A_k - B_0 B_k), \qquad k = 1, \dots n$$

and in particular

(1.10) $$W(0) = A_0 - B_0\ , \quad W'(0) = -(A_0 A_1 - B_0 B_1).$$

By n-order Taylor's formula with Lagrange's remainder, we have

(1.11) $$W(Y) = (A_0 - B_0) - (A_0 A_1 - B_0 B_1) Y + \frac{1}{2}(A_0 A_2 - B_0 B_2) Y^2 + \cdots +$$

. $$+ \frac{(-1)^n}{n!}(A_0 A_n - B_0 B_n) Y^n +$$

$$+ \frac{(-1)^{n+1}}{(n+1)!}\left[\sum_{h=1}^{n} (x_h - y_h) v(t, t_h)(t_h - t)^k\, e^{-\xi_Y (t_h - t)}\ \right] Y^{n+1}$$

. $$\xi_Y \ opportune\ such\ that \qquad |\xi_Y| \le |Y|,\ Y \in \boldsymbol{R}.$$

If the portfolio $\pi = \{x, -y\}$ satisfies the balance-sheet constraint

(1.12) $$A_0 = B_0\,.$$

(1.11) gives the formula

(1.14) $$W(Y) = -A_0 S_1 Y + \frac{1}{2} A_0 S_2 Y^2 \ + \cdots + \ \frac{(-1)^n}{n!} A_0 S_n\, Y^n +$$

. $$+ \ \frac{(-1)^{n+1}}{(n+1)!}\left[\sum_{h=1}^{n} (x_h - y_h) v(t, t_h)(t_h - t)^h\, e^{-\xi_Y (t_h - t)}\right] Y^{n+1},$$

. ξ_Y *opportune such that* $|\xi_Y| \leq |Y|,\ Y \in \boldsymbol{R},$

and if, also, duration constraint

(1.15) $$A_1 = B_1$$

is fulfilled, we obtain

(1.16) $$W(Y) = \frac{1}{2}A_0S_2Y^2 + \cdots + \frac{(-1)^n}{n!}A_0S_n\,Y^n +$$

. $$+\frac{(-1)^{n+1}}{(n+1)!}\left[\sum_{h=1}^{n}(x_h - y_h)v(t,t_h)(t_h - t)^h\, e^{-\xi_Y(t_h-t)}\right]Y^{n+1},$$

. ξ_Y *opportune such that* $|\xi_Y| \leq |Y|,\ Y \in \boldsymbol{R}.$

2. CLASSICAL RESULTS BASED ON II° ORDER'S APPROXIMATION.

For n=2 the formula (1.13) gives

(2.1) $$W(Y) = (A_0 - B_0) - (A_0A_1 - B_0B_1)Y + \frac{1}{2}(A_0A_2 - B_0B_2)Y^2 +$$

$$-\frac{1}{3!}\left[\sum_{h=1}^{n}(x_h - y_h)v(t,t_h)(t_h - t)^3\, e^{-\xi_Y(t_h-t)}\right]Y^3$$

. ξ_Y *opportune such that* $|\xi_Y| \leq |Y|,\ Y \in \boldsymbol{R}.$

then for $|Y|$ enough small we obtain the following approximation

(2.2) $$W(Y) \cong (A_0 - B_0) - (A_0A_1 - B_0B_1)Y + \frac{1}{2}(A_0A_2 - B_0B_2)Y^2$$

or, if (1.12) is true,

(2.3) $$W(Y) \cong -A_0S_1Y + \frac{1}{2}A_0S_2Y^2$$

and, if also (1.15) is true, we obtain

(2.4) $$W(Y) \cong \frac{1}{2}A_0S_2Y^2$$

Therefore

(2.5) $$W(Y) \geq 0 \quad \Leftrightarrow \quad S_2 \geq 0$$

if $|Y|$ is enough small. Then, the following classical result is true:

Theorem 2.1. *If the portfolio* $\pi = \{x,\ -y\}$ *satisfies in* t *the constraints* (1.12*) and (1.15), it results that:*

A) *the net value post-shift of portfolio is*

$$\textbf{(2.6)} \qquad \boldsymbol{W(Y) = \frac{1}{2}A_0 S_2 Y^2 - \frac{1}{3!}\left[\sum_{h=1}^{n}(x_h - y_h)v(t,t_h)(t_h - t)^3\, e^{-\xi_Y(t_h-t)}\right]Y^3}$$

. $Y \in \boldsymbol{R},\ \xi_Y$ *opportune such that* $|\xi_Y| \le |Y|$,

B) *if the duration constraint (1.15) is fulfilled and* $|Y|$ *is enough small, then the approximation*

$$\textbf{(2.7)} \qquad \boldsymbol{W(Y) \cong \frac{1}{2}A_0 S_2 Y^2}$$

is true, and for (2.7)

C) ***(Redington's theorem)*** *the portfolio* $\pi = \{x,\ -y\}$ *is immunized by infinitesimal shifts if and only if*

$$\text{(2.8)} \qquad S_2 \ge 0 \quad \text{or} \quad A_2 \ge B_2,$$

ie the cash flow dispersion for assets exceed the cash flow dispersion for liabilities.

Proof. A) is obtained directly substituting (1.12) and (1.15) in(2.1). B) follows obviously from A). Finally, C) follows from (2.3) and (2.5).
In the case C) of previous theorem 1.1, if $A_2 > B_2$ immunization is valid also for shifts Y with $|Y|$ enough small and it is possible to determine interval width of immunization, cfr. Montrucchio and Peccati (1991), Amato (1995).

3. AN INEQUALITY FOR THE NET ASSET VALUE POST-SHIFT.

Let us narrow our focus to the remainder in (2.6):

$$\text{(3.1)} \qquad R_3(Y) = \frac{1}{3!}\left[\sum_{h=1}^{n}(x_h - y_h)v(t,t_h)(t_h - t)^3\, e^{-\xi_Y(t_h-t)}\right]Y^3$$

$Y \in \boldsymbol{R},\ \xi_Y$ opportune such that $|\xi_Y| \le |Y|$. We can raise the rest on the nonrestrictive assumption that $|Y| \le c \le 1$, $A_0 = B_0$, $A_1 = B_1$

$$\text{(3.2)} \qquad |\,R_3(Y)| \le \frac{1}{3!}\left[\sum_{h=1}^{n}|(x_h - y_h)|v(t,t_h)(t_h - t)^3\, e^{|Y|(t_h-t)}\right]|Y|^3$$

and bearing in mind that $\omega \ge t_n - t \ge \cdots \ge t_1 - t$, we obtain

$$\text{(3.3)} \qquad |\,R_3(Y)| \le \frac{c^3\omega^3}{3!}e^{c\omega}\sum_{h=1}^{n}|x_h - y_h|v(t,t_h)$$

Considering the value in t

(3.4) $$V(t,|x-y|,\delta_t)=\sum_{h=1}^{n}|x_h-y_h|v(t,t_h)$$

of the bond

(3.5) $$|x-y|=(|x_1-y_1|,|x_2-y_2|,\ldots,|x_n-y_n|)/(t_1,t_2,\ldots,t_n),$$

we obtain the following result

Lemma 3.1. *Under the assumptions* $t_n-t\le\omega$, $|Y|\le c\le 1$, $A_0=B_0$, $A_1=B_1$, *we have*

(3.6) $$|\boldsymbol{R_3(Y)}|\le\frac{c^3\omega^3}{3!}\boldsymbol{e^{c\omega}}\ \ \boldsymbol{V(t,|x-y|,\delta_t)}.$$

Also, substituting (3.6) in (2.6), we have the following theorem:

Theorem 3.2. *Let* $\delta_t(s)=\delta(t,s)$ *be the force of interest at time* $t\ge 0$ *with maturity s such that* $t\le s\le\omega$ *and* $x=(x_1,x_2,\ldots,x_n)/(t_1,t_2,\ldots,t_n)$ *and* $y=(y_1,y_2,\ldots,y_n)/(t,t_2,.,t_n)$ *the assets and liabilities cash flows and assume that* $t_n-t\le\omega$, $|Y|\le c\le 1$, $A_0=B_0$, $A_1=B_1$. *Then, for the net asset post-shift value of portfolio* $\pi=\{x,\ -y\}$, *the following inequality is valid*

(3.7) $$\boldsymbol{W(Y)}\ge\frac{1}{2}\boldsymbol{A_0S_2Y^2}-\frac{c^3\omega^3}{3!}\boldsymbol{e^{c\omega}}\ \ \boldsymbol{V(t,|x-y|,\delta_t)}.$$

We note that the quantity at second member of inequality can be negative as well as the amount $W(Y)$. Therefore we obtain the following

Theorem 3.3. *Under the assumptions of the theorem 3.2, if* $W(Y)<0$ *the loss* $L(Y)=|W(Y)|$ *satisfies the inequality*

(3.8) $$\boldsymbol{L(Y)}\le\left|\frac{1}{2}\boldsymbol{A_0S_2Y^2}-\frac{c^3\omega^3}{3!}\boldsymbol{e^{c\omega}}\ \ \boldsymbol{V(t,|x-y|,\delta_t)}\right|,$$

i.e. the loss $L(Y)$ *not exceed the amount at second member.*

Also, we have

Corollary 3.2. *Under the assumptions of the theorem 3.2, if also* $A_2=B_2$ *for the net asset post-shift value of portfolio* π *the following inequality is valid*

(3.9) $$\boldsymbol{W(Y)}\ge-\frac{c^3\omega^3}{3!}\boldsymbol{e^{c\omega}}\ \ \boldsymbol{V(t,|x-y|,\delta_t)},$$

and

(3.10) $$\boldsymbol{L(Y)}\le\frac{c^3\omega^3}{3!}\boldsymbol{e^{c\omega}}\ \ \boldsymbol{V(t,|x-y|,\delta_t)}.$$

Now, we remark that the term of control of loss it depend by the value of the bond (3.5), which is much smaller as x and y are close (perfect matching). We also give some examples of the values considered in the inequality; if $V(0, |x-y|, \delta_t) \leq 0.1$ then

$$c\omega \leq 0.5 \Rightarrow |R_3(Y)| \leq 0.00343485 , \quad c\omega \leq 0.1 \Rightarrow |R_3(Y)| \leq 0.00018460 .$$

4. CONCLUSION

The established inequalities are important from a practical point of view, because you can set bounds on losses from investments in semi-deterministic framework. They seem relevant in a perspective of risk control, as required by the recent Basel agreements.

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